# A REMARK ON SPECTRAL GAP AND LOGARITHMIC SOBOLEV INEQUALITIES FOR CONSERVATIVE SPIN SYSTEMS

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ABSTRACT. We observe that a class of conditional probability measures for unbounded spin systems with convex interactions satisfies Poincaré and logarithmic Sobolev inequalities. For the corresponding conservative dynamics in a box of linear size $L$ we show that the inverse of the spectral gap and the logarithmic Sobolev constant scale as $L^2$ in any dimension.

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## 1. Introduction and results

Let $L$ be a positive integer and consider the probability measure $\mu_L$ on $\mathbb{R}^L$ given by the density

$$\frac{\mathrm{e}^{-H}}{Z}\,, \quad Z = \int \mathrm{e}^{-H(\eta)}\,\mathrm{d}\eta_1 \cdots \mathrm{d}\eta_L\,, \tag{1.1}$$

where $H \in \mathcal{C}^2(\mathbb{R}^L)$ is a real function such that $Z < \infty$. Given a real number $M$ we perturb the product measure $\mu_L$ by conditioning on the event that the "magnetization" equals $M$. This defines a new probability measure $\nu_{L,M}$ on $\mathbb{R}^L$

$$\nu_{L,M} = \mu_L\Big(\;\cdot\;\Big|\sum_{i=1}^{L}\eta_i = M\Big)\,. \tag{1.2}$$

There are several equivalent ways to give precise meaning to the above definition. For our purposes a convenient one is the following. Let $\mu^{\lambda}_{L,M}$, $\lambda > 0$, be the probability measure obtained via

$$\frac{\mathrm{d}\mu^{\lambda}_{L,M}}{\mathrm{d}\mu_L} = \frac{\mathrm{e}^{-\lambda\mathcal{W}_{L,M}}}{\mathcal{Z}^{\lambda}_{L,M}} \tag{1.3}$$

with

$$\mathcal{Z}^{\lambda}_{L,M} = \mathbb{E}_{\mu_L}\mathrm{e}^{-\lambda\mathcal{W}_{L,M}}\,, \quad \mathcal{W}_{L,M}(\eta) = \Big(\sum_{i=1}^{L}\eta_i - M\Big)^2\,.$$

The measure $\nu_{L,M}$ can then be defined as the weak limit of $\mu^{\lambda}_{L,M}$ as $\lambda \to \infty$.

The main result in this note gives sufficient conditions on the energy function $H$ in order that $\nu_{L,M}$ satisfy both a Poincaré inequality and a logarithmic Sobolev inequality, uniformly in the "volume" $L$ and in the "magnetization" $M$. It turns out that a uniform convexity requirement is sufficient. Before stating the theorem we recall some standard notations.

Given a probability measure $\nu$ on $\big(\mathbb{R}^L,\mathcal{B}\big)$, $\mathcal{B}$ denoting the Borel $\sigma$-field, and a measurable function $f:\mathbb{R}^L\to\mathbb{R}$, we write

$$\mathrm{Var}_\nu(f)=\mathbb{E}_\nu(f-\mathbb{E}_\nu f)^2$$

for the variance of $f$ with respect to $\nu$. The entropy of a nonnegative function $f$ with respect to $\nu$ is defined by

$$\mathrm{Ent}_\nu(f)=\mathbb{E}_\nu f\log f-\mathbb{E}_\nu f\log\mathbb{E}_\nu f\,.$$

If $f$ is smooth we write $\partial_i f$ for its partial derivative with respect to $\eta_i$. We also denote by $\mathrm{Hess}f$ the $L\times L$ matrix of second derivatives $\partial^2_{ij}f$. With $\mathbb{I}$ denoting the identity matrix $\mathbb{I}_{ij}=\delta_{ij}$, we shall write $\mathrm{Hess}f\geqslant\mathbb{I}$ for domination in the sense of quadratic forms.

