ABSOLUTELY CONTINUOUS SPECTRUM OF SCHRÖDINGER OPERATORS WITH SLOWLY DECAYING AND OSCILLATING POTENTIALS

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ABSTRACT. The aim of this paper is to extend a class of potentials for which the absolutely continuous spectrum of the corresponding multidimensional Schrödinger operator is essentially supported by $[0, \infty)$. Our main theorem states that this property is preserved for slowly decaying potentials provided that there are some oscillations with respect to one of the variables.

1. INTRODUCTION

In this paper we prove that the absolutely continuous spectrum of a class of Schrödinger operators $-\Delta + V$ in $L^2(\mathbb{R}^d)$, $d \geq 3$ is essentially supported by $[0, \infty)$. This means that the spectral projection corresponding to any subset of positive Lebesgue measure is not zero.

In particular, our results can be applied to the following classes of long range potentials:

Example I. Let $d = 3$ and

$$V(x, y, z) = v_1(x)v_2(y)v_3(z),$$

where $v_j(t) = \sum_n c_n^{(j)} \theta_j(t - x_n(j))$, $\theta_j \in C^\infty(\mathbb{R})$ and $\sum_n |c_n^{(j)}|^4 < \infty$, $j = 1, 2, 3$. Assume that functions θ_2 and θ_3 are of finite support and the support of the Fourier transform of θ_1 is finite and does not intersect a neighbourhood of zero. Now, choose the sequences $x_n(j)$ so that $v_j \in L^4(\mathbb{R})$ and the potentials v_j, $j = 1, 2, 3$, are sparse (see the papers of A.Kiselev, Y.Last and B.Simon[13] and also D.Pearson [30]). Note that in this case for each $j = 1, 2, 3$ there is a set of a positive Lebesgue measure containing only singular continuous spectrum of the corresponding one-dimensional operators with the potential v_j. However, it follows from our Theorem 2.1 that the absolutely continuous spectrum of the operator $-\Delta + V$ in $L^2(\mathbb{R}^3)$ is essentially supported by $[0, \infty)$.

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Example II. The same statement is true for a 3-dimensional operator with the potential

$$V(x,y,z) = v_1(x)v(y,z), \quad v \in L^4(\mathbb{R}^2).$$

Here v_1 is a so-called Wigner-von Neumann potential

$$v_1(x) = \sum_{j=1}^{m} c_j \frac{\sin(\omega_j x) + o(1)}{1 + |x|^{p_j}}, \qquad |x| \to \infty,$$

where $\omega_j > 0$, $p_j > 1/4$, $c_j \in \mathbb{R}$, $m \in \mathbb{N}$, is a function whose Fourier transform vanishes on a small interval containing zero. For example, one can consider

$$v_1(x) = \text{Re} \sum_{j=1}^{m} \int_{\omega_j}^{1+\omega_j} \frac{C_j \exp(ikx)}{(k-\omega_j)^{1-p_j}}\, dk,$$

with appropriate constants C_j.

Our work differs from the results obtained in the scattering theory, where the existence of wave operators is proved either for the potentials $|V(x)| \leq C(1+|x|)^{-1-\varepsilon}$, $\varepsilon > 0$ or $|V(x)| + |\nabla V(x)|(1+|x|) \leq C(1+|x|)^{-\varepsilon}$. In this case the corresponding a.c. property of the spectrum is a byproduct of much stronger results on the unitary equivalence of the operators $-\Delta$ and $-\Delta + V$.

As in our previous paper [15] the multidimensional case is reduced to a problem for a one-dimensional second order elliptic integro-differential operator. The "potential" type term appears to be a dissipative Fredholm integral operator depending on the spectral parameter. Such an operator might have poles appearing in an operator version of the so-called first Buslaev-Faddeev-Zakharov (BFZ) trace formula. Their contribution appears with the "right" sign and therefore can be ignored.

There are two new elements compared with [15]. One of them suggests new "spectrally local" Lieb-Thirring inequalities for the $3/2$ moments of the negative eigenvalues of Schrödinger operators (compare with O.Safronov [20]). Before applying this result we need an argument from A.Laptev and T.Weidl [16] lifting the corresponding eigenvalue estimates for their $1/2$-moments to $3/2$-moments by using an induction with respect to dimension. This argument forces us to consider the problem starting from dimension $d \geq 3$. The second new element is concerned with a parallel consideration of a couple of Schrödinger operators with potentials V and $-V$. This leads to the cancellation of the term $\int_0^\infty \int_{\mathbb{S}^{d-1}} V\, d\theta dr$ appearing in the BFZ first trace formula.

Note that the first result based on Buslaev-Faddeev-Zakharov trace formulae for the study of the a.c. properties of the spectrum of one-dimensional Schrödinger operators was suggested in the paper by P.Deift and R.Killip [10]. Their theorem gave a natural generalization of the results obtained by by M.Christ, A.Kiselev and C.Remling in [7], M.Christ, A.Kiselev [8] and C.Remling[19]. R.Killip [12] was first in proving a "local" one-dimensional result. That is if $\hat{V} \in L^2(2a, 2b)$, $a > 0$, and $V \in L^3(\mathbb{R})$, then the absolutely continuous spectrum fills the interval (a^2, b^2).

Our theorems requires only $V \in L^4(\mathbb{R})$ rather than the condition $V \in L^3(\mathbb{R})$. Note that the second L^2-condition (2.10) on the Fourier transform of V with respect to one of variables near the origin, becomes interesting if there are cancellations provided by oscillations of the potential V near infinity.

There is extensive literature concerning the properties of the spectrum of oscillating potentials starting from the classical Wigner-von Neumann construction [31], see also M.Skriganov [29] and H.Behncke [3], [4]. Some examples of oscillating potentials with respect to the radial variable were given in M.Reed and B.Simon [21], vol.3 Ch XI.

Our Theorems 2.1 and 2.2 are applied to a class of potentials described in terms of the Fourier transform with respect to one of the variables. Some related results for a class of Schrödinger operators with anisotropic behaviour of potentials at infinity were considered in the paper by V.G.Deich, E.L.Korotjaev and D.R. Yafaev [9].

This article is a natural development of our previous paper [15]. For the sake of completeness we recall the arguments of Section 3-4 and 8 from [15] which become in this text Sections 4-6 and 10 respectively.

2. THE MAIN RESULTS

Let us consider a Schrödinger operator $-\Delta + V$ in $L^2(\mathbb{R}^d)$, $d \geq 3$, where

$$V \in L^\infty(\mathbb{R}^d), \quad V(x) \to 0, \text{ as } |x| \to \infty. \tag{2.1}$$

Let $\hat{V}$ be the Fourier transform of V with respect to the first variable

$$\hat{V}(\xi, y) = \int_{\mathbb{R}} e^{-i\xi s} V(s, y)\, ds, \quad x = (s, y) \in \mathbb{R}^d. \tag{2.2}$$

Theorem 2.1. *Let $d \geq 3$ and let V be a real valued function on $\mathbb{R}^d$ obeying (2.1) and let for some $\delta > 0$*

$$\int_{\mathbb{R}^d} V^4(x)\, dx < \infty, \qquad \int_{\mathbb{R}^{d-1}} \Big(\int_{-\delta}^{\delta} |\hat{V}(\xi, y)|^2 \, d\xi \Big) dy < \infty.$$

Then the absolutely continuous spectrum of the operator $-\Delta + V$ *is essentially supported by* $[0, \infty)$.

The latter theorem gives some qualitative information about the absolutely continuous spectrum of Schrödinger operators. The next result is related to more delicate properties of the a.c. spectrum. It provides some quantitative characteristics of the spectral measure which is a multidimensional continuous analog of the well-known Szegő condition for orthogonal polynomials and Jacobi matrices (compare with [15]).

