Generalized boundary conditions of a spin-1/2 particle for the Aharonov-Bohm effect combined with a homogeneous magnetic field

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Exner et. al. have derived the most general admissible boundary conditions (MGABC) of the Schrödinger operator H for an idealized Aharonov-Bohm flux interacting the plane at the origin on the background of a homogeneous magnetic field [J. Math. Phys., **43**, p2151–2168 (2002)]. In this paper, we derive the MGABC of the Dirac-Weyl operator Q under the same situation. It is differ from H that the 'standard boundary condition,' $\lim_{r\to 0}\Psi(r) = 0$, gives no self-adjoint extension of Q and that we can obtain exactly spectra and eigenfunctions of all self-adjoint extensions of Q.

I. INTRODUCTION

The purpose of this paper is to identify the most general boundary conditions (BC) for the Aharonov-Bohm (AB) effect for a spin-1/2 particle in the plane on the background of a homogeneous magnetic field, and also to investigate the basic properties of Hamiltonians obtained this way.

It is well known that the history of the AB effect started from the celebrated paper by Aharonov and Bohm [2] and then a lot of physicists and mathematicians have been investigating this phenomenon under various situations. (For examples, see Refs. [8, 14] and the references therein.) In the spinless case, the general BC for pure AB effect is derived by Ref. [7] and Ref. [1] independently, and then, Exner et. al. have derived the general BC for AB effect with the background of a homogeneous magnetic field in Ref. [8]: The authors say that a natural question is what happens if such a system (a pure AB effect system) is placed into a background homogeneous magnetic field. From a similar interest, the author of this paper have investigated the properties of the Pauli operator (i.e., a spin 1/2 particle) for AB effect with the background of a magnetic field which vanishs at infinity [10]. In this paper, we investigate the properties of a spin-1/2 particle placed in the Exner et. al.'s situations. The pure AB effect case for a spin-1/2 particle has been treated in Refs. [6, 9].

We have the two choice of the Hamiltonian for a spin-1/2 particle, a Pauli operator or a Dirac-Weyl operator at least from the point of view of mathematics, and we select a Dirac-Weyl operator Q. The reason of this selection is the following: Let H_0 be a Hamiltonian for a spin-less particle, i.e., the operator treated by Exner et. al. Then, the Pauli operator $H_{1/2}$ for a spin-1/2 particle is

$$H_{1/2} = \begin{pmatrix} H_0 + \mathrm{e}B & 0 \\ 0 & H_0 - \mathrm{e}B \end{pmatrix} \quad \text{on } \oplus^2 C_0^\infty(\mathbb{R}^2 \setminus \{0\}) \tag{1}$$

with the magnetic strength B and the charge of the particle e. This means that the spin effect is appeared only as the constant additions in each of the two spin components. If we chose $H_{1/2}$, the solutions of our problem are directly and easily obtained from the Exner et. al.'s results. In the other hand, if we choose Q, since the two spin components are mixed by the action of Q, distinct phenomena happen as we shall see in the following sections. Our conclusion is that our model is exactly solvable: (1) we can obtain all self-adjoint extensions Q^e of Q; (2) we can exactly identify the spectrum for all Q^e; (2) we can obtain all eigenfunctions for all Q^e. (For solvable models on δ- and δ'-type interactions, see Ref. [11] and the references therein.)

We give one more remark. In this paper, we consider a massless particle in a two-dimensional space: Let $M \in \mathbb{R}$ be the mass of the particle and σ^j $(j = 1, 2, 3)$ be the Pauli matrices. Since the massless Dirac-Weyl operator, $Q_0 = \sum_{j=1}^2 \sigma^j P_j$, and the mass term $M\sigma^3$ are anti-commute, thus we can easily determine the self-adjoint extensions of the massive Dirac-Weyl operator, $Q_M = Q_0 + M\sigma^3$, from the ones of Q_0 and can obtain the spectrum of Q_M from the one of Q_0 (See, Refs. [5, 15]).

The plan of this parer is as follows: In Sec. II, we formulate the problem, give some notations and obtain the all generalized eigenfunctions of the Dirac-Weyl operator, Q. In Sec. III, we examine the spectral problem for the standard BC, $\lim_{r\to 0+} \Psi(r)$. In Sec. IV, we give the all self-adjoint extensions Q^τ parametrized by one parameter $\tau \in [0, 2\pi)$ and the most generalized BC for them. In Sec. V, we obtain spectra of all self-adjoint extensions Q^τ. In Sec. VI, we give some discussions.

