On the theory of resonances in non-relativistic QED and related models

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Abstract

We study the mathematical theory of quantum resonances for the standard model of non-relativistic QED and for Nelson's model. In particular, we estimate the survival probability of metastable states due to quantum resonances and relate the resonances to poles of an analytic continuation of matrix elements of the resolvent of the quantum Hamiltonian.

1 Introduction

One of the key triumphs of Quantum Mechanics has been to enable one to calculate the discrete energy spectrum and the corresponding stationary states - eigenstates of the quantum Hamiltonian - of atoms and molecules, neglecting their interactions with the quantized electromagnetic field. However, if these interactions are taken into account, stationary states corresponding to discrete energies are absent, except for the groundstate. The data of atomic and molecular spectroscopy can be interpreted in terms of the decay of meta-stable states with energies close to the discrete energies, or eigenvalues, of the non-interacting atoms or molecules. The decay of these states is accompanied by emission of photons with nearly discrete energies equal to differences between energies of stationary states; (Bohr's frequency condition). These meta-stable states are called "(quantum) resonances". Their analysis is the subject of this paper: We further develop some key ingredients of the mathematical theory of resonances for the standard model of "non-relativistic quantum electrodynamics" (QED) and for Nelson's model of electrons interacting with quantized (longitudinal lattice) vibrations, i.e., phonons. Due to the interactions of electrons with massless field quanta - photons or phonons - the standard techniques to analyze quantum-mechanical resonances developed during the past thirty or so years (see, e.g., [1]) cannot be applied to realistic models of atoms or molecules. Our goal, in this paper, is to modify these techniques to cover the present models.

Before introducing the models we explain the resonance problem in general terms. Let H_g be a quantum Hamiltonian, where g is a real parameter called the coupling constant. Assume there is a one-parameter unitary group, $\theta \to \mathcal{U}_\theta$, with $\theta \in \mathbb{R}$, s.t. the family $H_{g,\theta} := \mathcal{U}_\theta H_g \mathcal{U}_\theta^{-1}$ has an analytic continuation in θ to a disc $D(0, \theta_0)$ in the complex plane. We call such an analytic continuation a complex deformation of H_g. We note that, while the essential spectrum of $H_{g,\theta}$ usually changes dramatically under such an analytic continuation, the eigenvalues are locally independent of θ, for a certain set of groups $\mathcal{U}_\theta$. Moreover, the real eigenvalues of $H_{g,\theta}$ coincide with the eigenvalues of H_g. The complex eigenvalues of $H_{g,\theta}$, $\mathrm{Im}\theta > 0$, are called the (quantum) $\mathcal{U}-$resonance eigenvalues - or just resonance eigenvalues - of the Hamiltonian H_g. The group most commonly used is the group of dilatations of positions and momenta (see below), and the corresponding resonances are sometimes called "dilatation resonances".

It is clear from our definition that resonances - at least for weakly coupled systems (g small) - are closely related to eigenvalues of $H_{g=0}$. But what is their physical significance?

Physically, one thinks of quantum resonances as long-lived metastable states or as "bumps" in the scattering cross-section as a function of energy. The energies and life-times of metastable states are given by the bumps' centers and the inverse of the "bumps' widths". A known approach to establish such properties is as follows. Let $\mathcal{D} \subset \mathcal{H}$ denote the dense linear subspace of $\mathcal{U}-$entire vectors, i.e., vectors ψ for which the family $\{\psi_\theta := \mathcal{U}_\theta \psi\}_{\theta \in \mathbb{R}}$ has an analytic continuation to the entire complex plane. For such vectors one has the "Combes formula"

$$(\psi, (H_g - z)^{-1}\psi) = (\psi_{\bar{\theta}}, (H_{g,\theta} - z)^{-1}\psi_\theta). \tag{1.1}$$

If we continue the r.h.s. analytically, first in θ and then in z, then we see that matrix elements, $(\psi, (H_g - z)^{-1}\psi)$, of the resolvent, for $z \in \mathbb{C}$, $\mathrm{Im} z > 0$, and $\psi \in \mathcal{D}$, have an analytic continuation in z across the essential spectrum of H_g to the "second Riemann sheet" whenever the resolvent set of the operator $H_{g,\theta}$, $\mathrm{Im}\theta > 0$, contains a part of this essential spectrum. Clearly, eigenvalues of $H_{g,\theta}$, $\mathrm{Im}\theta > 0$, in the lower complex half-plane, $\mathbb{C}^-$, are poles of this analytic continuation, provided these eigenvalues are isolated.

In the latter case, the metastability property can be established (at least, for weakly coupled systems) by using the relation - via the Fourier transform - between the propagator and the resolvent, contour deformation and Cauchy's theorem (see [2, 1]). The "bumpiness" of the cross-section can be connected to the resonance poles. The real and imaginary parts of the resonance eigenvalues give the energy and the rate of decay, or the reciprocal life-time, of the metastable state.

The situation described above is exactly the one encountered in Quantum Mechanics. In non-relativistic QED and phonon models, the resonance eigenvalues are *not isolated*; more precisely, a branch of essential spectrum is attached to every complex eigenvalue of the deformed Hamiltonian $H_{g,\theta}$. This is due to the fact that photons and phonons are *massless*. As a result, establishing the property of metastability and the pole structure of the resolvent (and the related bumpiness of the cross-section) becomes a challenge. In this paper, we prove, for non-relativistic QED and Nelson's model, the metastability property of resonances and characterize them in terms of poles of a meromorphic continuation of the matrix elements of the resolvent on a dense set of vectors.

Next, we introduce the models considered in this paper. The Hamiltonian of the QED model is defined as

$$H_g^{SM} := \sum_{j=1}^{N} \frac{1}{2m_j}(p_j + gA(x_j))^2 + V(x) + H_f, \tag{1.2}$$

where $x = (x_1, \ldots, x_N)$, $p_j = i\nabla_j$ denotes the momentum of the j^{th} particle and m_j its mass, and $V(x)$ is the potential energy of the particle system. Furthermore, $A(y)$ denotes the quantized vector potential

$$A(y) = \sum_{\lambda \in \{-1,1\}} \int \frac{d^3k \chi(k)}{(2\pi)^3\sqrt{2|k|}}(e^{ik\cdot y}\epsilon_\lambda(k)a_\lambda(k) + e^{-ik\cdot y}\overline{\epsilon_\lambda(k)}a_\lambda^*(k)), \tag{1.3}$$

where $k \in \mathbb{R}^3$, χ is an ultraviolet cut-off that vanishes sufficiently fast at infinity, and $\epsilon_\lambda(k), \lambda = -1, 1$, are two transverse polarization vectors, i.e., orthonormal vectors in $\mathbb{R}^3 \otimes \mathbb{C}$ satisfying $k \cdot \epsilon_\lambda(k) = 0$; moreover, H_f is the photon (quantized electromagnetic field) Hamiltonian defined as

$$H_f = \sum_{\lambda=-1,1} \int_{\mathbb{R}^3} \omega(k) a_\lambda^*(k) a_\lambda(k) dk, \tag{1.4}$$

where $\omega(k) = |k|$.

The operator-valued distributions $a_\lambda(k)$ and $a_\lambda^*(k)$ are annihilation and creation operators acting on the symmetric Fock space $\mathcal{F}_s$ over $L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$. They obey the canonical commutation relations

$$[a_\lambda^*(k), a_{\lambda'}^*(k')] = [a_\lambda(k), a_{\lambda'}(k')] = 0 \quad , \quad [a_\lambda(k), a_{\lambda'}^*(k')] = \delta(k-k')\delta_{\lambda,\lambda'}, \tag{1.5}$$

and

$$a_\lambda(k)\Omega = 0,$$

where $\Omega \in \mathcal{F}_s$ is the vacuum vector.

The QED Hamiltonian H_g^{SM} acts on the Hilbert space $\mathcal{H}_p \otimes \mathcal{F}_s$, where $\mathcal{H}_p$ is the Hilbert space for N electrons, e.g. $\mathcal{H}_p = \mathrm{L}^2(\mathbb{R}^{3N})$, (neglecting permutation symmetry). In (1.2), Zeeman terms coupling the magnetic moments of the electrons to the magnetic field are neglected.

Nelson's model describes non-relativistic particles without spin interacting with a scalar, massless boson field. The Hamiltonian of the model acts on $\mathcal{H}_p \otimes \mathcal{F}_s$, where $\mathcal{F}_s$ is the symmetric Fock space over $\mathrm{L}^2(\mathbb{R}^3)$, and is given by

$$H_g^N := H_p \otimes I + I \otimes H_f + W_g. \tag{1.6}$$

Here, $H_p = \sum_{j=1}^N p_j^2/2m_j + V$ denotes an N-particle Schrödinger operator on $\mathcal{H}_p$. We assume that its spectrum, $\sigma(H_p)$, consists of a sequence of discrete eigenvalues, $\lambda_0, \lambda_1, \cdots$, below some real number Σ called the *ionization threshold*. We require the following assumptions.

(A) The potential $V(x)$ is dilatation analytic, i.e., the vector-function $\theta \mapsto V(e^\theta x)(-\Delta+1)^{-1}$ has an analytic continuation to a small complex disc $D(0,\theta_0) \subset \mathbb{C}$, for some $\theta_0 > 0$.

An example of a dilatation-analytic potential V is the Coulomb potential for N electrons and one fixed nucleus located at the origin. For a molecule in the Born-Oppenheimer approximation, the potential, $V(x)$ is not dilatation-analytic. In this case, one has to use a more general notion of distortion analyticity (see [1]), which can be easily accommodated in our analysis.

For k in $\mathbb{R}^3$, we denote by $a^*(k)$ and $a(k)$ the usual phonon creation and annihilation operators on $\mathcal{F}_s$. They are operator-valued distributions obeying the canonical commutation relations

$$[a^*(k), a^*(k')] = [a(k), a(k')] = 0 \quad , \quad [a(k), a^*(k')] = \delta(k-k'). \tag{1.7}$$

The operator associated with the energy of the free boson field, H_f, is given by the expression (1.4), except that the operators $a^*(k)$ and $a(k)$ now are scalar creation and annihilation operators as given above. The interaction W_g in (1.6) is assumed to be of the form

$$W_g = g\phi(G_x) \tag{1.8}$$

where

$$\phi(G_x) = \sum_{j=1}^N \int_{\mathbb{R}^3} \frac{\chi(k)}{|k|^{1/2-\mu}} \left[e^{-ik\cdot x_j} a^*(k) + e^{ik\cdot x_j} a(k)\right] dk. \tag{1.9}$$

As above, the function $\chi(k)$ denotes an ultraviolet cut-off, and the parameter μ is assumed to be positive. We pick $\chi(k)$ so as to satisfy the following condition.

(B) The function χ is dilatation analytic, i.e., $\theta \mapsto \chi(e^{-\theta}k)$ has an analytic continuation from $\mathbb{R}$ to the disc $D(0,\theta_0)$.

For instance, we can choose $\chi(k) := e^{-k^2/\Lambda^2}$, for some fixed, arbitrarily large ultraviolet cut-off $\Lambda > 0$.

Let H_g denote either H_g^{SM} or H_g^N. To define quantum resonances for the Hamiltonian H_g, we use the dilatations of electron positions and of photon momenta:

$$x_j \to e^{\theta} x_j \text{ and } k \to e^{-\theta} k,$$

where θ is a real parameter. Such dilatations are represented by the one-parameter group of unitary operators, $\mathcal{U}_\theta$, on the total Hilbert space $\mathcal{H} := \mathcal{H}_p \otimes \mathcal{F}_s$ of the system. This is a key example of the deformation groups mentioned above. Following the general prescription, we define, for $\theta \in \mathbb{R}$, the family of unitarily equivalent Hamiltonians

$$H_{g,\theta} := \mathcal{U}_\theta H_g \mathcal{U}_\theta^{-1}. \tag{1.10}$$

By the above assumptions on V and χ, the family $H_{g,\theta}$ can be analytically extended, as a type-A family in the sense of Kato, to all θ belonging to the disc $D(0, \theta_0)$ in the complex plane. The deformation resonances are now defined as complex eigenvalues of $H_{g,\theta}$, $\mathrm{Im}\theta > 0$.

Let $\lambda_0 = \inf(\sigma(H_{g=0}))$. We consider the eigenvalues λ_j of H_p, or of $H_0 := H_p \otimes I + I \otimes H_f$, with $\lambda_0 < \lambda_j < \Sigma$. By the renormalization group analysis in [3, 4, 5], we know that, as the interaction between the non-relativistic particles and the field is turned on, these eigenvalues turn into resonances $\lambda_{j,g}$, with $\mathrm{Im}\lambda_{j,g} < 0$; (see also [6] for a somewhat different model). Our goal is to investigate the properties of these resonances, as described above.

To simplify our presentation, we assume that λ_j is non-degenerate, and we denote by $\Psi_j = \psi_j \otimes \Omega$ the normalized, unperturbed eigenstate associated with λ_j. We also assume that

(C) *Fermi's Golden Rule* ([3, 4, 5]) holds.

This condition implies that $\mathrm{Im}\lambda_{j,g} \leq -\mathrm{c}_0 g^2$, for some positive constant c_0; see for example [3, 4, 5]. The main results of this paper are summarized in the following theorems.

Theorem 1.1 *Let H_g be either H_g^{SM} or H_g^N. Given Ψ_j, and $\lambda_{j,g}$ as above, and under Assumptions (A)-(C) formulated above, there exists some $g_0 > 0$ such that, for all $0 < g < g_0$ and times $t \geq 0$,*

$$\left(\Psi_j, e^{-itH_g}\Psi_j\right) = e^{-it\lambda_{j,g}} + O(g^\alpha), \tag{1.11}$$

where $\alpha := \frac{1+2\mu}{4+2\mu}$, with $\mu > 0$ appearing in (1.9).

Remark 1.2 *We expect that our approach extends to situations where Fermi's Golden Rule condition fails, as long as $\mathrm{Im}\lambda_{j,g} < 0$, and that we can improve the exponent of g in the error term by using an initial state that is a better approximation of the "resonance state"; see section 3.*

Remark 1.3 *The analysis below, together with Theorem 3.3 in [7], gives an adiabatic theorem for quantum resonances in non-relativistic QED.*

Theorem 1.1 estimates the survival probability, $\left(\Psi_j, e^{-itH_g}\Psi_j\right)$, of the state Ψ_j. Let $\gamma_{j,g} := -\mathrm{Im}\lambda_{j,g}$ and $T_{j,g} := 1/\gamma_{j,g}$. Theorem 1.1 implies that

$$\|e^{-itH_g}\Psi_j - e^{-it\lambda_{j,g}}\Psi_j\| = [1 - e^{-2t\gamma_{j,g}} + O(g^\alpha)]^{1/2}, \tag{1.12}$$

which is $\ll 1$, for $t \ll T_{j,g}$. This property is what we call the *"metastability"* of the resonance associated with the resonance eigenvalue $\lambda_{j,g}$.

