Projective metrics and mixing properties on towers

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Abstract

We study decay of correlations for towers. Using Birkhoff's projective metrics, we obtain a rate of mixing of the form: $c_n(f,g) \leq$ Ct $\alpha(n)\|f\|\|g\|_1$ where $\alpha(n)$ goes to zero in a way related to the asymptotic mass of upper floors, $\|f\|$ is some Lipschitz norm and $\|g\|_1$ is some L^1 norm. The fact that the dependence on g is given by a L^1 norm is useful to study asymptotic laws of successive entrance times.

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Introduction

A powerful method to study non uniformly hyperbolic systems or hyperbolic systems with singularities consists in conjugating the system to a tower (see below the description of towers). Indeed, this strategy was initiated by F. Hofbauer ([H]) to study piecewise expanding maps of the interval. It was then used to study piecewise expanding maps in higher dimension ([Buz]), stochastic stability of unimodal maps (([Ba,V], [Ba,Y]), to obtain exponential decay of correlations for unimodal maps ([K,N],[Y1]), some billiards ([Y1], [Ch]), Henon maps ([B,Y]), partially hyperbolic diffeomorphisms ([D], [Ca]). In a recent paper, L.-S. Young ([Y2]) relates the decay of correlations for some mixing towers to the asymptotics of return times on the base of the tower. This gives, for example, polynomial decay of correlations for some interval maps with neutral fixed points. To this aim, she uses a coupling method.

We present a new approach to study mixing properties on towers "à la Young". Using Birkhoff cones and projective metrics instead of coupling, we relate explicitly the rate of mixing to the mass of upper floors of the tower (Theorem 1.4). Moreover, our method gives strong mixing properties: the correlations depend of one observable solely through some L^1 norm. This is very useful to study asymptotic laws of return times and successive return times (see [C], [C,G,S], [G,S], [Sau], [P]).

Section 1 contains the setting and precise statement of the results on decay of correlations.
In section 2, we briefly recall definitions and properties of Birkhoff's cones and projective metrics.
Section 3 is devoted to the proof of the estimation of decay of correlations. The strategy is the following: we construct a sequence of cones of locally Lipschitz functions C_j such that for some $k \in \mathbb{N}$ the transfer operator $\mathcal{L}^k$ maps C_j into C_{j+1} with contraction $\gamma_j < 1$. The rate of mixing is given by the product of the γ_j's. This product is easily related to the mass of the upper floors. The rate of convergence of $\mathcal{L}^k$ to the spectral projection associated with the invariant density is obtained in the uniform norm on each floor.
In section 4, we discuss specific rates of mixing and on lower bounds for uniform speed of convergence.

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1 Setting, statement of the results

Let us describe the tower model. We follow Young's setting.

A tower Δ consists of floors Λ_ℓ, $\ell \in \mathbb{N}$. The base Λ_0 of the tower is a probability space, let m_0 be the probability measure on Λ_0. The tower structure is given by a partition $(\Lambda_0^j)_{j\in\mathbb{N}}$ (mod m_0) of Λ_0 and a return time function defined on the base Λ_0 of the tower $R : \Lambda_0 \longrightarrow \mathbb{N}$ which is constant on each Λ_0^j. The floors Λ_ℓ are just copies of a part of Λ_0:

$$\Lambda_\ell = \{(x, \ell) \; / x \in \Lambda_0, \; R(x) > \ell\}.$$

We will denote by Λ_ℓ^j the copies of the Λ_0^j:

$$\Lambda_\ell^j = \{(x,\ell)\ /x \in \Lambda_0^j,\ R(x) > \ell\} \subset \Lambda_\ell.$$

The dynamic on the tower is given by $f_0 : \Lambda_0 \longrightarrow \Lambda_0$ such that $f_0\ : \Lambda_0^j \longrightarrow \Lambda_0$ is bijective (mod m_0).

Remark 1.1 *The assumption that $f_0\ : \Lambda_0^j \longrightarrow \Lambda_0$ is bijective may be replaced by the* Markov *assumption: $f_0(\Lambda_0^j)$ is a union of some Λ_0^k's and some additional informations on the ratios $\frac{m_0(f_0\Lambda_0^j)}{m_0\Lambda_0^j}$.*

Let us consider $F\ : \Delta \longrightarrow \Delta$ defined by

$$\begin{cases} F(x,\ell) &= F(x,\ell+1) \text{ for } (x,\ell)\in\Lambda_\ell \text{ and } R(x) > \ell+1 \\ &= f_0(x,0) =: F^R(x,0) \text{ otherwise.} \end{cases}$$

On each floor Λ_ℓ we put the σ-algebra of Λ_0 pushed by F, so that Δ is a measurable space. We assume that the partition $\mathcal{R} = \{\Lambda_\ell^j\}$ generates in the sense that $\bigvee_{i=0}^{\infty} F^{-i}\mathcal{R} := \left\{\bigvee_{i=0}^{\infty} F^{-i}A_i,\ A_i \in \mathcal{R}\right\}$, is the partition into points.

For $k \in \mathbb{N}$, the elements of the partition $\mathcal{R}^{(k)} = \bigvee_{i=0}^{k} F^{-i}\mathcal{R}$ are called *cylinder or k-cylinders.* We denote by $C_k(x)$ the element of $\mathcal{R}^{(k)}$ which contains x.
The measure m_0 is pushed on each floor by F, we assume that $\int R dm_0 < \infty$, so that we obtain a finite measure on Δ. This measure will also be denoted by m_0 and we assume that it has been normalized ($m_0(\Delta) = 1$).
The space Δ may be endowed with a dynamical distance. For x and y in Λ_0, the separation time $s(x,y)$ is the greatest integer n such that $(F^R)^p(x)$ and $(F^R)^p(y)$ belong to the same Λ_0^j for all $p \leq n$. If x and y belong to Λ_ℓ, $\ell \geq 1$ then they have exactly one preimage x^0 and y^0 in Λ_0, the separation time between x and y is $s(x,y) = s(x^0,y^0)$, if x and y are not in the same Λ_ℓ then $s(x,y) = 0$. The distance on the tower is defined by : $d(x,y) = \beta^{s(x,y)}$ for some $0 < \beta < 1$. Endowed with this metric, the space Δ is separable (the topology is generated by the cylinders).
We will always assume the following regularity condition on F:

$$F^R : \Lambda_0^j \longrightarrow \Lambda_0$$

and its inverse are non singular, its Jacobian JF^R satisfies the following bounded distortion inequality: $\exists C > 0$ such that $\forall j \in \mathbb{N}\ \forall x, y \in \Lambda_0^j$

$$\left|\frac{JF^R(x)}{JF^R(y)} - 1\right| \leq C\ d(F^R(x), F^R(y)). \qquad \textbf{(BD)}$$

The *Ruelle-Perron-Frobenius operator or transfer operator* associated to F is defined in the standard way:

$$\mathcal{L}_0 f(x) = \sum_{F(y)=x} JF(y)^{-1} f(y),$$

where JF is the Jacobian of F. Remark that except on Λ_0, this Jacobian is 1. The bounded distortion property implies that for any $x \in \Lambda_\ell^j$ with $R(j) = \ell + 1$,

$$C^{-1} m_0(\Lambda_\ell^j) \leq JF(x)^{-1} \leq C m_0(\Lambda_\ell^j) \tag{1.1}$$

and more generally, for any $k \in \mathbb{N}$, for any $x \in \Delta$ such that $F^k(x) \in \Lambda_0$,

$$C^{-1} m_0(C_k(x)) \leq JF^k(x)^{-1} \leq C m_0(C_k(x)), \tag{1.2}$$

so that $\mathcal{L}_0$ is well defined and acts continuously on the space $C_u(\Delta)$ of uniformly continuous and bounded functions on Δ, it also acts on $L^1(m)$.
The following facts on $\mathcal{L}_0$ directly follow from the definitions:

- $\displaystyle\int \mathcal{L}_0 f \cdot g dm_0 = \int g \circ F \cdot f dm_0$ for all f and g in $C_u(\Delta)$ (i.e. m_0 is a *conformal measure*),
- a measure $\mu = h_0 m_0$, $h_0 \in L^1(m)$ is F invariant if and only if $\mathcal{L}_0 h_0 = h_0$.

