Explicit solitary-wave ground states in one dimension

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Abstract

We give explicit solutions, that decay to zero at infinity, to the class of equations

$$-\partial_x^2 Q + cQ - \beta Q^{2p+1} - \alpha Q^{p+1} = 0,$$

where $c > 0$, $\beta > 0$, $p > 0$ and $\alpha \in \mathbb{R}$. This class of equations appears as the equation for the ground state for a solitary wave in the generalized nonlinear Schrödinger equation in one dimension and in the generalized KdV equation.

Keywords: explicit solutions, solitons, solitary waves, ground state, nonlinear scalar field equations.

1 Explicit solutions

Consider the class of one dimensional equations

$$-\partial_x^2 Q + cQ - Q^{2p+1} - \alpha Q^{p+1} = 0, \tag{1}$$

with $p > 0$, $c > 0$, $\alpha \in \mathbb{R}$. See Remark 2 for the general case. These equations belong to the class of nonlinear scalar field equations see *e.g.*, Berestycki & Lions (1983) [1] and references therein. Applications include the ground state to the nonlinear Schrödinger equation see *e.g.*, Chaio *et al.* (1964) [2] and to the Korteweg–de Vries (1894) [3] equation. We have the following lemma:

Proposition 1. *For fixed $p > 0$, $c > 0$ and $\alpha \in \mathbb{R}$ the equation* (1) *has solutions that decay to zero as $|x| \to \infty$ of the form*

$$Q(x) = \left(\frac{\alpha}{c(2+p)} + \sqrt{\frac{1}{c(1+p)} + \frac{\alpha^2}{c^2(2+p)^2}} \cosh\left(p\sqrt{c}(x - x_0)\right) \right)^{-1/p}, \tag{2}$$

for any translation constant $x_0 \in \mathbb{R}$.

Proof. Equation (1) is translational invariant, and hence it suffice to consider the case $x_0 = 0$. To verify that (2) is a solution to (1) we substitute it into (1). First consider the term $\partial_x^2 Q$. We have

$$\partial_x Q(x) = -\sqrt{\frac{1}{1+p} + \frac{\alpha^2}{c(2+p)^2}} \sinh(p\sqrt{c}x) Q^{p+1}(x), \tag{3}$$

and hence

$$\begin{aligned} -\partial_x^2 Q(x) = p\sqrt{c}\sqrt{\frac{1}{1+p} + \frac{\alpha^2}{c(2+p)^2}} \cosh(p\sqrt{c}x) Q^{p+1}(x) \\ - \left(1 + \frac{\alpha^2(p+1)}{c(2+p)^2}\right) \sinh^2(p\sqrt{c}x) Q^{2p+1}(x). \end{aligned} \tag{4}$$

To the end of comparing $\partial_x^2 Q$ with the remaining terms in (1) we break out Q^{2p+1} and use the explicit form of Q^p to obtain

$$\begin{aligned} -\partial_x^2 Q(x) = Q^{2p+1}(x) \Bigg(\left(\frac{\alpha}{c(2+p)} + \sqrt{\frac{1}{c(1+p)} + \frac{\alpha^2}{c^2(2+p)^2}} \cosh(p\sqrt{c}x) \right) \cdot \\ p\sqrt{c}\sqrt{\frac{1}{1+p} + \frac{\alpha^2}{c(2+p)^2}} \cosh(p\sqrt{c}x) - \left(1 + \frac{\alpha^2(p+1)}{c(2+p)^2}\right) \sinh^2(p\sqrt{c}x) \Bigg). \end{aligned} \tag{5}$$

Recalling that $\cosh^2(y) - \sinh^2(y) = 1$ and collecting equal powers of $\cosh(\cdot)$ together, gives

$$\begin{aligned} -\partial_x^2 Q(x) = Q^{2p+1}(x) \Bigg(\frac{\alpha p}{\sqrt{c}(2+p)} \sqrt{\frac{1}{1+p} + \frac{\alpha^2}{c(2+p)^2}} \cosh(p\sqrt{c}x) \\ - \left(\frac{1}{1+p} + \frac{\alpha^2}{c(2+p)^2}\right) \cosh^2(p\sqrt{c}x) + 1 + \frac{\alpha^2(p+1)}{c(2+p)^2} \Bigg). \end{aligned} \tag{6}$$