There is a dense linear subspace $\mathcal{D} \subset \mathcal{H}$ of vectors with the property that, for every $\psi \in \mathcal{D}$, the family $\{\mathcal{U}_\theta \psi\}_{\theta\in\mathbb{R}}$ of vectors has an analytic extension in θ to the entire complex plane, with $\mathcal{U}_\theta\psi \in \mathcal{D}$, for any $\theta \in \mathbb{C}$. Vectors in $\mathcal{D}$ are called dilatation-entire vectors.

Next, for $z_* \in \mathbb{C}$ and $0 \leq \varphi_1 < \varphi_2 < 2\pi$, we define domains

$$W_{z_*}^{\varphi_1,\varphi_2} := \{z \in \mathbb{C}| \ \ |z - z_*| < \frac{1}{2}|\mathrm{Im} z_*|, \ \varphi_1 \leq \arg(z - z_*) \leq \varphi_2\}.$$

Our second main result is the following theorem.

Theorem 1.4 *Consider the Hamiltonian H_g^N. Let conditions (A) and (B) and (1.9), with $\mu > 0$, be satisfied, and let λ_j be an eigenvalue of H_p for which condition (C) above holds. Then there are a constant $g_* > 0$ and a dense set $\mathcal{D}' \subset \mathcal{D}$ s.t., for $g < g_*$ and for all $\psi \in \mathcal{D}'$, the function*

$$F_\psi(z) := (\psi, (H_g^N - z)^{-1}\psi)$$

has an analytic continuation in z from the upper half-plane, across a neighbourhood of λ_j, into the domain $W_{\lambda_{j,g}}^{\varphi_1,\varphi_2}$, for some $\varphi_1 < \pi/2$ and $\varphi_2 > \pi$, and this continuation satisfies the relations

$$F_\psi(w) = \frac{p(\psi)}{w - \lambda_{j,g}} + r(w; \psi), \tag{i}$$

with

$$|r(w; \psi)| \leq \mathrm{C}(\psi)|w - \lambda_{j,g}|^{-\beta}, \tag{ii}$$

for some $\beta < 1$. Here $p(\psi)$ and $r(w; \psi)$ are quadratic forms on the domain $\mathcal{D}' \times \mathcal{D}'$.

Remark 1.5 *Since we can rotate the essential spectrum of $H_{g,\theta}^N, \theta \in D(0, \theta_0)$, in $\mathbb{C}^-$ using dilatation analyticity, if $\theta_0 < \frac{\pi}{2}$ is large enough, we can analytically continue $F_\psi(z)$ in z from the upper half-plane into a neighbourhood in the vicinity of $\lambda_{j,g}$ that is larger than $W_{\lambda_{j,g}}^{\varphi_1,\varphi_2}$ given in Theorem 1.4. We note that the quadratic form $r(w; \psi)$ also depends on the homotopy class of the path along which $F_\psi(z)$ is analytically continued from the upper half-plane to the point w in the vicinity of $\lambda_{j,g}$.*

For an operator A on the one-particle space $L^2(\mathbb{R}^3)$, we denote by $d\Gamma(A)$ its "lifting" to the Fock space $\mathcal{F}_s$, (second quantization). The set $\mathcal{D}'$ in Theorem 1.4 can be chosen explicitly as

$$\mathcal{D}' := \{\psi \in \mathcal{D}| \ \ \|d\Gamma(\omega^{-1/2})(1 - P_\Omega)\psi\| < \infty\},$$

where P_Ω is the projection onto the vacuum Ω in $\mathcal{F}_s$. Since $\mathcal{U}_\theta d\Gamma(\omega^{-1/2}) = e^{\theta/2} d\Gamma(\omega^{-1/2})\mathcal{U}_\theta$, the set $\mathcal{D}'$ is dense in $\mathcal{D}$. In this case $\beta = (1 + \frac{2}{3}\mu)^{-1}$.

The main difficulty in the proofs of our main results comes from the fact that the unperturbed eigenvalue λ_j is the threshold of a branch of continuous spectrum. To overcome this difficulty, we introduce an infrared cut-off that opens a gap in the spectrum of $H_{g,\theta}$, and we control the error introduced by opening such a gap using the fact that the interaction between the electrons and the photons or phonons vanishes sufficiently fast at low photon/phonon energies (see [3, 4, 8] and Eqn. (2.7) below).

Our paper is organized as follows. In Sections 2-4 we prove Theorem 1.1 for the Nelson Hamiltonian, H_g^N. In Section 5 we extend this proof to the QED Hamiltonian, H_g^{SM}. Theorem 1.4 is proven in Section 6.

As we were completing this paper, there appeared an e-print [9] where lower and upper bounds for the lifetime of the metastable states considered in this paper are estimated by somewhat different techniques.

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2 Dilatation analyticity and IR cut-off Hamiltonians

Let $H_g = H_g^N$ be the Hamiltonian defined in (1.6). We begin this section with a discussion of the dilation deformation $H_{g,\theta}$ of H_g defined in the introduction. As was already mentioned above, by the above assumptions on V and χ, the family $H_{g,\theta}$ can be analytically extended to all θ belonging to a disc $D(0,\theta_0)$ in the complex plane. The relation $H_{g,\theta}^* = H_{g,\overline{\theta}}$ holds for real θ and extends by analyticity to $\theta \in D(0,\theta_0)$. A direct computation gives

$$H_{g,\theta} = H_{p,\theta} \otimes I + e^{-\theta} I \otimes H_f + W_{g,\theta},$$

where $H_{p,\theta} = \mathcal{U}_\theta H_p \mathcal{U}_\theta^{-1}$ and $W_{g,\theta} := \mathcal{U}_\theta W_g \mathcal{U}_\theta^{-1}$. Note that $W_{g,\theta} = g\phi(G_{x,\theta})$, with

$$G_{x,\theta}(k) = e^{-(1+\mu)\theta} \frac{\chi(e^{-\theta}k)}{|k|^{1/2-\mu}} e^{-ik.x}. \tag{2.1}$$

We now introduce an infra-red cut-off Hamiltonian

$$H_{g,\theta}^\sigma := H_{p,\theta} \otimes I + e^{-\theta} I \otimes H_f + W_{g,\theta}^{\geqslant\sigma}, \tag{2.2}$$

where $W_{g,\theta}^{\geqslant\sigma} := g\phi(G_{x,\theta}^{\leqslant\sigma})$, and $G_{x,\theta}^{\leqslant\sigma} := \kappa_\sigma G_{x,\theta}$. Here κ_σ is an infrared cut-off function that we can choose, for instance, as $\kappa_\sigma = \mathbf{1}_{|k|\geq\sigma}$. We also define

$$W_{g,\theta}^{\leqslant\sigma} := W_{g,\theta} - W_{g,\theta}^{\geqslant\sigma} = g\phi(G_{x,\theta}^{\geqslant\sigma}), \tag{2.3}$$

where $G_{x,\theta}^{\geqslant\sigma} := (1-\kappa_\sigma)G_{x,\theta}$. We then have that

$$H_{g,\theta} = H_{g,\theta}^\sigma + W_{g,\theta}^{\leqslant\sigma}. \tag{2.4}$$

We denote by $\mathcal{F}_s^{\geqslant\sigma}$ and $\mathcal{F}_s^{\leqslant\sigma}$ the symmetric Fock spaces over $\mathrm{L}^2(\{k \in \mathbb{R}^3 : |k| \geq \sigma\})$ and $\mathrm{L}^2(\{k \in \mathbb{R}^3 : |k| \leq \sigma\})$, respectively. It is well-known that there exists a unitary operator $\mathcal{V}$ that maps $\mathrm{L}^2(\mathbb{R}^{3N}; \mathcal{F}_s)$ to $\mathrm{L}^2(\mathbb{R}^{3N}; \mathcal{F}_s^{\geqslant\sigma}) \otimes \mathcal{F}_s^{\leqslant\sigma}$, so that

$$\mathcal{V} H_{g,\theta}^\sigma \mathcal{V}^{-1} = H_{g,\theta}^{\geqslant\sigma} \otimes I + e^{-\theta} I \otimes H_f^{\leqslant\sigma}. \tag{2.5}$$

Here, $H_{g,\theta}^{\geqslant\sigma}$ acts on $\mathrm{L}^2(\mathbb{R}^{3N}; \mathcal{F}_s^{\geqslant\sigma})$ and is defined by

$$H_{g,\theta}^{\geqslant\sigma} := H_{p,\theta} + e^{-\theta} H_f^{\geqslant\sigma} + W_{g,\theta}^{\geqslant\sigma}. \tag{2.6}$$

The operators $H_f^{\geqslant\sigma}$ and $H_f^{\leqslant\sigma}$ denote the restrictions of H_f to $\mathcal{F}_s^{\geqslant\sigma}$ and $\mathcal{F}_s^{\leqslant\sigma}$ respectively. We note the following estimate that will often be used in this paper:

$$\left\| W_{g,\theta}^{\leqslant\sigma} \left[H_f + 1 \right]^{-1} \right\| \leq \mathrm{C} g \sigma^{1/2+\mu}, \tag{2.7}$$

where $\mu > 0$, C is a positive constant, and $\theta \in D(0,\theta_0)$.

We now consider an unperturbed isolated eigenvalue λ_j of H_0. To simplify our analysis, we assume that λ_j is non-degenerate. Let

$$\mathrm{d}_j := \mathrm{dist}(\lambda_j; \sigma(H_p)\backslash\{\lambda_j\}), \tag{2.8}$$

which is positive. It is shown in [3, 4, 10] that, as the perturbation W_g is turned on, the eigenvalue λ_j turns into a resonance $\lambda_{j,g}$ of H_g. In other words, for $\theta \in D(0, \theta_0)$ with $\mathrm{Im}(\theta) > 0$, there exists a non-degenerate eigenvalue $\lambda_{j,g}$ of $H_{g,\theta}$ *not* depending on θ, with $\mathrm{Re}\lambda_{j,g} = \lambda_j + O(g^2)$, $\mathrm{Im}\lambda_{j,g} = O(g^2)$, and, if Fermi's Golden Rule condition holds, $\mathrm{Im}\lambda_{j,g} \le -\mathrm{c}_0 g^2$, for some positive constant c_0. Similarly, the operator $H_{g,\theta}^{\geqslant\sigma}$ has an eigenvalue $\lambda_{j,g}^{\geqslant\sigma}$ bifurcating from the eigenvalue λ_j of H_0 having the same properties as $\lambda_{j,g}$, with the important exception that $\lambda_{j,g}^{\geqslant\sigma}$ depends on θ. The reason for this is that $H_{g,\theta+r}^{\geqslant\sigma} \neq \mathcal{U}_r H_{g,\theta}^{\geqslant\sigma} \mathcal{U}_{-r}, \quad r \in \mathbb{R}$. Furthermore, we have the crucial property (see Proposition 4.1) that the eigenvalue $\lambda_{j,g}^{\geqslant\sigma}$ of $H_{g,\theta}^{\geqslant\sigma}$ is isolated from the rest of the spectrum of $H_{g,\theta}^{\geqslant\sigma}$. More precisely,

$$\mathrm{dist}\left(\lambda_{j,g}^{\geqslant\sigma}, \sigma(H_{g,\theta}^{\geqslant\sigma}) \setminus \{\lambda_{j,g}^{\geqslant\sigma}\}\right) \ge \mathrm{C}\sigma, \tag{2.9}$$

for some positive constant C independent of σ.

It is tempting to treat $H_{g,\theta}$ as a perturbation of $H_{g,\theta}^{\geqslant\sigma}$. However, we have to take care of the difference between $\lambda_{j,g}$ and $\lambda_{j,g}^{\geqslant\sigma}$. In order to deal with this problem, we "renormalize" the unperturbed part $H_{g,\theta}^{\sigma}$ by setting

$$\widetilde{H}_{g,\theta}^{\sigma} = H_{g,\theta}^{\sigma} + \left(\lambda_{j,g} - \lambda_{j,g}^{\geqslant\sigma}\right) \mathcal{V}^{-1}(P_{g,\theta}^{\geqslant\sigma} \otimes I)\mathcal{V}. \tag{2.10}$$

Here $P_{g,\theta}^{\geqslant\sigma}$ denotes the spectral projection onto the eigenspace associated with the eigenvalue $\lambda_{j,g}^{\geqslant\sigma}$ of $H_{g,\theta}^{\geqslant\sigma}$. As in (2.5), we have the representation

$$\mathcal{V}\widetilde{H}_{g,\theta}^{\sigma}\mathcal{V}^{-1} = \widetilde{H}_{g,\theta}^{\geqslant\sigma} \otimes I + e^{-\theta} I \otimes H_f^{\leqslant\sigma}, \tag{2.11}$$

where we have set

$$\widetilde{H}_{g,\theta}^{\geqslant\sigma} = H_{g,\theta}^{\geqslant\sigma} + \left(\lambda_{j,g} - \lambda_{j,g}^{\geqslant\sigma}\right) P_{g,\theta}^{\geqslant\sigma}. \tag{2.12}$$

By (2.12), we see that $\lambda_{j,g}$ is a non-degenerate eigenvalue of $\widetilde{H}_{g,\theta}^{\geqslant\sigma}$. In Proposition 4.3 we will show that, for $g^2 \ll \sigma < g^{\frac{3}{2+\mu}}$, there exists a positive constant C such that

$$\left|\lambda_{j,g} - \lambda_{j,g}^{\geqslant\sigma}\right| \le \mathrm{C} g^{4-\frac{3}{2+\mu}}, \tag{2.13}$$

and that the operator $\widetilde{H}_{g,\theta}^{\geqslant\sigma}$ still has a gap of order $O(\sigma)$ around $\lambda_{j,g}$. Then the decomposition (2.2) is replaced by

$$H_{g,\theta} = \widetilde{H}_{g,\theta}^{\sigma} + \widetilde{W}_{g,\theta}^{\leqslant\sigma}, \tag{2.14}$$

where

$$\widetilde{W}_{g,\theta}^{\leqslant\sigma} = W_{g,\theta}^{\leqslant\sigma} - \left(\lambda_{j,g} - \lambda_{j,g}^{\geqslant\sigma}\right) \mathcal{V}^{-1} P_{g,\theta}^{\geqslant\sigma} \otimes I\mathcal{V}. \tag{2.15}$$

Let $H_\star^{\#}$ denote one of the operators H_g, $H_{g,\theta}$, $H_{g,\theta}^{\sigma}$ or $H_{g,\theta}^{\geqslant\sigma}$. We write its resolvent by using the notation $R_\star^{\#}(z) = \left[H_\star^{\#} - z\right]^{-1}$. Similarly, we define $\widetilde{R}_\star^{\#}(z) = \left[\widetilde{H}_\star^{\#} - z\right]^{-1}$.