We expect that the mixing properties of an invariant measure absolutely continuous with respect to m_0 are related with the spectral properties of $\mathcal{L}_0$. To be more precise, let us assume that there exists a fixed point $h_0 \in C_u(\Sigma)$ for $\mathcal{L}_0$ which is normalized ($m_0(h_0) = 1$) and let $\mu = h_0 m_0$. For $f \in C_u(\Sigma)$ and $g \in L^1(m_0)$, the *correlations* of f and g measure the lack of independence between f and $g \circ F^n$ with respect to the invariant measure μ: for $n \in \mathbb{N}$,

$$c_n(f,g) = \left| \int f(g \circ F^n) d\mu - \int f d\mu \int g d\mu \right|.$$

The measure μ is *mixing* if and only if the coefficients $c_n(f,g)$ go to zero for any $f \in C_u(\Sigma)$ and $g \in L^1(m_0)$. In this case, estimates on the speed of convergence to zero of $c_n(f,g)$ or equivalently estimates on the *decay of correlations* may lead to the Central Limit Theorem (see [Li2]) and to the determination of asymptotic laws for entrance times (see [G,S] and [Sau]). The following trivial computation relates the decay of correlations to the asymptotic behavior of the iterates of $\mathcal{L}_0$:

$$c_n(f,g) = \left| \int [\mathcal{L}_0^n(fh_0) - h_0 m_0(fh_0)] g dm_0 \right| \tag{1.3}$$

so that if $\mathcal{L}_0^n f \to h_0 m_0(f)$ in some reasonable way then μ is mixing and estimates on the speed of this convergence would precise the decay of correlations.

We will denote by L the space of functions that are *uniformly Lipschitz (u.L.)* and bounded on Δ. That is, bounded functions f satisfying the u.L. condition:
there exists $L(f) > 0$, such that for x and y in the same Λ_ℓ,

$$|f(x) - f(y)| \leq L(f)d(x, y).$$

Let $\|f\| = \max(\|f\|_\infty, L(f))$, this defines a norm on L which turns L into a Banach space.

1.1 Invariant measure and decay of correlations

Under some additional assumptions, we will prove existence and uniqueness on an F-invariant probability, absolutely continuous with respect to m_0. We will also give some estimations on the decay of correlations when this invariant measure is mixing.

Following [A,D,U], we will say that the system (Δ, F) is *irreducible* if for all $i,\ j,\ \ell,\ \ell'$, there exists an integer n such that $F^{-n}\Lambda_\ell^j \cap \Lambda_{\ell'}^i \neq \emptyset$; we will say that the system (Δ, F) is *aperiodic* if for all $i,\ j,\ \ell,\ \ell'$, there exists an integer N such that for all $n \geq N$, $F^{-n}\Lambda_\ell^j \cap \Lambda_{\ell'}^i \neq \emptyset$. Following Markov chains and positive matrices methods ([Se]), it may be proved that aperiodicity is equivalent to L.-S. Young's condition: g.c.d.$R = 1$ ([Y2]).
To study mixing properties of $\mathcal{L}$, we will need to deal with more general potentials.
Let $\Phi\ : \Delta \longrightarrow \mathbb{R}$ satisfy the *uniformly locally Lipschitz (u.l.L.) condition*:
there exists a constant $L > 0$ such that, for all x and y in the same Λ_ℓ^j,

$$|\Phi(x) - \Phi(y)| \leq Ld(x, y)$$

Let $\mathcal{L}_\Phi$ be the associated transfer operator:

$$\mathcal{L}_\Phi f(x) = \sum_{Fy=x} e^{\Phi(y)} f(y).$$

Under some general conditions, the operator $\mathcal{L}_\Phi$ admits a unique non zero positive fixed point.

THEOREM 1.2 *Let Φ satisfy the u.l.L condition, have a conformal measure m' (i.e. $m'(\mathcal{L}_\Phi f) = m'(f)\ \forall f \in C_u(\Delta)$), satisfy the "weak contribution of infinity" condition:*

$$\exists n_0 \ / \ \forall n \geq n_0 \ \sup_{x\in\Lambda_n} |\mathcal{L}_\Phi \mathbb{1}(x)| \leq 1, \tag{\textbf{K}}$$

and $\|\mathcal{L}_\Phi \mathbb{1}\|_\infty < \infty$, then $\mathcal{L}_\Phi$ admits a non zero positive fixed point $h_\Phi \in L$. If (Δ, F) is irreducible then $h_\Phi > 0$, it is unique up to a multiplicative factor and the invariant measure $\mu = h_\Phi m'$ is ergodic.
If (Δ, F) is aperiodic then μ is mixing. Moreover $\mathcal{L}^n_\Phi f$ converges to $h_\Phi m'(f)$ uniformly on each compact subset of Δ and in $L^1(m')$ for any $f \in C_u(\Delta)$.

The proof of theorem 1.2 is now more or less standard. It follows arguments from [Sa] and [Ma1, Ma2] (see also [Y2] for a proof without transfer operator). We will just sketch the different steps. We begin with the following lemma.

LEMMA 1.3 *Let Φ satisfy hypothesis of theorem 1.2, then there exists $M > 0$ such that $\forall\ k \in \mathbb{N}$, $\|\mathcal{L}^k_\Phi \mathbb{1}\|_\infty \leq M$.*

Proof: First, let us note that the u.l.L condition implies the following bounded distortion property: $\exists C > 0$ such that $\forall x, y$ in the same Λ_ℓ,

$$\forall k \in \mathbb{N},\ \left|\frac{\mathcal{L}^k_\Phi \mathbb{1}(x)}{\mathcal{L}^k_\Phi \mathbb{1}(y)} - 1\right| \leq Cd(x,y).$$

Let Φ satisfy the "weak contribution of infinity condition". For any $k \in \mathbb{N}$, $n \in \mathbb{N}$ and $x \in \Lambda_n$, the bounded distortion property gives:

$$\begin{aligned}
\mathcal{L}^k_\Phi \mathbb{1}(x) &\leq C\frac{1}{m'(\Lambda_n)} \int_{\Lambda_n} \mathcal{L}^k_\Phi \mathbb{1} dm' \\
&\leq C\frac{1}{m'(\Lambda_n)} \int_{\Delta} \mathcal{L}^k_\Phi \mathbb{1} dm' \\
&= C\frac{1}{m'(\Lambda_n)} \text{ because } m' \text{ is a conformal probability.}
\end{aligned}$$