II. FORMULATION OF THE PROBLEM AND GENERALIZED EIGENFUNCTIONS

Give a vector potential A of the idealized AB magnetic field with the background homogeneous magnetic field by

$$A = A_x\,dx + A_y\,dy = -\frac{\mathrm{i}}{2}\left(B + \frac{\alpha}{r^2}\right)(-y\,dx + x\,dy)$$

with $r^2 = x^2 + y^2$. We can assume that $B > 0$ and $\alpha \in (0,1)$ without loss of generality [8]. Here, we use the natural unit; $c = \mathrm{e} = \hbar = 1$. Define a Dirac-Weyl operator Q on $\mathcal{H}_{\mathrm{total}} = \oplus^2 L^2(\mathbb{R}^2; dx\,dy)$ with the domain $D(Q) = C_0^\infty(\mathbb{R}^2 \setminus \{0\}) \oplus C_0^\infty(\mathbb{R}^2 \setminus \{0\})$ by

$$Q = -\mathrm{i}\left(\frac{\partial}{\partial x} - A_x\right)\sigma^x - \mathrm{i}\left(\frac{\partial}{\partial y} - A_y\right)\sigma^y$$

with the Pauli matrices,

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}.$$

We denote the domain of a operator T by $D(T)$. In the polar coordinate, $x + \mathrm{i}y = re^{\mathrm{i}\theta}$, we have that

$$Q \begin{pmatrix} f(r)e^{\mathrm{i}m\theta} \\ g(r)e^{\mathrm{i}(m+1)\theta} \end{pmatrix} = \begin{pmatrix} q_{m+1}^+ g(r)e^{\mathrm{i}m\theta} \\ q_m^- f(r)e^{\mathrm{i}(m+1)\theta} \end{pmatrix}$$

with

$$q_m^\pm = -\mathrm{i}\left[\frac{\partial}{\partial r} \pm \left(\frac{Br}{2} + \frac{\alpha + m}{r}\right)\right]$$

for $f, g \in C_0^\infty((0,\infty))$. We denote $L^2((0,\infty); r\,dr)$ by $\mathcal{L}^2$ and let

$$\mathcal{H}_m = \left\{ \begin{pmatrix} f(r)e^{\mathrm{i}m\theta} \\ g(r)e^{\mathrm{i}(m+1)\theta} \end{pmatrix} ; f, g \in \mathcal{L}^2, \theta \in [0, 2\pi) \right\}$$

for $m \in \mathbb{Z}$. We call $\mathcal{H}_m$ m-sector. We can easily see that $\mathcal{H}_{\mathrm{total}} = \oplus_{m\in\mathbb{Z}}\mathcal{H}_m$ and Q^* is reduced on each $\mathcal{H}_m$. Therefore, we examine the self-adjoint extensions of the symmetric operator Q_m defined by

$$Q_m = \begin{pmatrix} 0 & q_m^- \\ q_{m+1}^+ & 0 \end{pmatrix}$$

on $\mathcal{H} = \mathcal{L}^2 \oplus \mathcal{L}^2$ with $D(Q_m) = C_0^\infty((0,\infty)) \oplus C_0^\infty((0,\infty))$. We remark that q_m^- and q_{m+1}^+ are formal adjoint as operators acting on $\mathcal{L}^2$.

We note that the sequence of self-adjoint extensions of Q_m's, $\{Q_m^e; m \in \mathbb{Z}\}$, are corresponding to a self-adjoint extension Q^e of Q: Let P_m be the projection operator from $\mathcal{H}_{\text{total}}$ to $\mathcal{H}_m$ given by

$$P_m \begin{pmatrix} f(r,\theta) \\ g(r,\theta) \end{pmatrix} = \begin{pmatrix} (2\pi)^{-1} \int_0^{2\pi} f(r,t)\, e^{-\mathrm{i}m(t-\theta)}\, dt \\ (2\pi)^{-1} \int_0^{2\pi} g(r,t)\, e^{-\mathrm{i}(m+1)(t-\theta)}\, dt \end{pmatrix}$$

and U be the natural unitary operator from $\mathcal{H}_m$ to $\mathcal{H}$. Then we have that $Q^e = \oplus_{m\in\mathbb{Z}} (UP_m)^{-1} Q_m^e UP_m$, where $(UP_m)^{-1}$ stands for the lifting from $\mathcal{H}$ to $\mathcal{H}_m$.

In the rest of this section, we examine the generalized eigenfunctions of Q_m, i.e., we find the vectors $\Psi \in D(Q_m^*)$ which satisfies that there exists a constant $E \in \mathbb{C}$ such that $\langle \psi, Q_m^* \Psi \rangle_{\mathcal{H}} = E \langle \psi, \Psi \rangle_{\mathcal{H}}$ for all $\psi \in D(Q_m)$. Let $k = m + \alpha$. We have that

$$Q_m^2 = \begin{pmatrix} L_k + B & 0 \\ 0 & L_{k+1} - B \end{pmatrix} \text{ with } L_k = -\frac{1}{r}\partial_r r \partial_r + \frac{1}{r^2}\left(k + \frac{Br^2}{2}\right)^2. \tag{2}$$