3 Proof of Theorem 1.1

We begin with some notation. We consider an interval I of size δ, containing λ_j, such that $\delta < \frac{1}{2}\mathrm{d}_j$. For concreteness, let

$$\mathrm{I} = \left(\lambda_j - \frac{\delta}{2}, \lambda_j + \frac{\delta}{2}\right). \tag{3.1}$$

Define, in addition,

$$\mathrm{I}_1 = \left(\lambda_j - \frac{\delta}{4}, \lambda_j + \frac{\delta}{4}\right). \tag{3.2}$$

We consider a smooth function $f \in \mathrm{C}_0^\infty(\mathrm{I})$, $\mathrm{Ran}(f) \in [0,1]$, such that $f = 1$ on I_1. It is known that there exists an almost analytic extension $\widetilde{f}$ of f such that

$$\widetilde{f} = 1 \text{ on } \{z \in \mathbb{C} | \ \mathrm{Re}(z) \in \mathrm{I}_1\} \quad , \quad \mathrm{supp}(\widetilde{f}) \subset \{z \in \mathbb{C} | \ \mathrm{Re}(z) \in \mathrm{I}\}, \tag{3.3}$$

and $\left|(\partial_{\bar{z}}\widetilde{f})(z)\right| = O(\delta^{-1}|\mathrm{Im}(z)/\delta|^n)$, for any $n \in \mathbb{N}$. We shall use these properties of $\widetilde{f}$ in the sequel.

We begin with the following proposition.

Proposition 3.1 *Given H_g, Ψ_j, $\lambda_{j,g}$ and f as above, there exists $g_0 > 0$ such that, for all $0 < g \leq g_0$, $\delta = \mathrm{C}\sigma$, $\mathrm{C} > 1$, and $\sigma = g^{2-\frac{1+2\mu}{4+2\mu}}$,*

$$\left(\Psi_j, e^{-itH_g} f(H_g)\Psi_j\right) = e^{-it\lambda_{j,g}} + O(g^{\frac{1+2\mu}{4+2\mu}}), \tag{3.4}$$

for all times $t \geq 0$.

We subdivide the proof of Proposition 3.1 into several steps, deferring the proof of some technical ingredients to the following section. We extend a method due to Hunziker to prove Proposition 3.1, see [2] or [1]. Let $\mathcal{N}(\theta)$ be a punctured neighbourhood of λ_j such that $\mathcal{N}(\theta) \cap \sigma(\widetilde{H}_{g,\theta}^{\geqslant\sigma}) = \lambda_{j,g}$ and $\mathrm{I} \subset \mathcal{N}(\theta) \cup \{\lambda_j\}$. Let $\Gamma \subset \mathcal{N}(\theta)$ be a contour that encloses I and $\lambda_{j,g}$. For z inside Γ, we have that

$$\widetilde{R}_{g,\theta}^{\geqslant\sigma}(z) = \frac{P_{g,\theta}^{\geqslant\sigma}}{\lambda_{j,g} - z} + \widehat{R}_{g,\theta}^{\geqslant\sigma}(z), \tag{3.5}$$

where $P_{g,\theta}^{\geqslant\sigma}$ denotes the spectral projection onto the eigenspace associated to the eigenvalue $\lambda_{j,g}$ of $\widetilde{H}_{g,\theta}^{\geqslant\sigma}$, that is

$$P_{g,\theta}^{\geqslant\sigma} = \frac{1}{2\pi i} \oint_{\mathcal{C}} \widetilde{R}_{g,\theta}^{\geqslant\sigma}(z) dz, \tag{3.6}$$

where $\mathcal{C}$ denotes a circle centered at $\lambda_{j,g}$ with radius chosen so that $\mathcal{C} \subset \rho(H_{g,\theta}^{\geqslant\sigma}) \cap \mathcal{N}(\theta)$, and the *regular part*, $\widehat{R}_{g,\theta}^{\geqslant\sigma}(z)$, is given by

$$\widehat{R}_{g,\theta}^{\geqslant\sigma}(z) = \frac{1}{2\pi i} \oint_{\Gamma} \widetilde{R}_{g,\theta}^{\geqslant\sigma}(w)(w-z)^{-1} dw, \tag{3.7}$$

where z is inside Γ. Note that

$$\widehat{R}_{g,\theta}^{\geqslant\sigma} P_{g,\theta}^{\geqslant\sigma} = P_{g,\theta}^{\geqslant\sigma} \widehat{R}_{g,\theta}^{\geqslant\sigma} = 0, \tag{3.8}$$

and

$$(P_{g,\theta}^{\geqslant\sigma})^2 = P_{g,\theta}^{\geqslant\sigma}. \tag{3.9}$$

We will need the following easy lemma, which follows from dilatation analyticity and Stone's theorem.

Lemma 3.2 *Assume that the infrared cut-off parameter σ is chosen such that $g^2 \ll \sigma < g^{\frac{3}{2+\mu}} \ll 1$. Then*

$$\left(\Psi_j, e^{-itH_g} f(H_g)\Psi_j\right) = A(t,\overline{\theta}) - A(t,\theta) + B(t,\overline{\theta}) - B(t,\theta), \tag{3.10}$$

for $\theta \in D(0,\theta_0)$, $\mathrm{Im}\theta > 0$, *where*

$$A(t,\theta) = \frac{1}{2\pi i}\int_{\mathbb{R}} e^{-itz} f(z)\left(\Psi_{j,\overline{\theta}}, \widetilde{R}^{\sigma}_{g,\theta}(z)\Psi_{j,\theta}\right) dz, \tag{3.11}$$

$$B(t,\theta) = \frac{1}{2\pi i}\int_{\mathbb{R}} e^{-itz} f(z)\left(\Psi_{j,\overline{\theta}}, \widetilde{R}^{\sigma}_{g,\theta}(z)\sum_{n\geq 1}\left(-\widetilde{W}^{\leqslant\sigma}_{g,\theta}\widetilde{R}^{\sigma}_{g,\theta}(z)\right)^n \Psi_{j,\theta}\right) dz. \tag{3.12}$$

Proof. By Stone's theorem,

$$\left(\Psi_j, e^{-itH_g} f(H_g)\Psi_j\right) = \lim_{\varepsilon\searrow 0}\frac{1}{2\pi i}\int_{\mathbb{R}} e^{-itz} f(z)\left(\Psi_j, \left[R_g(z-i\varepsilon) - R_g(z+i\varepsilon)\right]\Psi_j\right) dz. \tag{3.13}$$

Since H_g and Ψ_j are dilatation analytic, this implies

$$\left(\Psi_j, e^{-itH_g} f(H_g)\Psi_j\right) = F(t,\overline{\theta}) - F(t,\theta), \tag{3.14}$$

where

$$F(t,\theta) = \frac{1}{2\pi i}\int_{\mathbb{R}} e^{-itz} f(z)\left(\Psi_{j,\overline{\theta}}, R_{g,\theta}(z)\Psi_{j,\theta}\right) dz, \tag{3.15}$$

for $\theta \in D(0,\theta_0)$. It follows from Lemma 4.4, below, that we can expand $R_{g,\theta}(z)$ into a Neumann series, which is convergent under our assumptions on g and σ if Fermi's Golden Rule holds. We obtain

$$F(t,\theta) = A(t,\theta) + B(t,\theta), \tag{3.16}$$

for $\theta \in D(0,\theta_0)$, $\mathrm{Im}\theta > 0$, and hence the claim of the lemma is proven. □

In what follows, we fix $\theta \in D(0,\theta_0)$ with $\mathrm{Im}\theta > 0$. We estimate $A(t,\overline{\theta}) - A(t,\theta)$ and $B(t,\overline{\theta}) - B(t,\theta)$ in the following two lemmata.

Lemma 3.3 *For $g^2 \ll \sigma < \delta < g^{\frac{3}{2+\mu}} \ll 1$, we have*

$$A(t,\overline{\theta}) - A(t,\theta) = e^{-it\lambda_{j,g}} + O(\delta g^2/\sigma^{-2}) + O(g^2\sigma^{-1}),$$

for all $t \geq 0$.

Proof. It follows from the spectral theorem that

$$\mathcal{V}\widetilde{R}^{\sigma}_{g,\theta}(z)\mathcal{V}^{-1} = \int_{\sigma(H_f^{\leqslant\sigma})} \widetilde{R}^{\geqslant\sigma}_{g,\theta}(z - e^{-\theta}x)\otimes dE_{H_f^{\leqslant\sigma}}(x), \tag{3.17}$$

where $E_{H_f^{\leqslant\sigma}}$ are the spectral projections of $H_f^{\leqslant\sigma}$; see for example [11]. Furthermore, $\mathcal{V}\Psi_{j,\theta} = \psi_{j,\theta}\otimes\Omega^{\geqslant\sigma}\otimes\Omega^{\leqslant\sigma}$, where $\Omega^{\geqslant\sigma}$ (respectively $\Omega^{\leqslant\sigma}$) denotes the vacuum in $\mathcal{F}_s^{\geqslant\sigma}$ (in $\mathcal{F}_s^{\leqslant\sigma}$). Inserting this into (3.11) and using (3.17), we get

$$A(t,\theta) = \frac{1}{2i\pi}\int_{\mathbb{R}} e^{-itz} f(z)\left(\psi_{j,\overline{\theta}}\otimes\Omega^{\geqslant\sigma}, \widetilde{R}^{\geqslant\sigma}_{g,\theta}(z)\psi_{j,\theta}\otimes\Omega^{\geqslant\sigma}\right) dz. \tag{3.18}$$

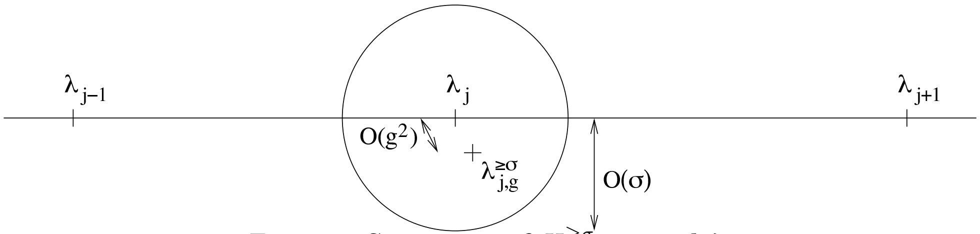

Figure 1: **Spectrum of $H_{g,\theta}^{\geqslant\sigma}$ around λ_j**

From Proposition 4.1, we know that the spectrum of $H_{g,\theta}^{\geqslant\sigma}$ is of the form pictured in figure 1. In particular, a gap of order σ opens between the non-degenerate eigenvalue $\lambda_{j,g}^{\geqslant\sigma}$ and the essential spectrum of $H_{g,\theta}^{\geqslant\sigma}$. By Proposition 4.3, the same holds for $\widetilde{H}_{g,\theta}^{\geqslant\sigma}$ instead of $H_{g,\theta}^{\geqslant\sigma}$, with $\lambda_{j,g}$ replacing $\lambda_{j,g}^{\geqslant\sigma}$, since $|\lambda_{j,g}-\lambda_{j,g}^{\geqslant\sigma}|\leq \mathrm{C}g^{4-\frac{3}{2+\mu}}$, and we assumed that $g^2\ll\sigma<g^{\frac{3}{2+\mu}}$.

Let us begin to estimate $A(t,\overline{\theta})-A(t,\theta)$ by considering the contribution of the regular part, $\widehat{R}_{g,\theta}^{\geqslant\sigma}(z)$, in $A(t,\theta)$. By applying Green's theorem, we find that

$$
\begin{aligned}
R(t,\theta) &:= \frac{1}{2i\pi}\int_{\mathbb{R}} e^{-itz}f(z)\left(\Psi_{j,\overline{\theta}},\widehat{R}_{g,\theta}^{\geqslant\sigma}(z)\Psi_{j,\theta}\right)dz\\
&= \frac{1}{2i\pi}\int_{\Gamma(\gamma_1)} e^{-itz}\widetilde{f}(z)\left(\Psi_{j,\overline{\theta}}^{\geqslant\sigma},\widehat{R}_{g,\theta}^{\geqslant\sigma}(z)\Psi_{j,\theta}^{\geqslant\sigma}\right)dz \qquad (3.19)\\
&\quad+\frac{1}{2i\pi}\iint_{\mathrm{D}(\gamma_1)} e^{-itz}(\partial_{\bar z}\widetilde{f})(z)\left(\Psi_{j,\overline{\theta}}^{\geqslant\sigma},\widehat{R}_{g,\theta}^{\geqslant\sigma}(z)\Psi_{j,\theta}^{\geqslant\sigma}\right)dzd\overline{z},
\end{aligned}
$$

where $\Psi_{j,\theta}^{\geqslant\sigma}=\psi_{j,\theta}\otimes\Omega^{\geqslant\sigma}$, and $\Gamma(\gamma_1)$ and $\mathrm{D}(\gamma_1)$ denote respectively the curve and the domain pictured in figure 2, such that the interval I_0 strictly contains I.

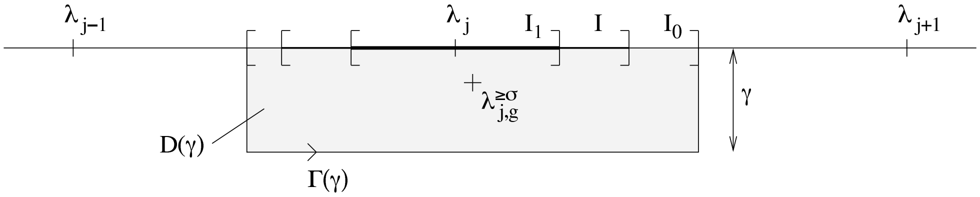

Figure 2: **Deformation of the path of integration**

By Proposition 4.1 and (3.7), the regular part $\widehat{R}_{g,\theta}^{\geqslant\sigma}(z)$ in (3.5) is analytic in $z\in\mathrm{D}(\gamma_1)$ and satisfies

$$
\left\|\widehat{R}_{g,\theta}^{\geqslant\sigma}(z)\right\|\leq\frac{\mathrm{C}}{\mathrm{dist}(z,\sigma(\widetilde{H}_{g,\theta}^{\geqslant\sigma})\setminus\{\lambda_{j,g}\})}, \qquad (3.20)
$$

where C is a positive constant. We also have from (3.8) that

$$
P_{0,\theta}^{\geqslant\sigma}\widehat{R}_{g,\theta}^{\geqslant\sigma}(z)P_{0,\theta}^{\geqslant\sigma}=(P_{0,\theta}^{\geqslant\sigma}-P_{g,\theta}^{\geqslant\sigma})\widehat{R}_{g,\theta}^{\geqslant\sigma}(z)(P_{0,\theta}^{\geqslant\sigma}-P_{g,\theta}^{\geqslant\sigma}), \qquad (3.21)
$$

and from (3.9) that

$$
P_{0,\theta}^{\geqslant\sigma}=P_{0,\theta}^{\geqslant\sigma}P_{g,\theta}^{\geqslant\sigma}P_{0,\theta}^{\geqslant\sigma}-(P_{g,\theta}^{\geqslant\sigma}-P_{0,\theta}^{\geqslant\sigma})(P_{g,\theta}^{\geqslant\sigma}-1)(P_{g,\theta}^{\geqslant\sigma}-P_{0,\theta}^{\geqslant\sigma}), \qquad (3.22)
$$

and from Proposition 4.2, below, that

$$
\|P_{g,\theta}^{\geqslant\sigma}-P_{0,\theta}^{\geqslant\sigma}\|\leq \mathrm{C}g\sigma^{-1/2}, \qquad (3.23)
$$

for some positive constant C. Thus, by (3.19) – (3.23), our assumptions on $\widetilde{f}$ and the fact that $|I_0| = O(\delta)$, we get

$$|R(t,\theta)| = O(\delta g^2 \sigma^{-2} e^{-t\gamma_1}) + O(|\gamma_1/\delta|^n), \tag{3.24}$$

where $0 < \gamma_1 < \sigma \sin(\mathrm{Im}\theta)$, and any $n \in \mathbb{N}$. Similarly, for the contribution of the regular part $\widehat{R}^{\geqslant\sigma}_{g,\overline{\theta}}$ in $A(t,\overline{\theta})$, we have the following estimate

$$\left|R(t,\overline{\theta})\right| = O(\delta g^2 \sigma^{-2} e^{-t\gamma_2}) + O(|\gamma_2/\delta|^n), \tag{3.25}$$

where $0 < \gamma_2 < \sin(\mathrm{Im}\theta)\sigma$, and any $n \in \mathbb{N}$.