**Theorem 1.** *Assume there exists $\gamma>0$ such that*

$$\mathrm{Hess}H(\eta)\geqslant\gamma\mathbb{I}\,,\quad \eta\in\mathbb{R}^L\,. \tag{1.4}$$

*Then any smooth function $f$ satisfies the following inequalities*

$$\gamma\,\mathrm{Var}_{\nu_{L,M}}(f)\leqslant\sum_{i=1}^L\mathbb{E}_{\nu_{L,M}}(\partial_i f)^2\,, \tag{1.5}$$

$$\gamma\,\mathrm{Ent}_{\nu_{L,M}}(f^2)\leqslant 2\sum_{i=1}^L\mathbb{E}_{\nu_{L,M}}(\partial_i f)^2\,. \tag{1.6}$$

The proof of this theorem is given in the next section. It relies on the powerful techniques developed for models with convex interactions, see the recent review by M. Ledoux [7] and references therein. A basic example to keep in mind is the case where the reference measure $\mu_L$ defined in (1.1) is a product, i.e. the energy takes the form

$$H(\eta)=\sum_{i=1}^L V_i(\eta_i) \tag{1.7}$$

with functions $V_i\in\mathcal{C}^2(\mathbb{R})$ satisfying $V_i''\geqslant\gamma$.

As a simple corollary to Theorem 1 we shall obtain estimates of spectral gaps and logarithmic Sobolev constants associated to a class of locally interacting diffusion processes which we introduce below. We consider the discrete lattice $\mathbb{Z}^d$, with $d\geqslant 1$ an integer. Given a finite subset $\Lambda\subset\mathbb{Z}^d$, we denote by $\Lambda^*$ the set of oriented bonds $b$ contained in $\Lambda$, i.e. the couples $b=(x,y)$, $x,y\in\Lambda$ with $x=y+e$, $e$ a unit vector in $\mathbb{Z}^d$. Denoting $\Lambda_L=\{1,2,\dots,L\}^d$, the $L-$hypercube in $\mathbb{Z}^d$, for each $x\in\Lambda_L$ we have a real variable $\eta_x$. The energy function is now $H:\mathbb{R}^{\Lambda_L}\to\mathbb{R}$. If $\mu_{\Lambda_L}$ stands for the probability measure with density proportional to $\exp -H$, we denote $\nu_{\Lambda_L,M}$ the probability measure obtained from $\mu_{\Lambda_L}$ by conditioning on $\sum_{x\in\Lambda_L}\eta_x=M$. The dynamics is defined by the Dirichlet form

$$\mathcal{E}_{L,M}(f,f)=\frac12\sum_{b\in\Lambda_L^*}\mathbb{E}_{\nu_{\Lambda_L,M}}\big(\widetilde{\nabla}_b f\big)^2 \tag{1.8}$$

where we used the notation

$$\widetilde{\nabla}_b f=\partial_y f-\partial_x f\,,\quad b=(x,y)\,.$$

**Corollary 1.** *In the hypothesis of Theorem 1, any smooth function $f$ satisfies the following inequalities with a constant $C$ only depending on $d$ :*

$$\gamma\,\mathrm{Var}_{\nu_{\Lambda_L,M}}(f)\leqslant CL^2\mathcal{E}_{L,M}(f,f)\,, \tag{1.9}$$

$$\gamma\, \mathrm{Ent}_{\nu_{\Lambda_L,M}}(f^2) \leqslant 2CL^2 \mathcal{E}_{L,M}(f,f)\,. \tag{1.10}$$

*Remark 1.* The $O(L^2)$ growth of constants appearing in (1.9) and (1.10) is the characteristic diffusive scaling of conservative dynamics for systems satisfying a mixing condition, in any dimension. These estimates should be compared with the famous results for spin-exchange (Kawasaki) dynamics of discrete spin systems, [2, 3, 8, 10]. Moreover, as in the case of discrete spins it is not hard to see that functions of type

$$f(\eta) = \sum_{x\in\Lambda_L} g_L(x)\eta_x\,, \tag{1.11}$$

where $g_L(x) = g(x/L)$ for some smooth $g : [0,1]^d \to \mathbb{R}$ with $\int g = 0$ saturate the bound in (1.9), in the sense that the ratio variance/Dirichlet form is $O(L^2)$. The latter holds whenever the system is sufficiently mixing, in particular if $H$ is of type (1.7). Note, however, that (1.9) and (1.10) may be applied to non-mixing situations as well, provided some convexity holds. Consider for instance the massless gaussian field, corresponding to the quadratic function $H$ with $\mathrm{Hess}H = (-\Delta)$, $\Delta$ being the discrete Laplace operator. Imposing e.g. zero boundary conditions on $\Lambda_L$ we have $\mathrm{Hess}H \geqslant \gamma\mathbb{I}$, with $\gamma = O(L^{-2})$ which implies a growth $O(L^4)$ of the ratio variance/Dirichlet form. In this case it is also easy to see that a function of type (1.11) saturates the bound.

