Non-equilibrium steady states of finite quantum systems coupled to thermal reservoirs

V. Jakšić[1] and C.-A. Pillet[2,3]

[1]Department of Mathematics and Statistics
McGill University
805 Sherbrooke Street West
Montreal, QC, H3A 2K6, Canada

[2]PHYMAT
Université de Toulon, B.P. 132
F-83957 La Garde Cedex, France

[3]CPT-CNRS Luminy, Case 907
F-13288 Marseille Cedex 9, France

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Abstract

We study the non-equilibrium statistical mechanics of a 2-level quantum system, $\mathcal{S}$, coupled to two independent free Fermi reservoirs $\mathcal{R}_1$, $\mathcal{R}_2$, which are in thermal equilibrium at inverse temperatures $\beta_1 \neq \beta_2$. We prove that, at small coupling, the combined quantum system $\mathcal{S}+\mathcal{R}_1+\mathcal{R}_2$ has a unique non-equilibrium steady state (NESS) and that the approach to this NESS is exponentially fast. We show that the entropy production of the coupled system is strictly positive and relate this entropy production to the heat fluxes through the system.

A part of our argument is general and deals with spectral theory of NESS. In the abstract setting of algebraic quantum statistical mechanics we introduce the new concept of C-Liouvillean, L, and relate the NESS to zero resonance eigenfunctions of L^*. In the specific model $\mathcal{S}+\mathcal{R}_1+\mathcal{R}_2$ we study the resonances of L^* using the complex deformation technique developed previously by the authors in [JP1].

1 Introduction

1.1 The framework

This paper deals with some thermodynamical aspects of a class of models in non-equilibrium quantum statistical mechanics which are commonly used to describe interaction of a small quantum system $\mathcal{S}$ with finitely many heat reservoirs $\mathcal{R}_i$. We will study the simplest non-trivial model, namely in our work $\mathcal{S}$ is an 2-level atom (spin 1/2) and each reservoir $\mathcal{R}_i$ is a free Fermi gas in thermal equilibrium at inverse temperature β_i. Various generalizations of our results will be discussed in Section 1.3 and in the forthcoming paper [JP4].

We will work in the framework of algebraic quantum statistical mechanics [BR1, BR2, Ha]. For the reader convenience and notational purposes, in this section we review some basic notions of this framework.

In the algebraic formalism a physical system is described either by a C^*- or W^*-dynamical system. The advantage of Fermi reservoirs is that we can deal with C^*-systems which are conceptually simpler. A C^*-dynamical system is a pair $(\mathcal{O}, \tau)$, where $\mathcal{O}$ is a C^*-algebra with identity and τ is a strongly continuous group of automorphisms of $\mathcal{O}$ (that is, the map $\mathbb{R} \ni t \mapsto \tau^t(A)$ is norm continuous for each $A \in \mathcal{O}$). The elements of $\mathcal{O}$ describe observables of the physical system under consideration and the group τ specifies their time evolution. A physical state is described by a mathematical state on $\mathcal{O}$, that is, a positive linear functional ω such that $\omega(\mathbf{1}) = 1$. The set $E(\mathcal{O})$ of all states is a convex, weak-$*$ compact subset of the dual $\mathcal{O}^*$. A state ω is called faithful if $\omega(A^*A) = 0 \Rightarrow A = 0$ and τ-invariant if $\omega \circ \tau^t = \omega$ for all t.

The thermal equilibrium states of $(\mathcal{O}, \tau)$ are characterized by the KMS condition. Let $\beta \neq 0$ be the inverse temperature (although the physically relevant case is $\beta > 0$, it is mathematically convenient to define KMS-states for all non-zero β). The state ω is (τ, β)-KMS if for any pair $A, B \in \mathcal{O}$ there exists a complex function $F_{A,B}$, analytic inside the strip $\{z \,|\, 0 < \mathrm{sign}(\beta)\mathrm{Im} z < |\beta|\}$, bounded and continuous on its closure, and satisfying the KMS boundary conditions

$$F_{A,B}(t) = \omega(A\tau^t(B)), \qquad F_{A,B}(t+\mathrm{i}\beta) = \omega(\tau^t(B)A).$$

A (τ,β)-KMS state is faithful and τ-invariant.

Let $(\mathcal{O},\tau)$ be a C^*-dynamical system and let δ be the generator of τ ($\tau^t = \mathrm{e}^{t\delta}$). The operator δ is a $*$-derivation: Its domain $\mathcal{D}(\delta)$ is a $*$-subalgebra of $\mathcal{O}$ and for $A, B \in \mathcal{D}(\delta)$,

$$\delta(A)^* = \delta(A^*), \qquad \delta(AB) = \delta(A)B + A\delta(B).$$

Let $V = V^* \in \mathcal{O}$ be a perturbation (such perturbations are called local). The generator of the perturbed dynamics is $\delta_V(A) = \delta(A) + \mathrm{i}[V,A]$. The operator δ_V is also a $*$-derivation and $\mathcal{D}(\delta_V) = \mathcal{D}(\delta)$. The perturbed dynamics is described by

$$\begin{aligned}\tau_V^t(A) &:= \mathrm{e}^{t\delta_V}(A) \\ &= \tau^t(A) + \sum_{n\geq 1} \mathrm{i}^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n [\tau^{t_n}(V), [\cdots[\tau^{t_1}(V), \tau^t(A)]]].\end{aligned}$$

Until the end of this section we fix a C^*-dynamical system $(\mathcal{O},\tau)$, a state ω, and a local perturbation V.

The non-equilibrium steady states (NESS) of the locally perturbed system $(\mathcal{O},\tau_V)$ associated to the initial state ω are the weak-* limit points of the set of states

$$\frac{1}{T}\int_0^T \omega \circ \tau_V^t \mathrm{d}t, \tag{1.1}$$

for $T > 0$. In other words, ω_V^+ is a NESS if there is a sequence $T_n \to \infty$ such that for all $A \in \mathcal{O}$

$$\lim_{n\to\infty} \frac{1}{T_n}\int_0^{T_n} \omega \circ \tau_V^t(A)\mathrm{d}s = \omega_V^+(A).$$

The set $\Sigma_V^+(\omega)$ of NESS associated to ω is a non-empty weak-* compact subset of $E(\mathcal{O})$ whose elements are τ_V-invariant.

One of the key concept of non-equilibrium thermodynamics is the notion of entropy production. Within the framework of algebraic quantum statistical mechanics this notion has been precisely defined in the recent works [Ru2, JP3], see also [Sp1, O1, O2, OHI] . We recall the definitions and the results we will need.

For positive linear functionals $\eta, \xi \in \mathcal{O}^*$, let $\mathrm{Ent}(\eta\,|\,\xi)$ be the relative entropy of Araki (we use the ordering and the sign convention of Bratelli-Robinson [BR2, Don]). For definition and properties of Araki's relative entropy we refer the reader to [Ar1, Ar2, BR2, Don, OP].

We make the following assumption.

E(1) There exists a C^*-dynamics σ_ω such that ω is a $(\sigma_\omega, -1)$-KMS state.

The choice of reference temperature $\beta = -1$ is made for mathematical convenience. If (E1) holds, then for any $\beta \neq 0$ there is a C^*-dynamics $\sigma_{\omega,\beta}$ such that ω is $(\sigma_{\omega,\beta}, \beta)$-KMS state (set $\sigma^t_{\omega,\beta} = \sigma_\omega^{-\beta t}$).

Let δ_ω be the generator of σ_ω. Our second assumption concerns the local perturbation V.

(E2) $V \in \mathcal{D}(\delta_\omega)$.

Until the end of this section we assume that (E1) and (E2) hold. We set $\sigma_V := \delta_\omega(V)$ and call $\mathrm{Ep}(\eta) = \eta(\sigma_V)$ the entropy production (w.r.t. the reference state ω) of the perturbed system $(\mathcal{O}, \tau_V)$ in the state $\eta \in E(\mathcal{O})$.

The following identity was proven in [JP3]:

$$\mathrm{Ent}(\omega \circ \tau_V^t \,|\, \omega) = -\int_0^t \omega(\tau_V^s(\sigma_V))\mathrm{d}s. \tag{1.2}$$

This identity motivates the definition of entropy production and is the starting point for study of this notion [JP3, JP4]. In particular, since the relative entropy is non-positive, Relation (1.2) yields that for any $\omega_V^+ \in \Sigma_V^+(\omega)$, $\mathrm{Ep}(\omega_V^+) \geq 0$.

The NESS ω_V^+ is thermodynamically non-trivial if $\mathrm{Ep}(\omega_V^+) > 0$. One of the central problems of mathematical theory of non-equilibrium quantum statistical mechanics is to show that the NESS of concrete physically relevant models are thermodynamically non-trivial. We describe below one simple criterion which ensures strict positivity of entropy production and which will be used in this paper.

Let $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ be the GNS-representation of the algebra $\mathcal{O}$ associated to ω. The states in $\mathcal{O}^*$ which are represented by density matrices on $\mathcal{H}_\omega$ are called ω-normal. The set of all ω-normal states is a norm closed subset of $E(\mathcal{O})$ which we denote by $\mathcal{N}_\omega$. One can show that the entropy production of ω-normal NESS is zero, see [JP4].

Theorem 1.1 *Assume that NESS ω_V^+ satisfies the following:*
(a) $\omega_V^+ \notin \mathcal{N}_\omega$.
(b) $\sup_{T>0} \left| \int_0^T (\omega(\tau_V^t(\sigma_V)) - \omega_V^+(\sigma_V))\mathrm{d}t \right| < \infty$.
Then $\mathrm{Ep}(\omega_V^+) > 0$.

We will prove this theorem in Section 5.

One of the main results of this paper is that the class of systems we study has strictly positive entropy production.

For additional information about NESS and entropy production we refer the reader to [JP4].

1.2 The model and the results

We now describe the specific model we will study in this paper.

The C^*-algebra of observables of the system $\mathcal{S}$ is $\mathcal{O}_\mathrm{s} \equiv M(\mathbb{C}^2)$, the matrix algebra on $\mathfrak{H}_\mathrm{s} \equiv \mathbb{C}^2$. Let $\sigma_x, \sigma_y, \sigma_z$ be the usual Pauli matrices. The dynamics is specified by the automorphisms

$$\tau_\mathrm{s}^t(A) = \mathrm{e}^{\mathrm{i}tH_\mathrm{s}} A \mathrm{e}^{-\mathrm{i}tH_\mathrm{s}}, \tag{1.3}$$

where $H_\mathrm{s} \equiv \sigma_z$ is the Hamiltonian of the system $\mathcal{S}$.

Let $\mathfrak{h}$ be the Hilbert space of a single fermion and h its energy operator. Let $\mathfrak{H}_\mathrm{f} \equiv \Gamma_-(\mathfrak{h})$ be the Fermi Fock space and $a(f)$, $a^*(f)$ the corresponding annihilation and creation operators on $\mathfrak{H}_\mathrm{f}$. In the sequel $a^\#$ stands either for a or a^*. It follows from CAR (canonical anti-commutation relations) that $\|a^\#(f)\| = \|f\|$. The algebra of observables of the free Fermi gas, $\mathcal{O}_\mathrm{f}$, is the C^*-algebra of operators generated by $\{a^\#(f) \mid f \in \mathfrak{h}\}$. The field operators are defined by

$$\varphi(f) \equiv \frac{1}{\sqrt{2}}(a(f) + a^*(f)).$$

The Hamiltonian and the dynamics are specified by $H_\mathrm{f} = \mathrm{d}\Gamma(h)$ and

$$\begin{aligned}\tau_\mathrm{f}^t(a^\#(f)) &= \mathrm{e}^{\mathrm{i}tH_\mathrm{f}} a^\#(f) \mathrm{e}^{-\mathrm{i}tH_\mathrm{f}} \\ &= a^\#(\mathrm{e}^{\mathrm{i}th} f).\end{aligned}$$

The pair $(\mathcal{O}_\mathrm{f}, \tau_\mathrm{f})$ is a C^*-dynamical system describing a free Fermi gas. For each $\beta > 0$ there exists a unique (τ_f, β)-KMS state $\omega_{\mathrm{f},\beta}$ on $\mathcal{O}_\mathrm{f}$. $\omega_{\mathrm{f},\beta}$ is a quasi-free, gauge-invariant state uniquely determined by the two point function

$$\omega_{\mathrm{f},\beta}(a^*(f)a(f)) = (f, (\mathrm{e}^{\beta h} + 1)^{-1} f).$$

Notation. In the sequel, whenever the meaning is clear within the context, we denote by A the operators $A \otimes \mathbf{1}$, $\mathbf{1} \otimes A$.