Using this, we obtain the two linearly independent generalized eigenfunctions of Q_m with a generalized eigenvalue $E \in \mathbb{C}$ as

$$\Psi^1(E;r) = \begin{pmatrix} 2(k+1)\, \Psi_+^1(E;r) \\ \mathrm{i}E\, \Psi_-^1(E;r) \end{pmatrix},$$
$$\Psi^2(E;r) = \begin{pmatrix} \mathrm{i}E\, \Psi_+^2(E;r) \\ -2k\, \Psi_-^2(E;r) \end{pmatrix},$$

where

$$\Psi_+^1(E;r) = r^k F\left(k + 1 - \frac{E^2}{2B}, 1 + k, \frac{Br^2}{2}\right) e^{-Br^2/4},$$
$$\Psi_-^1(E;r) = r^{k+1} F\left(k + 1 - \frac{E^2}{2B}, 2 + k, \frac{Br^2}{2}\right) e^{-Br^2/4},$$
$$\Psi_+^2(E;r) = r^{-k} F\left(1 - \frac{E^2}{2B}, 1 - k, \frac{Br^2}{2}\right) e^{-Br^2/4},$$
$$\Psi_-^2(E;r) = r^{-k-1} F\left(-\frac{E^2}{2B}, -k, \frac{Br^2}{2}\right) e^{-Br^2/4}.$$

Here, $F(\beta,\gamma,z)$ is the confluent hypergeometric function [4]. We know that if $\gamma \notin \mathbb{N}$ then $F(\beta,\gamma,z)$ and $z^{1-\gamma}F(\beta-\gamma+1, 2-\gamma, z)$ are linearly independent. Note that $k \notin \mathbb{N}$, and we can rewrite $\Psi^2_\pm$ as

$$\begin{aligned}\Psi^2_+ =&\text{const.}r^k\left(\frac{Br^2}{2}\right)^{1-(1+k)}\times F\left(k+1-\frac{E^2}{2B}-(1+k)+1, 2-(1+k), \frac{Br^2}{2}\right)e^{-Br^2/4},\\ \Psi^2_- =&\text{const.}r^{k+1}\left(\frac{Br^2}{2}\right)^{1-(2+k)}\times F\left(k+1-\frac{E^2}{2B}-(2+k)+1, 2-(2+k), \frac{Br^2}{2}\right)e^{-Br^2/4}.\end{aligned}$$

Since these $\Psi^j_\pm$'s and Ψ^j's are depend on m, we should write a subscript m as $\Psi^j_{m,\pm}$, but we omit it.

Remark 1. The operator L_k is the Schrödinger operator on the m-sector treated in Ref. [8]. Using Eq. (2), we have that

$$Q^2 = \begin{pmatrix} H_0 + B & 0 \\ 0 & H_0 - B \end{pmatrix},$$

where, H_0 is the formal Schrödinger operator on the whole space treated in Ref. [8]. Thus, Q^2 is the operator $H_{1/2}$ given by Eq. (1).

Remark 2. For later use, we summarize the facts on the confluent hypergeometric function [4].

1. $F(\beta, \gamma, z)$ is an entire function and $F(\beta, \gamma, 0) = 1$.

2. If $\beta \in -\mathbb{N}_0$ then $F(\beta, \gamma, z)$ is a polynomial with the leading term $\Gamma(\gamma)/\Gamma(\gamma-\beta)(-z)^{-\beta}$. Indeed, it holds that

$$F(-n, 1+\sigma, z) = \text{const.}L^\sigma_n(z) \tag{3}$$

for $n \in \mathbb{N}_0$ and $L^\sigma_n(z)$ is a Laguerre polynomial.

3. If $\beta \notin -\mathbb{N}_0$, we have that

$$F(\beta, \gamma, z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)}e^z z^{\beta-\gamma}(1 + O(z^{-1})) \quad \text{for} \quad z \to +\infty. \tag{4}$$

4. Let

$$G(\beta, \alpha, z) = \frac{\Gamma(1-\gamma)}{\Gamma(\beta-\gamma+1)}F(\beta, \gamma, z) + \frac{\Gamma(\gamma-1)}{\Gamma(\beta)}z^{1-\gamma}F(\beta-\gamma+1, 2-\gamma, z).$$

Then, we have that

$$G(\beta, \gamma, z) = z^{-\beta}(1 + O(z^{-1})) \quad \text{for } z \to +\infty \tag{5}$$

and

$$G(\beta,\gamma,z)=\begin{cases}\dfrac{\Gamma(1-\gamma)}{\Gamma(\beta-\gamma+1)}+O(z^{1-\gamma}) & \text{if } \gamma\le 1,\\ \dfrac{\Gamma(\gamma-1)}{\Gamma(\beta)}z^{1-\gamma}+O(z^{2-\gamma}) & \text{if } 1<\gamma<2\end{cases} \tag{6}$$

for $z\to 0$.