The measures $\nu_{L,M}$ involve the global constraint in the magnetization. A second result of this note concerns systems with local constraints which are of interest in applications to statistical mechanics. They are referred to as fields satisfying the *plaquette* constraint. Let us consider variables $\eta$ which are not indexed by sites but rather by bonds of the lattice $\mathbb{Z}^d$. We then have a collection of variables $\eta_b$ and an energy function $H : \mathbb{R}^{\Lambda_L^*} \to \mathbb{R}$. For $b = (x,y)$ we write $\tilde b = (y,x)$. A sequence of bonds $b_1,\dots,b_n$ is called a path connecting $x$ and $y$, denoted $\gamma_{xy}$, if there exists a sequence of sites $y_0 = x, y_1,\dots,y_{n-1}, y_n = y$ such that $|y_{i+1} - y_i| = 1$ and $b_i = (y_{i-1}, y_i)$, $i = 1,\dots,n$. A path $\gamma_{xy}$ is called a closed loop if $x = y$. A plaquette is a closed loop $\mathcal{P} = \{b_1, b_2, b_3, b_4\}$ consisting of four bonds connecting four different sites. We consider the box $\Lambda_L$ as before, and write $\mathcal{P} \subset \Lambda_L$ if all bonds composing the plaquette $\mathcal{P}$ belong to $\Lambda_L^*$. The plaquette constraint in $\Lambda_L$ is defined by the following event, which we call $\Xi_L$:

$$\eta_b = -\eta_{\tilde b} \ \text{ for all } b\in\Lambda_L^* \quad \text{and} \quad \sum_{b\in\mathcal{P}} \eta_b = 0 \ \text{ for all } \mathcal{P}\subset\Lambda_L \tag{1.12}$$

Let $\mu_{\Lambda_L^*}$ denote the usual measure on $\mathbb{R}^{\Lambda_L^*}$ with density proportional to $\exp -H$ and define the conditional measure

$$\nu_L = \mu_{\Lambda_L^*}\big(\ \cdot \ \big|\, \Xi_L\big)\,. \tag{1.13}$$

This measure can be obtained as follows. Let $\mathcal{L}$ be the set of closed loops $\gamma$ contained in $\Lambda_L$ which are either of the form $\gamma = (b,\tilde b)$ or $\gamma = \mathcal{P}$. Consider the measure $\mu^\lambda_{\Lambda_L^*}$, $\lambda > 0$, defined via

$$\frac{\mathrm{d}\mu^\lambda_{\Lambda_L^*}}{\mathrm{d}\mu_{\Lambda_L^*}} = \frac{\mathrm{e}^{-\lambda\mathcal{W}_L}}{\mathcal{Z}_L^\lambda} \tag{1.14}$$

with

$$\mathcal{Z}_L^\lambda = \mathbb{E}_{\mu_{\Lambda_L^*}} \mathrm{e}^{-\lambda\mathcal{W}_L}\,, \quad \mathcal{W}_L(\eta) = \sum_{\gamma\in\mathcal{L}}\Big(\sum_{b\in\gamma}\eta_b\Big)^2.$$

Then $\nu_L$ is the weak limit of $\mu^\lambda_{\Lambda_L^*}$ as $\lambda\to\infty$.

In the following theorem we consider smooth functions $f : \mathbb{R}^{\Lambda_L^*} \to \mathbb{R}$ and denote partial derivative with respect to $\eta_b$ by $\partial_b f$.

**Theorem 2.** *Assume uniform convexity of $H$ as in Theorem 1. Then any smooth function $f$ satisfies the following inequalities*

$$\gamma \operatorname{Var}_{\nu_L}(f) \leqslant \sum_{b\in\Lambda_L^*} \mathbb{E}_{\nu_L}(\partial_b f)^2\,, \tag{1.15}$$

$$\gamma \operatorname{Ent}_{\nu_L}(f^2) \leqslant 2\sum_{b\in\Lambda_L^*} \mathbb{E}_{\nu_L}(\partial_b f)^2\,. \tag{1.16}$$

The examples we have in mind are so-called effective interface models, described by height variables $\phi_x$, $x \in \Lambda_L$, interacting via nearest neighbour convex pair-potentials $V_{xy}(\phi_x - \phi_y)$, $x, y \in \Lambda_L$, see e.g. [4]. The marginal field on height-difference variables $\eta_b = \phi_y - \phi_x$, $b = (x, y)$, is then a probability measure of type $\nu_L$. Here the reference measure $\mu_{\Lambda_L^*}$ is a product with the corresponding energy being of the form

$$H = \sum_{b\in\Lambda_L^*} V_b\,, \quad V_b = V_{\tilde b}\,, \ \ V_b'' \geqslant \gamma > 0\,, \ \ b \in \Lambda_L^*\,.$$