We consider now two identical reservoirs $(\mathcal{O}_\mathrm{f}^{(i)}, \tau_\mathrm{f}^{(i)})$, $i = 1, 2$. The C^*-algebra of observables of the combined system $\mathcal{S} + \mathcal{R}_1 + \mathcal{R}_2$ is

$$\mathcal{O} \equiv \mathcal{O}_\mathrm{s} \otimes \mathcal{O}_\mathrm{f}^{(1)} \otimes \mathcal{O}_\mathrm{f}^{(2)}, \tag{1.4}$$

the tensor product algebra of operators on $\mathfrak{H} \equiv \mathfrak{H}_\mathrm{s} \otimes \mathfrak{H}_\mathrm{f} \otimes \mathfrak{H}_\mathrm{f}$. The free dynamics is given by the group of automorphisms $\tau = \tau_\mathrm{s} \otimes \tau_\mathrm{f}^{(1)} \otimes \tau_\mathrm{f}^{(2)}$. The pair $(\mathcal{O}, \tau)$ is a C^*-dynamical system describing the combined system in absence of interaction. Note that

$$\tau^t(A) = \mathrm{e}^{\mathrm{i}tH} A \mathrm{e}^{-\mathrm{i}tH},$$

where

$$H = H_\mathrm{s} + H_\mathrm{f}^{(1)} + H_\mathrm{f}^{(2)}.$$

We now describe the interaction of $\mathcal{S}$ with the reservoirs. Choose form-factors $\alpha_i \in \mathfrak{h}$, $i = 1, 2$, and set

$$\begin{aligned} V_1 &= \sigma_x \otimes \varphi(\alpha_1) \otimes \mathbf{1}, \\ V_2 &= \sigma_x \otimes \mathbf{1} \otimes \varphi(\alpha_2), \\ V &= V_1 + V_2. \end{aligned} \tag{1.5}$$

Obviously, $V = V^* \in \mathcal{O}$. The Hamiltonian and the dynamics of the interacting system are specified by

$$\begin{aligned} H_\lambda &= H + \lambda V, \\ \tau_\lambda^t(A) &= \mathrm{e}^{\mathrm{i}tH_\lambda} A \mathrm{e}^{-\mathrm{i}tH_\lambda}, \end{aligned}$$

where λ is a real coupling constant. The pair $(\mathcal{O}, \tau_\lambda)$ is a C^*-dynamical system.

In what follows we fix the inverse temperatures $\beta_i > 0$ of the reservoirs. Let ω_{s} be a state on $\mathcal{O}_{\mathrm{s}}$ and $\omega_{\mathrm{f},\beta_i}$ be the $(\tau_{\mathrm{f}}^{(i)}, \beta_i)$-KMS state on $\mathcal{O}_{\mathrm{f}}^{(i)}$ describing the thermal equilibrium state of the i-th reservoir. Consider first the initial states of the form

$$\omega = \omega_{\mathrm{s}} \otimes \omega_{\beta_1} \otimes \omega_{\beta_2}, \qquad \omega_{\mathrm{s}} \in E(\mathcal{O}_{\mathrm{s}}). \tag{1.6}$$

We denote the set of all such states by $\mathcal{N}_{\mathrm{s}}$. For $\omega \in \mathcal{N}_{\mathrm{s}}$, let $\mathcal{N} \subset E(\mathcal{O})$ be the set of all ω-normal states ($\mathcal{N}$ does not depend on the choice of $\omega \in \mathcal{N}_{\mathrm{s}}$). Our goal is to study NESS of $(\mathcal{O}, \tau_\lambda)$ associated to initial states in $\mathcal{N}$.

For technical reasons related to use of the complex deformation technique of [JP1], we impose some regularity assumptions on the reservoirs and form factors. Our first assumption is:

A(1) $\mathfrak{h} \equiv L^2(\mathbf{R}^+; \mathfrak{G})$ for some auxiliary Hilbert space $\mathfrak{G}$, and h is the operator of multiplication by $s \in \mathbf{R}^+$.

Let $I(\delta) \equiv \{z \in \mathbb{C} : |\mathrm{Im} z| < \delta\}$. We denote by $H^2(\delta)$ the Hardy class of all analytic functions $f : I(\delta) \mapsto \mathfrak{G}$ such that

$$\|f\|_{H^2(\delta)} \equiv \sup_{|\theta|<\delta} \int_{\mathbb{R}} \|f(s + \mathrm{i}\theta)\|_{\mathfrak{G}}^2 \mathrm{d}s < \infty.$$

We fix a complex conjugation $f \mapsto \overline{f}$ on $\mathfrak{h}$ which commutes with h. To any $f \in \mathfrak{h}$ we associate a function $\tilde{f} : \mathbb{R} \mapsto \mathfrak{G}$ by

$$\tilde{f}(s) = \begin{cases} f(s) & \text{if } s \geq 0, \\ \overline{f}(|s|) & \text{if } s < 0. \end{cases} \tag{1.7}$$

Our second regularity assumption is:

(A2) For some $\delta > 0$, $\mathrm{e}^{-\beta_i s/2}\tilde{\alpha}_i \in H^2(\delta)$ for $i = 1, 2$.

Our third assumption ensures that the small system $\mathcal{S}$ is effectively coupled to the reservoirs.

(A3) $\|\alpha_i(2)\|_{\mathfrak{G}} > 0$ for $i = 1, 2$.

To illustrate the above assumptions with a concrete example, assume that $\mathfrak{h} = L^2(\mathbb{R}^d, \mathrm{d}k)$ and that h is operator of multiplication by $k^2/2m$. Passing to polar coordinates and changing the variable one sees that (A1) holds with $\mathfrak{G} = L^2(S^{d-1}, \mathrm{d}\sigma)$, where S^{d-1} is the unit sphere in $\mathbb{R}^d$ and $\mathrm{d}\sigma$ is the surface measure. If $\alpha_i(k) = |k|^{\frac{d-2}{2}}\mathrm{e}^{-k^4}$, then (A2) and (A3) hold (in this example (A2) holds for all δ and β_i).

Our first result is:

Theorem 1.2 *Assume that* (A1)-(A3) *hold. Then, for some* $\Lambda > 0$ *and* $0 < |\lambda| < \Lambda$*, there is a state* ω_λ^+ *on* $\mathcal{O}$ *so that the following hold:*
(i) *For all* $\eta \in \mathcal{N}$ *and* $A \in \mathcal{O}$,

$$\lim_{t\to\infty} \eta(\tau_\lambda^t(A)) = \omega_\lambda^+(A). \tag{1.8}$$

(ii) *The limit (1.8) is exponentially fast in the following sense: There exist* $\gamma(\lambda) > 0$*, a norm dense set of states* $\mathcal{N}_0 \subset \mathcal{N}$*, and a norm-dense* $*$*-subalgebra* $\mathcal{O}_0 \subset \mathcal{O}$ *such that for* $\eta \in \mathcal{N}_0$*,* $A \in \mathcal{O}_0$*, and* $t > 0$,

$$|\eta(\tau^t(A)) - \omega_\lambda^+(A)| \le C_{A,\eta,\lambda}\mathrm{e}^{-\gamma(\lambda)t}. \tag{1.9}$$

Moreover, $\mathcal{N}_{\mathrm{s}} \subset \mathcal{N}_0$ *and* $\mathcal{O}_{\mathrm{s}} \subset \mathcal{O}_0$.
(iii) *For* $A \in \mathcal{O}_0$*, the functions* $\lambda \mapsto \omega_\lambda^+(A)$ *are analytic for* $|\lambda| < \Lambda$.

Remark 1. Our proof gives that $\Lambda = O(\min 1/\beta_i)$, and thus the above theorem is a high-temperature result. It is an interesting question whether the techniques of [BFS] or [DJ1, DJ2] can be adopted to prove the above theorem for Λ independent of the temperatures β_i.
Remark 2. If $\beta_1 \neq \beta_2$, then ω_λ^+ *is not* a (τ_λ, β)-KMS state for any β.
Remark 3. In the thermal equilibrium case $\beta_1 = \beta_2 = \beta$ Theorem 1.2 was proven in [JP1, JP2] (ω_λ^+ is then the unique (τ_λ, β)-KMS state of $(\mathcal{O}, \tau_\lambda)$). The method of this paper is suited to non-equilibrium situations and, when restricted to thermal equilibrium case, differs from the method of [JP1, JP2]. In particular, here we require a stronger regularity condition than [JP1, JP2] (there it suffices that $\alpha_i \in H^2(\delta)$) but we also obtain a slightly stronger result (the method of [JP1, JP2] fails to show that $\mathcal{N}_{\mathrm{s}} \subset \mathcal{N}_0$ and $\mathcal{O}_{\mathrm{s}} \subset \mathcal{O}_0$).
Remark 4. Our proof gives that

$$\gamma(\lambda) = \gamma_0\lambda^2 + O(\lambda^4),$$

where

$$\gamma_0 = \frac{\pi}{2}\left(\|\alpha_1(2)\|_{\mathfrak{G}}^2 + \|\alpha_2(2)\|_{\mathfrak{G}}^2\right).$$

Remark 5. Regarding (iii), it follows from our arguments that there exists linear functionals $\omega_k^+ : \mathcal{O}_0 \mapsto \mathbb{C}$, $k \geq 0$, such that for $A \in \mathcal{O}_0$,

$$\omega_\lambda^+(A) = \sum_{k=0}^{\infty} \lambda^k \omega_k^+(A). \tag{1.10}$$

The first term ω_0 is computed from a linear eigenvalue problem on $\mathfrak{H}_{\rm s}$. This eigenvalue problem is determined by the second order correction (Fermi's Golden Rule) for the resonances of a suitable non-self-adjoint operator (C-Liouvillean). Although formulas for the higher order terms become quickly very complicated, in principle it is possible to compute all terms in the expansion (1.10). We will discuss this point at the end of Section 4.

Theorem 1.2 establishes the basic thermodynamical property of the system $\mathcal{S}+\mathcal{R}_1+\mathcal{R}_2$, namely that the set of initial states $\mathcal{N}$ belongs to the basin of attraction of a single NESS ω_λ^+. We now discuss the other thermodynamical properties of this system.

The first question is whether ω_λ^+ belongs to the set $\mathcal{N}$ of normal states.

Theorem 1.3 *Assume that* $({\rm A1}) - ({\rm A3})$ *hold and that* $\beta_1 \neq \beta_2$. *Then there is* $\ell > 0$ *such that for* $0 < |\lambda| < \ell$ *there are no* τ_λ*-invariant states in* $\mathcal{N}$. *In particular, if* $0 < |\lambda| < \min(\Lambda, \ell)$, *then* $\omega_\lambda^+ \notin \mathcal{N}$.

Remark 1. This result can be proven under more general condition then (A2), see [DJ1, DJ2].
Remark 2. The constant ℓ differs from the constant Λ in Theorem 1.2. In contrast to Λ, ℓ can be chosen independently of the size of β_i's as $\beta_i \to \infty$ (see [DJ2] for details). On the other hand, ℓ depends on $d = |\beta_1 - \beta_2|$ and $\ell \downarrow 0$ as $d \downarrow 0$. The constant Λ can be chosen independently of d as long as $0 \leq d \leq {\rm const}$.

Recall that the entropy production depends on the choice of the initial state ω. Let $\widehat{\mathcal{N}}_{\rm s}$ be the set of states in $\mathcal{N}_{\rm s}$ with the property that $\omega_{\rm s} > 0$ and is $\tau_{\rm s}$-invariant. The assumption (E1) of Section 1.1 hold for all $\omega \in \widehat{\mathcal{N}}_{\rm s}$. If (A2) holds, then (E2) holds for the perturbation V.