Let Ω_1 be the unit ball in $\mathbb{R}^d$, $\partial\Omega_1 = \mathbb{S}^{d-1}$, and V be a real valued function on $\mathbb{R}^d \setminus \Omega_1$. We consider the operator H in $L^2(\mathbb{R}^d \setminus \Omega_1)$ with the Dirichlet boundary conditions on $\mathbb{S}^{d-1}$

$$Hu = H_0 u + Vu = -\Delta u + Vu, \qquad u|_{\partial\Omega_1} = 0. \tag{2.3}$$

Let us assume for the sake of simplicity that there is $c_1 > 1$ such that

$$V + \frac{\alpha_d}{|x|^2} = 0 \quad \text{for} \quad 1 < |x| < c_1, \tag{2.4}$$

where $\alpha_d = \frac{(d-1)^2}{4} - \frac{d-1}{2}$. Let $E_H(\omega)$, $\omega \subset \mathbb{R}$, be the spectral projection of the operator H. We construct a measure μ on the real line such that for spherically symmetric functions f

$$(E_H(\omega)f, f) = \int_\omega |F(\lambda)|^2 d\mu(\lambda), \quad \omega \subset \mathbb{R}_+ = (0, \infty), \tag{2.5}$$

where

$$F(\lambda) = \frac{1}{k}\int_0^{c_1} \sin(k(r-1)) f(r)\, r^{(d-1)/2} dr, \quad \operatorname{supp} f \subset \{x : 1 < |x| < c_1\}. \tag{2.6}$$

and $k^2 = \lambda > 0$.

Let us extend V by zero into Ω_1 and then define $\hat{V}$ as in (2.2).

The following theorem is the main result of the paper.

Theorem 2.2. *Let* $d \geq 3$ *and let* V *be a real valued function on* $\mathbb{R}^d \setminus \Omega_1$ *obeying* (2.1) *and* (2.4). *Let*

$$\int_{\mathbb{R}^d\setminus\Omega_1} V^4(x)\, dx < \infty, \qquad \int_{\mathbb{R}^{d-1}} \Big(\int_{-\delta}^{\delta} |\hat{V}(\xi, y)|^2\, d\xi\Big) dy < \infty \tag{2.7}$$

for some $\delta > 0$. *Then*

$$\int_0^\infty \frac{\log(1/\mu'(t))\, dt}{(1 + t^{3/2})\sqrt{t}} < \infty, \tag{2.8}$$

where μ *is defined in* (2.5)*. If* (2.4) *is satisfied then* (2.8) *is equivalent to*

$$\int_0^\infty \frac{\log\left(\frac{d}{d\lambda}(E_H(\lambda)f,f)\right)\,d\lambda}{(1+\lambda^{3/2})\sqrt{\lambda}} > -\infty, \tag{2.9}$$

for any bounded spherically symmetric function $f \neq 0$ *with* $\mathrm{supp} f \subset \{x : 1 < |x| < c_1\}$.

Remark 1. The inequality (2.8) guarantees that the a.c. spectrum of H is essentially supported by $[0,\infty)$, since $\mu' > 0$ almost everywhere and gives quantitative information about the measure μ.
Remark 2. If $d = 1$, then the conditions (2.7) do not provide existence of the absolutely continuous spectrum on R_+. This is confirmed by examples of sparse potentials constructed in [13]. The validity of Theorem 2.2 in dimension $d = 2$ remains open.
Remark 3. The equivalence of (2.8) and (2.9) follows from the fact that if F is defined as in (2.6), then the function $(1+\lambda^2)^{-1}\log(|F(\lambda)|)$ is in $L^1(\mathbb{R}_+)$ see, for example, P. Koosis [14] (section IIIG2).
Remark 4. When proving Theorem 2.2 we use the projection operator P_0 on the spherical function Y_0 which leads us to a scalar one-dimensional problem (4.2) with an operator valued potential Q_z. Had we used instead of P_0 the projection $\sum_{j=1}^n P_j$, where P_j are projections on the spherical functions Y_j, then we would have obtained the corresponding system of one-dimensional equations with an operator valued potential which could be treated similarly. This would imply that the multiplicity of the a.c. spectrum is not smaller than n. Since n is arbitrary, we obtain the a.c. spectrum is of infinite multiplicity.

Denote by $\tilde V$ the Fourier transform of the V with respect to all variables

$$\tilde V(\eta) = \int_{\mathbb{R}^d} V(x) e^{-i(\eta,x)}\,dx.$$

Theorem 2.3. *Let* $d \geq 3$ *and let* V *be a real valued function on* $\mathbb{R}^d \setminus \Omega_1$ *obeying* (2.1) *and* (2.4)*. Let*

$$\int_{\mathbb{R}^d\setminus\Omega_1} V^4(x)\,dx < \infty, \qquad \int_{\mathbb{R}^d} \frac{|\tilde V(\eta)|^2}{1+|\eta|}\,d\eta < \infty. \tag{2.10}$$

Then the statements (2.8) *and* (2.9) *of Theorem 2.2 hold true.*

Remark 5. Theorems 2.2 and 2.3 do not allow us to resist the temptation of formulating the following conjecture:
Let $V \in L^4(\mathbb{R}^d)$, $\Omega \subset \mathbb{R}^d$ be an open domain. Assume that $\tilde V \in L^2(\Omega)$. Then the set $\{\lambda :\ \lambda = |\eta|^2,\ 2\eta \in \Omega\}$ is contained in the essential support of the absolutely continuous spectrum of the Schrödinger operator $-\Delta + V$.

3. Estimates for the discrete spectrum

Throughout the paper, $T_\pm$ denotes the positive and negative part of a self adjoint operator T, i.e. $2T_\pm = |T| \pm T$. Denote by $\mathfrak{S}_p$, $p > 0$ the standard Neumann-Schatten classes of compact operators

$$\mathfrak{S}_p = \{T \ : \ \text{tr}\,(T^*T)^{p/2} < \infty\}.$$

Consider a one dimensional Schrödinger operator $J = -\frac{d^2}{dx^2} + V(x)$ in $L^2(\mathbb{R})$ with a real valued potential $V \in C_0^\infty(\mathbb{R})$.

Theorem 3.1. *Let $V \in C_0^\infty(\mathbb{R})$. Then for any $\delta > 0$*

$$\text{tr}\,\Big(-\frac{d^2}{dx^2} + V(x)\Big)_-^{3/2} \leq C\Big(\int_{\mathbb{R}} V^4 dx + \int_{-\delta}^{\delta} |\hat{V}(\xi)|^2\, d\xi\Big), \tag{3.1}$$

where the constant $C = C(\delta, ||V||_\infty)$ and $\hat{V}(\xi) = \int \exp(-i\xi x) V(x)\, dx$.

Proof. For each $T \in \mathfrak{S}_1$, one can define a complex-valued function $\det(1+T)$, so that

$$|\det(1+T)| \leq \exp(\|T\|_{\mathfrak{S}_1}).$$

For $T \in \mathfrak{S}_4$ one defines

$$\det{}_4(1+T) = \det((1+T)e^{-T+T^2/2-T^3/3}). \tag{3.2}$$

It is proved in [27], Section 9, Theorem 9.2(b), that there is a constant $c > 0$ such that

$$|\det{}_4(1+T)| \leq \exp(c\|T\|_{\mathfrak{S}_4}^4), \quad c > 0. \tag{3.3}$$

Note that if J_0 is the operator $J_0 = -\frac{d^2}{dx^2}$ in $L^2(\mathbb{R})$, then

$$\lim_{\varepsilon \to 0} |\det\Big(I + V(J_0 - (\lambda \pm i\varepsilon))^{-1}\Big)| \geq 1.$$

In order to prove Theorem 3.1 we need the following auxiliary statement:

Lemma 3.1. *Let $V(x)$ be a smooth real valued function of finite support. For every $\delta > 0$ there is a constant $C = C(\delta, ||V||_\infty)$ such that for all z:*

$$|z - ||V||_\infty| = ||V||_\infty + \delta^2$$

it holds

$$|\log\det{}_4(I + V(J_0 - z)^{-1})| \leq \frac{C}{|\text{Im}\, z|^4} ||V||_{L^4}^4. \tag{3.4}$$

Proof. Let $z = \lambda + i\eta$, where λ and η are real. One can repeat the arguments of R.Killip and B.Simon, Proposition 5.2 [11], in order to show that

$$
\begin{aligned}
&|\frac{d}{d\eta}\log\det{}_4(I+V(J_0-z)^{-1})| = \\
(3.5)\qquad &|\text{tr}\,(i[(J_0-z)^{-1}V]^4(J-z)^{-1})| \leq \\
&\quad ||V(J_0-z)^{-1}||^4_{\mathfrak{S}_4}||(J-z)^{-1}||.
\end{aligned}
$$

On the other hand,

$$\lim_{\eta\to\infty}\det{}_4(I+V(J_0-z)^{-1}) = 1.$$

Therefore the estimate (3.4) follows from (3.5) by the Fundamental Theorem of Calculus. $\square$