Next, we estimate the singular part of $A(t,\overline{\theta}) - A(t,\theta)$. It is given by

$$S(t,\overline{\theta}) - S(t,\theta) := \frac{C_g^\sigma(\overline{\theta})}{2i\pi} \int_{\mathbb{R}} e^{-itz} f(z) \left(z - \overline{\lambda_{j,g}}\right)^{-1} dz - \frac{C_g^\sigma(\theta)}{2i\pi} \int_{\mathbb{R}} e^{-itz} f(z) \left(z - \lambda_{j,g}\right)^{-1} dz, \tag{3.26}$$

where we use the notation

$$C_g^\sigma(\theta) = \left(\Psi_j^{\geqslant\sigma}(\overline{\theta}), P_{g,\theta}^{\geqslant\sigma} \Psi_{j,\theta}^{\geqslant\sigma}\right). \tag{3.27}$$

By Proposition 4.2, we know that

$$C_g^\sigma(\theta) = 1 + O(g^2\sigma^{-1}). \tag{3.28}$$

We deform the path of integration as we did above, adding a circle $\mathcal{C}_\rho$ of radius ρ around $\lambda_{j,g}$. This yields

$$\begin{aligned} S(t,\overline{\theta}) - S(t,\theta) =& \frac{1}{2i\pi} \int_{\Gamma(\gamma_3)} e^{-itz} \widetilde{f}(z) \left[\frac{C_g^\sigma(\overline{\theta})}{z - \overline{\lambda_{j,g}}} - \frac{C_g^\sigma(\theta)}{z - \lambda_{j,g}} \right] dz \\ &+ \frac{1}{2i\pi} \int_{\mathcal{C}_\rho} e^{-itz} \widetilde{f}(z) \left[\frac{C_g^\sigma(\overline{\theta})}{z - \overline{\lambda_{j,g}}} - \frac{C_g^\sigma(\theta)}{z - \lambda_{j,g}} \right] dz \\ &+ \frac{1}{2i\pi} \iint_{\mathrm{D}(\gamma_3)\setminus \mathrm{D}_\rho} e^{-itz} (\partial_{\bar{z}} \widetilde{f})(z) \left[\frac{C_g^\sigma(\overline{\theta})}{z - \overline{\lambda_{j,g}}} - \frac{C_g^\sigma(\theta)}{z - \lambda_{j,g}} \right] dz d\overline{z}, \end{aligned} \tag{3.29}$$

for all $\rho > 0$ sufficiently small, where D_ρ denotes the disc of radius ρ centered at $\lambda_{j,g}$, and $0 < \gamma_3 < \sin(\mathrm{Im}\theta)\sigma$. The first integral can be estimated by using arguments similar to those used to estimate the regular part, (3.28), and the fact that

$$\mathrm{Im}\lambda_{j,g} = O(g^2).$$

We then obtain that

$$\left| \frac{1}{2i\pi} \int_{\Gamma(\gamma_3)} e^{-itz} \widetilde{f}(z) \left[\frac{C_g^\sigma(\overline{\theta})}{z - \overline{\lambda_{j,g}}} - \frac{C_g^\sigma(\theta)}{z - \lambda_{j,g}} \right] dz \right| = O(\delta g^2 \sigma^{-2} e^{-t\gamma_3}), \tag{3.30}$$

for $0 < g^2 < \sigma \ll 1$. Similarly, since $(\partial_{\overline{z}} \widetilde{f}) = 0$ on $\{z| \ \mathrm{Re}(z) \in \mathrm{I}_1\}$, we see that the third integral in the rhs of (3.29) is independent of ρ, for ρ sufficiently small, and that

$$\left| \frac{1}{2i\pi} \iint_{\mathrm{D}(\gamma_3)\setminus \mathrm{D}_\rho} e^{-itz} (\partial_{\bar{z}} \widetilde{f})(z) \left[\frac{C_g^\sigma(\overline{\theta})}{z - \overline{\lambda_{j,g}}} - \frac{C_g^\sigma(\theta)}{z - \lambda_{j,g}} \right] dz \right| = O(|\gamma_3/\delta|^n), \tag{3.31}$$

for any $n \in \mathbb{N}$. It remains to estimate the second integral on the right hand side of (3.29). Taking the limit as $\rho \to 0$ leads to the "residue" $C_g^\sigma(\theta)e^{-it\lambda_{j,g}}\widetilde{f}(\lambda_{j,g})$. Since, by construction, $\widetilde{f} = 1$ on $\{z| \ \mathrm{Re}(z) \in \mathrm{I}_1\}$, we get

$$\lim_{\rho\to 0}\frac{1}{2i\pi}\int_{\mathcal{C}_\rho} e^{-itz}\widetilde{f}(z)\left[\frac{C_g^\sigma(\overline{\theta})}{\overline{z-\overline{\lambda_{j,g}}}} - \frac{C_g^\sigma(\theta)}{z-\lambda_{j,g}}\right]dz = C_g^\sigma(\theta)e^{-it\lambda_{j,g}}. \tag{3.32}$$

The claim of the lemma follows from (3.24) – (3.32). □

Lemma 3.4 *Assume that the infrared cutoff parameter σ is chosen such that $g^2 \ll \sigma < g^{\frac{3}{2+\mu}} \ll 1$. Then, for all times $t \geq 0$, we have that*

$$\left|B(t,\overline{\theta}) - B(t,\theta)\right| = O(\delta g^{-2}(\sigma^{\frac{1}{2}+\mu} + g^{2-\frac{3}{2+\mu}})), \tag{3.33}$$

where $B(t,\theta)$ is defined in (3.12).

Proof. Recall that

$$B(t,\theta) = \sum_{n\geq 1} B^n(t,\theta), \tag{3.34}$$

where

$$B^n(t,\theta) = \frac{1}{2i\pi}\int_{\mathbb{R}} e^{-itz}f(z)\left(\Psi_j(\overline{\theta}), \widetilde{R}_{g,\theta}^\sigma(z)\left(-\widetilde{W}_{g,\theta}^{\leqslant\sigma}\widetilde{R}_{g,\theta}^\sigma(z)\right)^n\Psi_j(\theta)\right)dz. \tag{3.35}$$

It follows from (3.35) and Lemma 4.4 that[1]

$$|B^n(t,\theta)| = O(\delta g^{-2}(\sigma^{\frac{1}{2}+\mu} + g^{2-\frac{3}{2+\mu}})^n), \tag{3.36}$$

uniformly in $t \geq 0$. Together with (3.34) and the assumption on σ and g, it follows that

$$|B(t,\theta)| = O(\delta g^{-2}(\sigma^{\frac{1}{2}+\mu} + g^{2-\frac{3}{2+\mu}})), \tag{3.37}$$

uniformly in t. One can similarly show that

$$|B(t,\overline{\theta})| = O(\delta g^{-2}(\sigma^{\frac{1}{2}+\mu} + g^{2-\frac{3}{2+\mu}})), \tag{3.38}$$

and hence the claim of the lemma follows. □

Proof of Proposition 3.1 . It follows from Lemmata 3.2, 3.3 and 3.4 that

$$\left(\Psi_j, e^{-itH_g}f(H_g)\Psi_j\right) = e^{-it\lambda_{j,g}} + O(\delta g^2\sigma^{-2}) + O(\delta g^{-2}\sigma^{1/2+\mu}) + O(\delta g^{-\frac{3}{2+\mu}}) + O(g^2\sigma^{-1}), \quad t \geq 0. \tag{3.39}$$

Let $\delta = C\sigma$, for some $C > 1$. We optimize the estimate on the error term by choosing

$$\sigma = g^{2-\frac{1+2\mu}{4+2\mu}}, \tag{3.40}$$

and hence the claim of the proposition is proven. □

[1] We note that it follows from (2.15) that estimate (3.36) can be improved if one uses instead of Ψ_j a state that is a better approximation to the resonance eigenstate.

Proof of Theorem 1.1. Proposition 3.1 implies that, for $t = 0$,

$$(\Psi_j, (1 - f(H_g))\Psi_j) = \|\sqrt{1 - f(H_g)}\Psi_j\|^2 = O(g^{\frac{1+2\mu}{4+2\mu}}), \tag{3.41}$$

which, together with the boundedness of the unitary operator e^{-itH_g} and Proposition 3.1, for arbitrary $t > 0$, yields

$$\begin{aligned}(\Psi_j, e^{-itH_g}\Psi_j) &= (\Psi_j, e^{-itH_g}(1 - f(H_g) + f(H_g))\Psi_j)\\ &= (\Psi_j, e^{-itH_g}f(H_g)\Psi_j) + O(\|\sqrt{1 - f(H_g)}\Psi_j\|^2)\\ &= e^{-it\lambda_{j,g}} + O(g^{\frac{1+2\mu}{4+2\mu}}).\end{aligned}$$

□

4 The Hamiltonian $H^{\sigma}_{g,\theta}$

In this section, we study the operator $H^{\sigma}_{g,\theta}$ used in the previous section as an approximation of $H_{g,\theta}$. We use the Feshbach map, [3, 4], defined for a projection P and a closed operator H whose domain is contained in $\mathrm{Ran}(P)$, by

$$\mathcal{F}_P(H) = PHP - PH\overline{P}\left[\overline{P}H\overline{P}\right]^{-1}\overline{P}HP, \tag{4.1}$$

where $\overline{P} = 1 - P$. Note that the domain of $\mathcal{F}_P$ consists of operators H such that

$$\left[\overline{P}H\overline{P}\right]^{-1}|_{\mathrm{Ran}(\overline{P})}, \quad PH\overline{P}\left[\overline{P}H\overline{P}\right]^{-1}|_{\mathrm{Ran}(\overline{P})}, \quad \left[\overline{P}H\overline{P}\right]^{-1}\overline{P}HP \tag{4.2}$$

extend to bounded operators. We begin with the following proposition.

Proposition 4.1 *Suppose* $0 < g^2 \ll \sigma \ll 1$. *Then, for* $\theta \in D(0, \theta_0)$ *such that* $\mathrm{Im}\theta \neq 0$, *the spectrum of* $H^{\geqslant\sigma}_{g,\theta}$ *in the disc* $D(\lambda_j, \sigma/2)$ *consists of a single eigenvalue*

$$\sigma(H^{\geqslant\sigma}_{g,\theta}) \cap D(\lambda_j, \sigma/2) = \{\lambda^{\geqslant\sigma}_{j,g}\}. \tag{4.3}$$

Furthermore, there exists $\varepsilon > 0$ *such that, for all* z *in* $D(\lambda_j, \sigma/3)$ *such that* $\left|z - \lambda^{\geqslant\sigma}_{j,g}\right| \gg g^{2+\varepsilon}$,

$$\left\|R^{\geqslant\sigma}_{g,\theta}(z)\right\| \leq \frac{\mathrm{C}}{\mathrm{dist}(z, \sigma(H^{\geqslant\sigma}_{g,\theta}))}, \tag{4.4}$$

for some positive constant C.

Proof. Let $P_\theta := P^{\geqslant\sigma}_{0,\theta} = P_{p,j,\theta} \otimes P^{\geqslant\sigma}_{\Omega}$. For $z \in D(\lambda_j, \sigma/2)$, the operator $H^{\geqslant\sigma}_{g,\theta} - z$ is in the domain of $\mathcal{F}_{P_\theta}$. Indeed, from $g\sigma^{-1/2} \ll 1$, we have that for any z in $D(\lambda_j, \sigma/2)$

$$\|(H_f + 1)^n(H^{\geqslant\sigma}_{g,\theta} - z)^{-1}\overline{P}_\theta\| \leq \mathrm{C}\sigma^{-1}, \quad n = 0, 1, \tag{4.5}$$

for some positive constant C. Hence the operator $\overline{P}_\theta(H^{\geqslant\sigma}_{g,\theta} - z)\overline{P}_\theta$ is invertible on $\mathrm{Ran}\overline{P}_\theta$. Moreover, $W^{\geqslant\sigma}_{g,\theta}$ is $(H_f + 1)-$bounded.

Note that our choice of P_θ yields $P_\theta W_{g,\theta}^{\geqslant\sigma} P_\theta = 0$. Therefore

$$\mathcal{F}_{P_\theta}(H_{g,\theta}^{\geqslant\sigma} - z) = (\lambda_j - z)\, P_\theta - P_\theta W_{g,\theta}^{\geqslant\sigma} \overline{P}_\theta \left[\overline{P}_\theta H_{g,\theta}^{\geqslant\sigma} \overline{P}_\theta - z\right]^{-1} \overline{P}_\theta W_{g,\theta}^{\geqslant\sigma} P_\theta. \tag{4.6}$$

The non-degeneracy of λ_j implies that $\mathcal{F}_{P_\theta}(H_{g,\theta}^{\geqslant\sigma} - z)$ can be written as $[\lambda_j - z + a(z)]P_\theta$, where $a(z)$ is an analytic function from $D(\lambda_j, \sigma/2) \to \mathbb{C}$. Following [3, 5] (see also Proposition 4.3 below), we have

$$a(z) = g^2 Z_{j,\theta} + O(g^{2+\varepsilon}) \tag{4.7}$$

for some $\varepsilon > 0$, where $Z_{j,\theta} := Z_{j,\theta}^{\mathrm{od}} + Z_{j,\theta}^{\mathrm{d}}$ with

$$Z_{j,\theta}^{\mathrm{od}} = \int_{\mathbb{R}^3} \mathcal{U}_\theta P_{p,j} \overline{G_x}(k) \overline{P}_{p,j} \left[H_p - \lambda_j + \omega(k) - i0\right]^{-1} \overline{P}_{p,j} G_x(k) P_{p,j} \mathcal{U}_\theta^{-1} dk, \tag{4.8}$$

$$Z_{j,\theta}^{\mathrm{d}} = \int_{\mathbb{R}^3} \mathcal{U}_\theta P_{p,j} \overline{G_x}(k) P_{p,j} G_x(k) P_{p,j} \mathcal{U}_\theta^{-1} \frac{dk}{\omega(k)}. \tag{4.9}$$

Using the Leibniz rule and the fact that

$$\frac{d}{dz}\left[\overline{P}_\theta H_{0,\theta}^{\geqslant\sigma} - z\right]^{-1} = \left[\overline{P}_\theta H_{0,\theta}^{\geqslant\sigma} - z\right]^{-2}, \tag{4.10}$$

one can prove, by differentiating (4.6) with respect to z, that $z \mapsto b(z) := \lambda_j - z + a(z)$ is an analytic function on $D(\lambda_j, \sigma/2)$, and that $|db(z)/dz - 1| < 1$, provided that $g^2\sigma^{-1}$ is sufficiently small. This implies that b is a bijection on $D(\lambda_j, \sigma/2)$.