So, it exists $M' > 0$ such that $\sup_{x\in\Lambda_n} \mathcal{L}^k_\Phi \mathbb{1}(x) \leq M'$ for all $k \in \mathbb{N}$ and for all $n \leq n_0$. It remains to control $\mathcal{L}^k_\Phi \mathbb{1}(x)$ for $k \in \mathbb{N}$ and $x \in \Lambda_n$, $n \geq n_0$. Let us denote by $M_k = \sup_{x\in\Delta} \mathcal{L}^k_\Phi \mathbb{1}(x) < \infty$ because of $\|\mathcal{L}^k \mathbb{1}\| < \infty$ for all k. Since Φ satisfies the weak contribution of infinity assumption, if x belong to Λ_n, $n \geq n_0$ then $\mathcal{L}_\Phi \mathbb{1}(x) \leq 1$ and

$$\mathcal{L}^{k+1}_\Phi \mathbb{1}(x) \leq \mathcal{L}_\Phi \mathbb{1}(x) \cdot \sup_\Delta \mathcal{L}^k_\Phi \mathbb{1} \leq M_k.$$

So, $M_{k+1} \leq \max(M', M_k)$ which leads to $M_k \leq \max(M', 1)$ which concludes the proof. ∎

Sketch of proof of theorem 1.2 : From lemma 1.3, there exists $M > 0$ such that $\|\mathcal{L}_\Phi^k \mathbb{1}\|_\infty \leq M$ for all $k \in \mathbb{N}$. It is then easy to see that $\mathcal{L}_\Phi$ acts on L. Indeed, let f belong to L we have the following inequality: $\forall k \in \mathbb{N}$

$$|\mathcal{L}_\Phi^k f(x) - \mathcal{L}_\Phi^k f(y)| \leq d(x,y) \text{ Ct } [\|f\|_\infty + L(f)] \ \forall x,\ y \text{ in the same floor.}$$

Ascoli theorem on separable spaces imply that there exists a sequence n_k such that $Q_{n_k} = \frac{1}{n_k}\sum_{i=0}^{n_k-1} \mathcal{L}_\Phi^i \mathbb{1}$ converges, uniformly on compact subsets of Δ and in $L^1(m')$, to some limit $h_\Phi \in L$. Using the fact that m' is conformal, we get that this limit is non zero (indeed, $m'(h_\Phi) = 1$ by Lebesgue theorem) and satisfies $\mathcal{L}_\Phi h_\Phi = h_\Phi$. Let $\mu = h_\Phi m'$, this is a F-invariant probability.
If (Δ, F) is irreducible then [A,D,U] (theorem 2.5) implies that $\mu = hm$ is ergodic and $h > 0$.
Using [A,D,U] theorem 3.2 (see also Young's arguments [Y2]), aperiodicity of the system implies that (F, μ) is exact. So μ is mixing this implies that $\mathcal{L}_\Phi^k f$ converges to $h_\Phi m'(f)$ uniformly on each compact subset of Δ (any accumulation point of $\mathcal{L}_\Phi^k f$ must be $h_\Phi m'(f)$ because of mixing). ∎

Recall that $\mathcal{L}_0$ is the transfer operator associated to the potential $-\log JF$.

THEOREM 1.4 *The operator $\mathcal{L}_0$ admits a non zero positive fixed point $h_0 \in L$.*
If (Δ, F) is irreducible then $h_0 > 0$, it is unique up to a multiplicative factor and the invariant measure $\mu = h_0 m_0$ is ergodic.
If (Δ, F) is aperiodic then μ is mixing. Moreover $\mathcal{L}_0^n f$ converges to $h_0 m_0(f)$ uniformly on each compact subset of Δ and in $L^1(m_0)$ for any $f \in C_u(\Delta)$.
For any increasing sequence $(v_n)_{n\in\mathbb{N}}$ such that $\sum_{\ell\in\mathbb{N}} m_0(\Lambda_\ell)\cdot v_\ell < \infty$, for any $f \in L$ and g such that $gv \in L^1(m_0)$ (gv is defined by $gv(x) = g(x)v_\ell$ if x belong to Λ_ℓ), there exists $0 < \gamma < 1$ such that

$$c_n(f,g) \leq Ct \ \max[(v_n)^{-1},\ \gamma^n]\|f\| \ \|gv\|_1.$$

We also estimate the speed of convergence of $\mathcal{L}_0^n f$ to $h_0 m_0(f)$ in the uniform norm on each floor.
In particular,
• *if there exist constants $0 < e_1$ and $0 < \theta < 1$ such that for all $\ell \in \mathbb{N}$, $m(\Lambda_\ell) \leq e_1\theta^\ell$ then there exists $0 < \gamma < 1$ such that for any $\theta' < \theta$, for g such that $\frac{g}{\theta} \in L^1(m)$ (where $\frac{g}{\theta'}(x) = \frac{g(x)}{\theta'^\ell}$ for $x \in \Lambda_\ell$),*

$$c_n(f,g) \leq \ Ct\ (\theta')\ \gamma^n \ \|f\| \ \left\|\frac{g}{\theta'}\right\|_1,$$

• *if there exist constants* $0 < e_1$, $0 < \theta < 1$ *and* $0 < \beta < 1$ *such that for all* $\ell \in \mathbb{N}$, $m(\Lambda_\ell) \leq e_1 \theta^{\ell^\beta}$ *then for any* $\beta' < \beta$, *let* $v_\ell = \theta^{-\ell^{\beta'}}$,

$$c_n(f,g) \leq Ct\ (\beta')\theta^{n^{\beta'}}\ \|f\|\ \|gv\|_1,$$

• *if there exist constants* $0 < e_1$, $\beta > 1$ *such that for all* $\ell \in \mathbb{N}$, $m(\Lambda_\ell) \leq e_1 \ell^{-\beta}$, *then for any* $\gamma > 1$, *let* $v_\ell = \left[\frac{(\ln \ell)^\gamma}{\ell^{\beta-1}}\right]^{-1}$,

$$c_n(f,g) \leq Ct\ (\gamma)\frac{(\ln n)^\gamma}{n^{\beta-1}}\ \|f\|\ \|gv\|_1.$$

Remark 1.5 *On one hand, L-S. Young obtained a decay of correlations in* $O(n^{-(\beta-1)})$ *in the polynomial case above. Our result is a little bit slower in this case. Except this, we recover her results. On the other hand, we get that the decay of correlations is of the form* $c_n(f,g) \leq (v_n)^{-1}C(f,g)$ *where the dependence of* $C(f,g)$ *on* g *is given by* $\|gv\|_1$. *This kind of property is very useful to study asymptotic laws of successive return times (see [C,G,S], [G,S], [Sau], [P]) and was not obtained using coupling methods as in [Y2].*

Existence and mixing properties of an invariant measure absolutely continuous with respect to m_0 follows form theorem 1.2. Remark that (1.1) and (**BD**) imply that the hypothesis of lemma 1.3 are satisfied for $\Phi = -\log JF$ (obviously from definitions, m_0 is a conformal probability).

Our estimation of the decay of correlations is based on Birkhoff's cones and projective metrics. Let us briefly recall basic definitions and facts on this theory.

2 Birkhoff's cones and projective metrics

The theory of cones and projective metrics of G. Birkhoff [Bi1] is a powerful tool to study linear operators. P. Ferrero and B. Schmitt [F,S1] applied it to estimate the correlation decay for random dynamical systems. Then, this strategy had been used by many authors. Let us mention C. Liverani [Li1] and M. Viana [V] for Anosov and Axiom A diffeomorphisms. They used Birkhoff cones to obtain exponential decay of correlations. In [K,M,S] the Birkhoff cones techniques were used in a different way to obtain sub-exponential decay of correlations. The way we use cone techniques to prove theorem 1.4 follows some ideas of P. Ferrero and B. Schmitt ([F,S2]) and [Ma1, Ma2].