It was established in [12] that for $z = k^2$, $k \in \mathbb{R}$,

$$-\text{Re tr}\,(V(J_0-z)^{-1})^2 = \frac{|\hat{V}(2k)|^2}{2k^2}.$$

Therefore for $z = k^2$, $k \in \mathbb{R}$,

(3.6)

$$
\begin{aligned}
0 &\leq \log|\det(I+V(J_0-z)^{-1})| + \log|\det(I-V(J_0-z)^{-1})| \\
&= -\text{Re tr}\,(V(J_0-z)^{-1})^2 + \log|\det{}_4(I+V(J_0-z)^{-1})| \\
&+ \log|\det{}_4(I-V(J_0-z)^{-1})| = \frac{|\hat{V}(2k)|^2}{2k^2} + \log|\det{}_4(I+V(J_0-z)^{-1})| \\
&\qquad + \log|\det{}_4(I-V(J_0-z)^{-1})|.
\end{aligned}
$$

Let now

$$\sigma(k) = k^2(k^2-\delta^2)^4, \qquad \mathfrak{L} = \{k:\ |k^2 - ||V||_\infty| = ||V||_\infty + \delta^2\}.$$

Then applying (3.4) we obtain

$$\Big|\int_{\mathfrak{L}} \log\det{}_4(I+V(J_0-k^2)^{-1})\sigma(k)\,dk\Big| \leq C||V||^4_{L^4}. \tag{3.7}$$

Now let $i\beta_j(V)$ be the zeros of $\log\det_4(I+V(J_0-k^2)^{-1})$ and let $\mathfrak{B}(k,V)$ be the Blaschke product

$$\mathfrak{B}(k,V) = \prod_j \frac{k - i\beta_j(V)}{k + i\beta_j(V)}.$$

Then
(3.8)

$$\mathrm{Re}\int_{-\delta}^{\delta}\log\det{}_4(I+V(J_0-z)^{-1})\sigma(k)\,dk$$

$$=\mathrm{Re}\int_{\mathfrak{L}}\log\det{}_4(I+V(J_0-z)^{-1})\sigma(k)\,dk-\mathrm{Re}\int_{\mathfrak{L}}\log(\mathfrak{B}(k,V))\sigma(k)\,dk.$$

Thus, combining the inequality (3.6) with the estimate (3.7) and the relation (3.8), we obtain

$$\sum_j f(\beta_j(V))+\sum_j f(\beta_j(-V))\le C\Big(\int_{\mathbb{R}}V^4dx+\int_{-\delta}^{\delta}|\hat V(2\xi)|^2\,d\xi\Big), \tag{3.9}$$

where

$$f(t)=\mathrm{Re}\int_{\mathfrak{L}}\log\Big(\frac{k-it}{k+it}\Big)\sigma(k)\,dk,\quad t>0.$$

Integrating by parts and using the fact that σ is even we obtain

$$f(t)=\mathrm{Re}\int_{\mathfrak{L}}\Big(\frac{1}{k-it}-\frac{1}{k+it}\Big)\Xi(k)\,dk=\mathrm{Re}\int_{|k|=2t}\frac{1}{k-it}\Xi(k)\,dk=2\pi\Xi(it),$$

where

$$\Xi(k)=\int_0^k\sigma(\tau)\,d\tau.$$

This implies

$$f(t)\ge\frac{2\pi\delta^8}{3}t^3.$$

The proof is complete. □

4. The beginning of the proof of Theorem 2.2

In this section we reduce problem (2.3) to a one-dimensional problem with an operator valued potential. Such a reduction has been already used in [15].

Assume that $V\in C_0^\infty$ and introduce polar coordinates $(r,\theta),\ x=r\theta\in\mathbb{R}^d$, $\theta\in\mathbb{S}^{d-1}$. Denote by $\{Y_j\}_{j=0}^\infty$ the orthonormal in $L^2(\mathbb{S}^{d-1})$ basis of (real) spherical functions, i.e. eigenfunctions of the Laplace-Beltrami operator $-\Delta_\theta$, and let P_j be the orthogonal projection given by

$$P_ju(r,\theta)=Y_j(\theta)\int_{\mathbb{S}^{d-1}}Y_j(\theta')u(r,\theta')\,d\theta'.$$

Clearly P_0u depends only on r. Denote

$$V_1=P_0VP_0,\quad H_{0,1}=P_0H_0P_0,$$

$$V_{1,2}=P_0V(I-P_0),\quad V_{2,1}=V_{1,2}^*,$$

$$V_2 = (I - P_0)V(I - P_0), \quad H_{0,2} = (I - P_0)H_0(I - P_0).$$

Then the operator $H - z$ can be represented as a matrix:

$$H - z = \begin{pmatrix} H_{0,1} + V_1 - z & V_{1,2} \\ V_{2,1} & H_{0,2} + V_2 - z \end{pmatrix},$$

and the equation

$$(H - z)u = P_0 f, \quad \text{Im}\, z \neq 0,$$

is equivalent to

$$(H_{0,1} + T_z - z)P_0 u = P_0 f, \quad (H_{0,2} + V_2 - z)^{-1} V_{2,1} P_0 u = (P_0 - I)u. \tag{4.1}$$

Here the operator T_z is defined by

$$T_z = V_1 - V_{1,2}(H_{0,2} + V_2 - z)^{-1} V_{2,1}$$

on $L^2((1,\infty), r^{d-1}\, dr)$.

By using the unitary operator from $L^2((1,\infty), dr)$ to $L^2((1,\infty), r^{d-1}\, dr)$,

$$Uu(r) = r^{-(d-1)/2} u,$$

we reduce (4.1) to the problem for the following one-dimensional Schrödinger operator in $L^2(1,\infty)$

$$L_z u(r) = -\frac{d^2 u}{dr^2} + Q_z u, \quad u \in L^2(1,\infty),\ u(1) = 0, \tag{4.2}$$

where

$$Q_z = V_1 + \frac{\alpha_d}{r^2} - V_{1,2}(U^* H_{0,2} U + V_2 - z)^{-1} V_{2,1}, \quad \alpha_d = \frac{(d-1)^2}{4} - \frac{d-1}{2}.$$

We are going to approximate the problem by the corresponding problem with a smooth compactly supported potential V and the term α_d/r^2 substituted by $\zeta_\varepsilon(r)\alpha_d/r^2$, where $\zeta_\varepsilon/r^2 \to 1/r^2$, as $\varepsilon \to 0$, in the both spaces $L^1(1,\infty)$ and $L^2(1,\infty)$ and $\zeta_\varepsilon \in C_0^\infty(1,+\infty)$. The same should be done with the term $\Delta_\theta u/r^2$, i.e. it should be substituted by $\zeta_\varepsilon(r)\Delta_\theta u/r^2$.

So when approximating the problem we always assume that

$$Q_z = V_1 + \zeta_\varepsilon(r)\frac{\alpha_d}{r^2} - V_{1,2}(S_\varepsilon + V_2 - z)^{-1} V_{2,1}, \tag{4.3}$$

where

$$S_\varepsilon u = -\frac{d^2 u}{dr^2} - \zeta_\varepsilon(r)\frac{\Delta_\theta u}{r^2}, \quad u(1,\theta) = 0. \tag{4.4}$$

According to (4.1) we obtain

$$P_0(H - z)^{-1} P_0 = U(L_z - z)^{-1} U^*. \tag{4.5}$$

We see also that if $\operatorname{supp} V \cup \operatorname{supp} \zeta_\varepsilon(|\cdot|) \subset \{x \in \mathbb{R}^d :\ c_1 < |x| < c_2\}$, $c_1 > 1$, then for the operator (4.3) we have

$$Q_z = Q_z \chi = \chi Q_z,$$

where χ is an operator of multiplication by the characteristic function of the interval (c_1, c_2), $c_1 > 0$. It is important for us that Q_z is an analytic operator valued function of z with a negative imaginary part in the upper half plane and which has a positive imaginary part in the lower half plane.

5. GREEN'S FUNCTION.

In sections 5-7 we assume that V is not a potential but the operator PVP, $P = \sum_{j=0}^{n} P_j$, which approximates V for large n. It can be interpreted as an operator of multiplication by a matrix valued function of r. In this case the function V_1 remains the same as before. Since P_j are projections on real spherical functions, this matrix is real. Recall that the factor $1/r^2$ in front of $-\Delta_\theta$ and α_d is also substituted by a smooth compactly supported function ζ_ε/r^2.