The isospectrality of the Feshbach map (see [3, 4]) tells us that

$$z \in \sigma(H_{g,\theta}^{\geqslant\sigma}) \iff 0 \in \mathcal{F}_{P_\theta}(H_{g,\theta}^{\geqslant\sigma} - z) \iff b(z) = 0.$$

On the other hand, it follows from the usual perturbation theory, applied to the isolated non-degenerate eigenvalue λ_j of $H_{0,\theta}^{\geqslant\sigma}$, that the spectrum of $H_{g,\theta}^{\geqslant\sigma}$ is not empty in $D(\lambda_j, \sigma/2)$, for g sufficiently small. Hence there exists a unique $\lambda_{j,g}^{\geqslant\sigma}$ in $D(\lambda_j, \sigma/2)$ such that $b(\lambda_{j,g}^{\geqslant\sigma}) = 0$, that is

$$\sigma(H_{g,\theta}^{\geqslant\sigma}) \cap D(\lambda_j, \frac{\sigma}{2}) = \{\lambda_{j,g}^{\geqslant\sigma}\}. \tag{4.11}$$

Note that for $g\sigma^{-1/2} \ll (\mathrm{d}_j \sin(\mathrm{Im}\theta))^2$ and any z in $D(\lambda_j, \sigma/2)$ the Neumann series

$$\left[\overline{P}_\theta H_{0,\theta}^{\geqslant\sigma} - z\right]^{-1} \sum_{n\geq 0} \left(-W_{g,\theta}^{\geqslant\sigma}\left[\overline{P}_\theta H_{0,\theta}^{\geqslant\sigma} - z\right]^{-1}\right)^n = \left[\overline{P}_\theta H_{g,\theta}^{\geqslant\sigma} \overline{P}_\theta - z\right]^{-1} \tag{4.12}$$

is convergent.

To prove (4.4), we use the following identity (see [3]):

$$\begin{aligned} \left[H_{g,\theta}^{\geqslant\sigma} - z\right]^{-1} = &\left[P_\theta - \left[\overline{P}_\theta\left(H_{g,\theta}^{\geqslant\sigma} - z\right)\overline{P}_\theta\right]^{-1} \overline{P}_\theta W_{g,\theta}^{\geqslant\sigma} P_\theta\right] \\ &\left[\mathcal{F}_{P_\theta}(H_{g,\theta}^{\geqslant\sigma} - z)\right]^{-1} \left[P_\theta - P_\theta W_{g,\theta}^{\geqslant\sigma} \overline{P}_\theta \left[\overline{P}_\theta\left(H_{g,\theta}^{\geqslant\sigma} - z\right)\overline{P}_\theta\right]^{-1}\right] \\ &+ \left[\overline{P}_\theta\left(H_{g,\theta}^{\geqslant\sigma} - z\right)\overline{P}_\theta\right]^{-1} \overline{P}_\theta, \end{aligned} \tag{4.13}$$

which holds for z in $\rho(H^{\geqslant\sigma}_{g,\theta})\cap D(\lambda_j,\sigma/2)$. The simple form of $\mathcal{F}_{P_\theta}(H^{\geqslant\sigma}_{g,\theta}-z)$, (4.12) and the fact that $|a(z)-a(\lambda^{\geqslant\sigma}_{j,g})|=O(g^{2+\varepsilon})$ by (4.7) lead to

$$\left\|\left[H^{\geqslant\sigma}_{g,\theta}-z\right]^{-1}\right\|\leq \mathrm{C}_1\left(\frac{1+g^2\sigma^{-1}}{|z-\lambda^{\geqslant\sigma}_{j,g}|-\mathrm{C}_2 g^{2+\varepsilon}}+\sigma^{-1}\right), \tag{4.14}$$

for some positive constants $\mathrm{C}_1,\mathrm{C}_2$. Hence the proposition is proven for z in $D(\lambda_j,\sigma/3)$ such that $|z-\lambda^{\geqslant\sigma}_{j,g}|\gg g^{2+\varepsilon}$. □

Recall that, for $g\geq 0$, $P^{\geqslant\sigma}_{g,\theta}$ denotes the projection onto the eigenspace associated with the eigenvalue $\lambda^{\geqslant\sigma}_{j,g}$ of $H^{\geqslant\sigma}_{g,\theta}$.

Proposition 4.2 *Let g,σ as in Proposition 4.1 and choose $\theta\in D(0,\theta_0)$ such that* $\mathrm{Im}\theta\neq 0$. *Then, for g small enough,*

$$\left\|P^{\geqslant\sigma}_{g,\theta}-P^{\geqslant\sigma}_{0,\theta}\right\|\leq \mathrm{C}_0 g\sigma^{-1/2}, \tag{4.15}$$

where C_0 *is a positive constant.*

Proof. Let $\mathcal{C}_j$ denote a circle centered at λ_j, with radius $\sigma/3$, so that $\mathcal{C}_j\subset\rho(H^{\geqslant\sigma}_{g,\theta})$. Since we have assumed $g^2\ll\sigma$, for g sufficiently small, $\mathcal{C}_j$ contains both $\lambda^{\geqslant\sigma}_{j,g}$ and λ_j. Thus,

$$P^{\geqslant\sigma}_{g,\theta}-P^{\geqslant\sigma}_{0,\theta}=\oint_{\mathcal{C}_j}\left[R^{\geqslant\sigma}_{g,\theta}(z)-R^{\geqslant\sigma}_{0,\theta}(z)\right]dz. \tag{4.16}$$

We expand $R^{\geqslant\sigma}_{g,\theta}(z)$ into a Neumann series

$$R^{\geqslant\sigma}_{g,\theta}(z)=R^{\geqslant\sigma}_{0,\theta}(z)\sum_{n\geq 0}\left(-W^{\geqslant\sigma}_{g,\theta}R^{\geqslant\sigma}_{0,\theta}(z)\right)^n, \tag{4.17}$$

and we claim that, for all $n\geq 1$,

$$\left\|R^{\geqslant\sigma}_{0,\theta}(z)\left(-W^{\geqslant\sigma}_{g,\theta}R^{\geqslant\sigma}_{0,\theta}(z)\right)^n\right\|\leq\frac{\mathrm{C}_1}{\sigma}\left(\mathrm{C}_2 g\sigma^{-1/2}\right)^n. \tag{4.18}$$

Here C_1 and C_2 denote positive constants. The last estimate (4.18) could be proven by using the method of [5, Lemma 3.14]. Inserting this in (4.16) and using the fact that the radius of $\mathcal{C}_j$ is equal to $\sigma/3$, we obtain

$$\left\|P^{\geqslant\sigma}_{g,\theta}-P^{\geqslant\sigma}_{0,\theta}\right\|=\left\|\oint_{\mathcal{C}_j}R^{\geqslant\sigma}_{0,\theta}(z)\sum_{n\geq 1}\left(-W^{\geqslant\sigma}_{g,\theta}R^{\geqslant\sigma}_{0,\theta}(z)\right)^n dz\right\|\leq \mathrm{C}_0 g\sigma^{-1/2}, \tag{4.19}$$

provided that $g\sigma^{-1/2}$ is sufficiently small. Hence the proposition is proven. □

We now estimate the difference between the eigenvalues $\lambda_{j,g}$ and $\lambda^{\geqslant\sigma}_{j,g}$ of $H_{g,\theta}$ and $H^{\sigma}_{g,\theta}$.

Proposition 4.3 *Suppose* $0<g^2\ll\sigma<g^{\frac{3}{2+\mu}}$. *Then*

$$\lambda_{j,g}-\lambda^{\geqslant\sigma}_{j,g}=O\left(g^{4-\frac{3}{2+\mu}}\right). \tag{4.20}$$

Proof. For g and σ small enough, we choose $\theta \in D(0,\theta_0)$, $\mathrm{Im}(\theta) \neq 0$, such that $0 < g\sigma^{-1/2} \ll (\mathrm{d}_j \sin(\mathrm{Im}\theta))^2 < 1$. For ρ such that $g\rho^{-1/2} \ll (\mathrm{d}_j \sin(\mathrm{Im}\theta))^2$, let $P_\theta := P_{p,j,\theta} \otimes \mathbf{1}_{H_f<\rho}$. Following [3, 5], $\lambda_{j,g}$ satisfies

$$\left|\lambda_{j,g} - \lambda_j - g^2 Z_{j,\theta}\right| \leq \sum_{i=1}^{5} \|\mathrm{Rem}_i\|, \tag{4.21}$$

where $Z_{j,\theta} := Z^{\mathrm{od}}_{j,\theta} + Z^{\mathrm{d}}_{j,\theta}$, with $Z^{\mathrm{od}}_{j,\theta}$ and $Z^{\mathrm{d}}_{j,\theta}$ given by (4.8) – (4.9), and

$$\mathrm{Rem}_1 = P_\theta W_{g,\theta} P_\theta, \tag{4.22}$$

$$\mathrm{Rem}_2 = P_\theta W_{g,\theta} \overline{P}_\theta \left[\overline{P}_\theta H_{0,\theta} - \lambda_{j,g}\right]^{-1} \overline{P}_\theta W_{g,\theta} P_\theta - g^2 Q_\theta \tag{4.23}$$

$$\mathrm{Rem}_3 = g^2 \left[Q_\theta - Z_{j,\theta}\right], \tag{4.24}$$

$$\mathrm{Rem}_4 = P_\theta W_{g,\theta} \left(\overline{P}_\theta \left[\overline{P}_\theta H_{0,\theta} - \lambda_{j,g}\right]^{-1} \overline{P}_\theta W_{g,\theta}\right)^2 P_\theta, \tag{4.25}$$

$$\mathrm{Rem}_5 = P_\theta W_{g,\theta} \sum_{n\geq 3} \left(\overline{P}_\theta \left[\overline{P}_\theta H_{0,\theta} - \lambda_{j,g}\right]^{-1} \overline{P}_\theta W_{g,\theta}\right)^n P_\theta. \tag{4.26}$$

Here we have set

$$Q_\theta = \int_{\mathbb{R}^3} P_\theta \overline{G_{x,\theta}}(k) \left[\frac{\overline{P}_\theta(\omega(k))}{H_{0,\theta} + e^{-\theta}\omega(k) - \lambda_{j,g}}\right] G_{x,\theta}(k) P_\theta dk. \tag{4.27}$$

where $\overline{P}_\theta(\omega(k)) := \mathbf{1} - P_\theta(\omega(k))$, and $P_\theta(\omega(k)) := P_{p,j,\theta} \otimes \mathbf{1}_{H_f+\omega(k)<\rho}$. Using the expression (2.1) of $G_{x,\theta}$ and estimates similar to [3, Lemmas IV.6-IV.12] or [5, Lemma 3.16], we claim that

$$\|\mathrm{Rem}_1\| = O\left(g\rho^{1+\mu}\right), \quad \|\mathrm{Rem}_2\| = O\left(g^2\rho^{1+\mu}\right), \quad \|\mathrm{Rem}_3\| = O\left(g^2\rho\right). \tag{4.28}$$

The first bound in (4.28) easily follows from

$$\int_{\mathbb{R}^3} \|P_\theta \overline{G_{x,\theta}(k)} \otimes a^*(k)\| dk = \int_{\mathbb{R}^3} \|G_{x,\theta}(k) \otimes a(k) P_\theta\| dk \leq \mathrm{C}\rho^{1+\mu}. \tag{4.29}$$

The second one follows from normal-ordering (4.23) and using again (4.29). Finally, the last bound in (4.28) follows from computing the difference in (4.24) and using the estimate

$$\left\|\frac{\overline{P}_\theta(\omega(k))}{H_{0,\theta} + e^{-\theta}\omega(k) - \lambda_{j,g}}\right\| \leq \frac{\mathrm{C}_1}{(\mathrm{d}_j \sin \mathrm{Im}\theta)\omega(k)}, \tag{4.30}$$

for some positive constant C_1. Now it is proved in [3, 5] that $\|\mathrm{Rem}_4 + \mathrm{Rem}_5\| = O\left(g^3\rho^{-1/2}\right)$. Let us estimate these terms more precisely: we claim that

$$\|\mathrm{Rem}_4\| = O\left(g^3\rho^{\mu}\right), \quad \|\mathrm{Rem}_5\| = O\left(g^4\rho^{-1}\right). \tag{4.31}$$

To prove the first bound in (4.31), we decompose $W_{g,\theta}$ into $W_{g,\theta} = g(a^*(G_{x,\theta}) + a(G_{x,\theta}))$ and estimate each term separately by normal ordering. For instance, let us compute

$$\begin{aligned}
&P_\theta a(G_{x,\theta}) \frac{\overline{P}_\theta}{H_{0,\theta} - \lambda_{j,g}} a^*(G_{x,\theta}) \frac{\overline{P}_\theta}{H_{0,\theta} - \lambda_{j,g}} a(G_{x,\theta}) P_\theta \\
&= P_\theta \int_{\mathbb{R}^9} G_{x,\theta}(k_1) \otimes a(k_1) \frac{\overline{P}_\theta}{H_{0,\theta} - \lambda_{j,g}} \overline{G}_{x,\theta}(k_2) \otimes a^*(k_2) \frac{\overline{P}_\theta}{H_{0,\theta} - \lambda_{j,g}} \\
&\qquad G_{x,\theta}(k_3) \otimes a(k_3) P_\theta dk_1 dk_2 dk_3.
\end{aligned} \tag{4.32}$$

It follows from a pull-through formula and the canonical commutation rules that the "worst" term we have to estimate from the rhs of (4.32) is

$$\begin{aligned} \mathrm{T}(\rho,\theta) := P_\theta \int_{\mathbb{R}^6} G_{x,\theta}(k_1)\otimes \mathbf{1}\frac{\overline{P}_\theta(\omega(k_1))}{H_{0,\theta}+e^{-\theta}\omega(k_1)-\lambda_{j,g}}\overline{G}_{x,\theta}(k_1)\otimes\mathbf{1}\frac{\overline{P}_\theta}{H_{0,\theta}-\lambda_{j,g}} \\ G_{x,\theta}(k_3)\otimes a(k_3)P_\theta dk_1 dk_3. \end{aligned} \tag{4.33}$$

One can see that

$$\left\| \frac{\overline{P}_\theta}{H_{0,\theta}-\lambda_{j,g}}\right\| \leq \frac{\mathrm{C}_1}{(\mathrm{d}_j \sin \mathrm{Im}\theta)\rho}, \tag{4.34}$$

for some positive constant C_1. Inserting this together with (4.30) into (4.33), we get

$$\begin{aligned} \|\mathrm{T}(\rho,\theta)\| &\leq \frac{\mathrm{C}_1^2}{(\mathrm{d}_j\sin\mathrm{Im}\theta)^2}\rho^{-1}\int_{\mathbb{R}^3}\frac{|G_{x,\theta}(k_1)|^2}{\omega(k_1)}dk_1\int_{\mathbb{R}^3}\|G_{x,\theta}(k_3)\otimes a(k_3)P_\theta\|dk_3 \\ &\leq \frac{\mathrm{C}_2}{(\mathrm{d}_j\sin\mathrm{Im}\theta)^2}\rho^{\mu}, \end{aligned} \tag{4.35}$$

where C_2 is a positive constant. Since the other terms could be estimated in the same way, the first bound in (4.31) follows; the second bound in (4.31) can be obtained by using similar computations (see also [5, Lemma 3.16]).