III. SPECTRAL PROBLEM FOR THE STANDARD AB HAMILTONIAN

We examine the spectral problem for the standard AB Hamiltonian. This means to solve the eigenvalue problem

$$Q^{*}\Psi=E\Psi \tag{7}$$

with BC

$$\lim_{r\to 0+}\Psi(r)=0 \quad \text{and} \quad \Psi\in\mathcal{H}_{\text{total}}. \tag{8}$$

We solve this problem on each $\mathcal{H}_m$. Put $\mathbb{N}_k=\{n\in\mathbb{Z};n\ge k\}$. Then we obtain the following theorem.

Theorem 3. *The eigenvalue problem (7) with BC (8) has the following solutions.*

(1) If $m=-1$, we have no nontrivial solution.

(2) If $m\in-\mathbb{N}_2$, the solutions are $\Phi^2(E;r)$ with $E=\pm\sqrt{2Bn}$ for all $n\in\mathbb{N}_0$.

(3) If $m\in\mathbb{N}_0$, the solutions are $\Phi^1(E;r)$ with $E=\pm\sqrt{2B(m+\alpha+n+1)}$ for all $n\in\mathbb{N}_0$.

Proof. Part (1): Since $-1<k<0$, we have that

$$\lim_{r\to 0+}\Psi^1(E;r)=\begin{pmatrix}\infty\\ 0\end{pmatrix} \quad \text{and} \quad \lim_{r\to 0+}\Psi^2(E;r)=\begin{pmatrix}0\\ \infty\end{pmatrix}.$$

Therefore, we have that $\lim_{r\to 0+}\Psi(r)\ne 0$ for all nonzero linear combinations Ψ of Ψ^1 and Ψ^2. Thus, we have no nontrivial solution.

Part (2): We have $k+1<0$. In the case when $E\ne 0$,

$$\lim_{r\to 0+}\Psi^1(E;r)=\begin{pmatrix}\infty\\ \infty\end{pmatrix} \quad \text{and} \quad \lim_{r\to 0+}\Psi^2(E;r)=0.$$

Thus, Ψ^2 is a candidate of the solutions. If $1-E^2/2B\notin-\mathbb{N}_0$, then Ψ^2_+ behaves as a product of a polynomial of r and $\exp(Br^2/4)$ for $r\to+\infty$, and thus $\Psi^2_+\notin\mathcal{L}^2$. On the contrary, if $1 - E^2/2B \in -\mathbb{N}_0$ (and thus $-E^2/2B \in -\mathbb{N}_0$), the F parts in $\Psi^2_\pm$ are polynomials of r. Since $\Psi^2_\pm \in \mathcal{L}^2$, we have $\Psi^2 \in \mathcal{H}$. In the case when $E = 0$, we have that

$$\Psi^2(0;r) = \begin{pmatrix} 0 \\ -2kr^{-k-1}e^{-Br^2/4} \end{pmatrix}$$

and thus $\Psi^2(0;r) \in \mathcal{H}$. Consequently, $\Psi^2 \in \mathcal{H}$ if and only if $E = \pm\sqrt{2Bn}$ with $n \in \mathbb{N}_0$. Thus, we can obtain the desired result.

Part (3): In a way similar as in the proof of Part (2), we can see that Ψ^2 does not satisfy the standard BC and that $\Psi^1 \in \mathcal{H}$ if and only if $E = \pm\sqrt{2B(m+\alpha+1+n)}$ with $n \in \mathbb{N}_0$. Thus, we can obtain the desired result. □

The part (1) of Theorem 3 says that the eigenvalue problem (7) with BC (8) has no solution in -1-sector. It is differ from the spin-less case treated in Ref. [8]: In that case, the spectral problem for the standard AB Hamiltonian has at least one solution in all sectors.

We give an important proposition.

Proposition 4. *If $m \in -\mathbb{N}$, the set*

$$\mathcal{S}_2 = \left\{\Psi^2(E;r); E = \pm\sqrt{2Bn},\ n \in \mathbb{N}_0\right\}$$

is complete in $\mathcal{H}$. If $m \in \mathbb{N}_{-1}$, the set

$$\mathcal{S}_3 = \left\{\Psi^1(E;r); E = \pm\sqrt{2B(m+\alpha+1+n)},\ n \in \mathbb{N}_0\right\}$$

is complete in $\mathcal{H}$.

Proof. By Eq. (3), we have

$$\Psi^2(\sqrt{2Bn};r) - \Psi^2(-\sqrt{2Bn};r) = \begin{pmatrix} \mathrm{const.}r^{-k}L^{-k}_{n-1}(Br^2/2)e^{-Br^2/4} \\ 0 \end{pmatrix}$$

for $n \in \mathbb{N}$. Since the set $\{L^\sigma_n(z); n \in \mathbb{N}_0\}$ with $\sigma > -1$ is complete in $L^2((0,\infty); z^\sigma e^{-z}\,dz)$, we can see that the set

$$\left\{\Psi^2(\sqrt{2Bn},r) - \Psi^2(-\sqrt{2Bn},r); n \in \mathbb{N}\right\}$$

is complete in $\mathcal{L}^2 \oplus \{0\}$. In a way similar as above, we can see that the set

$$\left\{\Psi^2(\sqrt{2Bn},r) + \Psi^2(-\sqrt{2Bn},r); n \in \mathbb{N}\right\}$$

is complete in $\{0\} \oplus \mathcal{L}^2$. Consequently, $\mathcal{S}_2$ is complete in $\mathcal{H} = \mathcal{L}^2 \oplus \mathcal{L}^2$. In the case when $m \in \mathbb{N}_{-1}$, we can obtain the desired result as in a way similar as above. □