Let us consider the equation

$$-\frac{d^2}{dr^2}\psi(r) + (Q_z\psi)(r) = z\psi(r), \quad r \geq 1,\ z \in \mathbb{C}, \tag{5.1}$$

with Q_z given by (4.3) and let $\psi_k(r)$ be the solution of the equation (5.1) satisfying

$$\psi_k(r) = \exp(ikr), \quad k^2 = z,\ \operatorname{Im} k > 0,\ \forall r > c_2.$$

Then this solution also satisfies the following "integral" equation

$$\psi_k(r) = e^{ikr} - k^{-1}\int_r^\infty \sin k(r-s)(Q_z\psi_k)(s)\,ds. \tag{5.2}$$

In order to describe the properties of $\psi_k(r)$ we systematically use the following analytic Fredholm theorem (see, for example, M.Reed and B.Simon [21], Theorem VI.14 or D.Yafaev Ch.I, Section 8):

Theorem 5.1. *Let $D \subset \mathbb{C}$ be an open connected set and let $\mathfrak{T}(k)$ be an analytic operator valued function on D such that $\mathfrak{T}(k)$ is a compact operator in a Hilbert space for each $k \in D$. Then*
(1) *either $(I - \mathfrak{T}(k))^{-1}$ exists for no $k \in D$,*
(2) *or $(I - \mathfrak{T}(k))^{-1}$ exists for all $k \in D \setminus D_0$, where D_0 is a discrete subset of D. In this case $(I - \mathfrak{T}(k))^{-1}$ is meromorphic in D with possible poles belonging to D_0.*

We first apply this theorem in order to prove the statement which is quite standard in the resonance theory.

Lemma 5.1. *The operator Q_z has a meromorphic continuation into the second sheet of the complex plane.*

Proof. Let S_ε be the same operator as in (4.4) and let $\tilde S = -d^2/dr^2$ be an operator in $L^2((1,\infty), PL^2(\mathbb{S}^{d-1}))$ with the Dirichlet boundary condition at 1. Let $\phi \in C_0^\infty(\mathbb{R}_+)$ be a function which is identically equal to one on the support of the matrix-function V and ζ_ε. Then

$$\phi(S_\varepsilon + V_2 - z)^{-1}\phi = \Big(I + \phi(\tilde S - z)^{-1}\Big(V_2 + \zeta_\varepsilon \frac{\Delta_\theta}{r^2}\Big)\Big)^{-1} \phi(\tilde S - z)^{-1}\phi.$$

Obviously both operators $\phi(\tilde S - z)^{-1}(V_2 + \zeta_\varepsilon \frac{\Delta_\theta}{r^2})$ and $\phi(\tilde S - z)^{-1}\phi$ have an analytic continuation into the second sheet of the complex plane through the positive semi-axis. By using Theorem 5.1 we obtain that the operator

$$\Big(I + \phi(\tilde S - z)^{-1}\Big(V_2 + \zeta_\varepsilon \frac{\Delta_\theta}{r^2}\Big)\Big)^{-1}$$

and thus the operator Q_z defined in (4.3) have meromorphic continuations into the second sheet of the complex plane. □

Let us now apply Theorem 5.1 to the operator

$$\mathfrak{T}(k)\psi(r) = -k^{-1}\int_r^\infty \sin k(r-s)(Q_z\psi)(s)\,ds$$

in $L^2(1, c_2)$. We conclude that the equation (5.2) is uniquely solvable for all k except perhaps a discrete sequence of points and that its solution ψ_k is a meromorphic with respect to k function with values in $L^2(1, c_2)$, in a neighbourhood of every Im $k \geq 0$, $k \neq 0$. Clearly

$$\psi_k(x) = a(k)e^{ikx} + b(k)e^{-ikx}, \quad 1 < x < c_1, \tag{5.3}$$

and therefore both $a(k)$ and $b(k)$ are meromorphic functions (even in some neighborhoods of points $k \neq 0$ of the real axis).

Consider the resolvent operator $R(z) = (L_z - z)^{-1}$, where L_z is defined in (4.2). If χ_{c_1} is the operator of multiplication by the characteristic function of $(1, c_1)$. Then $R(z)\chi_{c_1}$ is an integral operator with the kernel:

$$G_z(r,s) = \begin{cases} \frac{\psi_k(s)}{\psi_k(1)}\frac{\sin(k(r-1))}{k}, & \text{for } r < s < c_1, \\ \frac{\psi_k(r)}{\psi_k(1)}\frac{\sin(k(s-1))}{k}, & \text{for } s < \min\{c_1, r\}. \end{cases} \tag{5.4}$$

Indeed, assuming that supp$(f) \subset (1, c_1)$ we can easily check that the function

$$u(r) = \frac{1}{\psi_k(1)}\Big\{\int_r^\infty \frac{\sin(k(r-1))}{k}\psi_k(s)f(s)\,ds + \int_1^r \psi_k(r)\frac{\sin(k(s-1))}{k}f(s)\,ds\Big\}$$

satisfies the equation

$$-\frac{d^2}{dr^2}u(r) + (Q_z u)(r) - zu(r) = f(r), \quad r \geq 1,\ z \in \mathbb{C}, \tag{5.5}$$

and moreover $u(1) = 0$.

Here we should also mention that since $\psi_k(1)$ is meromorphic in k in a neighborhood of any $k \neq 0$, we conclude that $\psi_k(1) = 0$ only on a discrete subset of the closed upper half plane, having no accumulation points except perhaps zero.

6. Wronskian and properties of the M-function.

Let as in (4.3)

$$Q_z = V_1 - V_{1,2}(S_\varepsilon + V_2 - z)^{-1}V_{2,1}.$$

The function

$$M(k) = \frac{\psi_k'(1)}{\psi_k(1)}, \qquad \text{Im } k \geq 0, \tag{6.1}$$

is now well defined and called the Weyl M-function of the operator (5.1). Let us consider the Wronskian

$$W[\overline{\psi_k}, \psi_k](r) = \overline{\psi_k'}(r)\psi_k(r) - \overline{\psi_k}(r)\psi_k'(r). \tag{6.2}$$

Note that $\overline{\psi_k}$ satisfies the equation (5.1) with $Q_{\overline{z}}$ and $\overline{z}$ instead of Q_z and z. Since ψ_k is a solution of the equation (5.1) we find

$$\frac{d}{dr}W[\overline{\psi_k}, \psi_k](r) = (z-\overline{z})\overline{\psi_k}(r)\psi_k(r)+(Q_{\overline{z}}\overline{\psi_k})(r)\psi_k(r)-\overline{\psi_k}(r)(Q_z\psi_k)(r).$$

So we obtain

$$\pm\text{Im } \{W[\overline{\psi_k}, \psi_k](c_2) - W[\overline{\psi_k}, \psi_k](c_1)\} \geq 0, \quad \text{for } \pm \text{Im } z \geq 0+, \tag{6.3}$$

which means that for all real k we have the following inequality

$$\frac{k}{\text{Im } M(k)} \leq |\psi_k(1)|^2.$$

Moreover, if we represent the solution ψ_k for real k in the form

$$\psi_k(x) = a(k)e^{ikx} + b(k)e^{-ikx}, \quad x < c_1,$$

then it follows from (6.3) that

$$|a|^2 - |b|^2 \geq 1, \qquad k = \overline{k}.$$

Clearly

$$M(k) = \psi_k'(1)(\psi_k(1))^{-1} = ik(1 - \rho(k))(1 + \rho(k))^{-1},$$

where

$$\rho(k) := e^{-2ik}b(k)a(k)^{-1}.$$

The latter implies

$$\rho(k) = (ik - M(k))(ik + M(k))^{-1}.$$

Since $|a|^2 - |b|^2 \geq 1$ we obtain that for real k

$$|a(k)|^{-2} \leq 1 - |\rho(k)|^2 = \frac{4k\mathrm{Im}\, M}{|ik + M(k)|^2}.$$

Note that since Im $M \geq 0$, then for any $k > 0$ we have

$$|ik + M(k)|^2 = k^2 + |M|^2 + 2k\mathrm{Im}\, M \geq k^2$$

and therefore

$$|a(k)|^{-2} \leq 4k^{-1}\big(\mathrm{Im}\, M\big), \quad k > 0. \tag{6.4}$$

From (6.1) and (6.2) we obtain

$$\mathrm{Im}\, M(k) > 0 \quad \text{if } \mathrm{Im}\, k^2 > 0. \tag{6.5}$$

Thus, there are constants $C_0 \in \mathbb{R}$ and $C_1 \geq 0$ and a positive measure μ, such that

$$\int_{-\infty}^{\infty} \frac{d\mu(t)}{1+t^2} < \infty,$$

where

$$M(k) = C_0 + C_1 z + \int_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t}{1+t^2}\Big) d\mu(t), \quad k^2 = z. \tag{6.6}$$