For $\rho > \sigma$ the eigenvalue $\lambda_{j,g}^{\geqslant\sigma}$ of $H_{g,\theta}^{\geqslant\sigma}$ is given by the formulas (4.21)−(4.27), except that $W_{g,\theta}$ and $G_{x,\theta}(k)$ are replaced respectively by $W_{g,\theta}^{\geqslant\sigma}$ and $G_{x,\theta}^{\geqslant\sigma}(k)$. For the terms analogous to $Z_{j,\theta}^{\mathrm{od}}$ and $Z_{j,\theta}^{\mathrm{d}}$ we have by a straightforward computations that

$$\int_{|k|\leq\sigma}\mathcal{U}_\theta P_{p,j}\overline{G_x}(k)\overline{P}_{p,j}\left[H_p-\lambda_j+\omega(k)-i0\right]^{-1}\overline{P}_{p,j}G_x(k)P_{p,j}\mathcal{U}_\theta^{-1}dk=0, \tag{4.36}$$

$$\int_{|k|\leq\sigma}\mathcal{U}_\theta P_{p,j}\overline{G_x}(k)P_{p,j}G_x(k)P_{p,j}\mathcal{U}_\theta^{-1}\frac{dk}{\omega(k)} = O\left(\sigma^{1+2\mu}\right), \tag{4.37}$$

where in (4.36) we used the fact that $\sigma < \mathrm{d}_j$. Hence, with the obvious notation, $Z_{j,\theta} = Z_{j,\theta}^{\geqslant\sigma} + O(\sigma^{1+2\mu})$. Furthermore, Eqns. (4.28)−(4.31) still hold for $\lambda_{j,g}^{\geqslant\sigma}$. Hence remembering the assumptions $\sigma<\rho$, $g^2\ll\rho$ we obtain

$$\lambda_{j,g}-\lambda_{j,g}^{\geqslant\sigma} = O\left(g\rho^{1+\mu}\right)+O\left(g^2\rho\right)+O\left(g^4\rho^{-1}\right). \tag{4.38}$$

Optimizing with respect to ρ leads to the claim of the proposition. □

The following lemma was used in the proof of Lemmata 3.2 and 3.4.

Lemma 4.4 *Let θ in $D(0,\theta_0)$, $\mathrm{Im}\theta>0$ and let g,σ be such that $0<g^2\ll\sigma<g^{\frac{3}{2+\mu}}\ll 1$. Then for all $z\in\mathrm{I}$ and $n\geq 1$, we have the estimate:*

$$\left\|\widetilde{R}_{g,\theta}^{\sigma}(z)\left(\widetilde{W}_{g,\theta}^{\leqslant\sigma}\widetilde{R}_{g,\theta}^{\sigma}(z)\right)^n\right\| \leq \mathrm{C}_1 g^{-2}\left(\mathrm{C}_2(\sigma^{1/2+\mu}+g^{2-\frac{3}{2+\mu}})\right)^n, \tag{4.39}$$

where $\mathrm{C}_1,\mathrm{C}_2$ are positive constants.

Proof. Recall that

$$\widetilde{W}_{g,\theta}^{\leqslant\sigma} = ga^*(G_{x,\theta}^{\leqslant\sigma})+ga(G_{x,\theta}^{\leqslant\sigma})+\left(\lambda_{j,g}-\lambda_{j,g}^{\geqslant\sigma}\right). \tag{4.40}$$

Let $\Gamma_\sigma = \Gamma(\mathbf{1}_{|k|\le\sigma})$ be the second quantization of $\mathbf{1}_{|k|\le\sigma}$. Note that $\mathcal{V}\Gamma_\sigma H_f \mathcal{V}^{-1} = I \otimes H_f^{\leqslant\sigma}$. Hence, from the spectral representation (3.17) and the decomposition (3.5), we can write

$$\widetilde{R}^\sigma_{g,\theta}(z) = \widehat{R}^\sigma_{g,\theta}(z) + \mathcal{V}^{-1}(P^{\geqslant\sigma}_{g,\theta} \otimes I)\mathcal{V}\left[z - \lambda^{\geqslant\sigma}_{j,g} - e^{-\theta}\Gamma_\sigma H_f\right]^{-1}, \tag{4.41}$$

where $\widehat{R}^\sigma_{g,\theta}(z) := (I - \mathcal{V}^{-1}(P^{\geqslant\sigma}_{g,\theta} \otimes I)\mathcal{V})\widetilde{R}^\sigma_{g,\theta}(z)$. We have

$$\mathcal{V}\widehat{R}^\sigma_{g,\theta}(z)\mathcal{V}^{-1} = \int_{\sigma(H_f^{\leqslant\sigma})} \widehat{R}^{\geqslant\sigma}_{g,\theta}(z - e^{-\theta}x) \otimes dE_{H_f^{\leqslant\sigma}}(x). \tag{4.42}$$

It follows from Proposition 4.1 that $\|\widehat{R}^\sigma_{g,\theta}(z)\| = O(\sigma^{-1})$ for all z in I. Besides,

$$\left\|\left[z - \lambda^{\geqslant\sigma}_{j,g} - e^{-\theta}\Gamma_\sigma H_f\right]^{-1}\right\| \le \left|\mathrm{Im}(\lambda^{\geqslant\sigma}_{j,g})\right|^{-1} = O(g^{-2}), \tag{4.43}$$

provided that Fermi's Golden Rule holds. Since Γ_σ commutes with $a(G^{\leqslant\sigma}_{x,\theta})$ and $a^*(G^{\leqslant\sigma}_{x,\theta})$, one can show by using a pull-through formula that

$$\left\|a(G^{\leqslant\sigma}_{x,\theta})\left[z - \lambda^{\geqslant\sigma}_{j,g} - e^{-\theta}\Gamma_\sigma H_f\right]^{-1}\right\| = O(g^{-1}\sigma^{1/2+\mu}). \tag{4.44}$$

Similarly (4.42) leads to

$$\left\|a(G^{\leqslant\sigma}_{x,\theta})\widehat{R}^\sigma_{g,\theta}(z)\right\| = O(\sigma^\mu). \tag{4.45}$$

The claim of the lemma then follows from (4.40) – (4.45), the assumption $g^2 \ll \sigma < g^{\frac{3}{2+\mu}}$, and Proposition 4.3. □

5 Extension to non-relativistic QED

Now we extend the analysis above to the standard Hamiltonian of non-relativistic QED introduced in (1.2), Sect. 1. The results and proofs of Sections 3 - 4 go through without a change except for the proof of Lemma 3.4. In the non-relativistic QED case, $W_{g,\theta}$ is given by

$$W_{g,\theta} = \frac{g}{m}e^{-\theta}p \cdot A_\theta(x) + \frac{g^2}{2m}A_\theta(x)\cdot A_\theta(x) - \frac{g^2}{2m}\Lambda,$$

where we used the notation $p \cdot A_\theta(x) := i\sum_{j=1}^N \nabla_j \cdot A_\theta(x_j)$, and similarly for $A_\theta(x)\cdot A_\theta(x)$. The quantized vector potential $A_\theta(x_j)$ is given by (1.3), and $\Lambda := 2N\int \chi(k)^2/|k|d^3k$. Here we set

$$W^{\leqslant\sigma}_{g,\theta} = \frac{g}{m}e^{-\theta}p\cdot A^{\leqslant\sigma}_\theta(x) + \frac{g^2}{2m}A^{\leqslant\sigma}_\theta(x)\cdot A^{\geqslant\sigma}_\theta(x) + \frac{g^2}{2m}A^{\leqslant\sigma}_\theta(x)\cdot A^{\leqslant\sigma}_\theta(x) - \frac{g^2}{2m}\Lambda^{\leqslant\sigma},$$

where $\Lambda^{\leqslant\sigma} := 2N\int_{|k|\le\sigma}\chi(k)^2/|k|d^3k$. Hence the QED Hamiltonian satisfies the condition (2.7) with $\mu = 0$. We show now how to overcome this difficulty (a technique alternative to the one used here can be found in [12]).

In our sketch of the proof of Lemma 3.4, we begin with the most singular term B^1 of the expansion (3.34), Section 3. Thus we have to bound the term $\widetilde{R}^\sigma_{g,\theta}\widetilde{W}^{\leqslant\sigma}_{g,\theta}\widetilde{R}^\sigma_{g,\theta}$. The part of $\widetilde{W}^{\leqslant\sigma}_{g,\theta}$

involving the difference of the eigenvalues is estimated in the same way as before, and its norm is of order $O(\sigma g^{-3/2})$. The remaining part can be written as

$$\begin{aligned} W_{g,\theta}^{\leqslant\sigma} &= g[H_{g,\theta}^{\sigma}, x]\cdot A_{\theta}^{\leqslant\sigma}(x) + \frac{g^2}{2m}\Big[A_{\theta}^{\leqslant\sigma}(x)\cdot A_{\theta}^{\leqslant\sigma}(x) - \Lambda^{\leqslant\sigma}\Big] \\ &= g[H_{g,\theta}^{\sigma}, x]\cdot A_{\theta}^{\leqslant\sigma}(0) + \underbrace{g[H_{g,\theta}^{\sigma}, x]\cdot (A_{\theta}^{\leqslant\sigma}(x) - A_{\theta}^{\leqslant\sigma}(0))}_{I} + \underbrace{\frac{g^2}{2m}\Big[A_{\theta}^{\leqslant\sigma}(x)\cdot A_{\theta}^{\leqslant\sigma}(x) - \Lambda^{\leqslant\sigma}\Big]}_{II}, \end{aligned}$$

We can now rewrite the operator $\widetilde{R}_{g,\theta}^{\sigma} W_{g,\theta}^{\leqslant\sigma} \widetilde{R}_{g,\theta}^{\sigma}$ as

$$\begin{aligned} & gx\cdot A_{\theta}^{\leqslant\sigma}(0)\widetilde{R}_{g,\theta}^{\sigma} + g\widetilde{R}_{g,\theta}^{\sigma}\{-x\cdot A_{\theta}^{\leqslant\sigma}(0) - e^{-\theta}\underbrace{gx\cdot[H_f, A_{\theta}^{\leqslant\sigma}(0)]}_{III}\} + \widetilde{R}_{g,\theta}^{\sigma}(I+II)\widetilde{R}_{g,\theta}^{\sigma} \\ &= -g[\widetilde{R}_{g,\theta}^{\sigma}, x\cdot A_{\theta}^{\leqslant\sigma}(0)] + \widetilde{R}_{g,\theta}^{\sigma}(I+II)\widetilde{R}_{g,\theta}^{\sigma} + \widetilde{R}_{g,\theta}^{\sigma} III \\ &= g\widetilde{R}_{g,\theta}^{\sigma}[H_{g,\theta}^{\sigma}, x\cdot A_{\theta}^{\leqslant\sigma}(0)]\widetilde{R}_{g,\theta}^{\sigma} + \widetilde{R}_{g,\theta}^{\sigma}(I+II)\widetilde{R}_{g,\theta}^{\sigma} + \widetilde{R}_{g,\theta}^{\sigma} III. \end{aligned}$$

The factor $\widetilde{R}_{g,\theta}^{\sigma} II \widetilde{R}_{g,\theta}^{\sigma}$ is bounded from above by $O(\sigma)$, see (2.7). In order to estimate the factors $g\widetilde{R}_{g,\theta}^{\sigma}[H_{g,\theta}^{\sigma}, x\cdot A_{\theta}^{\leqslant\sigma}(0)]\widetilde{R}_{g,\theta}^{\sigma}$, $\widetilde{R}_{g,\theta}^{\sigma} I \widetilde{R}_{g,\theta}^{\sigma}$ and $\widetilde{R}_{g,\theta}^{\sigma} III$, we pull $e^{-\delta\langle x\rangle}$ from Ψ_j until it hits $x\cdot A_{\theta}^{\leqslant\sigma}(0)$, I or III, which, using estimate (2.7), gives a term whose norm is of order $O(g^{-1}\sigma^{3/2})$. Therefore, $|B^1|$ is of order $O(\sigma g^{-3/2})$.

For the terms B^n with $n \geq 2$, one does a symmetric version of the estimates in Lemma 3.4. For example, for the term $B^2(t,\theta)$, we have the bound

$$\begin{aligned} & |(\Psi_{j,\overline{\theta}}, \widetilde{R}_{g,\theta}^{\sigma} W_{g,\theta}^{\leqslant\sigma} \widetilde{R}_{g,\theta}^{\sigma} W_{g,\theta}^{\leqslant\sigma} \widetilde{R}_{g,\theta}^{\sigma}\Psi_{j,\theta})| \leq \mathrm{C} g^3 \| \underbrace{\widetilde{R}_{g,\theta}^{\sigma}\langle p_\theta\rangle}_{O(g^{-2})} \underbrace{A_{\theta}^{\leqslant\sigma}(0)\widetilde{R}_{g,\theta}^{\sigma} A_{\theta}^{\geqslant\sigma}(x) A_{\theta}^{\leqslant\sigma}(0)\widetilde{R}_{g,\theta}^{\sigma}}_{O(\sigma^{\frac{3}{2}}g^{-2})} \| \\ & \leq \mathrm{C}\sigma^{\frac{3}{2}} g^{-1}. \end{aligned}$$

Moreover, the most singular term in $B^2(t,\theta)$ involving the difference of the eigenvalues is

$$g|(\Psi_{j,\overline{\theta}}, \underbrace{\widetilde{R}_{g,\theta}^{\sigma}(\lambda_{j,g} - \lambda_{j,g}^{\geqslant\sigma})\mathcal{V}^{-1}P_{g,\theta}^{\geqslant\sigma}\otimes I\mathcal{V}}_{O(g^{1/2})} \underbrace{\widetilde{R}_{g,\theta}^{\sigma}\langle p_\theta\rangle A_{\theta}^{\leqslant\sigma}(0)\widetilde{R}_{g,\theta}^{\sigma}}_{O(\sigma^{1/2}g^{-3})}\Psi_{j,\theta})| \leq \mathrm{C}\sigma^{1/2}g^{-3/2},$$

which, together with the factor of σ gained from integration, gives that $|B^2|$ is of order $O(\sigma^{3/2}g^{-3/2})$. Higher terms are evaluated similarly.