IV. SELF-ADJOINT EXTENTIONS

We calculate the deficiency index of Q. Let

$$\Psi^{\pm\mathrm{i}}(r) = \begin{pmatrix} \Psi_+^{\pm\mathrm{i}}(r) \\ \Psi_-^{\pm\mathrm{i}}(r) \end{pmatrix} = -2(k+1) \begin{pmatrix} \pm r^k\, G\left(k+1+\dfrac{1}{2B}, 1+k, \dfrac{Br^2}{2}\right) e^{-Br^2/4} \\ Br^{k+1}\, G\left(k+1+\dfrac{1}{2B}, 2+k, \dfrac{Br^2}{2}\right) e^{-Br^2/4} \end{pmatrix},$$

where G is given in Remark 2. Note that we can rewrite $\Psi^{\pm\mathrm{i}}$ as follows:

$$\Psi^{\pm\mathrm{i}}(r) = \mp\frac{\Gamma(-k)}{\Gamma(1+1/2B)}\Psi^1(\pm\mathrm{i}; r) + \frac{2^{k+1}}{B^k}\frac{\Gamma(k+1)}{\Gamma(k+1+1/2B)}\Psi^2(\pm\mathrm{i}; r).$$

Thus, we have that $Q_m^*\Psi^{\pm\mathrm{i}} = \pm\mathrm{i}\,\Psi^{\pm\mathrm{i}}$.

Lemma 5. *The deficiency index of Q_{-1} is $(1,1)$ and the deficiency spaces $\mathcal{N}_\pm$ of Q_{-1} for $E = \pm\mathrm{i}$ are*

$$\mathcal{N}_\pm = \{c\,\Psi^{\pm\mathrm{i}}; c\in\mathbb{C}, m=-1\}.$$

If $m \neq -1$, the deficiency index of Q_m is $(0,0)$. Thus, the deficiency index of Q is $(1,1)$.

Proof. Let $E = \pm\mathrm{i}$. By Eq. (4), we have that for both $j = 1, 2$

$$\Psi_\pm^j(E; r) = \text{‘a polynomial of } r\text{’} \times \exp\left(Br^2/4\right)\left(1 + O(r^{-2})\right)$$

for $r \to \infty$. Thus $\Psi^j \notin \mathcal{H}$. However, by Eq. (5), we have that

$$\Psi_\pm^{\pm\mathrm{i}}(r) = \text{‘a polynomial of } r\text{’} \times \exp\left(-Br^2/4\right)\left(1 + O(r^{-2})\right)$$

for $r \to \infty$. Therefore $\Psi_\pm^{\pm\mathrm{i}}(r)$ are L^2 with respect to the measure $r\,dr$ at infinity. In the other hand, since $F(\beta,\gamma,z)$ is an entire function, we have that

$$\begin{aligned} \Psi_+^{\pm\mathrm{i}}(r) &= c_1 r^k + c_2 r^{-k} + O(r^2), \\ \Psi_-^{\pm\mathrm{i}}(r) &= c_3 r^{-k-1} + c_4 r^{k+1} + O(r^2) \end{aligned}$$

with constants c_j's for $r \to 0+$. Since $k \in (m, m+1)$, $\Psi_\pm^{\pm\mathrm{i}}(r)$ are L^2 with respect to $r\,dr$ at the origin if and only if $m = -1$. Consequently, the deficiency index of Q_{-1} is $(1,1)$ and the one of Q_m with $m \neq 1$ is $(0,0)$. $\square$

We examine the boundary conditions of all self-adjoint extensions of Q. In the case when $m \neq -1$, by Lemma 5, Q_m is essentially self-adjoint. We denote the self-adjoint extension of

Q_m by $\overline{Q_m}$. Consider Q_{-1}. We use the deficiency space technique (Theorem X.2 in Ref. [12]). Since the deficiency index of it is $(1,1)$, we can specify self-adjoint extensions by one real parameter $\tau \in [0, 2\pi)$. We introduce a quantity which characterizes the boundary condition of self-adjoint extensions Q^{τ}_{-1} of Q_{-1}. We put

$$\Phi_{-1}(\Psi) = \lim_{r\to 0+} \begin{pmatrix} r^{1-\alpha}\Psi_{+}(r) \\ r^{\alpha}\Psi_{-}(r) \end{pmatrix}$$

up to a nonzero constant multiplication. The following is the main theorem in this section.