Finally, note that $R(z) = P_0(U^*H_0U + V - z)^{-1}P_0$ and hence we can formally write that

$$M(k) = \frac{\partial^2}{\partial r \partial s} G_z(r,s)|_{(1,1)} = (P_0(U^*H_0U + V - z)^{-1}P_0\delta_1', \delta_1'),$$

where δ_1' is the derivative of the delta function $\delta(r-1)$. Let χ_{c_1} be the characteristic function of $(1, c_1)$. The representation (5.4) for the resolvent operator gives us the representation for the operator $\chi_{c_1}P_0E_{U^*H_0U+V}(\omega)P_0\chi_{c_1}$, where $E_{U^*H_0U+V}(\omega)$ is the spectral measure of $U^*H_0U + V$:

$$(P_0E_{U^*H_0U+V}(\omega)P_0f, f) = \int_\omega |F(\lambda)|^2 d\mu(\lambda) \tag{6.7}$$

and where

$$F(\lambda) = \frac{1}{k}\int_0^{c_1} \sin(k(r-1))f(r)\,dr, \quad \mathrm{supp} f \subset (1, c_1), \;\; k^2 = \lambda.$$

Since F is a boundary value of an analytic function, we obtain that $F(\lambda) \neq 0$ for a.e. λ. This means that $E_H(\omega) \neq 0$ if $\mu' > 0$ a.e. on ω.

7. Trace inequalities

Recall that we assume that V is not a potential but the operator $\sum_{j=0}^n P_j V \sum_{j=0}^n P_j$, which approximates V for large n. As before we substitute the term $-\Delta_\theta/r^2$ and α_d/r^2 on $(1,\infty)$ by a "compactly supported" approximations $-\zeta_\varepsilon(r)\Delta_\theta/r^2$ and $\zeta_\varepsilon(r)\alpha_d/r^2$, where $\zeta_\varepsilon \in C_0^\infty(1,\infty)$ and $\zeta_\varepsilon(r)/r^2 \to 1/r^2$ in $L^1(1,\infty)$ and $L^2(1,\infty)$ as $\varepsilon \to 0$. Then the coefficient $a(k)$ introduced in (5.3) will depend on ε and we shall write $a_\varepsilon(k)$ instead of $a(k)$. From (5.2) and (4.3) we find that

$$\exp(-ikr)\psi_k(r) = 1 - \frac{1}{2ik}\int_r^\infty (1-e^{2ik(s-r)})(\zeta_\varepsilon(s)\alpha_d/s^2 + V_1(s))\,ds + o(1/k)$$

and thus

(7.1)
$$a_\varepsilon(k) = \lim_{r\to-\infty} \exp(-ikr)\psi_k(r) = 1 - \frac{1}{2ik}\int (\zeta_\varepsilon(r)\alpha_d/r^2 + V_1)\,dr + o(1/k),$$

as $k \to \infty$. Now let $i\beta_m$ and γ_j be zeros and poles of $a_\varepsilon(k)$ in the open upper half plane. Note that $-\overline{\gamma_j}$ are also poles of $a_\varepsilon(k)$ (this will follow from (7.5)). We shall show in Proposition 7.1 that $\{-\beta_m^2\}$ are the eigenvalues of a certain self-adjoint operator of a Schrödinger type. Therefore we choose $\beta_m > 0$. Let $\mathfrak{B}$ be the corresponding Blaschke product

$$\mathfrak{B}(k) = \prod_m \frac{(k - i\beta_m)}{(k + i\beta_m)} \prod_j \frac{(k - \overline{\gamma_j})}{(k - \gamma_j)}.$$

Clearly $|\mathfrak{B}(k)| = 1$, $\overline{\mathfrak{B}(k)} = \mathfrak{B}(-k)$, $k \in \mathbb{R}$, and we obtain

(7.2)
$$\int_{-\infty}^{+\infty} \log(a_\varepsilon(k)/\mathfrak{B}(k))\,dk$$

$$= \frac{\pi}{2}\int (\zeta_\varepsilon(r)\alpha_d/r^2 + V_1(r))\,dr + 2\pi\Big(\sum \beta_m - \sum \operatorname{Im}\gamma_j\Big),$$

provided that for some integer $l \geq 0$ the coefficient $a_\varepsilon(k)$ has an expansion $a_\varepsilon(k) = \sum_{j\geq -l} c_j k^j$ at zero. The existence of such an expansion as well as the condition $|a_\varepsilon(k)| - 1 = O(1/|k|^2)$ as $k \to \pm\infty$ will be proven in Appendix.

Let $P = \sum_{j=0}^n P_j$ and let $\hat{H}_\varepsilon$ be the operator in $L^2(\mathbb{R}, PL^2(\mathbb{S}^{d-1}))$ such that

(7.3)
$$\hat{H}_\varepsilon u = -\frac{d^2 u}{dr^2} - \zeta_\varepsilon \frac{\Delta_\theta u}{r^2}, \quad (I - P_0)u(1,\cdot) = 0, \quad u(r,\cdot) \in PL^2(\mathbb{S}^{d-1}),\ \forall r,$$

where ζ_ε is the same as above.

Proposition 7.1. *Each $-\beta_m^2$ is one of the eigenvalues $-\beta_m^2(V)$ of the operator $\hat{H}_\varepsilon + V$. Moreover,*

$$\int_{-\infty}^{+\infty} \log |a_\varepsilon(k)|\, dk \le 2\pi\Big(\sum \beta_m(V) + \sum \beta_m(-V)\Big) \tag{7.4}$$

$$+\pi \int_0^\infty \frac{\zeta_\varepsilon(r)\alpha_d}{r^2}\, dr.$$

Proof. Obviously, if $s < c_1 < c_2 < r$, then the kernel of the operator $P_0(\hat{H}_\varepsilon + V - z)^{-1}P_0$ equals

$$g(r,s,k) = -\frac{\exp ik(r-s)}{2ika_\varepsilon(k)}. \tag{7.5}$$

The proof of the latter relation is a counterpart of the proof of (5.4). On the other hand we can consider the expansion of g near the eigenvalue $-\beta_m^2$. Denote by $\phi_{m,j}(r,\theta)$, $j = 1, 2 \dots n$ the orthonormal system of eigenfunctions corresponding to $-\beta_m^2$. If $\phi_{m,j}^{(0)} = \int_{\mathbb{S}^{d-1}} \phi_{m,j}(r,\theta)\, d\theta$, then
(7.6)

$$g(r,s,k) = \frac{\sum_{j=1}^n \phi_{m,j}^{(0)}(r)\overline{\phi_{m,j}^{(0)}(s)}}{k^2 + \beta_m^2} + g_0(r,s,k), \qquad s < c_1 < c_2 < r,$$

where $g_0(r,s,k) = O(1)$, as $k \to i\beta_m$. This proves that $a_\varepsilon(k)$ is a meromorphic function in the upper half plane and its zeros correspond to the eigenvalues $-\beta_m^2$ of the operator $\hat{H}_\varepsilon + V$. Comparing (7.5) and (7.6) we find that the multiplicities of these zeros are equal to one. For the latter arguments see [16]. Taking into account the estimate $|a_\varepsilon(k)| \ge 1$, we obtain the statement of the proposition if we add to (7.2) the same identity with $-V$ instead of V. $\square$

Observe that when $\varepsilon \to 0$ the eigenvalues of $\hat{H}_\varepsilon + V$ converge to the eigenvalues of the operator $\hat{H} + V$, where $\hat{H}$ is the following operator in $L^2(\mathbb{R}, L^2(\mathbb{S}^{d-1}))$

$$\hat{H} = -\frac{d^2u}{dr^2} + \frac{1}{r^2}(-\Delta_\theta u + \alpha_d u), \quad (I - P_0)u(1,\cdot) = 0.$$

Denote the eigenvalues of $\hat{H} + V$ by $-\big(\beta_m^{(0)}\big)^2$, where $\beta_m^{(0)} > 0$ and let $\hat{V}$ be the Fourier transform of V with respect to the first variable as in Theorem 2.2.