Instead of (3.39), Section 3, we have in the case of non-relativistic QED

$$\big(\Psi_j, e^{-itH_g}f(H_g)\Psi_j\big) = e^{-it\lambda_{j,g}} + O(g^2\sigma^{-1}) + O(\sigma g^{-3/2}), \quad t \geq 0.$$

Optimizing and removing the f dependence as in the proof of Theorem 1.1 gives

$$\big(\Psi_j, e^{-itH_g}\Psi_j\big) = e^{-it\lambda_{j,g}} + O(g^{1/4}), \quad t \geq 0.$$

6 Proof of Theorem 1.4

We prove Theorem 1.4 for the set $\mathcal{D}'$ chosen explicitly as

$$\mathcal{D}' := \{\psi \in \mathcal{D} | \ \ \|d\Gamma(\omega^{-1/2})(1-P_\Omega)\psi\| < \infty\}$$

where P_Ω is the projection on the vacuum Ω in $\mathcal{F}_s$. Since $\mathcal{U}_\theta d\Gamma(\omega^{-1/2}) = e^{\theta/2} d\Gamma(\omega^{-1/2})\mathcal{U}_\theta$, the set $\mathcal{D}'$ is dense in $\mathcal{D}$.

Proof. The RG analysis [3, 4, 10, 13] shows that given $\delta > 0$, there exist $g_* > 0$ and $\varphi_* \in (0, \varphi_0)$ s.t. for $g \leq g_*$ and $\mathrm{Im}\theta \in (\varphi_*, \varphi_0)$, the spectrum of the operator $H_{g,\theta}$ in the half-plane $\{\mathrm{Re} z \leq \Sigma - \delta\}$ lies in the union of wedges

$$S_j := \lambda_{j,g} + \{z \in \mathbb{C} | \ |\arg(z) - \mathrm{Im}\theta| \leq \epsilon\},$$

where $\lambda_{j,g} = \lambda_j + O(g^2)$, $\mathrm{Im}\lambda_{j,g} \leq 0$, $\mathrm{Im}\lambda_{j,g} = O(g^2)$ and $\epsilon < |\mathrm{Im}\theta|$ is a positive number. Moreover, the apices, $\lambda_{j,g}$, of these wedges are the eigenvalues of $H_{g,\theta}$. If, in addition, condition (C) holds for λ_j then $\mathrm{Im}\lambda_{j,g} \leq -const.\ g^2$.

We take $z \in W^{\varphi_1,\varphi_2}_{\lambda_{j,g}}$ with $\varphi_1 = \pi/2 - \varphi_0$ and $\varphi_2 > 3\pi/2 - \varphi_*$. We want to estimate $(\psi, (H_{g,\theta} - z)^{-1}\psi)$. Using an infrared cut-off as in section 2, we decompose

$$H_{g,\theta} = \widetilde{H}^\sigma_{g,\theta} + \widetilde{W}^{\leqslant\sigma}_{g,\theta}, \tag{6.1}$$

see (2.14). The infrared cut-off Hamiltonian $\widetilde{H}^\sigma_{g,\theta}$ has an eigenvalue at $\lambda_{j,g}$. We use the second resolvent equation

$$(H_{g,\theta} - z)^{-1} = (\widetilde{H}^\sigma_{g,\theta} - z)^{-1} + (\widetilde{H}^\sigma_{g,\theta} - z)^{-1}\widetilde{W}^{\leqslant\sigma}_{g,\theta}(H_{g,\theta} - z)^{-1}. \tag{6.2}$$

Let $\widetilde{R}^\sigma_{g,\theta}(z) := (\widetilde{H}^\sigma_{g,\theta} - z)^{-1}$ and let $P^{\leqslant\sigma}_\Omega$ be the projection onto the vacuum state of $\mathcal{F}^{\leqslant\sigma}_s$ and $\overline{P} = 1 - P$. Then

$$\widetilde{R}^\sigma_{g,\theta}(z) = \frac{P^{\geqslant\sigma}_{g,\theta} \otimes P^{\leqslant\sigma}_\Omega}{\lambda_{j,g} - z} + \frac{P^{\geqslant\sigma}_{g,\theta} \otimes \overline{P}^{\leqslant\sigma}_\Omega}{\lambda_{j,g} + e^{-\theta} H^{\leqslant\sigma}_f - z} + \widehat{R}^\sigma_{g,\theta}(z), \tag{6.3}$$

where, as above,

$$\widehat{R}^\sigma_{g,\theta}(z) := (\overline{P}^{\geqslant\sigma}_{g,\theta} \otimes I^{\leqslant\sigma})\widetilde{R}^\sigma_{g,\theta}(z). \tag{6.4}$$

By our condition on z we can pick θ so that

$$\mathrm{Re}(e^\theta(\lambda_{j,g} - z)) \geq 0, \tag{6.5}$$

i.e. $|\mathrm{Im}\theta + \arg(\lambda_{j,g} - z)| \leq \pi/2$. Then

$$|(\psi, \frac{P^{\geqslant\sigma}_{g,\theta} \otimes \overline{P}^{\leqslant\sigma}_\Omega}{\lambda_{j,g} + e^{-\theta} H^{\leqslant\sigma}_f - z}\psi)| \leq \|(H^{\leqslant\sigma}_f)^{-1/2}\overline{P}^{\leqslant\sigma}_\Omega \psi\|^2. \tag{6.6}$$

(More generally, the l.h.s. is bounded by $|\lambda_{j,g} - z|^{-\alpha}\|(H^{\leqslant\sigma}_f)^{-(1-\alpha)/2}\overline{P}^{\leqslant\sigma}_\Omega \psi\|^2$ for $0 \leq \alpha \leq 1$.) Furthermore, an elementary analysis of the $n-$photon sectors shows that

$$\|(H^{\leqslant\sigma}_f)^{-1/2}\overline{P}^{\leqslant\sigma}_\Omega \psi\| \leq \|d\Gamma(\omega^{-1/2})\overline{P}_\Omega \psi\|. \tag{6.7}$$

Hence, by the definition of $\mathcal{D}'$, we have that, for all $\psi \in \mathcal{D}'$,

$$|(\psi, \frac{P^{\geqslant\sigma}_{g,\theta} \otimes \overline{P}^{\leqslant\sigma}_\Omega}{\lambda_{j,g} + e^{-\theta} H^{\leqslant\sigma}_f - z}\psi)| \leq \mathrm{C}. \tag{6.8}$$

Next, to estimate $\widehat{R}^{\sigma}_{g,\theta}(z)$, see Eq. (6.4), we use the representation (4.42) which implies that

$$\mathcal{V}(H_f^{\leqslant\sigma})^n\widehat{R}^{\sigma}_{g,\theta}(z)\mathcal{V}^{-1}=\int_{\sigma(H_f^{\leqslant\sigma})}\widehat{R}^{\geqslant\sigma}_{g,\theta}(z-e^{-\theta}\omega)\omega^n\otimes dE_{H_f^{\leqslant\sigma}}(\omega). \tag{6.9}$$

Now, applying to $H^{\geqslant\sigma}_{g,\theta}$ a renormalization group analysis as in [3, 5, 10, 13], one can show that for $|z-\lambda_{j,g}|\leq\sigma/2$ and $\omega\geq 0$

$$\|\widehat{R}^{\geqslant\sigma}_{g,\theta}(z-e^{-\theta}\omega)\|\leq \mathrm{C}(\sigma+\omega)^{-1}, \tag{6.10}$$

which, together with (6.9), implies, for $|z-\lambda_{j,g}|\leq\sigma/2$ and $n=0,1/2,1$, the estimate

$$\|(H_f^{\leqslant\sigma})^n\widehat{R}^{\sigma}_{g,\theta}(z)\|\leq \mathrm{C}\sigma^{n-1}, \tag{6.11}$$

for some constant C. Eqns (6.3), (6.8) and (6.11) imply that, for $\psi\in\mathcal{D}'$,

$$|(\psi,(\widetilde{R}^{\sigma}_{g,\theta}(z)-\frac{P^{\geqslant\sigma}_{g,\theta}\otimes P^{\leqslant\sigma}_{\Omega}}{\lambda_{j,g}-z})\psi)|\leq \mathrm{C}/\sigma, \tag{6.12}$$

Finally we estimate the last term on the r.h.s. Eq. (6.2). Recall that

$$\widetilde{W}^{\leqslant\sigma}_{g,\theta}=W^{\leqslant\sigma}_{g,\theta}-\left(\lambda_{j,g}-\lambda^{\geqslant\sigma}_{j,g}\right)\mathcal{V}^{-1}(P^{\geqslant\sigma}_{g,\theta}\otimes I)\mathcal{V}, \tag{6.13}$$

where

$$W^{\leqslant\sigma}_{g,\theta}:=W_{g,\theta}-W^{\geqslant\sigma}_{g,\theta}=g\phi(G^{\leqslant\sigma}_{x,\theta}). \tag{6.14}$$

Below, we let $\sigma\to 0$, as $|\lambda_{j,g}-z|\to 0$. Hence we have to estimate $\lambda_{j,g}-\lambda^{\geqslant\sigma}_{j,g}$ for any $\sigma>0$. We claim that

$$|\lambda_{j,g}-\lambda^{\geqslant\sigma}_{j,g}|=O\left((g\sigma^{1/2+\mu})^2\right). \tag{6.15}$$

This estimate is proven in the proposition at the end of this section.

Iterating the last term on the r.h.s. of Eq. (6.2) we see that the worst term is $\widetilde{R}^{\sigma}_{g,\theta}(z)\widetilde{W}^{\leqslant\sigma}_{g,\theta}\widetilde{R}^{\sigma}_{g,\theta}(z)$. We use the decomposition (6.3). Since the operator $W^{\leqslant\sigma}_{g,\theta}$ is in normal form, we see that the term coming from sandwiching it between the first term on the r.h.s. of (6.3) vanishes. Thus, it remains to consider the terms

$$\widetilde{R}^{\sigma}_{g,\theta}(z)\left(\lambda_{j,g}-\lambda^{\geqslant\sigma}_{j,g}\right)(P^{\geqslant\sigma}_{g,\theta}\otimes I)\widetilde{R}^{\sigma}_{g,\theta}(z), \tag{6.16}$$

$$\begin{aligned}&[\frac{P^{\geqslant\sigma}_{g,\theta}\otimes P^{\leqslant\sigma}_{\Omega}}{\lambda_{j,g}-z}+\frac{P^{\geqslant\sigma}_{g,\theta}\otimes \overline{P}^{\leqslant\sigma}_{\Omega}}{\lambda_{j,g}+e^{-\theta}H_f^{\leqslant\sigma}-z}+\widehat{R}^{\sigma}_{g,\theta}(z)]\\ &\times W^{\leqslant\sigma}_{g,\theta}[\frac{P^{\geqslant\sigma}_{g,\theta}\otimes \overline{P}^{\leqslant\sigma}_{\Omega}}{\lambda_{j,g}+e^{-\theta}H_f^{\leqslant\sigma}-z}+\widehat{R}^{\sigma}_{g,\theta}(z)]\end{aligned} \tag{6.17}$$

and the term obtained by switching the right and left factors in (6.17).

We note that, by the decomposition (6.3) and the definition of $\widehat{R}^{\sigma}_{g,\theta}(z)$, Eq. (6.16) can be written as

$$\left(\lambda_{j,g}-\lambda^{\geqslant\sigma}_{j,g}\right)\frac{P^{\geqslant\sigma}_{g,\theta}\otimes P^{\leqslant\sigma}_{\Omega}}{(\lambda_{j,g}-z)^2}+\left(\lambda_{j,g}-\lambda^{\geqslant\sigma}_{j,g}\right)\frac{P^{\geqslant\sigma}_{g,\theta}\otimes \overline{P}^{\leqslant\sigma}_{\Omega}}{(\lambda_{j,g}+e^{-\theta}H_f^{\leqslant\sigma}-z)^2}. \tag{6.18}$$

Using (6.15) we obtain the following estimate for (6.16):

$$(6.16) = O\left((g\sigma^{1/2+\mu}|\lambda_{j,g} - z|^{-1})^2\right). \tag{6.19}$$

To estimate (6.17), we first observe that, due to (6.5), we have that, for $n = 0, 1/2, 1$

$$\|(H_f^{\leqslant\sigma})^n(\lambda_{j,g} + e^{-\theta}H_f^{\leqslant\sigma} - z)^{-1}\| \leq \mathrm{C}|\lambda_{j,g} - z|^{n-1}. \tag{6.20}$$

Assume $\sigma \geq |z - \lambda_{j,g}|$. Using estimates (2.7), (6.11) and (6.20) (or (6.26)), the fact that $P_\Omega^{\leqslant\sigma} a^*(G_{x,\theta}^{\leqslant\sigma}) = 0$ and standard estimates on the creation and annihilation operators, and remembering the condition that $\mathrm{Re}(e^\theta(\lambda_{j,g} - z)) \geq 0$, we obtain the bound

$$\|\widetilde{R}^\sigma_{g,\theta}(z) W^{\leqslant\sigma}_{g,\theta} \widetilde{R}^\sigma_{g,\theta}(z)\| \leq \mathrm{C}\frac{1}{|z - \lambda_{j,g}|} g\sigma^{\frac{1}{2}+\mu}(\frac{1}{|z - \lambda_{j,g}|^{1/2}} + \frac{1}{\sigma^{1/2}}). \tag{6.21}$$

This together with (6.19) yields

$$\|\widetilde{R}^\sigma_{g,\theta}(z) \widetilde{W}^{\leqslant\sigma}_{g,\theta} \widetilde{R}^\sigma_{g,\theta}(z)\| \leq \mathrm{C}\frac{g\sigma^{\frac{1}{2}+\mu}}{|z - \lambda_{j,g}|}(\frac{1}{\sigma^{1/2}} + \frac{1}{|z - \lambda_{j,g}|^{1/2}} + \frac{g\sigma^{\frac{1}{2}+\mu}}{|z - \lambda_{j,g}|}). \tag{6.22}$$

Since, as we mentioned, the higher order iterates of (6.2) are estimated similarly and lead to improved estimates, we conclude, assuming $\sigma \geq |z - \lambda_{j,g}|$, that

$$\|\widetilde{R}^\sigma_{g,\theta}(z) \widetilde{W}^{\leqslant\sigma}_{g,\theta} R_{g,\theta}(z)\| \leq \mathrm{C}(\frac{g\sigma^{\frac{1}{2}+\mu}}{|z - \lambda_{j,g}|^{3/2}} + \frac{g^2\sigma^{1+2\mu}}{|z - \lambda_{j,g}|^2}), \tag{6.23}$$

where $R_{g,\theta}(z) := (H_{g,\theta} - z)^{-1}$.