As in the previous corollary, the above estimate can be used to study relaxation properties of locally interacting diffusions. Given $\eta \in \Xi_L$ we may write

$$\phi_x = L^{-d} \sum_{y\in\Lambda_L} \sum_{b\in\gamma_{yx}} \eta_b\,. \tag{1.17}$$

where $\gamma_{yx}$ are paths connecting $x$ and $y$ contained in $\Lambda_L$ (since $\eta \in \Xi_L$ the definition of $\phi_x$ does not depend on the choice of paths). Then $\eta = \eta(\phi)$ is given by $\eta_b = \phi_y - \phi_x$, $b = (x, y)$. If $f = f(\eta)$ is a smooth function we write $F(\phi) = f(\eta(\phi))$ and denote

$$\widetilde\nabla_x f = \frac{\partial F}{\partial \phi_x}\,.$$

We define the Dirichlet form

$$\mathcal{D}_L(f, f) = \sum_{x\in\Lambda_L} \mathbb{E}_{\nu_L}(\widetilde\nabla_x f)^2\,. \tag{1.18}$$

**Corollary 2.** *In the hypothesis of Theorem 2, any smooth function $f$ satisfies the following inequalities with a constant $C$ only depending on $d$:*

$$\gamma \operatorname{Var}_{\nu_L}(f) \leqslant CL^2 \mathcal{D}_L(f, f) \tag{1.19}$$

$$\gamma \operatorname{Ent}_{\nu_L}(f^2) \leqslant 2CL^2 \mathcal{D}_L(f, f) \tag{1.20}$$

*Remark 2.* Following [4] we see that in the case of effective interface models the Markov process corresponding to the Dirichlet form (1.18) is the projection on height-differences of the classical Langevin diffusion. Estimates (1.15) and (1.16) are thus a slightly stronger version of the well known spectral gap and logarithmic Sobolev inequalities for the Langevin dynamics of effective interfaces with convex pair-potentials.

Finally, we would like to point out two interesting questions which have not been solved in this work. The first concerns generalizations of Theorem 1 to energy functions $H$ which are perturbations of a uniformly convex function $H_0$. It is well known that if $F$ is a bounded function and $H = H_0 + F$ then - apart from constants depending on $\|F\|_\infty$ - nothing changes in our inequalities. On the other hand the situation becomes much more delicate already if e.g. $H$ is of type (1.7) with each potential $V$ of the form $V = V_0 + F$, $V_0$ strictly convex and $F$ bounded, see [1, 5, 7]. Recently the above problem was studied by Landim, Panizo and Yau in [6], where the estimates of Corollary 1 were obtained - by a different technique - in the quadratic case $V_0(\eta_i) = \eta_i^2$ and $F$ a bounded function with bounded derivative. The second question concerns so-called SOS models. These are effective interfaces with non strictly convex potentials $V(\eta_b) = |\eta_b|$. For SOS models a logarithmic Sobolev inequality cannot hold, in view of non-gaussian tails. Still, we conjecture that inequalities of type (1.5) and (1.9) hold for any $L$ and $|M| \leqslant L$, see also [9] for related results.

## 2. Proofs

Below we prove the two theorems. The path-counting arguments needed to prove the corollaries are given in the last subsection.

2.1. **Proof of Theorem 1.** Given $L \in \mathbb{Z}_+$, $M \in \mathbb{R}$, $\lambda > 0$ and a smooth function $f : \mathbb{R}^L \to \mathbb{R}$, we consider the logarithmic Sobolev inequality

$$\gamma \, \mathrm{Ent}_{\mu^\lambda}(f^2) \leqslant 2 \sum_{i=1}^{L} \mathbb{E}_{\mu^\lambda} \big(\partial_i f\big)^2 , \tag{2.21}$$

where $\mu^\lambda = \mu^\lambda_{L,M}$ is the measure defined in (1.3). We recall (see e.g. [7]) that (2.21) implies the Poincaré inequality

$$\gamma \, \mathrm{Var}_{\mu^\lambda}(f) \leqslant \sum_{i=1}^{L} \mathbb{E}_{\mu^\lambda} \big(\partial_i f\big)^2 . \tag{2.22}$$