Proposition 7.2. *For any $\delta > 0$ there is a constant $C = C(\delta, \|V\|_\infty) > 0$ such that*

$$\sum \beta_m^{(0)} \le C\Big(\int_{\mathbb{R}^d} V^4 dx + \int_{\mathbb{R}^{d-1}} \int_{-\delta}^{\delta} |\hat V(\xi, y)|^2 \, d\xi dy + \|V(x)\|_\infty^{1/2}\Big). \tag{7.7}$$

Proof. For any self-adjoint operator T and $t > 0$ denote $N(t,T) = \operatorname{rank} E_T(-\infty, -t)$. Then

$$\sum \beta_m^{(0)} = \int_0^{||V||_\infty} N(t, \hat H + V)\frac{dt}{2\sqrt{t}} \le$$

$$\int_0^{||V||_\infty} (1 + N(t, \hat H_D + V))\frac{dt}{2\sqrt{t}} = \text{tr}\,(\hat H_D + V)_-^{1/2} + ||V||_\infty^{1/2},$$

where

$$\hat H_D = -\frac{d^2 u}{dr^2} - \frac{\Delta_\theta u}{r^2} + \frac{\alpha_d}{r^2}u, \quad u(1,\cdot) = 0.$$

Let $-\Delta + V$ be the operator in $L^2(\mathbb{R}^d)$. Then the mini-max principle tells us that

$$\sum \beta_m^{(0)} \le \text{tr}\,(-\Delta + V)_-^{1/2} + \|V\|_\infty^{1/2}. \tag{7.8}$$

Applying the Lieb-Thirring inequality for operator valued potentials (see [16]) and Theorem 3.1 we obtain

$$\begin{aligned} \text{tr}\,(-\Delta + V)_-^{1/2} &\le C \int_{\mathbb{R}^{d-1}} \Big(-\frac{d^2}{ds^2} + V(s,y)\Big)_-^{d/2} dy \\ &\le C_0 \int_{\mathbb{R}^{d-1}} \Big(-\frac{d^2}{ds^2} + V(s,y)\Big)_-^{3/2} dy \\ &\le C_1\Big(\int_{\mathbb{R}^d} V^4 dx + \int_{\mathbb{R}^{d-1}} \int_{-\delta}^{\delta} |\hat V(\xi, y)|^2 \, d\xi dy\Big), \end{aligned}$$

where $C_0 = C(\|V\|_\infty)$ and $C_1 = C(\delta, \|V\|_\infty)$. The latter inequality together with (7.8) implies (7.7). $\square$

Now the trace formula (7.4) and the inequality (7.7) lead us to

$$\limsup_{\varepsilon\to 0} \int_{-\infty}^{+\infty} \log|a_\varepsilon(k)|\, dk \le C\Big(\int_{\mathbb{R}^d} V^4 dx + \int_{\mathbb{R}^{d-1}} \int_{-\delta}^{\delta} |\hat V(\xi, y)|^2 \, d\xi dy + \|V\|_\infty^{1/2} + 1\Big). \tag{7.9}$$

For a perturbation V satisfying the conditions of Theorem 2.2 the Weyl function M can also be defined as $M(k) = \frac{\partial^2}{\partial r \partial s} G_z(r,s)|_{(1,1)}$, where G_z is the integral kernel of the operator $P_0(U^*HU - z)^{-1}P_0$.

For any pair of finite numbers $r_2 > r_1 \geq 0$ and for $V \in C_0^\infty(\mathbb{R}^d \setminus \Omega_1)$ it follows from Corollary 5.3 [11] that

$$\frac{1}{2}\int_{r_1}^{r_2} \log \frac{k}{4\mathrm{Im}\, M(k)}\, dk \leq \limsup_{\varepsilon\to 0} \int_{-\infty}^{+\infty} \log |a_\varepsilon(k)|\, dk. \tag{7.10}$$

Therefore (7.9) and (7.10) imply

Proposition 7.3. *For any pair of finite numbers $r_2 > r_1 \geq 0$ and for $V \in C_0^\infty(\mathbb{R}^d \setminus \Omega_1)$*

$$\begin{aligned} &\frac{1}{2}\int_{r_1}^{r_2} \log \frac{k}{4\mathrm{Im}\, M(k)}\, dk \\ &\leq C\Big(\int_{\mathbb{R}^d} V^4 dx + \int_{\mathbb{R}^{d-1}} \int_{-\delta}^{\delta} |\hat{V}(\xi, y)|^2\, d\xi dy + \|V\|_\infty^{1/2} + 1\Big), \end{aligned} \tag{7.11}$$

where $C = C(\delta, \|V\|_\infty)$.

Obviously

$$\int_{\mathbb{R}^{d-1}} \int_{-\delta}^{\delta} |\hat{V}(\xi, y)|^2\, d\xi dy = \int_{|(\eta,\theta)|<\delta} |\tilde{V}(\eta)|^2\, d\eta,$$

where $\theta = (1, 0, \dots, 0)$ and $\tilde{V}$ is the Fourier transform of V with respect to all variables. Note that the left hand side of (7.11) is independent of θ. Integrating both sides of this inequality over $\mathbb{S}^{d-1}$ we obtain the following inequality:

Proposition 7.4. *For any pair of finite numbers $r_2 > r_1 \geq 0$ and for $V \in C_0^\infty(\mathbb{R}^d \setminus \Omega_1)$*

$$\begin{aligned} &\int_{r_1}^{r_2} \frac{1}{2} \log \frac{k}{4\mathrm{Im}\, M(k)}\, dk \\ &\leq C\Big(\int_{\mathbb{R}^d} V^4 dx + \int_{\mathbb{R}^d} \frac{|\tilde{V}(\eta)|^2}{1+|\eta|}\, d\eta + \|V\|_\infty^{1/2} + 1\Big), \end{aligned} \tag{7.12}$$

where $C = C(\|V\|_\infty)$.

8. The end of the proof of Theorems 2.2 and 2.3

Let $\mathbb{Q} = [0, 1)^d$. The the cubes $\mathbb{Q}_m = \mathbb{Q} + m$, $m \in \mathbb{Z}^d$, form a partition of $\mathbb{R}^d$ to which we associate classes of functions u such that the sequence of (quasi-) norms $||u||_{L^p(\mathbb{Q}_m)}$, $q > 0$, belongs to ℓ^∞. These classes are denoted by $\ell^\infty(\mathbb{Z}^d; L^p(\mathbb{Q}))$. It is clear that (2.1) implies

$$V \in \ell^\infty(\mathbb{Z}^d; L^p(\mathbb{Q})), \quad p > d, \tag{8.1}$$

and therefore by [6] it guarantees the boundedness of the operator $\sqrt{|V|}(-\Delta+1)^{-1/2}$.

The next proposition allows us to approximate V by compactly supported smooth functions V_n.

Proposition 8.1. *Let V satisfy the conditions of Theorem 2.2. Then there exists a sequence V_n of compactly supported smooth functions converging to V*

$$\int |V_n|^4\,dx < C(V), \qquad ||V_n||_\infty < C(V) \tag{8.2}$$

and

$$\int_{\mathbb{R}^{d-1}} \int_{-\delta/2}^{\delta/2} |\hat{V}_n(\xi,y)|^2\,d\xi dy < C(V) \tag{8.3}$$

such that the Weyl functions M_n corresponding to V_n converge uniformly to $M(k)$ when k^2 belongs to any compact subset of the upper half plane. Therefore the sequence of measures μ_n converges weakly to the spectral measure μ.