Theorem 6. *The all self-adjoint extensions Q^{τ}_{-1} are specified by the boundary condition with a parameter $\tau \in [0, 2\pi)$:*

$$\Phi_{-1}(\Psi) = \begin{pmatrix} (e^{\mathrm{i}\tau} - 1)\Gamma(1-\alpha) \\ (e^{\mathrm{i}\tau} + 1)\Gamma(\alpha)2^{\alpha}B^{1-\alpha} \end{pmatrix}. \tag{9}$$

Proof. Since $\|\Psi^{\pm\mathrm{i}}_{+}\| = \|\Psi^{\pm\mathrm{i}}_{-}\|$, the all self-adjoint extensions Q^{τ}_{-1} with a parameter $\tau \in [0, 2\pi)$ is given by

$$\begin{aligned} D(Q^{\tau}_{-1}) &= \{\Omega + c\,(\Psi^{+\mathrm{i}} + e^{\mathrm{i}\tau}\Psi^{-\mathrm{i}});\, \Omega \in D(\overline{Q_{-1}}), c \in \mathbb{C}\}, \\ Q^{\tau}_{-1}[\Omega + c(\Psi^{+\mathrm{i}} + e^{\mathrm{i}\tau}\Psi^{-\mathrm{i}})] &= \overline{Q_{-1}}\Omega + \mathrm{i}c(\Psi^{+\mathrm{i}} - e^{\mathrm{i}\tau}\Psi^{-\mathrm{i}}), \end{aligned}$$

with the aid of the deficiency space technique. Here, $\overline{Q_{-1}}$ is the closure of Q_{-1}. We first investigate the BC to satisfy that $\Omega \in D(\overline{Q_{-1}})$. We use the following fact: Let A be a symmetric operator and $\mathcal{N}_{\pm}$ are the deficiency spaces of A. For $f \in D(A^{*})$, $f \in D(\overline{A})$ if and only if $\langle f, A^{*}g\rangle = \langle A^{*}f, g\rangle$ for all $g \in \mathcal{N}_{+} + \mathcal{N}_{-}$. Let

$$\Omega = \begin{pmatrix} \Omega_{+} \\ \Omega_{-} \end{pmatrix} \in D(Q^{*}_{-1}).$$

Then, $\Omega \in D(\overline{Q_{-1}})$ if and only if

$$\begin{cases} \langle \Psi^{\pm\mathrm{i}}_{-},\, q^{+}_{0}\Omega_{+}\rangle_{\mathcal{L}^2} = \langle q^{-}_{-1}\Psi^{\pm\mathrm{i}}_{-},\, \Omega_{+}\rangle_{\mathcal{L}^2}, \\ \langle \Psi^{\pm\mathrm{i}}_{+},\, q^{-}_{-1}\Omega_{-}\rangle_{\mathcal{L}^2} = \langle q^{+}_{0}\Psi^{\pm\mathrm{i}}_{+},\, \Omega_{-}\rangle_{\mathcal{L}^2}. \end{cases}$$

Here, q^{-}_{-1} and q^{+}_{0} are defined in Sec. II. This is equivalent to

$$\begin{cases} \lim_{r\to 0+} \Omega_{+}(r)\Psi^{\pm\mathrm{i}}_{-}(r)\, r = \lim_{r\to 0+} \Omega_{+}(r)\, r^{1-\alpha} = 0, \\ \lim_{r\to 0+} \Omega_{-}(r)\Psi^{\pm\mathrm{i}}_{+}(r)\, r = \lim_{r\to 0+} \Omega_{-}(r)\, r^{\alpha} = 0. \end{cases}$$

Thus, $\Phi_{-1}(\Omega)=0$. Let $\Psi=\Omega+c(\Psi^{+\mathrm{i}}+e^{\mathrm{i}\tau}\Psi^{-\mathrm{i}})\in D(Q_{-1}^{\tau})$. Then, we have that $\Phi_{-1}(\Psi)=\Phi_{-1}(\Psi^{+\mathrm{i}}+e^{\mathrm{i}\tau}\Psi^{-\mathrm{i}})$. Thus, it holds that

$$\begin{aligned}\Phi_{-1}(\Psi) &= \lim_{r\to 0+}\begin{pmatrix}(e^{\mathrm{i}\tau}-1)\exp\left(-\dfrac{Br^2}{4}\right)G\left(\alpha+\dfrac{1}{2B},\alpha,\dfrac{Br^2}{2}\right)\\(e^{\mathrm{i}\tau}+1)\exp\left(-\dfrac{Br^2}{4}\right)G\left(\alpha+\dfrac{1}{2B},1+\alpha,\dfrac{Br^2}{2}\right)\end{pmatrix}\\ &= \frac{1}{\Gamma(1+1/2B)}\begin{pmatrix}(e^{\mathrm{i}\tau}-1)\Gamma(1-\alpha)\\(e^{\mathrm{i}\tau}+1)\Gamma(\alpha)B(B/2)^{-\alpha}\end{pmatrix}.\end{aligned}$$