Moreover, when $f$ is bounded together with its gradients, as $\lambda \to \infty$ weak convergence in (2.21) and (2.22) implies (1.6) and (1.5) respectively. A simple approximation allows to cover the general case. Therefore Theorem 1 follows provided we establish (2.21) for all $\lambda > 0$. We write (see (1.3))

$$H_\lambda(\eta) = H(\eta) + \lambda \mathcal{W}_{L,M}(\eta) .$$

By Proposition 1.5 in [7] it is sufficient to prove that any nonnegative smooth $f$ satisfies

$$\gamma \sum_{i=1}^{L} \mathbb{E}_{\mu^\lambda} f \big(\partial_i \log f\big)^2 \leqslant \mathbb{E}_{\mu^\lambda} f \Gamma_2\big(\log f\big) , \tag{2.23}$$

where we have introduced the operator

$$\Gamma_2(g) = \sum_{i,j=1}^{L} \big(\partial^2_{ij} g\big)^2 + \sum_{i,j=1}^{L} \partial_i g \big(\mathrm{Hess} H_\lambda\big)_{ij} \partial_j g . \tag{2.24}$$

Since $\partial^2_{ij} \mathcal{W}_{L,M}(\eta) = 2$ for all $i, j$, using $\mathrm{Hess} H \geqslant \gamma \mathbb{I}$ we have

$$\mathrm{Hess} H_\lambda \geqslant \gamma \mathbb{I} , \quad \lambda > 0 . \tag{2.25}$$

Therefore

$$\Gamma_2(g) \geqslant \gamma \sum_{i=1}^{L} \big(\partial_i g\big)^2$$

for any smooth function $g$. Choosing $g = \log f$, (2.23) follows for all smooth $f \geqslant 0$.

2.2. **Proof of Theorem 2.** Consider the function (see (1.14))

$$H_\lambda(\eta) = H(\eta) + \lambda \mathcal{W}_L(\eta)\,.$$

As before, it is sufficient to establish (2.25) for the Hessian of $H_\lambda$. Observe that for any $b, b' \in \Lambda_L^*$

$$\partial^2_{bb'} \mathcal{W}_L = 2 \sum_{\gamma \in \mathcal{L}} 1_{\{b\in\gamma\}} 1_{\{b'\in\gamma\}}\,.$$

Then for any collection of numbers $\{z_b\}_{b\in\Lambda_L^*}$,

$$\sum_{b,b'\in\Lambda_L^*} z_b \partial^2_{bb'} \mathcal{W}_L z_{b'} = 2 \sum_{\gamma\in\mathcal{L}} \Big(\sum_{b\in\gamma} z_b\Big)^2 \geqslant 0\,.$$

The conclusion follows as in Theorem 1.

2.3. **Corollaries.** We first make some observations about paths in $\Lambda_L$. We denote $\mathcal{C}_{xy}(L)$ the set of all paths $\gamma_{xy}$ connecting sites $x, y \in \Lambda_L$, which use only bonds in $\Lambda_L^*$. The length of a path, denoted $|\gamma_{xy}|$ is the number of bonds composing it. Given $x, y \in \Lambda_L$ we need a rule to select a single path $\gamma_{xy}$ from $\mathcal{C}_{xy}(L)$. We may choose $\gamma_{xy}$ as follows. Fix $x, y \in \Lambda_L$ and define points $x^{(i)}$, $i = 0, \dots, d$, such that $x^{(0)} = x$, $x^{(d)} = y$, and when $i = 1, \dots, d-1$

$$x_j^{(i)} = \begin{cases} y_j & j = 1, \dots, i \\ x_j & j = i+1, \dots, d \end{cases}$$

Call $\gamma^{(i)}$, $i = 1, \dots, d$, the straight line parallel to the $i$-th axis joining sites $x^{(i-1)}$ and $x^{(i)}$. The path $\gamma_{xy}$ is given by $\gamma^{(1)} \cup \dots \cup \gamma^{(d)}$. It is not difficult to prove the following properties: there exists a finite constant $k$ only depending on $d$ such that

- for every $x, y \in \Lambda_L$, $|\gamma_{xy}| \leqslant kL$,
- for every $b \in \Lambda_L^*$, $\sum_{x,y\in\Lambda_L} 1_{\{b\in\gamma_{xy}\}} \leqslant kL^{d+1}$, and
- for every $x \in \Lambda_L$, $\sum_{y,z\in\Lambda_L} \sum_{b\in\Lambda_L^*} 1_{\{b\in\gamma_{zy}\}} 1_{\{b\in\gamma_{xy}\}} \leqslant kL^{d+2}$

When we write $\gamma_{xy}$ below we always assume that this path has been chosen according to the above rule.