Proof. Let $W_\pm = \sqrt{V_\pm}$. Since the class C_0^∞ is dense in L^p for any $p > 0$, we can find a pair of sequences $W_n^- \in C_0^\infty$ and $W_n^+ \in C_0^\infty$ satisfying

$$W_n^\pm \to W_\pm \ \text{ in } L^8(\mathbb{R}^d);\ \ W_n^\pm \to W_\pm \ \text{in } \ell^\infty(\mathbb{Z}^d; L^p(\mathbb{Q})),\ \ 2p > d. \tag{8.4}$$

Introduce a sequence of functions $\{V_n\}_{n=1}^\infty$

$$V_n = (W_n^+)^2 - (W_n^-)^2.$$

The sequences $W_n^\pm$ can be chosen so that

$$\int_{-\delta/2}^{\delta/2} |\hat{V}_n(\xi,y)|^2\,d\xi dy < C(V).$$

Then $V_n \in C_0^\infty$ and the relations (8.2), (8.4) hold true. Let

$$S_0 u = -\frac{d^2u}{dr^2} - \frac{\Delta_\theta u}{r^2}, \quad u(1,\theta) = 0 \tag{8.5}$$

acting in $L^2\Big((1,\infty); L^2(\mathbb{S}^{d-1})\Big)$. Suppose now that $\Gamma_0(z)$ and $\Gamma_n(z)$ are the resolvent operators of S_0 and $S_0 + V_n$ respectively. Recall that by δ_1' we denote the derivative of the delta function $\delta(r-1)$. The expression $\Gamma_0(z)\delta_1'$, Im $z \neq 0$, can be understood as an exponentially decaying function (Hankel's function) which coincides with the corresponding solution of the equation

$$-\frac{d^2\psi}{dr^2} + \frac{\alpha_d}{r^2}\psi = z\psi, \quad \psi(1) = -1. \tag{8.6}$$

According to assumptions (8.4) we have that

$$W_n^{\pm}\Gamma_0(z)\delta_1' \to W_{\pm}\Gamma_0(z)\delta_1',$$

in $L^2(\mathbb{R}^d)$. Thus in order to prove that the Weyl functions

$$M_n(k) = \frac{\partial^2}{\partial r \partial s} G_{n,z}(r,s)|_{(1,1)} = (\Gamma_n(z)\delta_1', \delta_1')$$
$$= (\Gamma_0(z)\delta_1', \delta_1') - ((W_n^+ - W_n^-)\Gamma_0(z)\delta_1', (W_n^+ + W_n^-)\Gamma_n(\overline{z})\delta_1')$$

converge, it is sufficient to show that

$$(W_n^+ + W_n^-)\Gamma_n(\overline{z})\delta_1' \to (W_+ + W_-)(S_0 + V - \overline{z})^{-1}\delta_1' \tag{8.7}$$

in $L^2(\mathbb{R}^d)$.

Let us denote $W_n = W_n^+ + W_n^-$ and $W_n{}^{(0)} = W_n^+ - W_n^-$. Clearly, if $W_n^{\pm} \to W_{\pm}$ in the class (8.1) with $2p > d$, as $n \to \infty$, then

$$W_n\Gamma_0(\overline{z})W_n^{(0)} \to (W_+ + W_-)\Gamma_0(\overline{z})(W_+ - W_-) \tag{8.8}$$

in the operator norm topology.

Then (8.7) follows from the identity

$$W_n\Gamma_n(\overline{z})\delta_1' = (I + W_n\Gamma_0(\overline{z})W_n^{(0)})^{-1}W_n\Gamma_0(\overline{z})\delta_1'.$$

□

Similarly we can prove the following result which allows us to pass from $\sum_{j=0}^{l} P_j V \sum_{j=0}^{l} P_j$ to V.

Proposition 8.2. *Let V be a compactly supported smooth function. Then the Weyl functions M_l corresponding to the potential $\sum_{j=0}^{l} P_j V \sum_{j=0}^{l} P_j$ converge uniformly to M when k^2 belongs to any compact subset K of the upper half plane and therefore the sequence of measures μ_l converges weakly to the spectral measure μ constructed for the potential V.*

Proof. Let us denote $V_l = \sum_{j=0}^{l} P_j V \sum_{j=0}^{l} P_j$ let $\Gamma_0(z)$ and let $\Gamma_l(z)$ be the resolvent operators of S_0 defined in (8.5) and $S_0 + V_l$ respectively. As in Proposition 7.1 the expression $\Gamma_0(z)\delta_1'$, Im $z \neq 0$, is understood as the exponentially decaying solution of the equation (8.6). According to our assumptions

$$V_l\Gamma_0(z)\delta_1' = \sum_{j=0}^{l} P_j V\Gamma_0(z)\delta_1' \to V\Gamma_0(z)\delta_1'$$

in $L^2(\mathbb{R}^d)$. Thus in order to prove that the Weyl functions

$$M_l(k) = \frac{\partial^2}{\partial r \partial s} G_{n,z}(r,s)|_{(1,1)} = (\Gamma_l(z)\delta_1', \delta_1')$$
$$= (\Gamma_0(z)\delta_1', \delta_1') - (V_l\Gamma_0(z)\delta_1', \Gamma_l(\overline{z})\delta_1')$$

converge, it is sufficient to show that $\Gamma_l(\overline{z})\delta_1'$ converges to $(S_0+V-\overline{z})^{-1}\delta_1'$ in $L^2(\mathbb{R}^d)$ uniformly on compact subsets K of the complex plane. The latter follows from the identity

$$\Gamma_l(\overline{z})\delta_1' = (S_0+V-\overline{z})^{-1}\delta_1' - \Gamma_l(\overline{z})(V_l-V)(S_0+V-\overline{z})^{-1}\delta_1' =$$

$$= (S_0+V-\overline{z})^{-1}\delta_1' + \Gamma_l(\overline{z})(I-\sum_{j=0}^{l} P_j)V(S_0+V-\overline{z})^{-1}\delta_1' +$$

$$+\Gamma_l(\overline{z})\sum_{i=0}^{l} P_i V(I-\sum_{j=0}^{l} P_j)(S_0+V-\overline{z})^{-1}\delta_1'$$

and from the bound

$$||\Gamma_l(\overline{z})|| \le \frac{1}{\text{Im } z} \le C, \quad z \in K.$$

□

Finally according to inequality (7.11) and Propositions 8.1 and 8.2 we observe that there exists a sequence of measures μ_l weakly convergent to μ, such that for any fixed $c > 0$

$$\int_0^c \frac{\log(1/\mu_l'(t))\,dt}{(1+t^{3/2})\sqrt{t}} < C(V), \quad \forall l,$$

where $C(V)$ is independent of c. Therefore due to the statement on the upper semi-continuity of an entropy (see [11]) we obtain

$$\int_0^\infty \frac{\log(1/\mu'(t))\,dt}{(1+t^{3/2})\sqrt{t}} < \infty.$$

The proof of Theorem 2.2 is complete.

The proof of Theorem 2.3 is exactly the same as the proof of Theorem 2.2 but instead of Proposition 7.3 we apply Proposition 7.4.

9. Proof of Theorem 2.1

The proof is reduced to the references on [5], [2] and Theorem 2.2. Let $-\Delta$ be the Laplace operator in $L^2(\mathbb{R}^d)$. According to [5], if V satisfies the conditions of Theorem 2.2, then

$$(-\Delta+V-z)^{-n} - (H+V-z)^{-n} \in \mathfrak{S}_1$$

for some z and sufficiently large $n > 0$. The latter relation implies that $-\Delta+V$ and $H+V$ have the same a.c. spectrum. Now by Theorem 2.11 and Corollary 2.13 of [2], the a.c. spectrum does not change if we add to V any real L^∞-function V_0 with a finite support. Indeed, in this case

$$(-\Delta+V-z)^{-n} - (-\Delta+V+V_0-z)^{-n} \in \mathfrak{S}_1$$

for some z and sufficiently large $n > 0$. This proves Theorem 2.1.

10. APPENDIX

Here we show that $a_\varepsilon(k)$ appearing in (7.1) is a meromorphic function in a neighborhood of zero and $|a_\varepsilon(k)| = 1 + O(1/|k|^2)$, as $k \to \pm\infty$ which, in particular, means that $\log|a_\varepsilon(k)| \in L^1(\mathbb{R})$.