Theorem 1.4 *Under the assumptions of Theorem 1.3, for any initial state* $\omega \in \widehat{\mathcal{N}}_{\rm s}$,

$${\rm Ep}(\omega_\lambda^+) = \omega_\lambda^+(\delta_\omega(\lambda V)) > 0,$$

for $0 < |\lambda| < \min(\Lambda, \ell)$. *Moreover,* ${\rm Ep}(\omega_\lambda^+)$ *does not depend on the choice of the initial state* $\omega \in \widehat{\mathcal{N}}_{\rm s}$.

Remark. This theorem can be proven in two different ways. The short proof (the one we will give in this paper) is based on Theorem 1.1. This proof gives no estimate on the size of entropy production. The second proof is based on the perturbative expansion of the state ω_λ^+. Although computationally tedious, this proof has the advantage of showing that the entropy production is strictly positive to the lowest non-trivial order (the first non-trivial term can be also computed using the van Hove weak coupling limit, see [LS]). We will discuss the perturbative proof of Theorem 1.4 in [JP4].

We finish this section with a brief discussion of the heat fluxes. Let δ_i be the generator of $\tau_{\rm f}^{(i)}$. (A2) implies that $V_i \in \mathcal{D}(\delta_i)$. The observable describing the heat flux (energy transfer) from the rest of the system into the i-th reservoir is $\Phi_i := \delta_i(\lambda V_i)$.

Theorem 1.5 *Assume that* (A1)-(A3) *hold and that* $\beta_1 \neq \beta_2$. *Then, for* $0 < |\lambda| < \min(\Lambda, \ell)$*, the following relations hold:*

$$\begin{aligned} \omega_\lambda^+(\Phi_1) + \omega_\lambda^+(\Phi_2) &= 0, \\ \beta_1\omega_\lambda^+(\Phi_1) + \beta_2\omega_\lambda^+(\Phi_2) &= -\mathrm{Ep}(\omega_\lambda^+) < 0, \end{aligned} \tag{1.11}$$

where in the second relation the entropy production is computed w.r.t. any initial state in $\widehat{\mathcal{N}}_{\rm s}$.

Remark 1. Relations (1.11) are respectively the first and the second law of thermodynamics for the model $\mathcal{S} + \mathcal{R}_1 + \mathcal{R}_2$.
Remark 2. If $\beta_1 > \beta_2$, then $\omega_\lambda^+(\Phi_1) > 0$. Thus, in NESS ω_λ^+ there is a constant non-vanishing heat flow from the hotter to the colder reservoir across the system $\mathcal{S}$.
Remark 3. Except for the strict positivity of entropy production, the relations (1.11) follow only from a few structural properties of the model $\mathcal{S} + \mathcal{R}_1 + \mathcal{R}_2$, and can be proven in considerable generality, see [JP4] for details.

1.3 Remarks

Although in this paper we have chosen to study the simplest non-trivial model, our results can be easily extended to the case where $\mathcal{S}$ is an N-level atom, there are M-reservoirs instead of two, and V_i is a finite sum of terms of the form

$$Q_i \otimes \varphi(\alpha_{i1}) \dots \varphi(\alpha_{in}) \mathrm{i}^{n(n-1)/2},$$

(one assumes that $(\alpha_{ik}, \alpha_{ij}) = 0$ for $k \neq j$ and $Q_i = Q_i^* \in M(\mathbb{C}^N)$). In this case, the assumption (A3) has to be replaced with a more complicated algebraic condition which ensures that a suitable $N \times N$ matrix has zero as a simple eigenvalue. This condition is studied in detail in [DJ2] and is closely related to the non-degeneracy condition discussed in the context of master equation approach to the non-equilibrium thermodynamics [Da, LS, Sp2, Fr]. We will discuss both the more general model and the relation of our results with the master equation technique in the continuation of this paper [JP4].

If the Fermi reservoirs are replaced with Bose reservoirs, then the combined system has to be described within the framework of W^*-dynamical systems. In this case the perturbation V is an unbounded operator and this leads to some technical difficulties in the study of the L^∞-Liouvillean (the analog of C-Liouvillean for W^*-systems). It is an important open problem to prove the analog of Theorem 1.2 for Bose reservoirs.

Among the works related to ours, we mention the one of Davies [Da], where the dynamics of the system $\mathcal{S} + \sum_i \mathcal{R}_i$ is studied in the van Hove weak-coupling limit $\overline{t} = \lambda^2 t$, $\lambda \downarrow 0$, $t \uparrow \infty$. In particular, Davies proves the existence and uniqueness of NESS in the van

Hove limit (this state coincides with ω_0 in the expansion (1.10)). Lebowitz and Spohn [LS] have used Davies results to study the thermodynamics of the system $\mathcal{S} + \sum_i \mathcal{R}_i$ in the van Hove limit steady state ω_0. There is a substantial literature on the use of van Hove limit and Markovian master equations in statistical mechanics, see [GFV, Hak] for references and additional information. The results beyond van Hove limit are scarce. In [JP1, JP2] Theorem 1.2 was proven in thermal equilibrium case where $\beta_1 = \beta_2$. The method of the proof was based on quantum Koopmanism and the spectral analysis of the quantum Koopman operator - the (standard) Liouvillean - of the system $\mathcal{S} + \sum_i \mathcal{R}_i$. Various extensions and generalizations of these results are given in [BFS, DJ1, DJ2, M].

An alternative (abstract) approach to the study of non-equilibrium steady states of finite quantum systems coupled to thermal reservoirs was recently proposed in [Ru1]. This proposal is based on the scattering theory of C^*-dynamical systems and an ergodicity hypothesis called L^1-asymptotic abelianness. This hypothesis is difficult to verify in concrete models, and in particular it is not known whether it holds for the model studied in this paper.

We would like to add the following general remark. It is known that the ergodic properties of C^*-dynamical systems in thermal equilibrium are encoded in the spectrum of a suitable self-adjoint operator, the quantum Koopman operator or Liouvillean, see e.g. [JP2]. In non-equilibrium situations, the quantum Koopmanism is not applicable, and it has been generally believed that the understanding of NESS requires the development of scattering theory. In the models of physical interest this is a difficult task, and the progress has been slow (see however [DG1, DG2, FGS]). A perhaps surprising aspect of our method is that at least in some situations, the spectral approach to NESS is possible, and that the structure of NESS is encoded in the spectral resonances of a suitable non-selfadjoint operator, the C-Liouvillean.

The paper is organized as follows.

The method of the proof is described in the abstract setting in Section 2 where we introduce the concept of C-Liouvillean, L, and show how the NESS of an abstract C^*-dynamical system are related to the resonances of L^*. The results of Section 2 are quite general and, we believe, shed some light on the structure of non-equilibrium quantum statistical mechanics. In Sections 3 and 4 we apply the abstract formalism of Section 2 to the specific model $\mathcal{S} + \mathcal{R}_1 + \mathcal{R}_2$ – in Section 3 we explicitly compute the modular structure and C-Liouvillean L, and in Section 4 we study the resonances of L^* using the complex deformation technique previously developed in [JP1].

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2 Liouvilleans and NESS

The goal of this chapter is to introduce the basic new ingredient of our method, the C-Liouvillean.

In Section 2.1 we recall the basic notions of Tomita-Takesaki modular theory and in particular the notion of standard Liouvillean. In Section 2.2 we introduce C-Liouvilleans. In Section 2.3 we describe the relation between the C-Liouvilleans and NESS.

Throughout this section we adopt the following framework.

Let $(\mathcal{O}, \tau)$ be a C^*-dynamical system and ω a given faithful state. Let $(\mathcal{H}, \pi, \Omega)$ be the GNS-representation of the algebra $\mathcal{O}$ associated to ω (for simplicity, we write $\mathcal{H}$ for $\mathcal{H}_\omega$, etc). Since ω is faithful, π is an injection and we can identify $\mathcal{O}$ and $\pi(\mathcal{O})$ (with a slight abuse of notation, we write A for $\pi(A)$). We set $\mathfrak{M} = \pi(\mathcal{O})''$ and assume that Ω is a separating vector for the von Neumann algebra $\mathfrak{M}$ ($A \in \mathfrak{M}, A\Omega = 0 \Rightarrow A = 0$).

We denote by $\mathcal{N} \subset E(\mathcal{O})$ the set of all π-normal states, that is, the states represented by density matrices on $\mathcal{H}$. Every element of $\mathcal{N}$ extends uniquely to a state on $\mathfrak{M}$.

In what follows we assume that ω is τ-invariant. Then τ has a unique extension to a weakly continuous group of automorphisms of $\mathfrak{M}$ which we denote by the same letter. The state $\omega(A) = (\Omega, A\Omega)$ is a τ-invariant state on $\mathfrak{M}$.

Let $V \in \mathcal{O}$ be a local perturbation and τ_V the perturbed C^*-dynamics. The group τ_V also extends to a weakly continuous group of automorphisms of $\mathfrak{M}$ which we denote by the same letter.

2.1 The standard Liouvillean

There exists a unique self-adjoint operator $\mathcal{L}$ on $\mathcal{H}$ such that for $A \in \mathfrak{M}$

$$\tau^t(A) = \mathrm{e}^{\mathrm{i}t\mathcal{L}} A \mathrm{e}^{-\mathrm{i}t\mathcal{L}},$$

$$\mathcal{L}\Omega = 0.$$

We call the operator $\mathcal{L}$ the standard Liouvillean. Note that the perturbed time evolution τ_V also has a unitary implementation

$$\tau_V^t(A) = \mathrm{e}^{\mathrm{i}t(\mathcal{L}+V)} A \mathrm{e}^{-\mathrm{i}t(\mathcal{L}+V)}.$$

Let Δ, J and $\mathcal{P}$ be the modular operator, the modular conjugation and the natural cone of the pair $(\mathfrak{M}, \Omega)$. By definition of the modular structure, $\mathfrak{M}\Omega \subset \mathcal{D}(\Delta^{\frac{1}{2}})$ and for $A \in \mathfrak{M}$,

$$J\Delta^{\frac{1}{2}} A\Omega = A^*\Omega. \tag{2.12}$$

By Tomita-Takesaki theorem, $\Delta^{\mathrm{i}t}\mathfrak{M}\Delta^{-\mathrm{i}t} = \mathfrak{M}$, $J\mathfrak{M}J = \mathfrak{M}'$. For every normal state $\eta \in \mathcal{N}$ there is a unique vector $\Omega_\eta \in \mathcal{P}$ such that $\eta(A) = (\Omega_\eta, A\Omega_\eta)$.

Let

$$\mathcal{L}_V \equiv \mathcal{L} + V - JVJ.$$

We will call $\mathcal{L}_V$ the standard Liouvillean for the perturbation V. The operator $\mathcal{L}_V$ is the unique self-adjoint operator satisfying

$$\tau_V^t(A) = \mathrm{e}^{\mathrm{i}t\mathcal{L}_V} A \mathrm{e}^{-\mathrm{i}t\mathcal{L}_V},$$
$$\mathrm{e}^{-\mathrm{i}t\mathcal{L}_V}\mathcal{P} \subset \mathcal{P},$$

see [BR2, DJP]. An immediate consequence of these relations is:

Proposition 2.1 *The state $\eta \in \mathcal{N}$ is τ_V-invariant iff $\mathcal{L}_V \Omega_\eta = 0$.*

By this proposition, the the study of normal, τ_V-invariant states reduces to study of $\mathrm{Ker}\, \mathcal{L}_V$.

If ω is (τ, β)-KMS, then by the fundamental result of Araki there exists a state $\omega_V \in \mathcal{N}$ which is (τ_V, β)-KMS. Thus, in thermal equilibrium $\mathrm{Ker}\, \mathcal{L}_V$ is never empty. On the other hand, if ω is not a KMS-state, then typically $\mathrm{Ker}\, \mathcal{L}_V = \emptyset$ and to study NESS using spectral techniques we need new concepts.

2.2 C-Liouvillean

The vector space $\mathcal{O}\Omega = \{A\Omega \,|\, A \in \mathcal{O}\}$ equipped with the norm

$$\|A\Omega\|_\infty = \|A\|, \tag{2.13}$$

is a Banach space which we denote by $C(\mathcal{O}, \Omega)$. Note that every $A \in \mathcal{O}$ defines, by right multiplication, a bounded linear map on $C(\mathcal{O}, \Omega)$. This map we again denote by A.