Therefore we obtain the desired result. □

Remark 7. Lemma 5 and Lemma 6 imply that the self-adjoint extension Q^{τ} of Q is parameterized by one real parameter $\tau\in[0,2\pi)$ and specified by the boundary condition:

$$\Phi(\Psi)=\begin{pmatrix}(e^{\mathrm{i}\tau}-1)\Gamma(1-\alpha)\\(e^{\mathrm{i}\tau}+1)\Gamma(\alpha)2^{\alpha}B^{1-\alpha}\end{pmatrix}$$

with $\Phi(\Psi)=\Phi_{-1}(UP_{-1}\Psi)$. This Q^{τ} is given by

$$Q^{\tau}=\oplus_{\substack{m\in\mathbb{Z}\\ m\neq -1}}(UP_m)^{-1}\overline{Q_m}UP_m\oplus(UP_{-1})^{-1}Q_{-1}^{\tau}UP_{-1}. \tag{10}$$

Here the operators U and P_m are given in Sec. II.

V. SEPCTRUM OF SELF-ADJOINT EXTENSIONS

In this section, we examine the spectrum of the self-adjoint extensions Q^{τ}. We shall see that all of Q^{τ} expect Q^0 has same spectrum. This is differ from the spin-less case treated in Ref. [8]. We denote the spectrum of a operator T by $\sigma(T)$.

Theorem 8. *We have that*

$$\sigma(Q^0)=\left\{\pm\sqrt{2Bn};n\in\mathbb{N}_0\right\}\cup\left\{\pm\sqrt{2B(\alpha+n+1)};n\in\mathbb{N}_0\right\}.$$

The multiplicity of $\pm\sqrt{2Bn}$ *is infinity for all* $n\in\mathbb{N}_0$ *and the multiplicity of* $\pm\sqrt{2B(\alpha+n+1)}$ *is* $n+1$ *for all* $n\in\mathbb{N}_0$. *If* $\tau\in(0,2\pi)$, *we have that*

$$\sigma(Q^{\tau})=\left\{\pm\sqrt{2Bn};n\in\mathbb{N}_0\right\}\cup\left\{\pm\sqrt{2B(\alpha+n)};n\in\mathbb{N}_0\right\}.$$

The multiplicity of $\pm\sqrt{2Bn}$ *is infinity for all* $n\in\mathbb{N}_0$ *and the multiplicity of* $\pm\sqrt{2B(\alpha+n)}$ *is* $n+1$ *for all* $n\in\mathbb{N}_0$.

To prove this theorem, we must examine the spectra of $\overline{Q_m}$ and Q^τ_{-1}.

Proposition 9. *If $m \in -\mathbb{N}_2$,*

$$\sigma(\overline{Q_m}) = \left\{\pm\sqrt{2Bn}; n \in \mathbb{N}_0\right\}.$$

If $m \in \mathbb{N}_0$,

$$\sigma(\overline{Q_m}) = \left\{\pm\sqrt{2B(m+\alpha+n+1)}; n \in \mathbb{N}_0\right\}.$$

In both cases, the multiplicity of these eigenvalues is 1.

Proof. In the case when $m \in -\mathbb{N}_2$, we can easily see that $\mathcal{S}_2 \subset D(\overline{Q_m})$ and the set $\mathcal{S}_2$ is the set of the eigenfunctions of Q_m and complete in $\mathcal{H}$. Thus, we obtain the desired result. In the case when $m \in \mathbb{N}_0$, we can obtain the desired result in a similar way as above. $\square$

Proposition 10. *We have that*

$$\sigma(Q^0_{-1}) = \left\{\pm\sqrt{2Bn}; n \in \mathbb{N}_0\right\}, \tag{11}$$

$$\sigma(Q^\pi_{-1}) = \left\{\pm\sqrt{2B(\alpha+n)}; n \in \mathbb{N}_0\right\} \tag{12}$$

If $\tau \in (0,\pi) \cup (\pi, 2\pi)$, we have that

$$\sigma(Q^\tau_{-1}) = \left\{\pm\sqrt{2B(n+1)}, \pm\sqrt{2B(\alpha+n)}; n \in \mathbb{N}_0\right\}. \tag{13}$$

In all cases, the multiplicity of these eigenvalues is 1.

Proof. We first examine the point spectrum of Q^τ_{-1}. Note that

$$\Phi(\Psi^1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \Phi(\Psi^2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{14}$$

for all $E \in \mathbb{C}$.

In the case when $\tau = 0$, the functions $\Psi \in D(Q^0_{-1})$ must satisfy BC, $\Phi(\Psi) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and $\Psi^2 \in \mathcal{H}$ if and only if $E = \pm\sqrt{2Bn}$ with $n \in \mathbb{N}_0$ as we already see in Sec. III. Therefore, the set $\mathcal{S}_2$ defined in Proposition 4 is a subset of $D(Q^0_{-1})$. Moreover, the set $\mathcal{S}_2$ is complete in $\mathcal{H}$. Thus, we obtain Eq. (11). In the case when $\tau = \pi$, we can obtain Eq. (12) in a way similar as above.