Let us recall definitions and properties of cones and projective metrics (see [Li1] for a more complete presentation).

Let B be a vector space and $C \subset B$ a cone with the following properties.

- C is convex,
- $C \cap -C = \{0\}$,
- if α_n is a sequence of real numbers such that $\alpha_n \to \alpha$ and $x - \alpha_n y \in C \ \forall n$ then $x - \alpha y \in C$. This property is called "integral closure".

For such a cone, the pseudo-metric θ_C on C is defined in the following way. Let $x, y \in C$,

$$\mu(x,y) = \inf\{\beta > 0 \text{ such that } \beta x - y \in C\},$$

$$\lambda(x,y) = \sup\{\alpha > 0 \text{ such that } y - \alpha x \in C\},$$

with the convention: $\mu(x,y) = \infty$ and $\lambda(x,y) = 0$ if the corresponding sets are empty. Let $\theta_C(x,y) = \log \frac{\mu}{\lambda}$. θ_C is called pseudo-metric because it is not necessarily finite. Moreover, it is a projective pseudo-metric: if x and x_1 are proportional then for any $y \in C$, $\theta_C(x,y) = \theta_C(x_1,y)$.

The following two results reveal the usefulness of projective metrics.
Let C and C' be two cones, P a linear operator $P : C \to C'$. Let Δ denote the diameter of PC in C':

$$\Delta = \sup_{f,g \in C} \theta_{C'}(Pf, Pg).$$

THEOREM 2.1 [Bi1] *For any f, g in C, we have:*

$$\theta_{C'}(Pf, Pg) \leq \tanh\left(\frac{\Delta}{4}\right) \theta_C(f,g).$$

This theorem implies that $P : C \to C'$ is always a contraction (in wide sense) for the projective metrics. If $\Delta < \infty$ then it is a strict contraction.
The following result relies the metric θ_C to certain norms on B. A norm $\| \ \|$ on B is a *norm adapted to* C if for f and g in B such that if $f+g$ belongs to C and $f-g$ belongs to C then $\|g\| \leq \|f\|$. ρ is a *homogeneous form adapted to* C if ρ maps C to $\mathbb{R}^+$, for any $\lambda > 0$ and $f \in C$, $\rho(\lambda f) = \lambda \rho(f)$ and if $f - g \in C$ implies $\rho(g) \leq \rho(f)$.

THEOREM 2.2 [Bi1], [Li1] *Let C be a cone, let $\| \ \|$ and ρ be adapted to C. For any f and g in C such that $\rho(f) = \rho(g) \neq 0$ we have:*

$$\|f - g\| \leq (e^{\theta(f,g)} - 1)\min(\|f\|, \|g\|).$$

3 Proof of theorem 1.4

We will now prove theorem 1.4. Let us begin with the following remark.

Remark 3.1 *Let $v = (v_\ell)_{\ell\in\mathbb{N}}$ be an increasing sequence of positive numbers such that:*

$$\sum_{\ell\in\mathbb{N}} m_0(\Lambda_\ell)\cdot v_\ell < \infty.$$

We will consider such a v as a function on Δ: $v(x) = v_\ell$ if x belongs to Λ_ℓ. Let $\mathcal{L}$ be the transfer operator associated to the change of potential v: $\mathcal{L}f = v^{-1}\mathcal{L}_0(fv)$. The measure $m = vm_0$ is conformal for $\mathcal{L}$ and it follows easily from (1.1) and lemma 1.3 that $\sup_{k\in\mathbb{N}} \|\mathcal{L}^k \mathbb{1}\|_\infty < \infty$. Moreover, $h = \frac{h_0}{v}$ is an eigenvector for $\mathcal{L}$. Theorem 1.2 implies that for any $f \in C_u(\Delta)$,

$$\mathcal{L}^k f \longrightarrow hm(f) \text{ uniformly on each compact set.}$$

Throughout this section, we fix a strictly increasing sequence $v = (v_\ell)_{\ell\in\mathbb{N}}$ such that $m(\Delta) < \infty$ and the sequence $(\frac{v_\ell}{v_{\ell+1}})_{\ell\in\mathbb{N}}$ is also increasing. We will assume that m and h are normalized ($m(\Delta) = 1$ and $m(h) = 1$). We will work with the transfer operator $\mathcal{L}$.
As usual in these topics, the main tool is a "Lasota-Yorke" type inequality. Before proving it, let us study a bit the distance on Δ.

3.1 Some properties of the distance on Δ

Let us consider the following function on Δ: $\rho(x) = \beta$ if x belongs to Λ_0 and $\rho(x) = 1$ otherwise. For $k \in \mathbb{N}$, we will denote by $\rho^{(k)}(x) = \prod_{i=0}^{k} \rho(F^i x)$. The relation between $\rho^{(k)}$ and the distance on Δ is the following:
if x and y are in Λ_0, their preimages by F^k are paired. If x' is a preimage of x, we will denote by y' the preimage of y which belongs to $C_k(x')$. We have:

$$d(x', y') = \rho^{(k)}(x')\cdot d(x, y).$$

The following lemma proves that $\rho^{(k)}(x)$ is almost exponential.

LEMMA 3.2 *There exists $0 < \gamma < 1$ such that for all $\eta > 0$, there exists $\Omega_\eta \subset \Delta$ of m measure greater than $1 - \eta$ and $k_0(\eta)$ such that for all $k \geq k_0$ and $x \in \Omega_\eta$, $\rho^{(k)}(x) \leq \gamma^k$.*

Proof: This is an easy consequence of Birkhoff's ergodic theorem and Egoroff's theorem applied to $\log\rho$. ■

Remark 3.3 *A finer analysis on $\rho^{(k)}$ should allow one to avoid the use of Birkhoff's and Egoroff's theorems and so give a constructive bound for k_0.*

3.2 A Lasota-Yorke type inequality

We are now in position to prove a Lasota-Yorke type inequality.

LEMMA 3.4 *For any $\varepsilon > 0$, there exists $C > 0$, $N \in \mathbb{N}$ and $k_0 \in \mathbb{N}$ such that for all $k > k_0$, for all $f \in L$, we have:*

1. *for $\ell \geq k$, x and y in Λ_ℓ*
$$\left|\mathcal{L}^k f(x) - \mathcal{L}^k f(y)\right| \leq \frac{v_{\ell-k}}{v_\ell} d(x,y) L(f),$$
2. *for $N \leq \ell < k$, x and y in Λ_ℓ*
$$\left|\mathcal{L}^k f(x) - \mathcal{L}^k f(y)\right| \leq \varepsilon \; d(x,y) \left[L(f) + \sup|f|\right],$$
3. *for $\ell < N$, x and y in Λ_ℓ*
$$\left|\mathcal{L}^k f(x) - \mathcal{L}^k f(y)\right| \leq d(x,y) \left[\varepsilon L(f) + C \sup|f|\right].$$

Proof: Let f belong to L. Item 1. directly follows from the definition of $\mathcal{L}$ and $d(x,y)$.
Let x and y be in Λ_ℓ, $\ell < k$, let us denote by $x^0 \in \Lambda_0$ (resp. y^0) the preimage of x (resp. y) by $F^{k-\ell}$. We have:

$$\left|\mathcal{L}^k f(x) - \mathcal{L}^k f(y)\right| = \frac{v_0}{v_\ell} \left|\mathcal{L}^{k-\ell} f(x^0) - \mathcal{L}^{k-\ell} f(y^0)\right|. \tag{3.4}$$