Obviously, the map

$$\mathcal{O} \ni A \mapsto A\Omega \in C(\mathcal{O}, \Omega),$$

is a Banach space isomorphism. Under this isomorphism, the group τ_V^t is mapped into a continuous group T_V^t of isometries of $C(\mathcal{O}, \Omega)$. Clearly,

$$T_V^t A\Omega = \tau_V^t(A)\Omega, \tag{2.14}$$

and

$$\begin{aligned} T_V^t \Omega &= \Omega, \\ T_V^t A T_V^{-t} &= \tau_V^t(A). \end{aligned} \tag{2.15}$$

The generator of the group T_V^t we denote by L_V and call it C-Liouvillean. It is convenient to include the imaginary unit in the definition of L_V so that

$$T_V^t = \mathrm{e}^{\mathrm{i}tL_V}.$$

By (2.14),

$$\mathcal{D}(L_V) = \{A\Omega \,|\, A \in \mathcal{D}(\delta_V)\},$$

and

$$\mathrm{i}L_V A\Omega = \delta_V(A)\Omega.$$

We proceed to compute the operator L_V in terms of the modular structure.

Let $A \in \mathcal{D}(\delta_V) = \mathcal{D}(\delta)$ be given. Differentiating the relation

$$\mathrm{e}^{\mathrm{i}tL_V} A\Omega = \mathrm{e}^{\mathrm{i}t(\mathcal{L}+V)} A \mathrm{e}^{-\mathrm{i}t(\mathcal{L}+V)}\Omega,$$

and setting $t = 0$ we derive

$$L_V A\Omega = (\mathcal{L} + V)A\Omega - (VA^*)^*\Omega.$$

Applying (2.12) twice we obtain

$$(VA^*)^*\Omega = J\Delta^{\frac{1}{2}} V J \Delta^{\frac{1}{2}} A\Omega.$$

Since $J\Delta^{\frac{1}{2}} = \Delta^{-\frac{1}{2}} J$ on $\mathcal{O}\Omega$, the operator L_V has the form

$$L_V = \mathcal{L} + V - J\Delta^{\frac{1}{2}} V \Delta^{-\frac{1}{2}} J. \tag{2.16}$$

Note that

$$J\Delta^{\frac{1}{2}} V \Delta^{-\frac{1}{2}} J : C(\mathcal{O}, \Omega) \to C(\mathcal{O}, \Omega),$$

is a bounded operator with norm $\|V\|$.

We now identify conditions under which T_V^t extends to a strongly continuous group on $\mathcal{H}$.

The formula (2.16) implies that the operator L_V extends to a dense subspace $\mathfrak{D} := \mathcal{D}(\mathcal{L}) \cap \mathcal{O}\Omega$. Moreover, since $\mathfrak{D} \subset \mathcal{D}(L_V^*)$, the linear operator L_V with domain $\mathfrak{D}$ is closable. We denote its closure by the same letter. It follows that T_V^t extends to a strongly continuous group on $\mathcal{H}$ iff L_V satisfies the conditions of Hille-Yosida-Phillips theorem:

(R1) For some $a > 0$, $\sigma(L_V) \subset \{z \,|\, |\mathrm{Im} z| \le a\}$.

(R2) There is a $M > 0$ such that for all z with $|\mathrm{Im} z| > a$ and all integers $n \ge 1$,

$$\|(z - L_V)^{-n}\| \le M(|\mathrm{Im} z| - a)^{-n}.$$

In the next proposition we summarize some elementary consequences of the assumptions (R1) and (R2). In the sequel $L_V^{\#}$ stands either for L_V or L_V^*.

Proposition 2.2 *Assume that* (R1) *and* (R2) *hold. Then the operators* $\mathrm{i}L_V^{\#}$ *are generators of strongly continuous groups on* $\mathcal{H}$. *Moreover:*
(i) $\|\mathrm{e}^{\mathrm{i}tL_V^{\#}}\| \le M\mathrm{e}^{a|t|}$.
(ii) *If* $\mathrm{Im} z > a$, *then*

$$(z - L_V^{\#})^{-1} = \frac{1}{\mathrm{i}} \int_0^\infty \mathrm{e}^{\mathrm{i}zt} \mathrm{e}^{-\mathrm{i}tL_V^{\#}} \mathrm{d}t. \tag{2.17}$$

(iii) *For all $A \in \mathfrak{M}$,*

$$\tau_V^t(A) = \mathrm{e}^{\mathrm{i}tL_V} A \mathrm{e}^{-\mathrm{i}tL_V} = \mathrm{e}^{\mathrm{i}tL_V^*} A \mathrm{e}^{-\mathrm{i}tL_V^*}.$$

(iv) $L_V \Omega = 0$.

Proof. Parts (i) and (ii) are well-known properties of strongly continuous groups. Parts (iii) and (iv) follow from (2.15). □

It is convenient to introduce conditions on the perturbation V which can be easily checked in concrete models and which imply (R1) and (R2) above. We describe one such condition below. For self-adjoint $V \in \mathcal{O}$ and $t \in \mathbb{R}$ we set

$$V_t \equiv \Delta^{\mathrm{i}t} V \Delta^{-\mathrm{i}t}.$$

(R3) The function $\mathbb{R} \ni t \mapsto V_t \in \mathfrak{M}$ has an analytic continuation to the strip $\{z \,|\, |\mathrm{Im} z| < 1/2\}$ which is bounded and continuous on its closure.

Note that since V_t is self-adjoint we must have $V_z^* = V_{\overline{z}}$. Clearly, (R3) implies (R1) and (R2), and

$$L_V = \mathcal{L} + V - JV_{-\mathrm{i}/2}J,$$

$$L_V^* = \mathcal{L} + V - JV_{\mathrm{i}/2}J.$$

Moreover, if (R3) holds, then one can take $a = \|V_{\mathrm{i}/2}\| = \|V_{-\mathrm{i}/2}\|$ and $M = 1$ in (R1)-(R2).

If ω is a (τ, β)-KMS state, there is an important relation between standard Liouvillean $\mathcal{L}_V$ and C-Liouvillean L_V. A simple computation shows that for $t \in \mathbb{R}$,

$$\mathcal{L} + V - JV_t J = \mathrm{e}^{-\mathrm{i}\beta t(\mathcal{L}+V)} \mathcal{L}_V \mathrm{e}^{\mathrm{i}\beta t(\mathcal{L}+V)}.$$

If (R3) holds, then by analytic continuation the relation

$$L_V = \mathrm{e}^{\beta(\mathcal{L}+V)/2} \mathcal{L}_V \mathrm{e}^{-\beta(\mathcal{L}+V)/2}, \tag{2.18}$$

holds in quadratic form sense on a domain $\mathcal{D}(\mathrm{e}^{-\beta(\mathcal{L}+V)/2}) \cap \mathcal{D}(\mathrm{e}^{\beta(\mathcal{L}+V)/2})$. The identity (2.18) leads to a simpler proof of some fundamental results of Araki's theory of perturbations of W^*-dynamical systems (see [DJP] for details). It can also be used to relate the method of the proof of Theorem 1.2, restricted to thermal equilibrium case $\beta_1 = \beta_2 = \beta$, to the method of [JP1, JP2]. For reasons of space we omit the details.

If ω is not a KMS-state, then there is no direct relation between L_V and $\mathcal{L}_V$.

2.3 Spectral theory of NESS

Our goal is to study NESS using spectral theory of C-Liouvilleans. For this reason it is more convenient to deal with NESS defined using Abelian limits. The weak-* limit points of the set of states

$$\epsilon \int_0^\infty \mathrm{e}^{-\epsilon t} \omega \circ \tau_V^t \mathrm{d}t,$$

as $\epsilon \downarrow 0$ we denote by $\Sigma^+_{V,\mathrm{Ab}}(\omega)$. The set $\Sigma^+_{V,\mathrm{Ab}}(\omega)$ is a non-empty weak-* compact subset of $E(\mathcal{O})$ whose elements are τ_V-invariant. Moreover:

Proposition 2.3 *If either $\Sigma^+_{V,\mathrm{Ab}}(\omega)$ or $\Sigma^+_V(\omega)$ consists of a single state, then*

$$\Sigma^+_{V,\mathrm{Ab}}(\omega) = \Sigma^+_V(\omega).$$

The proof of this proposition follows from standard Abelian and Tauberian theorems [Si].

With a slight abuse of terminology we will also call the elements of $\Sigma^+_{V,\mathrm{Ab}}(\omega)$ the NESS of $(\mathcal{O}, \tau_V)$ associated to the initial state ω.

In what follows we assume that the assumptions (R1) and (R2) hold.

Our goal is to characterize NESS in $\Sigma^+_{V,\mathrm{Ab}}(\omega)$ in terms of the corresponding C-Liouvillean. To motivate this characterization, for $\mathrm{Im} z > a$ let $\Omega_z := (z - L_V^*)^{-1}\Omega$, and let $\omega_z \in \mathcal{O}^*$ be defined by $\omega_z(A) = (\Omega, A\Omega_z)$. Then, since

$$\omega_z(A) = \frac{1}{\mathrm{i}} \int_0^\infty \mathrm{e}^{\mathrm{i}zt} \omega(\tau_V^t(A)) \mathrm{d}t,$$

the functionals ω_z have weak-$*$ analytic extension to the half-plane $\mathrm{Im} z > 0$ and $\Sigma^+_{V,\mathrm{Ab}}(\omega)$ is the weak-$*$ limit point set of the set of states $\{\mathrm{i}\epsilon\omega_{\mathrm{i}\epsilon} \,|\, \epsilon > 0\}$ as $\epsilon \downarrow 0$. We wish to go further along these lines and characterize $\Sigma^+_{V,\mathrm{Ab}}(\omega)$ directly in terms of the vectors Ω_z. Our main tool is an axiomatic abstract version of the complex deformation technique.

Let $D \geq 0$ be a bounded operator on $\mathcal{H}$ such that $\mathrm{Ran} D$ is dense in $\mathcal{H}$ and $D\Omega = \Omega$. Set

$$\mathcal{R}_D(z) := D(z - L_V^*)^{-1} D.$$

Our first assumption is:

(DL1) The vector-valued function $z \mapsto \mathcal{R}_D(z)\Omega$, originally defined for $\mathrm{Im} z > a$, has an analytic continuation to the half-plane $\mathrm{Im} z > 0$ such that

$$\sup_{\epsilon > 0} \epsilon \|\mathcal{R}_D(\mathrm{i}\epsilon)\Omega\| < \infty. \tag{2.19}$$

Note that since $(\Omega, \mathcal{R}_D(\mathrm{i}\epsilon)\Omega) = (\mathrm{i}\epsilon)^{-1}$, $\inf_{\epsilon>0} \epsilon \|\mathcal{R}_D(\mathrm{i}\epsilon)\Omega\| \geq 1$.

We define a vector subspace $\mathcal{O}_D \subset \mathcal{O}$ by

$$\mathcal{O}_D = \left\{ A \in \mathcal{O} \,|\, A^*\Omega \in \mathcal{D}(D^{-1}) \right\}.$$

Let $\mathcal{O}_D^{\rm cl}$ be the norm closure of $\mathcal{O}_D$. Our next two assumptions are:

(DL2) $\mathcal{O}_D^{\rm cl} = \mathcal{O}$.

(DL3) The set $\{D^{-1}A^*\Omega \,|\, A \in \mathcal{O}_D\}$ is dense in $\mathcal{H}$.

Let $\mathfrak{W}_V^+$ be the weak limit point set of $\mathrm{i}\epsilon\mathcal{R}_D(\mathrm{i}\epsilon)\Omega$ as $\epsilon \downarrow 0$. Since the unit ball in a Hilbert space is weakly compact, (2.19) implies that $\mathfrak{W}_V^+$ is non-empty.

Proposition 2.4 *Assume that* (DL1) *and* (DL2) *hold. Then there is an injection*

$$\mathfrak{W}_V^+ \ni \Omega_V^+ \mapsto \omega_V^+ \in \Sigma_{V,\mathrm{Ab}}^+(\omega) \tag{2.20}$$

such that for $A \in \mathcal{O}_D$,

$$\omega_V^+(A) = (D^{-1}A^*\Omega, \Omega_V^+). \tag{2.21}$$

If in addition (DL3) *holds, then the map (2.20) is a bijection.*

Remark. The vectors in $\mathfrak{W}_V^+$ are naturally interpreted as the zero-resonance eigenvectors associated to the triple (L_V^*, D, Ω), and in this sense Theorem 2.4 identifies NESS with zero resonance eigenvectors of L_V^*.