We start with the proof of Corollary 1. Given $\eta \in \mathbb{R}^{\Lambda_L}$, $y \in \Lambda_L$ we write $\eta^{(y)}$ for the configuration

$$\eta_x^{(y)} = \begin{cases} \eta_x & x \neq y \\ M - \sum_{z\neq y} \eta_z & x = y \end{cases}\,.$$

For $f : \mathbb{R}^{\Lambda_L} \to \mathbb{R}$, we denote $f_y(\eta)$ the function $\eta \to f(\eta^{(y)})$. Clearly, for any $y \in \Lambda_L$ we have

$$\mathrm{Var}_{\nu_{\Lambda_L,M}}(f) = \mathrm{Var}_{\nu_{\Lambda_L,M}}(f_y)\,, \quad \mathrm{Ent}_{\nu_{\Lambda_L,M}}(f^2) = \mathrm{Ent}_{\nu_{\Lambda_L,M}}(f_y^2) \tag{2.26}$$

In view of Theorem 1 applied to $f_y$, it is then sufficient to show

$$L^{-d} \sum_{y\in\Lambda_L} \sum_{x\in\Lambda_L} \mathbb{E}_{\nu_{\Lambda_L,M}} \big(\partial_x f_y\big)^2 \leqslant C L^2 \mathcal{E}_{L,M}(f,f)\,. \tag{2.27}$$

For any $x, y \in \Lambda_L$ we have

$$\partial_x f_y = \big(\partial_x f\big)_y - \big(\partial_y f\big)_y$$

and therefore

$$\mathbb{E}_{\nu_{\Lambda_L,M}} \big(\partial_x f_y\big)^2 = \mathbb{E}_{\nu_{\Lambda_L,M}} \big(\partial_x f - \partial_y f\big)^2\,.$$

We write

$$\partial_x f - \partial_y f = \sum_{b\in\gamma_{yx}} \widetilde{\nabla}_b f \,.$$

Since $|\gamma_{yx}| \leqslant kL$, Schwarz' inequality gives

$$\big(\partial_x f - \partial_y f\big)^2 \leqslant kL \sum_{b\in\Lambda_L^*} \big(\widetilde{\nabla}_b f\big)^2 1_{\{b\in\gamma_{yx}\}} \,. \tag{2.28}$$

From the second property of our paths we see that (2.27) with $C = k^2$ follows from (2.28) when summing over $x, y$ and dividing by $L^d$.

We now turn to the setting of Corollary 2. Consider a smooth function $f : \mathbb{R}^{\Lambda_L^*} \to \mathbb{R}$. We show that

$$\sum_{b\in\Lambda_L^*} \mathbb{E}_{\nu_L} \big(\partial_b f\big)^2 \leqslant CL^2 \mathcal{D}_L(f,f) \,. \tag{2.29}$$

Defining $\phi$ as in (1.17) we see that

$$\frac{\partial \phi_x}{\partial \eta_b} = L^{-d} \sum_{y\in\Lambda_L} 1_{\{b\in\gamma_{yx}\}} \,.$$

We have

$$\partial_b f = \sum_{x\in\Lambda_L} \widetilde{\nabla}_x f \frac{\partial \phi_x}{\partial \eta_b} = L^{-d} \sum_{y\in\Lambda_L} \sum_{x\in\Lambda_L} \widetilde{\nabla}_x f 1_{\{b\in\gamma_{yx}\}} \,. \tag{2.30}$$

By Schwarz' inequality

$$\big(\partial_b f\big)^2 \leqslant \sum_{x\in\Lambda_L} \big(\widetilde{\nabla}_x f\big)^2 \Big\{ L^{-d} \sum_{z,y\in\Lambda_L} 1_{\{b\in\gamma_{yz}\}} 1_{\{b\in\gamma_{yx}\}} \Big\} \,. \tag{2.31}$$

Summing over $b \in \Lambda_L^*$ and using the third property of the paths we obtain

$$\sum_{b\in\Lambda_L^*} \big(\partial_b f\big)^2 \leqslant kL^2 \sum_{x\in\Lambda_L} \big(\widetilde{\nabla}_x f\big)^2 \,. \tag{2.32}$$

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