Let us consider x and y in Λ_0, their paired preimages will be denoted by x' and y'.

$$\begin{aligned}
|\mathcal{L}^k f(x) - \mathcal{L}^k f(y)| &= \frac{1}{v_0} \Big|\sum_{F^k x' = x} \frac{v(x')}{JF^k(x')} f(x') - \frac{v(y')}{JF^k(y')} f(y')\Big| \\
&\leq \frac{1}{v_0} \sum_{F^k x' = x} \frac{v(x')}{JF^k(x')} |f(x') - f(y')| \\
&+ \frac{1}{v_0} \sum_{F^k x' = x} \frac{v(x')}{JF^k(x')} |f(y')| \left|\frac{JF^k(x')}{JF^k(y')} - 1\right| \\
&\leq L(f) d(x,y) \underbrace{\sum_{F^k x' = x} v(x') JF^k(x')^{-1} \rho^{(k)}(x')}_{(1)} \\
&+ C \; d(x,y) \sup|f| \underbrace{\sum_{F^k x' = x} v(x') JF^k(x')^{-1}}_{(2)}.
\end{aligned}$$

The term (2) is bounded above by $\|\mathcal{L}^k \mathbb{1}\|_\infty \le M$.
Let us turn our attention to term (1). Let $\eta > 0$ to be fixed later, let Ω_η be given by lemma 3.2 and assume $k \ge k_0(\eta)$. Recall (1.2): for z such that $F^k(z) \in \Lambda_0$, $JF^k(z)^{-1} \le Cm_0(C_k(z))$ and remark that $\rho^{(k)}$ is constant on each k-cylinder. Using lemma 3.2 we get:

$$\begin{aligned}
(1) \quad &\le \sum_{\substack{F^k x' = x \\ \rho^{(k)}(x') \le \gamma^k}} v(x') JF^k(x')^{-1} \rho^{(k)}(x') + \sum_{\substack{F^k x' = x \\ x' \in (\Omega_\eta)^c}} v(x') JF^k(x')^{-1} \rho^{(k)}(x') \\
&\le \gamma^k M + Cm(\Omega_\eta^c) \le C(\gamma^k + \eta).
\end{aligned}$$

We have proved that for x and y in Λ_0, for any $\eta > 0$, provided k is large enough,

$$|\mathcal{L}^k f(x) - \mathcal{L}^k f(y)| \le Cd(x,y)\left[(\gamma^k + \eta)L(f) + \sup|f|\right].$$

Remark that we have the following weak inequality for all k:

$$|\mathcal{L}^k f(x) - \mathcal{L}^k f(y)| \le Cd(x,y)\left[L(f) + \sup|f|\right]$$

Let ε be fixed, choose :

- η such that $C\eta < \varepsilon/2$, let $k_1 = k_0(\eta)$ be given by lemma 3.2,
- $N > 0$ such that $C\frac{v_0}{v_N} < \varepsilon$,
- $k_0 > k_1 + N$ and for $k > k_0 - N$, $\gamma^k < \varepsilon/2$.

Let $k > k_0$ and $N \le \ell < k$, the choices above and (3.4) give

$$\left|\mathcal{L}^k f(x) - \mathcal{L}^k f(y)\right| \le \frac{v_0}{v_\ell} Cd(x,y)\left[L(f) + \sup|f|\right] \le \varepsilon d(x,y)\left[L(f) + \sup|f|\right].$$

Let $\ell \le N$, we have:

$$\begin{aligned}
\left|\mathcal{L}^k f(x) - \mathcal{L}^k f(y)\right| &\le \frac{v_0}{v_\ell} Cd(x,y)\left[(\gamma^{k-\ell} + \eta)L(f) + \sup|f|\right] \\
&\le d(x,y)[\varepsilon L(f) + C\|f\|_\infty].
\end{aligned}$$

■

3.3 Construction of cones

We turn out to the hearth of the proof of theorem 1.4. Let us first introduce a finite partition of Δ. Let η and t be given, let s be greater than $k_0(\eta)$ given by lemma 3.2. Let $\mathcal{P}$ be the finite partition $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ defined by:

- $\mathcal{P}_2 = \left\{ \bigcup_{R(j) \ge t} \bigcup_{\ell \in \mathbb{N}} \Lambda_\ell^j \ \cup \ \{x \ / \ \rho^{(s)}(x) > \gamma^s\} \right\} =: \{P_2\}$,

- $\mathcal{P}_1$ is the partition of P_2^c into s-cylinders.

Because of lemma 3.2, the m measure of P_2 is as small as we want, provided t is large enough and η is small enough. We will denote by D_1 the diameter of $\mathcal{P}_1$ (which is smaller than γ^s by definition) and by D_2 the m measure of P_2.
Theorem 1.2 implies the following mixing property: for $\alpha < 1 < \alpha'$ there exists q_0 such that for $k > q_0$, for P and P' in $\mathcal{P}$,

$$\alpha < \frac{m(P \cap F^{-k}P')}{m(P)\mu(P')} < \alpha'. \tag{3.5}$$

Let a, b, c and d be positive numbers, k and j integers to be chosen later. $\mathcal{C}(a, b, c, d, j)$ is the cone of functions in L such that:

1. $0 \leq \frac{1}{\mu(P)} \int_P f dm \leq a \int f dm \ \forall P \in \mathcal{P}$,
2. for $\ell \leq jk$, for x and y in Λ_ℓ,
$$|f(x) - f(y)| \leq d(x, y) b \int f dm,$$
3. for $\ell \leq jk$, for $x \in \Lambda_\ell \cap P_2$ $|f(x)| \leq c \int f dm$,
4. for $\ell > jk$, $|f(x)| \leq \frac{v_{kj}}{v_\ell} d \int f dm$ and $|f(x) - f(y)| \leq \frac{v_{kj}}{v_\ell} d \int f dm$, for all $x,\ y \in \Lambda_\ell$.

Note that the cone also depends on s, t, k. We will choose a, b, c, d, k, j, s and t in such a way that $\mathcal{L}^k$ in a contraction from $\mathcal{C}(a, b, c, j)$ to $\mathcal{C}(a, b, c, d, j+1)$. Let us denote by $\delta_j = \frac{v_{kj}}{v_{k(j+1)}} < 1$.

Remark 3.5 *It will be clear from the proofs that if there exists $0 < \beta < 1$ such that $0 < \delta_j < \beta$ then condition 4 in the definition of the cone is useless. This means that it is possible to work with the cone $\mathcal{C}(a, b, c)$ of functions f such that:*

1. $0 \leq \frac{1}{\mu(P)} \int_P f dm \leq a \int f dm \ \forall P \in \mathcal{P}$,
2. $L(f) \leq b \int f dm$,

3. for $x \in \Lambda_\ell \cap P_2$ $|f(x)| \leq c \displaystyle\int f dm.$

PROPOSITION 3.6 *For any* $0 < \sigma < \delta_j$*, for any positive numbers* a_0*,* b_0*,* c_0*,* d_0*, there exist an integer* k_0*, positive numbers* $a \geq a_0$*,* $b \geq b_0$*,* $c \geq c_0$*,* $d \geq d_0$*, integers* s *and* t *such that for all* $k > k_0$*,*

$$\mathcal{L}^k \mathcal{C}(a,b,c,d,j) \subset \mathcal{C}(\sigma a, \delta_j b, \delta_j c, \delta_j d, j+1).$$

Moreover, for $f \in \mathcal{L}^k\mathcal{C}(a,b,c,d,j)$*, we have*

$$\frac{1}{\mu(P)} \int_P f dm \geq A \text{ for all } P \in \mathcal{P},$$

where A *can be chosen arbitrarily provided it is strictly smaller than* $1/2$*.*

Remark 3.7 *In fact, it is clear from the proof that modifying slightly the argument the constant* A *above may be chosen as close as one which from* 1*. This may be important to optimize the rate of convergence in the exponential case.*

We begin with the following lemma which gives a control on the sup. of functions in $\mathcal{C}(a,b,c,d,j)$. When there is no ambiguity, we will forget about the dependence on a, b, c d and j and write $\mathcal{C}$ instead of $\mathcal{C}(a,b,c,d,j)$ or $\mathcal{C}(j)$ if we wish to emphasize the dependence on j.