1. Let $P = \sum_{j=0}^{n} P_j$, $V = PVP$. Introduce matrices $A(k)$ and $B(k)$ defined in the space $PL^2(\mathbb{S}^{d-1})$, such that the solution of the equation (for the matrix valued function Φ)

$$-\frac{d^2\Phi}{dr^2} + \frac{\zeta_\varepsilon}{r^2}\Big(-\Delta_\theta\Phi + \alpha_d\Phi\Big) + V\Phi = k^2\Phi \tag{10.1}$$

satisfies the following conditions

$$\Phi = \exp(ikr)P, \quad \text{for} \quad r > c_2,$$

and

$$\exp(ikr)A(k) + \exp(-ikr)B(k) \quad \text{for} \quad r < c_1.$$

We shall see that $A(k)$ and $B(k)$ both have at most a simple pole at zero and therefore by (10.2) $a_\varepsilon(k)$ could also have a pole at zero.

Proposition 10.1. *The following relation holds true:*

$$\frac{1}{a_\varepsilon(k)}P_0 = P_0\big(A(k) + (I - P_0)e^{-2ik}B(k)\big)^{-1}P_0. \tag{10.2}$$

Proof. Let $G(r,s,k)$ be the kernel of the operator $(\hat{H}_\varepsilon + V - z)^{-1}\chi_{c_1}$, where χ_{c_1} is the operator of multiplication by the characteristic function of $(1, c_1)$. Then

$$G(r,s,k) = \begin{cases} \Psi(r,k)Z_1(s,k), & \text{as} \quad r < s < c_1 \\ -\Phi(r,k)Z_2(s,k), & \text{as} \quad s < c_1,\ s < r. \end{cases}$$

Here $\Psi(r,k) = e^{-ikr}P_0 + k^{-1}\sin(k(r-1))(P - P_0)$ for $r < c_1$ and $\Phi(r,k) = e^{ikr}P$ for $r > c_2$. The matrices $Z_1(s,k)$ and $Z_2(s,k)$ are chosen such that $G(r,s,k)$ is continuous at the diagonal and

$$\lim_{r\to s-0} G'_r(r,s,k) = \lim_{r\to s+0} G'_r(r,s,k) + P.$$

The two latter equations are equivalent to

$$\begin{aligned} &[e^{-ikr}P_0 + k^{-1}\sin(k(r-1))(P - P_0)]Z_1 + \\ &\qquad [e^{-ikr}B(k) + e^{ikr}A(k)]Z_2 = 0; \\ &[-ike^{-ikr}P_0 + \cos(k(r-1))(P - P_0)]Z_1 + \\ &\qquad [-ike^{-ikr}B(k) + ike^{ikr}A(k)]Z_2 = P \end{aligned} \tag{10.3}$$

and are uniquely solvable if and only if k^2 is not an eigenvalue of $\hat{H}_\varepsilon + V$. The first equation of the system (10.3) gives

$$Z_1 = -\big[e^{ikr}P_0 + \frac{k}{\sin(k(r-1))}(P - P_0)\big]\big[e^{-ikr}B(k) + e^{ikr}A(k)\big]Z_2.$$

Therefore we obtain from the second equation of (10.3) that

$$\begin{aligned}\big[ikP_0 - k\,\mathrm{ctg}\,(k(r-1))(P-P_0)\big]\big[e^{-ikr}B(k) + e^{ikr}A(k)\big]Z_2 \\ +[-ike^{-ikr}B(k) + ike^{ikr}A(k)]Z_2 = P,\end{aligned} \tag{10.4}$$

or equivalently

$$(P-P_0)\Big[\big(-k\,\mathrm{ctg}(k(r-1))-ik\big)e^{-ikr}B(k)+\big(-k\,\mathrm{ctg}(k(r-1))+ik\big)e^{ikr}A(k)\Big]Z_2$$
$$+2ikP_0e^{ikr}A(k)Z_2 = P.$$

Obviously

$$-k\,\mathrm{ctg}(k(r-1)) \pm ik = -\frac{ke^{\mp ik(r-1)}}{\sin k(r-1)}.$$

This implies

$$(P-P_0)\Big[\frac{-k}{\sin k(r-1)}\Big(e^{-ik}B(k)+e^{ik}A(k)\Big)\Big]Z_2 + 2ikP_0e^{ikr}A(k)Z_2 = P.$$

Multiplying both sides of this identity by

$$\frac{-\sin k(r-1)}{k}e^{-ik}(P-P_0) + \frac{e^{-ikr}}{2ik}P_0$$

we derive

$$P_0Z_2(r,k)P_0 = (2ik)^{-1}e^{-ikr}P_0\big(A(k) + e^{-2ik}(P-P_0)B(k)\big)^{-1}P_0.$$

Finally, since

$$P_0Z_2(r,k)P_0 = (2ika_\varepsilon)^{-1}e^{-ikr}P_0$$

we obtain (10.2). □

2. In this subsection we adapt the argument from [16]. The solution $\Phi(r,k)$ of (10.1) satisfies the integral equation

$$\Phi(r,k) = e^{ikr}P - \int_r^\infty k^{-1}\sin k(r-s)V_*(s)\Phi(s,k)\,ds, \tag{10.5}$$

where $V_* = V - r^{-2}\zeta_\varepsilon\,P\,\Delta_\theta$. Denote

$$X(r,k) = e^{-ikr}\Phi(r,k) - P\,.$$

Then

$$X(r,k) = \int_r^\infty K(r,s,k)\,ds + \int_r^\infty K(r,s,k)X(s,k)\,ds, \tag{10.6}$$

where

$$K(r,s,k) = \frac{e^{2ik(s-r)}-1}{2ik} V_*(s)\,. \tag{10.7}$$

Note that

$$\|K(r,s,k)\| \le C_1(V_*,n)/(1+|k|) \tag{10.8}$$

for all k with Im $k \ge 0$ and all k with $1 < r \le s$. Here and below $\|\cdot\|$ denotes the norm of an operator in $PL^2(\mathbb{S}^{d-1})$.

Solving the Volterra equation (10.6) we obtain the following convergent series

$$X(r,k) = \sum_{m=1}^{\infty} \int_{r \le r_1 \le \cdots \le r_m} \cdots \int \prod_{l=1}^{m} K(r_{l-1}, r_l, k)\, dr_1 \cdots dr_m\,.$$

From (10.8) we see that $|X(r,k)| \le C_2(V_*)$ for all $1 < r$. Obviously $X(r,k)$ is an entire function in k. Inserting this estimate back into (10.6), we conclude that the inequality

$$\|X(r,k)\| \le C_3(V_*,n)(1+|k|)^{-1} \tag{10.9}$$

holds for all r with $1 < r$ and all k with Im $k \ge 0$.

If we rewrite (10.5) as follows

$$\begin{aligned} \Phi(r,k) = e^{ikr} &\left[P - \frac{1}{2ik}\int_r^\infty V_*(s)\,ds - \frac{1}{2ik}\int_r^\infty V_*(s)X(s,k)\,ds \right] \\ &+ \frac{e^{-ikr}}{2ik}\left[\int_r^\infty e^{2iks}V_*(s)\,ds + \int_r^\infty e^{2iks}V_*(s)X(s,k)\,dx \right], \end{aligned} \tag{10.10}$$

then the expressions in the brackets in the r.h.s. do not depend on r for $r \le 1$. From (10.10) it follows that

$$A(k) = P - \frac{1}{2ik}\int_{-\infty}^{+\infty} V_*(s)\,ds - \frac{1}{2ik}\int_{-\infty}^{+\infty} V_*(s)X(s,k)\,ds\,, \tag{10.11}$$

$$B(k) = \frac{1}{2ik}\int_{-\infty}^{+\infty} e^{2iks}V_*(s)\,ds + \frac{1}{2ik}\int_{-\infty}^{+\infty} e^{2iks}V_*(s)X(s,k)\,ds\,. \tag{10.12}$$

Recall that $\operatorname{supp} \tilde{V} \subset (1, \infty)$. Thus for sufficiently large $|k|$ the smoothness of V and (10.9) imply

(10.13)
$$\left\| A(k) - P + \frac{1}{2ik} \int_{-\infty}^{+\infty} V_*(s)ds \right\| \leq C_4(V_*, n)|k|^{-2}, \quad \operatorname{Im} k \geq 0\,,$$

(10.14)
$$\left\| e^{-2ik} B(k) \right\| \leq C_5(V_*, n)|k|^{-2}\,, \quad \operatorname{Im} k \geq 0\,.$$

Note that from (10.2), (10.13) and (10.14) we now obtain that $a_\varepsilon(k)$ is a meromorphic function in a neighborhood of zero and $|a_\varepsilon(k)|$ tends to 1 as $O(1/|k|^2)$ when $k \to \pm\infty$.

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