Re-writing the remaining terms of (1) using the explicit form of Q^p yields

$$\begin{aligned} cQ(x) - Q^{2p+1}(x) - \alpha Q^{p+1}(x) = Q^{2p+1}(x) \cdot \\ \Bigg(-1 + c\left(\frac{\alpha}{c(2+p)} + \sqrt{\frac{1}{c(1+p)} + \frac{\alpha^2}{c^2(2+p)^2}} \cosh(p\sqrt{c}x) \right)^2 \\ - \alpha \left(\frac{\alpha}{c(2+p)} + \sqrt{\frac{1}{c(1+p)} + \frac{\alpha^2}{c^2(2+p)^2}} \cosh(p\sqrt{c}x) \right) \Bigg). \end{aligned} \tag{7}$$

Expanding the square and collecting terms of equal powers in $\cosh(\cdot)$, we find

$$cQ(x) - Q^{2p+1}(x) - \alpha Q^{p+1}(x) = Q^{2p+1}(x)\Bigg(\bigg(\frac{1}{1+p} + \frac{\alpha^2}{c(2+p)^2}\bigg)\cosh^2(p\sqrt{c}x) \\ - \frac{p\alpha}{\sqrt{c}(2+p)}\sqrt{\frac{1}{1+p} + \frac{\alpha^2}{c(2+p)^2}}\cosh(p\sqrt{c}x) - 1 - \frac{\alpha^2(1+p)}{c(2+p)^2}\Bigg). \quad (8)$$

Since (6) is minus (8), summing them yields zero. □

Remark 1. *For $\alpha = 0$ we recover the well known solution, see von Sz.-Nagy (1941) [4], Titchmarsh (1946) [5]*

$$Q(x) = \big(c(1+p)\big)^{1/2p}\operatorname{sech}^{1/p}\big(p\sqrt{c}(x+m)\big). \quad (9)$$

Remark 2. *The class of equations*

$$-\partial_x^2 Q + cQ - \beta Q^{2p+1} - \alpha Q^{p+1} = 0, \quad (10)$$

with $\beta > 0$, $c > 0$, and $\alpha \in \mathbb{R}$ have solutions, that decay to zero as $|x| \to \infty$, of the form

$$Q(x) = \left(\frac{\alpha}{c(2+p)} + \sqrt{\frac{\beta}{c(1+p)} + \frac{\alpha^2}{c^2(2+p)^2}}\cosh\left(p\sqrt{c}(x-x_0)\right)\right)^{-1/p}, \quad (11)$$

for any translation constant x_0. This result follows directly from Proposition 1 as the rescaling transformation $\{\alpha, c, x - x_0\} \mapsto \{\alpha\beta, c\beta, (x-x_0)\beta^{-1/2}\}$ maps (10) to (1). Furthermore, in the limit $\beta \to 0$, $\alpha > 0$, using the 'half-angle formula' for $\cosh(\cdot)$ we once again recover the solution (9), with $p \mapsto p/2$.

Remark 3. *For the nonlinear eigenvalue parameter, c, the solution is a one bump solution for all positive values of c. Thus there are no excited states.*

Remark 4. *Consider the decaying-to-zero at infinity solutions to the class of equations*

$$-\partial_x^2 Q + cQ - \sum_{j\in I} a_j Q^{p_j} = 0, \quad (12)$$

for constants $\{a_j, p_j\}_{j\in I}$, with $a_j \in \mathbb{R}$ and $p_j > 0$ where $I \subset \mathbb{Z}$. That equation (12) is translational invariant suggest the change of variable $v = \partial_x Q$, $\partial_x v = \partial_Q v\,\partial_x Q = v\partial_Q v$. Replace $\partial_x^2 Q$ in terms of v yields a separable equation, integration gives

$$2^{-1}v^2 = \int cQ - \sum_{j\in I} a_j Q^{p_j}\ \mathrm{d}Q = 2^{-1}cQ^2 - \sum_{j\in I}(p_j+1)^{-2}a_j Q^{p_j+1} + k, \quad (13)$$

for some constant k. For functions that decay at infinity both Q and $v = \partial_x Q$ are zero at infinity, thus $k = 0$. When the solution, Q, exists and is smooth, the critical points of Q are at $\partial_x Q = 0$ and (13) *gives the explicit value of Q at these points as the solution to the polynomial equation*

$$2^{-1} c Q^2 = \sum_{j \in I} (p_j + 1)^{-2} a_j Q^{p_j+1}. \tag{14}$$

This has applications as the starting point for a shooting algorithm, when numerically solving (13).

References

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