LEMMA 3.8 *For any* $f \in \mathcal{C}$*, for* $\ell \in \mathbb{N}$ *and* $P \in \mathcal{P}_1$*, for* x *in* $\Lambda_\ell \cap P$ *we have:*

$$|f(x)| \leq \frac{1}{m(P)} \int_P f dm + \max[b,]|P| \int f dm \tag{3.6}$$

$$|f(x)| \leq [Da + D_1 \max[b,\ d]] \int f dm. \tag{3.7}$$

(where $D = \sup_{P \in \mathcal{P}} \frac{\mu(P)}{m(P)}$*). For any* $f \in \mathcal{C}$*, for* $x \in P_2 \cap \Lambda_\ell$*,* $\ell \in \mathbb{N}$*,*

$$\begin{aligned} f(x) &\geq \frac{1}{m(P_2)} \int_{P_2} f dm - 2\max[c,d] \int f dm \\ f(x) &\leq \max[c,d] \int f dm \end{aligned} \tag{3.8}$$

Proof: Let f belong to $\mathcal{C}$. For $x \in P \in \mathcal{P}_1$, equations (3.6) and (3.7) for $\ell \leq kj$ directly follow from definitions. Just write that for x and y in $P \in \mathcal{P}$,

$$f(y) - L(f)d(x,y) \leq f(x) \leq f(y) + L(f)d(x,y)$$

and integrate on P with respect to m these inequalities. Let $x,\ y \in P_2 \cap \Lambda_\ell$ with $\ell \leq kj$,

$$f(y) - 2 \sup_{\Lambda_\ell \cap P_2} |f| \leq f(x) \leq \sup_{\Lambda_\ell \cap P_2} |f|,$$

this leads to (3.8) for $\ell \leq kj$. Inequalities for $\ell > kj$ follow from 4 in the definition of the cone. ■

Let us begin with the proof of proposition 3.6.
Proof of proposition 3.6: We are going to check how the four conditions in the definition of $\mathcal{C}$ evolute under the action of $\mathcal{L}^k$. First of all, condition 4. of the cone is very easy to check: if $x \in \Lambda_\ell$, $\ell > k(j+1)$ then $\mathcal{L}^k f(x) = \frac{v_{\ell-k}}{v_\ell} f(x^-)$ so $|\mathcal{L}^k f(x)|$ and $|\mathcal{L}^k f(x) - \mathcal{L}^k f(y)|$ are less than $\frac{v_{k(j+1)}}{v_\ell} \delta_j d \int f dm$, for x and y in Λ_ℓ, $\ell > kj$. It just remains to check the first three conditions. Assume that d is fixed, b and c will be chosen to have $d \leq \min[b, c]$.
We take $f \in \mathcal{C}$. We fix $0 < \sigma < \delta_j$ and $\varepsilon < \sigma$, $\alpha < 1 < \alpha'$ and $q_0 \in \mathbb{N}$ such that (3.5) is satisfied for $k > q_0$. In what follows, k is greater than q_0.
The following remark will be used many times in the proof: if $f \in \mathcal{C}$ then since we choose $d \leq \max(b,\ c)$ we have that $\sup |f| \leq c \int f dm$ and $L(f) \leq b \int f dm$ (use also that if $\ell > kj$ then $\frac{v_{kj}}{v_\ell} \leq 1$).
<u>Condition 1.</u>
Let $P \in \mathcal{P}$,

$$\begin{aligned} \frac{1}{\mu(P)} \int_P \mathcal{L}^k f dm &= \frac{1}{\mu(P)} \int_{F^{-k}P} f dm = \sum_{P' \in \mathcal{P}} \frac{1}{\mu(P)} \int_{P' \cap F^{-k}P} f dm \\ &= \sum_{P' \in \mathcal{P}_1} \frac{1}{\mu(P)} \int_{P' \cap F^{-k}P} f dm + \frac{1}{\mu(P)} \int_{P_2 \cap F^{-k}P} f dm. \end{aligned}$$

Using lemma 3.8, we get for $P' \in \mathcal{P}_1$:

$$\frac{m(F^{-k}P \cap P')}{\mu(P) m(P')} \left[\int_{P'} f d\mu - m(P') D_1 b \int f dm \right] \leq \frac{1}{\mu(P)} \int_{P' \cap F^{-k}P} f dm$$

$$\frac{1}{\mu(P)} \int_{P' \cap F^{-k}P} f dm \leq \frac{m(F^{-k}P \cap P')}{\mu(P) m(P')} \left[\int_{P'} f d\mu + m(P') D_1 b \int f dm \right],$$

using (3.5), this leads to

$$\alpha \left[\int_{P'} f dm - \alpha' b D_1 m(P') \int f dm \right] \leq \frac{1}{\mu(P)} \int_{P' \cap F^{-k}P} f dm$$

$$\frac{1}{\mu(P)} \int\limits_{P' \cap F^{-k}P} f dm \leq \alpha' \left[\int\limits_{P'} f dm + bD_1 m(P') \int f dm \right].$$

For P_2, we get:

$$\alpha \int\limits_{P_2} f dm - 2cD_2\alpha' \int f dm \leq \frac{1}{\mu(P)} \int\limits_{P_2 \cap F^{-k}P} f dm \leq \alpha' c D_2 \int f dm.$$

Summing these inequalities for all $P \in P_1$ and for P_2, we get (one has to notice that $m(\mathbb{1}_{P_2} \cdot f) \geq 0$):

$$[\alpha - \alpha' D_1 b - 2\alpha' c D_2] \int f dm \leq \frac{1}{\mu(P)} \int\limits_P \mathcal{L}^k f dm$$

$$\frac{1}{\mu(P)} \int\limits_P \mathcal{L}^k f \leq [1 + D_1 b + cD_2]\alpha' \int f dm \qquad (3.9)$$

<u>Condition 2.</u>

Let x and y belong to Λ_ℓ, $\ell \leq k(j+1)$ using the fact that if $x \in \Lambda_\ell$, $\ell \leq k(j+1)$ then $F^k x' = x$ with $x' \in \Lambda_{\ell-k}$ and lemma 3.4, we find k_0 and $N \in \mathbb{N}$ such that for $k > k_0$,

$$\begin{aligned}
|\mathcal{L}^k f(x) - \mathcal{L}^k f(y)| &\leq \frac{v_{\ell-k}}{v_\ell} d(x,y) b \int f dm \text{ if } \ell \geq k \\
&\leq \varepsilon d(x,y)[b \int f dm + \sup |f|] \text{ if } N \leq \ell < k \\
&\leq d(x,y)[\varepsilon b \int f dm + C \sup |f|] \text{ otherwise.}
\end{aligned}$$