Proof. Proposition 2.2 yields that for $A \in \mathcal{O}_D$

$$\epsilon\int_0^\infty \mathrm{e}^{-\epsilon t}\omega(\tau_V^t(A))\mathrm{d}t = \mathrm{i}\epsilon(D^{-1}A^*\Omega, \mathcal{R}_D(\mathrm{i}\epsilon)\Omega). \tag{2.22}$$

Since $\mathcal{O}_D^{\rm cl} = \mathcal{O}$, from this relation it follows that each $\Omega_V^+ \in \mathfrak{W}_V^+$ determines a unique state $\omega_V^+ \in \Sigma_{V,\mathrm{Ab}}^+(\omega)$ and that (2.21) holds for $A \in \mathcal{O}_D$.

If in addition (DL3) holds, then Relation (2.22) and the uniform bound (2.19) imply that each $\omega_V^+ \in \Sigma_{V,\mathrm{Ab}}^+(\omega)$ determines a unique $\Omega_V^+ \in \mathfrak{W}_V^+$. $\Box$

An immediate consequence of Proposition 2.4 is that under the assumptions (DL1)-(DL3), $\Sigma_{V,\mathrm{Ab}}^+(\omega)$ consists of a single state ω_V^+ iff

$$\mathrm{w} - \lim_{\epsilon\downarrow 0} \mathrm{i}\epsilon\mathcal{R}_D(\mathrm{i}\epsilon)\Omega = \Omega_V^+,$$

and in this case for all $A \in \mathcal{O}$ we have

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \omega(\tau_V^t(A))\mathrm{d}t = \lim_{\epsilon\downarrow 0}\epsilon\int_0^\infty \mathrm{e}^{-\epsilon t}\omega(\tau_V^t(A))\mathrm{d}t = \omega_V^+(A).$$

To refine the above result, we need additional assumptions. Let

$$\mathfrak{M}_D' = \left\{C \in \mathfrak{M}' \,\middle|\, C^*C\Omega \in \mathcal{D}(D^{-1})\right\},$$

and let $(\mathfrak{M}_D'\Omega)^{\rm cl}$ be the closure of $\mathfrak{M}_D'\Omega$ in $\mathcal{H}$.

(DL4) $(\mathfrak{M}'_D\Omega)^{\rm cl} = \mathcal{H}$.

Note that since Ω is a separating vector for $\mathfrak{M}$, $(\mathfrak{M}'\Omega)^{\rm cl} = \mathcal{H}$. We denote by $\mathcal{N}_D$ the set of vector states

$$\eta(\,\cdot\,) = (C\Omega,\, \cdot\, C\Omega),$$

where $C \in \mathfrak{M}'_D$ and $\|C\Omega\| = 1$. (DL4) implies that $\mathcal{N}_D$ is norm-dense in $\mathcal{N}$.

We will replace assumption (DL1) with

(DL5) The operator-valued function $z \mapsto \mathcal{R}_D(z)$, originally defined for $\mathrm{Im} z > a$, has an analytic continuation to the region $\mathrm{Im} z > 0$ and there is a bounded operator P_V^+ such that

$$\mathrm{w} - \lim_{\epsilon \downarrow 0} \mathrm{i}\epsilon \mathcal{R}_D(\mathrm{i}\epsilon) = P_V^+.$$

Proposition 2.5 *Assume that the assumptions* (DL2), (DL4) *and* (DL5) *hold and that* $\dim \mathrm{Ran} P_V^+ = 1$. *Then, for all* $\eta \in \mathcal{N}$*,*

$$\Sigma^+_{V,\mathrm{Ab}}(\eta) = \Sigma^+_{V,\mathrm{Ab}}(\omega) = \{\omega_V^+\}.$$

Proof. Note that since $P_V^{+*}\Omega = \Omega$ and $\dim \mathrm{Ran} P_V^+ = 1$, $P_V^+(\,\cdot\,) = (\Omega,\,\cdot\,)\Omega_V^+$. To prove the proposition it suffices to show that for $\eta \in \mathcal{N}_D$ and $A \in \mathcal{O}_D$,

$$\lim_{\epsilon \downarrow 0} \epsilon \int_0^\infty \mathrm{e}^{-\epsilon t} \eta(\tau_V^t(A)) \mathrm{d}t = (D^{-1}A^*\Omega, \Omega_V^+). \tag{2.23}$$

Let $\eta \in \mathcal{N}_D$ and $A \in \mathcal{O}_D$ be given. Let $C \in \mathfrak{M}'_D$ be such that $\eta(\,\cdot\,) = (C\Omega,\, \cdot\, C\Omega)$. Since $[C, \tau_V^t(A)] = 0$, we derive from Proposition 2.2 that

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-\epsilon t} \eta(\tau_V^t(A)) \mathrm{d}t &= \int_0^\infty \mathrm{e}^{-\epsilon t} (C\Omega, \tau_V^t(A) C\Omega) \mathrm{d}t \\ &= \mathrm{i}(D^{-1}A^*\Omega, \mathcal{R}_D(\mathrm{i}\epsilon) D^{-1}C^*C\Omega).\end{aligned}$$

Therefore

$$\lim_{\epsilon \downarrow 0} \epsilon \int_0^\infty \mathrm{e}^{-\epsilon t} \eta(\tau_V^t(A)) \mathrm{d}t = (D^{-1}A^*\Omega, P_V^+ D^{-1}C^*C\Omega). \tag{2.24}$$

Relations $P_V^+(\,\cdot\,) = (\Omega,\,\cdot\,)\Omega_V^+$, $D^{-1}\Omega = \Omega$, and $\|C\Omega\| = 1$ yield

$$P_V^+ D^{-1} C^* C\Omega = \Omega_V^+, \tag{2.25}$$

and (2.23) follows from (2.24) and (2.25). □

The last result we wish to discuss concerns conditions under which the approach to NESS is exponentially fast. For $\mu \in \mathbb{R}$ let $\mathfrak{P}(\mu)$ be the half-plane $\{z \,|\, \mathrm{Im} z > \mu\}$. We replace (DL5) with:

(DL6) The operator-valued function $z \mapsto \mathcal{R}_D(z)$, originally defined for $z \in \mathfrak{P}(a)$, has a meromorphic continuation to a half-plane $\mathfrak{P}(\mu)$ for some $\mu < 0$.

Since $(\Omega, \mathcal{R}_D(z)\Omega) = 1/z$, zero is always a pole of $\mathcal{R}_D(z)$. It is not difficult to show that if in addition (DL3) holds, then zero is a simple pole of $\mathcal{R}_D(z)$ and all other poles are in the half-plane $\mathrm{Im} z \le 0$. In particular, (DL3) $\wedge$ (DL6) $\Rightarrow$ (DL5). We will not make use of assumption (DL3) below.

Assume in addition to (DL6) that the function $\mathcal{R}_D(z)$ has only finitely many poles $\{z_0, z_1, \ldots, z_n\}$ ($z_0 = 0$) in the half-plane $\mathfrak{P}(\mu)$ and let m_k be the order or the pole z_k. Then we can decompose $\mathcal{R}_D(z)$ as

$$\mathcal{R}_D(z) = \mathcal{R}_D^{\mathrm{a}}(z) + \mathcal{R}_D^{\mathrm{s}}(z), \tag{2.26}$$

where $\mathcal{R}_D^{\mathrm{a}}(z)$ is an analytic operator-valued function in the half-plane $\mathfrak{P}(\mu)$ and

$$\begin{aligned} \mathcal{R}_D^{\mathrm{s}}(z) &= \sum_{k=0}^{n} S_k(z) \\ S_k(z) &= \sum_{i=1}^{m_k} \frac{S_{ki}}{(z - z_k)^i}. \end{aligned} \tag{2.27}$$

Let P_V^+ be the residue of $\mathcal{R}_D(z)$ at $z = 0$. Then

$$P_V^+ = \frac{1}{2\pi \mathrm{i}} \oint_\gamma \mathcal{R}_D(z) \mathrm{d}z = S_{01},$$

where γ is a small circle around zero such that inside γ zero is the only singularity of $\mathcal{R}_D(z)$.

Theorem 2.6 *Assume the following:*
(a) *Assumptions* (DL2), (DL4) *and* (DL6) *hold.*
(b) *The function $\mathcal{R}_D(z)$ has only finitely many singularities $\{z_0, z_1, \ldots, z_n\}$ in $\mathfrak{P}(\mu)$, where $z_0 = 0$ and $\mathrm{Im} z_k < 0$ for $k \ge 1$.*
(c) $\dim \mathrm{Ran} P_V^+ = 1$.
(d) *For all $\Psi \in \mathcal{H}$ and $j = 0, 1$, $\sup_{y > \mu} \int_{\mathbb{R}} |\partial_x^j (\Psi, \mathcal{R}_D^{\mathrm{a}}(x + \mathrm{i}y)\Psi)|^{2-j} \mathrm{d}x < \infty$.*
Then,
(i) *For all $\eta \in \mathcal{N}$, $\Sigma_{V,\mathrm{Ab}}^+(\eta) = \Sigma_{V,\mathrm{Ab}}^+(\omega) = \{\omega_V^+\}$. Moreover, for all $A \in \mathcal{O}$,*

$$\lim_{t \to \infty} \eta(\tau_V^t(A)) = \omega_V^+(A). \tag{2.28}$$

(ii) *For all* $\eta \in \mathcal{N}_D$, $A \in \mathcal{O}_D$, *and* $t > 0$,

$$\left|\eta(\tau_V^t(A)) - \omega_V^+(A)\right| \le C_{\eta,A}\mathrm{e}^{-\gamma t}\langle t\rangle^{r-1},$$

where $\gamma \equiv \min_{1\le k\le n}|\mathrm{Im} z_k|$ *and* r *is the maximum order of dominant poles (the poles in* $\{z_1, \cdots, z_n\}$ *closest to the real axis).*

Proof. Since (ii) $\Rightarrow$ (i), we have to prove (ii) only.

Fix $\eta \in \mathcal{N}_D$, $\eta(\,\cdot\,) = (C\Omega, \cdot\, C\Omega)$, and $A \in \mathcal{O}_D$. Then,

$$\int_0^\infty \mathrm{e}^{\mathrm{i}zt}\eta(\tau_V^t(A))\mathrm{d}t = \mathrm{i}(D^{-1}A^*\Omega, \mathcal{R}_D(z)D^{-1}C^*C\Omega) \equiv \ell(z).$$

Fix $\delta > 0$ and μ' such that $\mu < \mu' < -\gamma$. Let $\alpha > 0$ be a large number and Γ_α the rectangle with vertices $\{\pm\alpha + \mathrm{i}\delta, \pm\alpha + \mathrm{i}\mu'\}$. Then, for any $\epsilon > 0$,

$$\begin{aligned}\frac{1}{2\pi}\int_{\Gamma_\alpha} \mathrm{e}^{-\mathrm{i}tz}\ell(z)\mathrm{d}z &= -\frac{1}{2\pi}\int_{-\alpha}^{\alpha} \mathrm{e}^{-\mathrm{i}t(x+\mathrm{i}\delta)}\ell(x+i\delta)\mathrm{d}x + S(\alpha) + B(\alpha)\\ &= -\omega_V^+(A) - \sum_{k=1}^{n}\sum_{i=1}^{m_k}\frac{(-\mathrm{i}t)^{i-1}}{(i-1)!}(D^{-1}A^*\Omega, S_{ki}D^{-1}C^*C\Omega)\mathrm{e}^{-\mathrm{i}tz_k},\end{aligned} \tag{2.29}$$

where $S(\alpha)$ is the integral of ℓ over the vertical sides of the rectangle Γ_α and $B(\alpha)$ is the integral over the bottom side. Integration by parts and (d) with $j = 1$ yield that for $t > 1$ and uniformly in α, $|B(\alpha)| = O(\mathrm{e}^{\mu' t})$. Using (d) with $j = 0$, a standard argument (see e.g. Theorem 19.2 in [Rud]) yields that for some sequence $\alpha_n \to \infty$, $|S(\alpha_n)| \to 0$. Moreover, the sequence α_n can be chosen independently of δ as long as $\delta < \mathrm{const}$. Pick a subsequence α_{n_k} such that

$$\lim_{k\to\infty}\frac{1}{2\pi}\int_{-\alpha_{n_k}}^{\alpha_{n_k}} \mathrm{e}^{-\mathrm{i}t(x+\mathrm{i}\delta)}\ell(x+i\delta)\mathrm{d}x = \eta(\tau_V^t(A)),$$

for Lebesgue a.e. $t > 0$ and set $\alpha = \alpha_{n_k}$ in (2.29). Taking $k \to \infty$ we derive that for a.e. $t > 0$,

$$\left|\eta(\tau_V^t(A)) - \omega_V^+(A)\right| \le C_{A,\eta}\mathrm{e}^{-\gamma t}\langle t\rangle^{r-1}. \tag{2.30}$$

Since both sides in (2.30) are continuous functions of t, the estimate (2.30) holds for all $t > 0$. □

3 Modular structure of the model

In this section we return to the model $\mathcal{S} + \mathcal{R}_1 + \mathcal{R}_2$. We explicitly compute the modular structure associated to $(\mathcal{O}, \tau)$ and the states in $\mathcal{N}_\mathrm{s}$. We then use these results to compute the standard and the C-Liouvillean of the locally perturbed system.