We consider the case when $\tau \neq 0$ and $\tau \neq \pi$. Since the deficiency index of Q^{τ}_{-1} is finite and the essential spectrum of Q^{0}_{-1} is empty, the Krein's formula [3] jointly with Weyl's theorem (Theorem XIII.14 in Ref. [13]) tells us that the essential spectrum of Q^{τ}_{-1} is empty. Thus, we examine the point spectrum of it. Let

$$\beta = k + 1 - \frac{E^2}{2B}, \quad \gamma_+ = 1 + k, \quad \gamma_- = 2 + k,$$

and

$$\Psi^3(E;r) = \mathrm{i}E\left(\frac{B}{2}\right)^{\alpha-1}\frac{\Gamma(1-\gamma_+)}{\Gamma(\beta-\gamma_+ + 1)}\Psi^1(E;r) + 2\alpha\frac{\Gamma(\gamma_+ - 1)}{\Gamma(\beta)}\Psi^2(E;r).$$

Then, we have that

$$\Psi^3(E;r) = \begin{pmatrix} 2\alpha\mathrm{i}E\,(B/2)^{\alpha-1}\, r^{\alpha-1}G(\beta,\gamma_+,Br^2/2)e^{-Br^2/4} \\ -4\alpha\,(B/2)^{\alpha}\, r^{\alpha}G(\beta,\gamma_-,Br^2/2)e^{-Br^2/4} \end{pmatrix}.$$

This Ψ^3 behaves well at infinity, that is, Ψ^3 is L^2 with respect to $r\,dr$ at infinity, for any $E \in \mathbb{C}$. However, in general, it does not satisfy the boundary condition (9) for Q^{τ}_{-1}:

$$\Phi(\Psi^3(E;r)) = \begin{pmatrix} \mathrm{i}E\dfrac{\Gamma(1-\alpha)}{\Gamma(1-E^2/2B)} \\ -B\dfrac{\Gamma(\alpha)}{\Gamma(\alpha-E^2/2B)} \end{pmatrix}.$$

In the other hand, since Eq. (14) holds, choosing good coefficients, the linear combinations of Ψ^1, Ψ^2 and Ψ^3 satisfy the boundary condition (9) at the origin. However, Ψ^1 behaves well at infinity if and only if $\alpha - E^2/2B \in -\mathbb{N}_0$ and Ψ^2 behaves well at infinity if and only if $1 - E^2/2B \in -\mathbb{N}_0$. Therefore, we have that

$$\sigma(Q^{\tau}_{-1}) \supset \left\{\pm\sqrt{2B(n+1)}, \pm\sqrt{2B(\alpha+n)}; n \in \mathbb{N}_0\right\}.$$

Since all eigenfunctions is given by a linear combination of Ψ^1, Ψ^2 and Ψ^3, we can see the reverse inclusion relation. □

Now, we are in the position to prove Theorem 8.

Proof of Theorem 8. Since we have that

$$\sigma(Q^{\tau}) = \overline{\bigcup_{\substack{m\in\mathbb{Z} \\ m\neq -1}} \sigma(\overline{Q_m}) \cup \sigma(Q^{\tau}_{-1})}$$

by Eq. (10), we can easily obtain the desired result using Proposition 9 and Proposition 10. □

Remark 11. We remark that $\Psi^{\pm\mathrm{i}}(r) = \text{const.} \times \Psi^3(\pm\mathrm{i};r)$.

VI. DISCUSSIOONS AND FUTURE PROBLEMS

In our case, the generalized BC of a Dirac-Weyl operator Q is given by Remark 7. This BC is very simpler than the generalized BC of the spin-less Schrödinger operator H_0 in Ref. [8]. Moreover, we can obtain $\sigma(Q^\tau)$ explicitly as in Theorem 8. However, the exact values of $\sigma(H_0)$ can not be calculated in general, as mentioned in Ref. [8]. Thus, so does the spectrum of $H_{1/2}$. However, we have naively that $H_{1/2} = Q^2$, thus for all $\tau \in [0, 2\pi)$ there exists a self-adjoint extension H^τ of $H_{1/2}$ such that $H^\tau = (Q^\tau)^2$. Therefore, we can obtain the spectrum of such H^τ explicitly. These situations suggest that the spin structure strongly restrict the self-adjoint extensions of a Dirac-Weyl operator. It is a natural and future problem to identify the relation between of Q^τ we obtained, and the self-adjoint extensions of $H_{1/2}$, which we can obtain from the Exner et. al.'s results.

Acknowledgement

This work is supported by the Grant-In-Aid 14740113 for Encouragement of Young Scientists from Japan Society for the Promotion of Science (JSPS).

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