Lemma 3.8 gives $\sup |f| \leq \max[c,\ bD_1 + aD] \int f dm$ so we get:

$$|\mathcal{L}^k f(x) - \mathcal{L}^k f(y)| \leq \frac{v_{\ell-k}}{v_\ell} d(x,y) b \int f dm \text{ if } \ell \geq k \qquad (3.10)$$

$$\leq \varepsilon d(x,y) \int f dm [b + \max(c,\ bD_1 + aD)] \text{ if } N \leq \ell < k \qquad (3.11)$$

$$\leq d(x,y) \int f dm [\varepsilon b + C \max(c,\ bD_1 + aD)] \text{ otherwise.} \qquad (3.12)$$

<u>Condition 3.</u>

Let $x \in \Lambda_\ell \cap P_2$ with $\ell \leq k(j+1)$, if $\ell \geq k$, we have

$$|\mathcal{L}^k f(x)| \leq \frac{v_{\ell-k}}{v_\ell} c \int f dm.$$

Let $\ell > k$, we denote by x^0 the preimage of x by F^ℓ, $x^0 \in \Lambda_0$,

$$\mathcal{L}^k f(x) = \frac{v_0}{v_\ell} \mathcal{L}^{k-\ell} f(x^0). \tag{3.13}$$

For $x \in \Lambda_0$ and $q \in \mathbb{N}$,

$$\begin{aligned} |\mathcal{L}^q f(x)| &\leq \sum_{\substack{F^q x' = x \\ x' \in P_2}} \frac{v(x')}{JF^q(x')} f(x') + \sum_{\substack{F^q x' = x \\ x' \notin P_2}} \frac{v(x')}{JF^q(x)} f(x') \\ &\leq c \int f dm \; C \sum_{\substack{F^q x' = x \\ x' \in P_2}} m_0(C_q(x')) v(x') + \int f dm [bD_1 + aD] \text{ using (3.7).} \end{aligned}$$

So we have for $x \in \Lambda_\ell$, $\ell < k$,

$$|\mathcal{L}^k f(x)| \leq [cCD_2 + bD_1 + aD] \int f dm. \tag{3.14}$$

We are now in position to chose the parameters. Chose:

- $d > 0$
- $a > \dfrac{\alpha' + 3/8\alpha}{\sigma}$,
- $c > \dfrac{1 + aD}{\sigma - \varepsilon}$, $c \geq d$
- $b > \dfrac{Cc}{\sigma - \varepsilon}$, $b \geq d$
- η and t such that $2\alpha' c D_2 < \alpha/4$, and $CD_2 < \varepsilon$,
- s such that $s > k_0(\eta)$ ($k_0(\eta)$ is given by lemma 3.2) and $\alpha' c D_1 = \alpha' c \gamma^s < \alpha/4$,
- $k > k_0(\eta)$ and $k > q_0$ given by (3.5).

With these choices, we have: for any $P \in \mathcal{P}$,

$$\begin{aligned} \alpha/2 \int f dm &\leq \frac{1}{\mu(P)} \int_P \mathcal{L}^k f dm &\leq (\alpha' + 3/8\alpha) \int f dm \text{ by (3.9)} \\ \alpha/2 \int f dm &\leq \frac{1}{\mu(P)} \int_P \mathcal{L}^k f dm &\leq \sigma a \int f dm, \end{aligned}$$

for x and y in Λ_ℓ^q, $\ell \leq k(j+1)$, using (3.10, 3.11, 3.12),

$$\begin{aligned} |\mathcal{L}^k f(x) - \mathcal{L}^k f(y)| &\leq \delta_j b \int f dm \text{ if } \ell \geq k \\ |\mathcal{L}^k f(x) - \mathcal{L}^k f(y)| &\leq \sigma b \int f dm \text{ otherwise.} \end{aligned}$$

for any $x \in P_2 \cap \Lambda_\ell$, $\ell \leq k(j+1)$, using (3.13, 3.14),

$$\begin{aligned} |\mathcal{L}^k f(x)| &\leq \delta_j c \int f dm \text{ if } \ell \geq k \\ |\mathcal{L}^k f(x)| &\leq \sigma c \int f dm \text{ otherwise.} \end{aligned} \tag{3.15}$$

This concludes the proof since $\sigma < \delta_j$. ∎

It is easily seen that all the cones $\mathcal{C}(a,b,c,d,j)$ satisfy the properties of section 2. In order to apply results from section 2, we have to estimate the projective diameter of $\mathcal{L}^k\mathcal{C}(a,b,c,d,j)$ into $\mathcal{C}(a,b,c,d,j+1)$ and to find adapted homogeneous form and norm.

LEMMA 3.9 *The projective diameter Δ_j of $\mathcal{L}^k\mathcal{C}(a,b,c,d,j)$ into $\mathcal{C}(a,b,c,d,j+1)$ is bounded above by*

$$2 \log \max \left[\frac{2(\alpha' + 3/8\alpha)}{\alpha}, \frac{1+\delta_j}{1-\delta_j} \right].$$

Moreover, the homogeneous form $f \mapsto \int f dm$ is adapted, so is the norm

$$\|f\|_e = \max \left[e \left| \int f dm \right|,\ m(P)^{-1} \int_P f dm \text{ for } P \in \mathcal{P}, \|f\|_\infty \right],$$

where $e \geq \max[c, D_1 b]$.

Proof: The proof is a straightforward computation, (see [K,M,S], [Li1], [Ma1, Ma2] for similar computations). ∎

The following lemma proves that the cones $\mathcal{C}(a,b,c,d,j)$ are far from being empty.

LEMMA 3.10 *There exists a_0, b_0, c_0, d_0 such that for all $a \geq a_0$, $b \geq b_0$ and $c \geq c_0$, $d \geq d_0$ and for all $j \in \mathbb{N}$, $h \in \mathcal{C}(j)$.*
For any $f \in L$, there exists $R(f) > 0$ such that $\frac{f}{v} + R(f)h$ belongs to $\mathcal{C}(j)$ for all $j \in \mathbb{N}$. Moreover, $R(f) \leq Ct\ \|f\|$.

Proof: Recall that $h = \frac{h_0}{v}$ where $h_0 \in L$ is the fixed point for $\mathcal{L}_0$ (see pages 4, 7 and 10). Condition 4. in the definition of the cone is satisfied for h provided $d \geq \|h_0\|$ (recall $m(h) = 1$). Condition 1. is satisfied for h provided $a \geq 1$. Condition 2. is satisfied for h provided $b \geq L(h)$. Condition 3. is satisfied for h provided $c \geq \|h\|_\infty$.
Let $f \in L$, chose

$$R(f) \;\geq\; \frac{\|f\| - d\displaystyle\int \frac{f}{v}dm}{d - \|h_0\|},$$

$$R(f) \;\geq\; \frac{\mu(P)^{-1}\displaystyle\int_P \frac{f}{v}dm - a\int \frac{f}{v}dm}{a-1} \text{ for all } P \in \mathcal{P}.$$

$$R(f) \geq \frac{L(\frac{f}{v}) - b\displaystyle\int \frac{f}{v}dm}{b} \text{ and } R(f) \geq \frac{\|\frac{f}{v}\|_\infty - c\displaystyle\int \frac{f}{v}dm}{c - \|h\|_\infty},$$

so $R(f)$ may be chosen such that there exists a constant (independent on f) such that $R(f) \leq \mathrm{Ct}\|f\|$. ■

We will denote by $\gamma_j = \tanh \frac{\Delta_j}{4}$.