Since the results of this section are either well-known or follow from simple computations we will omit the proofs.

Notation. If A is a linear operator on $\mathfrak{H}_{\rm s}$, we denote by $\overline{A}$ the linear operator

$$\overline{A}\psi = \overline{A\overline{\psi}},$$

where on the right-hand side $\overline{\cdot}$ is the usual complex conjugation on $\mathfrak{H}_{\rm s} = \mathbb{C}^2$.

We begin by computing the modular structure associated to the small system $\mathcal{S}$. Set

$$\mathcal{H}_{\rm s} = \mathfrak{H}_{\rm s} \otimes \mathfrak{H}_{\rm s},$$

$$\pi_{\rm s}(A) = A \otimes \mathbf{1},$$

$$\pi_{\rm s}^{\#}(A) = \mathbf{1} \otimes \overline{A}.$$

Let $\omega_{\rm s}$ be a state on $\mathcal{O}_{\rm s}$. Then there is a density matrix $\rho_{\rm s}$ such that $\omega_{\rm s}(A) = \mathrm{Tr}(\rho_{\rm s} A)$. The state $\omega_{\rm s}$ is faithful iff $\rho_{\rm s} > 0$ and $\tau_{\rm s}$-invariant iff $[H_{\rm s}, \rho_{\rm s}] = 0$. If $\rho_{\rm s}(\,\cdot\,) = \sum p_i(\psi_i, \,\cdot\,)\psi_i$, let

$$\Omega_{\rm s} := \sum p_i^{\frac{1}{2}} \psi_i \otimes \overline{\psi}_i.$$

Recall that the dynamics of $\mathcal{S}$ is specified by automorphisms (1.3). Let

$$\mathcal{L}_{\rm s} \equiv H_{\rm s} \otimes \mathbf{1} - \mathbf{1} \otimes H_{\rm s}.$$

Proposition 3.1 *The triple $(\mathcal{H}_{\rm s}, \pi_{\rm s}, \Omega_{\rm s})$ is the GNS representation of $\mathcal{O}_{\rm s}$ associated to $\omega_{\rm s}$. If $\omega_{\rm s}$ is $\tau_{\rm s}$-invariant, then $\mathcal{L}_{\rm s}$ is the corresponding standard Liouvillean. If $\omega_{\rm s}$ is faithful, consider the pair $(\pi_{\rm s}(\mathcal{O}_{\rm s}), \Omega_{\rm s})$.*
(i) *Its modular operator is*

$$\Delta_{\rm s} = \rho_{\rm s} \otimes \overline{\rho}_{\rm s}^{\,-1}.$$

(ii) *Its modular conjugation is $J_{\rm s}(\phi \otimes \psi) = \overline{\psi} \otimes \overline{\phi}$.*
(iii) $J_{\rm s}\pi_{\rm s}(A)J_{\rm s} = \pi_{\rm s}^{\#}(A)$.

We now discuss the modular structure associated to a free Fermi reservoir in thermal equilibrium at inverse temperature β. We fix a complex conjugation (an anti-unitary involution) $f \mapsto \overline{f}$ which commutes with the single particle Hamiltonian h. Let $\hat{\Omega}_{\rm f}$ be the Fock vacuum on $\mathfrak{H}_{\rm f}$, N the number operator,

$$\vartheta \equiv \Gamma(-\mathbf{1}) = (-\mathbf{1})^N,$$

and

$$\varrho_\beta \equiv \left(\mathrm{e}^{\beta h} + 1\right)^{-1}.$$

The complex conjugation $\overline{\cdot}$ on $\mathfrak{h}$ naturally extends to a complex conjugation on $\mathcal{H}_{\rm f}$ which we denote by the same symbol, i.e. $\Psi \mapsto \overline{\Psi}$. Let

$$\mathcal{H}_{\rm f} \equiv \mathfrak{H}_{\rm f} \otimes \mathfrak{H}_{\rm f},$$

$$\Omega_{\rm f} = \hat{\Omega}_{\rm f} \otimes \hat{\Omega}_{\rm f}.$$

The Araki-Wyss representation π_β of $\mathcal{O}_{\rm f}$ on $\mathcal{H}_{\rm f}$ is defined by

$$\pi_\beta(a(f)) = a((\mathbf{1} - \varrho_\beta)^{\frac{1}{2}} f) \otimes \mathbf{1} + \vartheta \otimes a^*(\varrho_\beta^{\frac{1}{2}} \overline{f}),$$

$$\pi_\beta(a^*(f)) = a^*((\mathbf{1} - \varrho_\beta)^{\frac{1}{2}} f) \otimes \mathbf{1} + \vartheta \otimes a(\varrho_\beta^{\frac{1}{2}} \overline{f}).$$

The dual representation $\pi_\beta^{\#}$ is defined by

$$\pi_\beta^{\#}(a^*(f)) = \vartheta a(\varrho_\beta^{\frac{1}{2}} f) \otimes \vartheta + \mathbf{1} \otimes a^*((1 - \varrho_\beta)^{\frac{1}{2}} \overline{f})\vartheta,$$

$$\pi_\beta^{\#}(a(f)) = a^*(\varrho_\beta^{\frac{1}{2}} f)\vartheta \otimes \vartheta + \mathbf{1} \otimes \vartheta a((1 - \varrho_\beta)^{\frac{1}{2}} \overline{f}).$$

The representations π_β and $\pi_\beta^{\#}$ were introduced for the first time in [AW] (see also Example 5.2.20 in [BR2]). Let

$$\mathcal{L}_{\rm f} \equiv H_{\rm f} \otimes \mathbf{1} - \mathbf{1} \otimes H_{\rm f}.$$

Proposition 3.2 *The triple $(\mathcal{H}_{\rm f}, \pi_\beta, \Omega_{\rm f})$ is the GNS representation of $\mathcal{O}_{\rm f}$ associated to the KMS-state $\omega_{{\rm f},\beta}$ and $\mathcal{L}_{\rm f}$ is the corresponding standard Liouvillean. The vector $\Omega_{\rm f}$ is separating for the enveloping von Neumann algebra*

$$\mathfrak{M}_{{\rm f},\beta} \equiv \pi_\beta(\mathcal{O}_{\rm f})''.$$

Consider the pair $(\mathfrak{M}_{{\rm f},\beta}, \Omega_{\rm f})$.
(i) *Its modular operator is $\Delta_{\rm f} = {\rm e}^{-\beta \mathcal{L}_{\rm f}}$.*
(ii) *Its modular conjugation is*

$$J_{\rm f}(\Phi \otimes \Psi) = u\overline{\Psi} \otimes u\overline{\Phi},$$

where $u \equiv (-\mathbf{1})^{N(N-1)/2}$.
(iii) $J_{\rm f}\pi_\beta(A)J_{\rm f} = \pi_\beta^{\#}(A)$.

If (A1) holds, then the GNS representation and modular structure of a free Fermi gas can be described in a somewhat different form which is more suitable for the spectral analysis. In what follows we assume that (A1) holds. Let $\tilde{\mathfrak{h}} = L^2(\mathbb{R}, \mathfrak{G})$. To any $f \in \mathfrak{h}$ we associate a pair of functions $f_\beta, f_\beta^{\#} \in \tilde{\mathfrak{h}}$ by

$$f_\beta(s) = \left({\rm e}^{-\beta s} + 1\right)^{-\frac{1}{2}} \tilde{f}(s),$$

$$f_\beta^{\#}(s) = {\rm i}{\rm e}^{-\beta s/2} f_\beta(s) = {\rm i}\overline{f}_\beta(-s),$$

($\tilde{f}$ is defined by (1.7)). For latter purposes we make the following remark. Assume that $\tilde{f} \in H^2(\delta)$ for some $0 < \delta < \pi/\beta$. Then

$$C(\delta, \beta) \equiv \sup_{|\mathrm{Im} z| < \delta} |1 + \mathrm{e}^{-\beta z}|^{-1/2} < \infty.$$

It then follows that $f_\beta, f_\beta^\# \in H^2(\delta)$,

$$\begin{aligned} \|f_\beta\|_{H^2(\delta)} = \|f_\beta^\#\|_{H^2(\delta)} = \|\mathrm{e}^{\beta s/2} f_\beta^\#\|_{H^2(\delta)} = \|\mathrm{e}^{-\beta s/2} f_\beta\|_{H^2(\delta)}, \\ \|f_\beta\|_{H^2(\delta)} \le C(\delta, \beta) \|\tilde{f}\|_{H^2(\delta)}, \\ \|\mathrm{e}^{-\beta s/2} f_\beta^\#\|_{H^2(\delta)} \le C(\delta, \beta) \|\mathrm{e}^{-\beta s/2} \tilde{f}\|_{H^2(\delta)}. \end{aligned} \tag{3.31}$$

We denote by s the operator of multiplication by $s \in \mathbb{R}$. Let $\tilde{\Omega}$ be the vacuum on $\Gamma_-(\tilde{\mathfrak{h}})$.

Theorem 3.3 *There exists a unitary map*

$$U : \mathcal{H}_\mathrm{f} \mapsto \mathrm{d}\Gamma_-(\tilde{\mathfrak{h}})$$

such that

$$U\Omega_\mathrm{f} = \tilde{\Omega}_\mathrm{f}$$

$$U\mathcal{L}_\mathrm{f}U^{-1} = \mathrm{d}\Gamma(s)$$

$$U\pi_\beta(\varphi(f))U^{-1} = \varphi(f_\beta)$$

$$U\pi_\beta^\#(\varphi(f))U^{-1} = \mathrm{i}\Gamma(-\mathbf{1})\varphi(f_\beta^\#).$$

Proof. This result follows from the identification $\mathfrak{h} \oplus \mathfrak{h} = L^2(\mathbb{R}, \mathfrak{G})$ and the exponential law for fermionic systems (see Theorem 3.2 in [BSZ]). □

In what follows we will work exclusively in the representation given by Theorem 3.3 and we identify the quantities related by U ($\mathcal{H}_\mathrm{f}$ now stands for $\Gamma_-(\tilde{\mathfrak{h}})$, Ω_f for $\tilde{\Omega}_\mathrm{f}$, $\mathcal{L}_\mathrm{f}$ for $\mathrm{d}\Gamma(s)$ etc.).