It follows from (6.2), (6.12) and (6.23) that, for g small enough,

$$|(\psi, ((H_{g,\theta} - z)^{-1} - \frac{1}{\lambda_{j,g} - z} P^{\geqslant\sigma}_{g,\theta} \otimes P^{\leqslant\sigma}_\Omega)\psi)| \leq \mathrm{C}(\frac{1}{\sigma} + \frac{g\sigma^\alpha}{r^{3/2}} + \frac{g^2\sigma^{2\alpha}}{r^2}), \tag{6.24}$$

where $r := |z - \lambda_{j,g}|$ and $\alpha := 1/2 + \mu$, for some constant C, provided $\sigma \geq |z - \lambda_{j,g}|$. We now pick $\sigma = r^\beta g^{-(3/2+\mu)^{-1}}$, where $\beta := (1 + \frac{2}{3}\mu)^{-1}$. By our assumption, $\beta < 1$ and therefore $\sigma > r$. Then, for this choice of σ,

$$|(\psi, ((H_{g,\theta} - z)^{-1} - \frac{1}{\lambda_{j,g} - z} P^{\geqslant\sigma}_{g,\theta} \otimes P^{\leqslant\sigma}_\Omega)\psi)| \leq \mathrm{C}g^{(3/2+\mu)^{-1}} r^{-\beta}.$$

Let $\psi_\theta := \mathcal{U}_\theta\psi$. The last estimate, together with the relation

$$(\psi, (H_g - z)^{-1}\psi) = (\psi_{\bar\theta}, (H_{g,\theta} - z)^{-1}\psi_\theta), \tag{6.25}$$

implies (i) and (ii) in Theorem 1.4, with $\beta = (1 + \frac{2}{3}\mu)^{-1}$. □

Remark 6.1 *The expression for β can be improved if one uses* (6.7) *to conclude that, for $n = 0, 1/2$,*

$$\|(H_f^{\leqslant\sigma})^n(\lambda_{j,g} + e^{-\theta}H_f^{\leqslant\sigma} - z)^{-1}\psi\| \leq \mathrm{C}|\lambda_{j,g} - z|^{n-1/2}\|d\Gamma(\omega^{-1/2})\overline{P}_\Omega\psi\|, \tag{6.26}$$

which is better than (6.20). *This estimate leads to the inequality*

$$|(\psi, \widetilde{R}^\sigma_{g,\theta}(z) W^{\leqslant\sigma}_{g,\theta} \widetilde{R}^\sigma_{g,\theta}(z)\psi)| \leq \mathrm{C}\frac{1}{|z - \lambda_{j,g}|} g\sigma^{\frac{1}{2}+\mu}(\frac{\sigma^{1/2}}{|z - \lambda_{j,g}|^{1/2}} + \frac{1}{\sigma^{1/2}}), \tag{6.27}$$

which has a better r.h.s than (6.21).

Proposition 6.2 *Under the conditions of Theorem 1.4, the estimate* (6.15) *holds for any* $\sigma > 0$.

Proof. To prove (6.15) we use the RG approach. Here we only point out particularities of the present problem and outline the general strategy; technical details can be found in [3, 4, 10, 13]. Since we do not go into details, we use the Feshbach-Schur map, rather than the smooth Feshbach-Schur map, to underpin our construction. The former ([3, 4]) is simpler to formulate but the latter ([10, 14, 13]) is easier to handle technically. Our strategy follows ([13]).

First we apply the Feshbach-Schur map $\mathcal{F}_{\rho_0}$ associated to the projection $P_\rho := P_{g,\theta}^{\geqslant\sigma} \otimes \chi_\rho^{\leqslant\sigma}$, where $\chi_\rho^{\leqslant\sigma} := \chi_{H_f^{\leqslant\sigma} \leqslant \rho}$. For $z \in D(\lambda_{j,g}^{\geqslant\sigma}, \sigma/2)$ and $\rho_0 = \sigma$, the operator $H_{g,\theta} - z$ is in the domain of $\mathcal{F}_{P_{\rho_0}}$. Indeed, an easy estimate shows that the operator $\overline{P}_{\rho_0} H_{g,\theta}^{\sigma} \overline{P}_{\rho_0} - z$ is invertible on $\mathrm{Ran}\overline{P}_{\rho_0}$ and $\|[H_f^{\leqslant\sigma} + \sigma]\overline{P}_{\rho_0}\left[\overline{P}_{\rho_0} H_{g,\theta}^{\sigma} \overline{P}_{\rho_0} - z\right]^{-1} \overline{P}_{\rho_0}\| \leq \mathrm{C}$. Since $\|[H_f^{\leqslant\sigma} + \sigma]^{-1/2} W_{g,\theta}^{\leqslant\sigma} [H_f^{\leqslant\sigma} + \sigma]^{-1/2}\| \leq \mathrm{C} g\sigma^{\mu}$, we see by Neumann series expansion that the operator $\overline{P}_{\rho_0} H_{g,\theta} \overline{P}_{\rho_0} - z$ is invertible on $\mathrm{Ran}\overline{P}_{\rho_0}$ and $\|\overline{P}_{\rho_0}\left[\overline{P}_{\rho_0} H_{g,\theta} \overline{P}_{\rho_0} - z\right]^{-1} \overline{P}_{\rho_0}\| \leq \mathrm{C}/\sigma$. Hence the operator $H_{g,\theta} - z$ is in the domain of $\mathcal{F}_{P_{\rho_0}}$, as claimed. Next, we note that

$$\mathcal{F}_{P_{\rho_0}}(H_{g,\theta} - z) = P_{g,\theta}^{\geqslant\sigma} \otimes H_z,$$

where the operator H_z acts on $\mathrm{Ran}\chi_{\rho_0}^{\leqslant\sigma} \subset \mathcal{F}_s^{\leqslant\sigma}$ and is given by

$$H_z := \chi_{\rho_0}^{\leqslant\sigma}(\psi_{g,\theta}^{\geqslant\sigma}, (\lambda_{j,g}^{\geqslant\sigma} - z + H_f^{\leqslant\sigma} + W_{g,\theta}^{\leqslant\sigma} + U)\psi_{g,\theta}^{\geqslant\sigma})\chi_{\rho_0}^{\leqslant\sigma}, \tag{6.28}$$

where $U := -W_{g,\theta}^{\leqslant\sigma} \overline{P}_{\rho_0}\left[\overline{P}_{\rho_0} H_{g,\theta} \overline{P}_{\rho_0} - z\right]^{-1} \overline{P}_{\rho_0} W_{g,\theta}^{\leqslant\sigma}$.

By the isospectrality of the Feshbach-Schur map (see [3, 4, 5, 10, 13]), we have that $z \in D(\lambda_{j,g}^{\geqslant\sigma}, \sigma/2)$ is an eigenvalue of $H_{g,\theta}$ iff 0 is an eigenvalue of H_z. To investigate the spectral properties of H_z, we make use of the renormalization group method.

As a first step, we rewrite the operator H_z in a generalized normal form. To this end we expand the resolvent on the r.h.s. in a Neumann series in $W_{g,\theta}^{\leqslant\sigma}$ and normal order the creation and annihilation operators not entering the expression for $H_f^{\leqslant\sigma}$. This brings the operator H_z to the form (see [3, 4, 10, 13])

$$H_z := \chi_{\rho_0}^{\leqslant\sigma}(E_z + T_z + W_z)\chi_{\rho_0}^{\leqslant\sigma}, \tag{6.29}$$

where E_z is a number (more precisely, a complex function of z and other parameters), T_z is a differentiable function of $H_f^{\leqslant\sigma}$ and W_z is an operator in the generalized normal form that is a sum of terms with at least one creation or annihilation operator. A standard computation gives that $E_z := \lambda_{j,g}^{\geqslant\sigma} - z + \Delta E_z$, with

$$\Delta E_z := -\int (\psi_{g,\theta}^{\geqslant\sigma}, G_{x,\theta}^{\leqslant\sigma}(k) \overline{P}_{g,\theta}^{\geqslant\sigma}\left[\overline{P}_{g,\theta}^{\geqslant\sigma} H_{g,\theta}^{\geqslant\sigma} \overline{P}_{g,\theta}^{\geqslant\sigma} + e^{-\theta}\omega - z\right]^{-1} \overline{P}_{g,\theta}^{\geqslant\sigma} G_{x,\theta}^{\leqslant\sigma}(k)\psi_{g,\theta}^{\geqslant\sigma})dk + h.o.t.,$$

$$T_z := H_f^{\leqslant\sigma} - \int (\psi_{g,\theta}^{\geqslant\sigma}, G_{x,\theta}^{\leqslant\sigma}(k) f(H_f^{\leqslant\sigma} + \omega) G_{x,\theta}^{\leqslant\sigma}(k)\psi_{g,\theta}^{\geqslant\sigma})dk + h.o.t.,$$

$$W_z := (\psi_{g,\theta}^{\geqslant\sigma}, (W_{g,\theta}^{\leqslant\sigma} - \int\int G_{x,\theta}^{\leqslant\sigma}(k) a^*(k) f(H_f^{\leqslant\sigma} + \omega + \omega') a(k') G_{x,\theta}^{\leqslant\sigma}(k') dk dk')\psi_{g,\theta}^{\geqslant\sigma}) + h.o.t.,$$

where $f(H_f^{\leqslant\sigma}) := \overline{P}_{\rho_0}\left[\overline{P}_{\rho_0}(H_{g,\theta}^{\geqslant\sigma} + e^{-\theta} H_f^{\leqslant\sigma})\overline{P}_{\rho_0} - z\right]^{-1} \overline{P}_{\rho_0}$. Clearly,

$$\Delta E_z = O\left((g\sigma^{1/2+\mu})^2\right) \text{ and } \chi_{\rho_0}^{\leqslant\sigma} W_z \chi_{\rho_0}^{\leqslant\sigma} = O\left(g\sigma^{1+\mu}\right). \tag{6.30}$$

We define the *scaling transformation* $S_\rho : \mathcal{B}[\mathcal{F}_s^{\leqslant\sigma}] \to \mathcal{B}[\mathcal{F}_s^{\leqslant\sigma/\rho}]$, by

$$S_\rho(\mathbf{1}) := \mathbf{1}, \quad S_\rho(a^{\#}(k)) := \rho^{-3/2}\, a^{\#}(\rho^{-1}k), \tag{6.31}$$

where $a^{\#}(k)$ is either $a(k)$ or $a^*(k)$ and $k \in \mathbb{R}^3$, and the dilatation transform, by $A_\rho(A) := \rho^{-1}A$. Now we rescale the operator H_z as $H_z^{(0)} := A_\sigma(S_\sigma(H_z))$. The new operator acts on $\mathrm{Ran}\chi_1^{\leqslant 1} \subset \mathcal{F}_s^{\leqslant 1}$. The last estimate in (6.30) and an estimate on the derivative of T_z as a function of $H_f^{\leqslant\sigma}$, which we do not display here, show that the operator $H_z^{(0)}$ is in the domain of the Feshbach-Schur map $\mathcal{F}_{\chi_\rho^{\leqslant\sigma}}$ and therefore in the domain the renormalization map $\mathcal{R}_\rho$, provided $1/2 \geq \rho \gg g\sigma^\mu$ and $\rho \gg |E_z|/\sigma$ (the latter inequality is also considered as a restriction on z).

If we neglect the term W_z in $H_z^{(0)}$ (see (6.29)) then the remaining operator has the vacuum Ω as an eigenvector corresponding to the eigenvalue 0, provided z solves the equation $E_z^{(0)} := (\Omega, H_z^{(0)}\Omega\rangle = E_z/\sigma = 0$. One can show ([13]) that this equation has a unique solution $\lambda_{j,g}^{(1)} = \lambda_{j,g}^{\geqslant\sigma} + O\left((g\sigma^{1/2+\mu})^2\right)$. By the isospectrality mentioned above, this is our first approximation to $\lambda_{j,g}$.

Now we introduce the decimation map $F_\rho := \mathcal{F}_{\chi_\rho^{\leqslant\sigma}}$. On the domain of the decimation map F_ρ we define the renormalization map $\mathcal{R}_\rho$ as [2]

$$\mathcal{R}_\rho := A_\rho \circ S_\rho \circ F_\rho. \tag{6.32}$$

By the above, the operator $H_z^{(0)}$ is in the domain of the decimation map $\mathcal{F}_\rho$ and therefore in the domain the renormalization map $\mathcal{R}_\rho$, provided $1/2 \geq \rho \gg g\sigma^\mu$ and $\rho \gg |E_z|/\sigma$. Iterating this map as in [13] we obtain a sequence of operators $H_z^{(n)}$, $n = 0, 1, 2, ...$, (Hamiltonians on scales $0, 1, ...$) acting on the space $\mathrm{Ran}\chi_1^{\leqslant 1} \subset \mathcal{F}_s^{\leqslant 1}$. Again, one argues that 0 is an approximate eigenvalue of the operators $H_z^{(n)}$, provided z satisfies the equations $E_z^{(n)} := (\Omega, H_z^{(n)}\Omega) = 0$. These equations have unique solutions $\lambda_{j,g}^{(n)} = \lambda_{j,g}^{\geqslant\sigma} + O\left((g\sigma^{1/2+\mu})^2\right)$, and, by the isospectrality of $\mathcal{R}_\rho$, this produces our approximations, $\lambda_{j,g}^{(n)}$, to $\lambda_{j,g}$. Finally one shows that $\lambda_{j,g}^{(n)} \to \lambda_{j,g}$, as $n \to \infty$, which allows us to conclude that that the operator $H_z^{(0)}$ has a simple eigenvalue 0, provided $z = \lambda_{j,g}$. Hence, by isospectrality of the Feshbach-Schur map, the operator $H_{g,\theta}$ has a unique eigenvalue $\lambda_{j,g}$ in the disc $D(\lambda_{j,g}^{\geqslant\sigma}, \sigma/2)$ and this eigenvalue satisfies (6.15). $\square$

References

[1] W. Hunziker and I. M. Sigal: The quantum N-body problem. J. of Math. Phys. **41**(6), 3448-3510, (2000).

[2] W. Hunziker: Resonances, metastable states and exponential decay laws in perturbation theory. Comm. Math. Phys. **132**, 177-182, (1990).

[3] V. Bach, J. Fröhlich and I. M. Sigal: Quantum electrodynamics of confined non-relativistic particles. Adv. in Math. **137**, 299-395, (1998).

[4] V. Bach, J. Fröhlich and I. M. Sigal: Renormalization group analysis of spectral problems in quantum field theory. Adv. in Math. **137**, 205-298, (1998).

[5] V. Bach, J. Fröhlich and I. M. Sigal: Spectral analysis for systems of atoms and molecules coupled to the quantized radiation fields. Comm. Math. Phys. **207**(2), 249-290, (1999).

[6] J. Faupin: Resonances of the confined hydrogen atom and the Lamb-Dicke effect in non-relativistic qed. Preprint mp_arc 06-344, (2006).

[7] W. Abou Salem and J. Fröhlich, Adiabatic theorems for quantum resonances. Comm. Math. Phys. **273**, 651-675, (2007).

[2]In principle the rescaling is not needed for the argument that follows, but we use it, since the machinery developed in [3, 4, 10, 13] and used here, introduces it.

[8] V. Bach, J. Fröhlich and A.Pizzo: Infrared-finite algorithms in QED: the groundstate of an atom interacting with the quantized radiation field. Comm. Math. Phys. **264**, 145-165, (2006).

[9] D. Hasler, I. Herbst and M. Huber: On the lifetime of quasi-stationary states in non-relativistic QED. Preprint arXiv:0709.3856, (2007).

[10] V. Bach, T. Chen, J. Fröhlich and I. M. Sigal: Smooth Feshbach map and operator-theoretic renormalization group methods. J. Funct. Anal. **203**, 44-92, (2002).

[11] T. Kato: *Perturbation theory for linear operators*, (Springer-Verlag, 1966).

[12] I.M. Sigal: Ground state and resonances in the standard model of the non-relativistic QED. Preprint.

[13] J. Fröhlich, M. Griesemer, I.M. Sigal: On spectral renormalization group. Preprint.

[14] M. Griesemer and D. Hasler: On the Smooth Feshbach-Schur Map, Preprint mp_arc 07-102, (2007).

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