Remark 3.11 *If the ratio v_j/v_{j+1} goes to 1 when j goes to infinity then, for any $j \in \mathbb{N}$,*

$$\gamma_j \leq \frac{v_{jk}}{v_{(j+1)k}}.$$

If the ration v_j/v_{j+1} is smaller than some $0 < \beta < 1$ then $\gamma_j < \gamma$ for some $\gamma < 1$.

In what follows, we assume that a, b, c and d are large enough to ensure that h and $\frac{f}{v} + R(f)h_0$ belong to $\mathcal{C}(j)$ for all $f \in L$. For $j > 0$, define $\alpha(j) = \frac{1}{v_{kj}}$ if the ratio v_j/v_{j+1} goes to 1, $\alpha(j) = \gamma^j$ (γ is given by the above remark) if the ratio v_j/v_{j+1} is smaller than some $0 < \beta < 1$.

PROPOSITION 3.12 *Let $f \in \mathcal{C}(1)$, $p = kj + r$, $r < k$ then*

$$\|\mathcal{L}^p f_v - hm(f)\|_\infty \leq Ct\alpha(j)m(f).$$

For $f \in L$, let $f_v = \frac{f}{v}$, then

$$\|\mathcal{L}^p f_v - hm(f)\|_\infty \leq Ct\alpha(j)\|f\|.$$

Proof: Let f belong to $\mathcal{C}(1)$, proposition 3.6 implies that $\mathcal{L}^{jk}f$ and $\mathcal{L}^{jk}h$ belong to $\mathcal{C}(j)$. Applying successively theorem 2.1, we get

$$\theta_{\mathcal{C}(j)}(\mathcal{L}^{kj}f, \mathcal{L}^{kj}h) \leq \gamma_j \theta_{\mathcal{C}(j-1)}(\mathcal{L}^{k(j-1)}f, \mathcal{L}^{k(j-1)}h) \leq \cdots \leq \prod_{i=2}^{j} \gamma_{+i} \cdot \Delta_1.$$

Since the adapted norm $\| \ \|_e$ is equivalent to $\| \ \|_\infty$ and the form $f \mapsto m(f)$ is adapted, we get

$$\|\mathcal{L}^{kj}f - h\|_\infty \leq \mathrm{Ct}\alpha(j)m(f).$$

This inequality leads to the estimate, writing $p = kj + r$ and using that $\sup \mathcal{L}^r \mathbb{1} \leq M$, for functions in $\mathcal{C}(1)$. To get the estimate for all functions in L, it suffices to apply the previous argument to $\frac{f}{v} + R(f)h$ (use lemma 3.10). ■

The decay of correlations follows from proposition 3.12 by using (1.3) with $\mathcal{L}$ instead of $\mathcal{L}_0$ and m instead of m_0.

COROLLARY 3.13 [decay of correlations] *If f belongs to L and $g \in L^1(m)$ then*

$$c_p(f,g) \leq Ct\alpha([p/k])\|f\|\|g\|_1.$$

Proof: It suffices to remember that

$$c_p(f,g) = \int g\left[\mathcal{L}^p(fh) - hm(fh)\right] dm$$

and apply proposition 3.12 to $fh = \frac{fh_0}{v}$ with $fh_0 \in L$. ■

We conclude this section with the estimations for functions in L and for the initial transfer operator $\mathcal{L}_0$.

COROLLARY 3.14 *For any $f \in L$ and g such that $gv \in L^1(m_0)$, for $p \in \mathbb{N}$, and any increasing sequence $(v_\ell)_{\ell\in\mathbb{N}}$ such that $\sum_\ell v_\ell m_0(\Lambda_\ell) < \infty$, for any $\ell \in \mathbb{N}$*

$$\sup_{\Lambda_\ell} |\mathcal{L}_0^p f - h_0 m_0(f)| \leq Ct\ v_\ell \alpha([p/k])\|f/v\|,$$

and

$$c_p(f,g) \leq Ct\ \alpha([p/k])\|f\|\|gv\|_1.$$

Proof: It suffices to use that $\mathcal{L}_0 f = v_\ell \mathcal{L}(f/v)$ for $x \in \Lambda_\ell$. ■

4 Further discussions

To conclude, we will give some specific rates of mixing and some lower bounds for the speed of convergence in the uniform norm.

4.1 Exponential decay

If there exist constants $0 < e_1$ and $0 < \theta < 1$ such that for all $\ell \in \mathbb{N}$, $\leq m(\Lambda_\ell) \leq e_1\theta^\ell$ then it may be taken $v_\ell = \Theta^\ell$ provided $\Theta^{-1} < \theta$. In this situation, $\delta_j \leq \Theta^{-1} < 1$. Following remark 3.5, it is sufficient to work with cones $\mathcal{C}(a,b,c)$, we have: $\mathcal{L}^k\mathcal{C}(a,b,c) \subset \mathcal{C}(\sigma a, \Theta^{-1}b, \Theta^{-1}c)$ and the projective diameter of $\mathcal{L}^k\mathcal{C}(a,b,c)$ into $\mathcal{C}(a,b,c)$ is bounded by $2\log\max\left[\frac{2(\alpha'+3/8\alpha)}{\alpha}, \frac{1+\Theta^{-1}}{1-\Theta^{-1}}\right]$, where $0 < \alpha < 1 < \alpha'$ are arbitrary. We obtain exponential decay of correlations for functions in L. Moreover the norm $\| \ \|'_e = \max[\| \ \|_e, L(\ )]$ is adapted to the cone $\mathcal{C}(a,b,c)$. This implies that the transfer operator $\mathcal{L}$ on L is quasi-compact. This quasi-compactness result has been proved by L.-S. Young ([Y1]) by approximating $\mathcal{L}$ with finite rank operators. The cone method gives an constructive bound for the second eigenvalue if it exists.
Some unimodal maps and Henon maps may be conjugating with tower with such asymptotics for the measure of upper floors ([B,Y], [Y1]).

4.2 stretched exponential decay

If there exist constants $0 < e_1$, $0 < \theta < 1$ and $0 < \beta < 1$ such that for all $\ell \in \mathbb{N}$, $m(\Lambda_\ell) \leq e_1\theta^{\ell^\beta}$ then it may be taken $v_\ell = \left(\frac{1}{\theta}\right)^{\ell^{\beta'}}$ provided $\beta' < \beta$. We obtain decay of correlations in $\theta^{n^{\beta'}}$ for functions in L.

4.3 Polynomial decay

If there exist constants $0 < e_1$, $\beta > 1$ such that for all $\ell \in \mathbb{N}$, $m(\Lambda_\ell) \leq e_1\ell^{-\beta}$, it may be take $v_\ell = \frac{\ell^{\beta-1}}{(\ln \ell)^\gamma}$ provided $\gamma > 1$. We obtain decay of correlations in $\frac{(\ln n)^\gamma}{n^{\beta-1}}$ for functions in L. L.-S. Young obtained decay of correlations in $1/n^{\beta-1}$.

4.4 Lower bounds

We use a modification of L.-S. Young's argument ([Y2]). Let $f = h + c/v_\ell$ on Λ_ℓ with $\ell \geq 1$, c is such that $m(f) = 1$, clearly $f = \frac{f_0}{v}$ with $f_0 \in L$. We have $\mathcal{L}^n f = h + c/v_\ell$ on Λ_ℓ for $\ell \geq n$. So, for $\ell \geq n$, $\mathcal{L}^n f - hm(f) = c/v_\ell$ on Λ_ℓ. This implies that the bound given in proposition 3.12 is optimal if the ratio v_j/v_{j+1} goes to 1.

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