Consider now two identical reservoirs $(\mathcal{O}_\mathrm{f}^{(i)}, \tau_i)$ and let $\mathcal{O}$ be given by (1.4). Let ω_{β_i} be (τ_i, β_i)-KMS on $\mathcal{O}_\mathrm{f}^{(i)}$ for some $\beta_i > 0$. Set

$$\mathcal{H} = \mathcal{H}_\mathrm{s} \otimes \mathcal{H}_\mathrm{f}^{(1)} \otimes \mathcal{H}_\mathrm{f}^{(2)}$$

$$\Omega = \Omega_\mathrm{s} \otimes \Omega_\mathrm{f}^{(1)} \otimes \Omega_\mathrm{f}^{(2)},$$

$$\pi = \pi_\mathrm{s} \otimes \pi_{\beta_1} \otimes \pi_{\beta_2},$$

$$\pi^\# = \pi_\mathrm{s}^\# \otimes \pi_{\beta_1}^\# \otimes \pi_{\beta_2}^\#,$$

$$\mathcal{L} = \mathcal{L}_\mathrm{s} + \mathcal{L}_\mathrm{f}^{(1)} + \mathcal{L}_\mathrm{f}^{(2)}.$$

Proposition 3.4 *The GNS representation of $\mathcal{O}$ associated to $\omega_{\rm s} \otimes \omega_{\beta_1} \otimes \omega_{\beta_2}$ is $(\mathcal{H}, \pi, \Omega)$. If $\omega_{\rm s}$ is $\tau_{\rm s}$-invariant, then $\mathcal{L}$ is the corresponding standard Liouvillean. If $\omega_{\rm s}$ is faithful, then Ω is a separating vector for the enveloping von Neumann algebra*

$$\mathfrak{M} \equiv \pi(\mathcal{O})'' = \pi_{\rm s}(\mathcal{O}_{\rm s}) \otimes \mathfrak{M}_{{\rm f},\beta_1} \otimes \mathfrak{M}_{{\rm f},\beta_2}.$$

For $\omega_{\rm s}$ faithful, consider the pair $(\mathfrak{M}, \Omega)$.
(i) *Its modular operator is $\Delta = \Delta_{\rm s} \otimes \Delta_{\rm f}^{(1)} \otimes \Delta_{\rm f}^{(2)}$.*
(ii) *Its modular conjugation is $J = J_{\rm s} \otimes J_{\rm f} \otimes J_{\rm f}$.*
(iii) *$J\pi(A)J = \pi^{\#}(A)$.*

Let now V be the perturbation (1.5). The standard Liouvillean $\mathcal{L}_V$ for the perturbed dynamics is now easily computed in the representation π. With a slight abuse of notation we identify V and $\pi(V)$. Moreover, we denote the field and number operators on $\mathcal{H}_{\rm f}^{(i)}$ by $\varphi^{(i)}$ and N_i. Then,

$$V = (\sigma_x \otimes \mathbf{1}) \otimes \varphi^{(1)}(\alpha_{1\beta_1}) + (\sigma_x \otimes \mathbf{1}) \otimes \varphi^{(2)}(\alpha_{2\beta_2}),$$
$$JVJ = (\mathbf{1} \otimes \sigma_x) \otimes \left({\rm i}(-\mathbf{1})^{N_1}\varphi^{(1)}(\alpha_{1\beta_1}^{\#})\right) + (\mathbf{1} \otimes \sigma_x) \otimes \left({\rm i}(-\mathbf{1})^{N_2}\varphi^{(2)}(\alpha_{2\beta_2}^{\#})\right).$$

Proposition 3.5 *The standard Liouvillean of the perturbed system $(\mathcal{O}, \tau_\lambda)$ in the representation π is*

$$\mathcal{L}_\lambda = \mathcal{L} + \lambda V - \lambda JVJ.$$

Assume now that (A2) holds. Then, the assumption (R3) of the Section 2.2 holds and

$$JV_{-{\rm i}/2}J = {\rm i}\sum_i (\mathbf{1} \otimes \overline{\rho}_{\rm s}^{1/2}\sigma_x\overline{\rho}_{\rm s}^{\,-1/2}) \otimes \frac{1}{\sqrt{2}}(-\mathbf{1})^{N_i}\left(a^{(i)}({\rm e}^{-\beta_i s/2}\alpha_{i\beta_i}^{\#}) + a^{(i)*}({\rm e}^{\beta_i s/2}\alpha_{i\beta_i}^{\#})\right),$$

$$JV_{{\rm i}/2}J = {\rm i}\sum_i (\mathbf{1} \otimes \overline{\rho}_{\rm s}^{\,-1/2}\sigma_x\overline{\rho}_{\rm s}^{1/2}) \otimes \frac{1}{\sqrt{2}}(-\mathbf{1})^{N_i}\left(a^{(i)}({\rm e}^{\beta_i s/2}\alpha_{i\beta_i}^{\#}) + a^{(i)*}({\rm e}^{-\beta_i s/2}\alpha_{i\beta_i}^{\#})\right).$$

Proposition 3.6 *If $\omega_{\rm s}$ is faithful and Hypothesis* (A2) *holds, then the Hypothesis* (R3) *of Section 2.2 holds for the perturbation V and the C-Liouvillean is*

$$L_\lambda = \mathcal{L} + \lambda V - \lambda JV_{-{\rm i}/2}J.$$

The adjoint of L_λ is

$$L_\lambda^* = \mathcal{L} + \lambda V - \lambda JV_{{\rm i}/2}J.$$

Although the standard Liouvillean does not depend on the choice of the initial state of the small system, the C-Liouvillean does through the term $JV_{-{\rm i}/2}J$. It is often convenient to take a simple choice for the initial state $\omega_{\rm s}$, namely

$$\omega_{\rm s}(A) = {\rm Tr}(A)/2, \tag{3.32}$$

whose density matrix is $\rho_{\rm s} = \mathbf{1}/2$. In this case L_λ takes a slightly simpler form and

$$\|V_{\pm{\rm i}/2}\| \leq 2\sum_i \|{\rm e}^{\beta_i s/2}\alpha_i\|.$$

4 Spectral analysis

The spectral analysis of the operators $\mathcal{L}_\lambda$ and L_λ^* follows closely [JP1]. In this section we will state the main results of this analysis and discuss some of their consequences. We will only indicate the main steps of the proofs and the interested reader should consult [JP1] for details. Throughout this section we assume that the assumptions (A1) and (A2) hold.

Recall that $\tilde{\mathfrak{h}} = L^2(\mathbb{R}, \mathfrak{G})$ and $\mathcal{H}_\mathrm{f} = \Gamma_-(\tilde{\mathfrak{h}})$. Let $p \equiv \mathrm{i}\partial_\mathrm{s}$ be the generator of the group of translations on $\tilde{\mathfrak{h}}$ and $P = \mathrm{d}\Gamma(p)$ its second quantization. We adopt the shorthand $\langle P\rangle = (1+P^2)^{\frac{1}{2}}$. Let $\delta > 0$ be as in (A2). In what follows we fix κ such that

$$0 < \kappa < \min(\pi/\beta_1, \pi/\beta_2, \delta).$$

Let

$$D := \mathbf{1} \otimes \mathrm{e}^{-\kappa\langle P\rangle} \otimes \mathrm{e}^{-\kappa\langle P\rangle}.$$

Obviously, $\mathrm{Ran}D$ is dense in $\mathcal{H}$ and the vectors of the form $\psi \otimes \Omega_\mathrm{f}^{(1)} \otimes \Omega_\mathrm{f}^{(2)}$, $\psi \in \mathcal{H}_\mathrm{s}$, are invariant under D. Recall that $\mathfrak{P}(\mu) \equiv \{z \,|\, \mathrm{Im} z > \mu\}$.

We deal first with the standard Liouvillean and Theorem 1.3.

Theorem 4.1 *For any $\mu > -\kappa$ there is a constant $\Lambda > 0$ such that for $|\lambda| < \Lambda$ the operator-valued function*

$$z \mapsto D(z - \mathcal{L}_\lambda)^{-1} D, \tag{4.33}$$

originally defined for $\mathrm{Im} z > 0$, has a meromorphic continuation to the half-plane $\mathfrak{P}(\mu)$. The function (4.33) has at most four poles in $\mathfrak{P}(-\mu)$. If in addition (A3) *holds and $\beta_1 \neq \beta_2$, then there is a constant $\ell > 0$ such that for $0 < |\lambda| < \ell$ none of the poles is on the real axis. In particular, for $0 < |\lambda| < \ell$ the spectrum of $\mathcal{L}_\lambda$ is purely absolutely continuous and there are no τ_λ-invariant states in the set $\mathcal{N}$ of normal states.*

The last part of Theorem 4.1, the absence of τ_λ-invariant states in $\mathcal{N}$, is the statement of Theorem 1.3.

The proof of Theorem 4.1 follows the argument in [JP1, JP2]. Although in these works the Bose reservoirs are studied, the same (in fact, slightly simpler) argument applies to Fermi reservoirs. For the reader convenience and for latter applications, we recall the main steps of the argument in [JP1, JP2].

Sketch of the proof of Theorem 4.1. Let

$$u(\theta) \equiv \mathrm{e}^{-\mathrm{i}\theta P} = \Gamma(\mathrm{e}^{-\mathrm{i}\theta p}),$$

be the second quantization of the group of translations and $U(\theta) = \mathbf{1} \otimes u(\theta) \otimes u(\theta)$. We set

$$\mathcal{L}_\lambda(\theta) \equiv U(\theta)\mathcal{L}_\lambda U(-\theta).$$

Let $N = N_1 + N_2$. Note that

$$U(\theta)\mathcal{L}U(-\theta) = \mathcal{L} + \theta N,$$

$$U(\theta)NU(-\theta) = N,$$

and

$$U(\theta)VU(-\theta)=\sum_i(\sigma_x\otimes\mathbf{1})\otimes\varphi^{(i)}(\mathrm{e}^{-\mathrm{i}\theta p}\alpha_{i\beta_i}),$$

$$U(\theta)JVJU(-\theta)=\mathrm{i}\sum_i(\mathbf{1}\otimes\sigma_x)\otimes(-\mathbf{1})^{N_i}\varphi^{(i)}(\mathrm{e}^{-\mathrm{i}\theta p}\alpha^{\#}_{i\beta_i}).$$

If

$$V_{\mathrm{tot}}(\theta)=U(\theta)(V-JVJ)U(-\theta),$$

then

$$\mathcal{L}_\lambda(\theta)=\mathcal{L}+\theta N+\lambda V_{\mathrm{tot}}(\theta).$$

By (A2) and (3.31) the operator $V_{\mathrm{tot}}(\theta)$ is defined for all $\theta\in I(\kappa)$ and the map $I(\kappa)\ni\theta\mapsto V_{\mathrm{tot}}(\theta)$ is an analytic operator-valued function satisfying

$$C:=\sup_{\theta\in I(\kappa)}\|V_{\mathrm{tot}}(\theta)\|\le 2\sqrt{2}\sum_i C(\kappa,\beta_i)\|\tilde{\alpha}_i\|_{H^2(\kappa)}.$$

Obviously, the operator $\mathcal{L}_\lambda(\theta)$ is also defined for $\theta\in I(\kappa)$. For $\mathrm{Im}\theta\neq 0$, $\mathcal{L}_\lambda(\theta)$ is a closed operator with domain $\mathcal{D}(\mathcal{L})\cap\mathcal{D}(N)$. Let $I^-(\kappa)=\{z\,|\,-\kappa<\mathrm{Im}z<0\}$. The function $I^-(\kappa)\times\mathbb{C}\ni(\theta,\lambda)\mapsto\mathcal{L}_\lambda(\theta)$, with values in the closed operators on $\mathcal{H}$, is an analytic family of type A in each variable separately. Note that the spectrum of $\mathcal{L}_0(\theta)$ consists of two simple eigenvalues ±2, a double degenerate eigenvalue 0, and of the sequence of lines $\{\mathrm{i}n\mathrm{Im}\theta+\mathbb{R}\,|\,n\ge1\}$.

Let Λ be such that $\Lambda C<(\kappa-|\mu|)/4$. Then, for $|\lambda|<\Lambda$ and $-\kappa<\mathrm{Im}\theta<-(\kappa+|\mu|)/2$, the essential spectrum of $\mathcal{L}_\lambda(\theta)$ is contained in the half-plane $\{z\,|\,\mathrm{Im}z<\mu\}$. The location of the discrete spectrum inside $\mathfrak{P}(\mu)$ can be computed using regular perturbation theory. By possibly taking Λ smaller, one can show that this discrete spectrum consists of four points (resonances) $\{e_{\pm2}(\lambda),e_0^{1,2}(\lambda)\}$, where $\mathrm{e}_{\pm2}(\lambda)$ are near ±2 and $e_0^{1,2}(\lambda)$ are near 0, see Figure 1 below. These resonances do not depend on θ. Moreover, the functions $\lambda\mapsto e_{\pm2}(\lambda)$ are analytic for $|\lambda|<\Lambda$,

$$e_{\pm2}(\lambda)=\pm2+\sum_{j=1}^{\infty}\lambda^{2j}a_{2j}^{\pm},$$

and one can compute $a_2^{\pm}$ explicitly:

$$a_2^{\pm}=\frac{1}{2}\sum_i\left(-\mathrm{i}\pi\|\tilde{\alpha}_i(2)\|_{\mathfrak{G}}^2\pm\mathrm{PV}\int_{\mathbb{R}}\frac{\|\tilde{\alpha}_i(s)\|_{\mathfrak{G}}^2}{s-2}\mathrm{d}s\right),$$

where PV stands for Cauchy's principal value.

The resonances $e_0^{1,2}(\lambda)$ are the eigenvalues of a 2×2 matrix $\Sigma(\lambda)$ which is analytic for $|\lambda|<\Lambda$,

$$\Sigma(\lambda)=\sum_{j=1}^{\infty}\lambda^{2j}\Sigma_{2j},$$

and one can compute Σ_2 explicitly:

$$\Sigma_2 = -\mathrm{i}\pi \sum_i \|\tilde{\alpha}_i(2)\|^2_{\mathfrak{G}}\ T_i,$$

where

$$T_i = \frac{1}{2\cosh\beta_i}\begin{pmatrix} \mathrm{e}^{\beta_i} & -1 \\ -1 & \mathrm{e}^{-\beta_i} \end{pmatrix}.$$

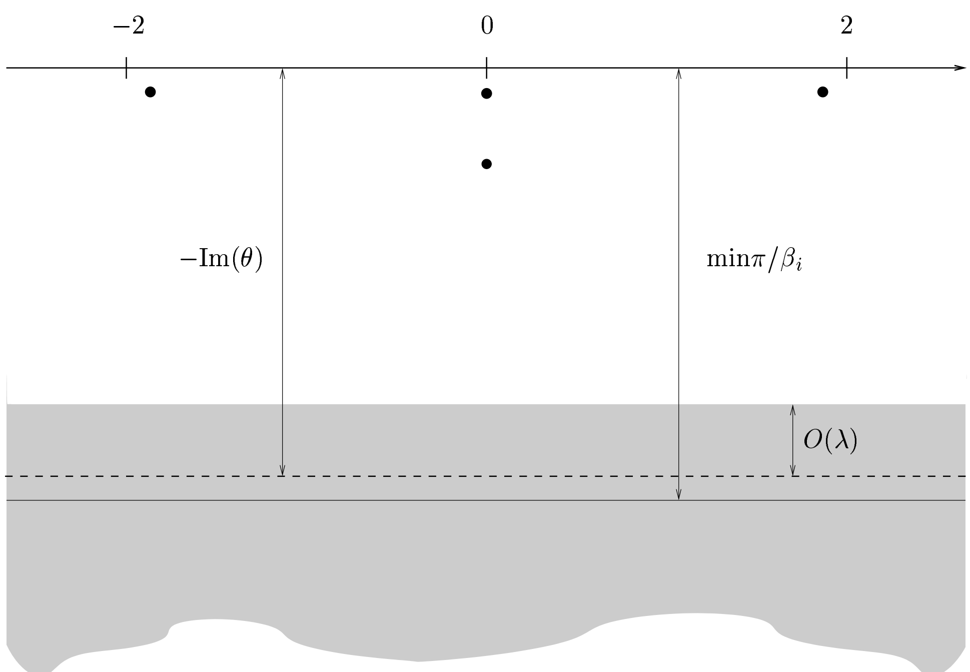

Figure 1: Resonances of the standard Liouvillean $\mathcal{L}_\lambda$

If (A3) holds, then $\mathrm{Im} a_2^{\pm} < 0$ and for λ non-zero and sufficiently small, $\mathrm{Im}\, e_{\pm 2}(\lambda) < 0$. Notice also that the matrices T_i are self-adjoint and non-negative with a simple eigenvalue 0 and corresponding eigenvector

$$\psi_{\beta_i} = \begin{pmatrix} \mathrm{e}^{-\beta_i/2} \\ \mathrm{e}^{\beta_i/2} \end{pmatrix}.$$

Thus, unless $\beta_1 = \beta_2$, $\mathrm{i}\Sigma_2 > 0$, and for λ non-zero and sufficiently small, $\mathrm{Im}\, e_0^{1,2}(\lambda) < 0$.

To finish the proof, we have to relate $D(z - \mathcal{L}_\lambda)^{-1}D$ and $(z - \mathcal{L}_\lambda(\theta))^{-1}$. To do so, we fix z with $\mathrm{Im} z$ large enough. Then, one shows that

$$\mathrm{s} - \lim_{\mathrm{Im}\theta \uparrow 0} (z - \mathcal{L}_\lambda(\theta))^{-1} = (z - \mathcal{L}_\lambda(\mathrm{Re}\theta))^{-1}. \tag{4.34}$$

Let

$$D(\theta)=\mathbf{1}\otimes \mathrm{e}^{-\kappa\langle P\rangle+\theta P}\otimes \mathrm{e}^{-\kappa\langle P\rangle+\theta P},$$

and consider the function

$$I^{-}(\kappa)\ni\theta\mapsto D(\theta)(z-\mathcal{L}_\lambda(\theta))^{-1}D(-\theta).$$

By analyticity, this function is constant in θ. By (4.34) and continuity, the relation

$$D(z-\mathcal{L}_\lambda)^{-1}D=D(\theta)(z-\mathcal{L}_\lambda(\theta))^{-1}D(-\theta), \tag{4.35}$$

holds for $-\kappa<\mathrm{Im}\theta\le 0$. If θ in (4.35) satisfies $-\kappa<\mathrm{Im}\theta<-(\kappa+|\mu|)/2$, then the right-hand side in (4.35) provides the desired meromorphic continuation of the function $D(z-\mathcal{L}_\lambda)^{-1}D$.

Since $\mathrm{Ran}D$ is dense in $\mathcal{H}$ and $D(z-\mathcal{L}_\lambda)^{-1}D$ has no poles on the real axis, the spectrum of $\mathcal{L}_\lambda$ is purely absolutely continuous for $0<|\lambda|<\ell$. In particular, $\mathrm{Ker}\mathcal{L}_\lambda=0$, and, by Proposition 2.1, there are no τ_λ-invariant states in the set $\mathcal{N}$ of normal states. □

In the proof of Theorem 4.1 we have not used the full strength of the assumption (A2) and for this theorem it suffices that $\tilde{\alpha}_i\in H^2(\kappa)$. In fact, if the complex deformation technique is replaced with Mourre theory, then the main conclusion of the theorem can be derived under much weaker regularity condition on $\tilde{\alpha}_i$, see [DJ1, DJ2].

We now deal with the C-Liouvillean and Theorem 1.2. As we have remarked at the end of the last section, it is convenient to take for the initial state of the small system the state $\omega_{\rm s}$ defined by (3.32). In what follows L_λ is the C-Liouvillean associated to $\omega=\omega_{\rm s}\otimes\omega_{\beta_1}\otimes\omega_{\beta_2}$. Let

$$\mathcal{R}_D(z)=D(z-L_\lambda^*)^{-1}D.$$

Theorem 4.2 *For any $\mu>-\kappa$ there is a constant $\Lambda>0$ such that for $|\lambda|<\Lambda$ the operator-valued function $\mathcal{R}_D(z)$, originally defined for $z\in\mathfrak{P}(a)$, has a meromorphic continuation to the half-plane $\mathfrak{P}(\mu)$. The function $\mathcal{R}_D(z)$ has at most four poles in $\mathfrak{P}(\mu)$, and zero is one of its poles. Let P_λ^+ be the residue of $\mathcal{R}_D(z)$ at 0. If in addition (A3) holds and $\lambda\neq 0$, then $\dim\mathrm{Ran}P_\lambda^+=1$ and all singularities of $\mathcal{R}_D(z)$ except zero are contained in the half-plane $\mathrm{Im}z<0$. Moreover, P_λ^+ is analytic function of λ for $|\lambda|<\Lambda$.*

The proof of this theorem is a slight elaboration of the arguments in [JP1, JP2] which we have already sketched above. We give below an outline of the proof.

Sketch of the proof of Theorem 4.2. We use the notation introduced in the proof of Theorem 4.1. For real θ let

$$L_\lambda^*(\theta)\equiv U(\theta)L_\lambda^*U(-\theta),$$

$$\tilde{V}_{\rm tot}(\theta)\equiv U(\theta)VU(-\theta)-U(\theta)JV_{{\rm i}/2}JU(-\theta).$$

Clearly,

$$L_\lambda^*(\theta)=\mathcal{L}+\theta N+\lambda\tilde{V}_{\rm tot}(\theta).$$

Assumption (A2) implies that $I(\kappa) \ni \theta \mapsto \tilde{V}_{\mathrm{tot}}(\theta)$ is an analytic operator-valued function satisfying

$$\tilde{C} := \sup_{\theta \in I(\kappa)} \|\tilde{V}_{\mathrm{tot}}(\theta)\| \leq \frac{1}{\sqrt{2}} \sum_i C(\kappa, \beta_i) \left(3\|\tilde{\alpha}_i\|_{H^2(\kappa)} + \|\mathrm{e}^{-\beta_i s/2} \tilde{\alpha}_i\|_{H^2(\kappa)} \right). \quad (4.36)$$

The function $\mathbf{C} \times I^-(\kappa) \ni (\lambda, \theta) \mapsto L_\lambda^*(\theta)$, with values in the closed operators on $\mathcal{H}$, is an analytic family of type A in each variable separately.

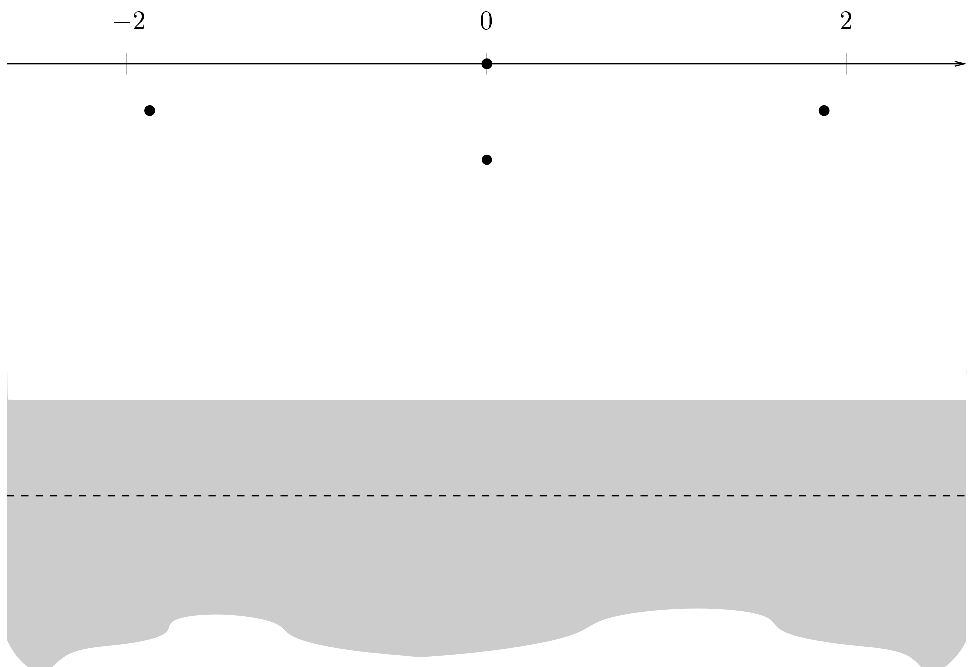

Figure 2: Resonances of L_λ^*

One now repeats the analysis outlined in the proof of Theorem 4.1. For $\Lambda\tilde{C} < (\kappa - |\mu|)/4$ and $|\lambda| < \Lambda$ the essential spectrum of $L_\lambda^*(\theta)$ is contained in the half-plane $\{z \,|\, \mathrm{Im} z < \mu\}$. Here, again, the location of the discrete spectrum inside $\mathfrak{P}(\mu)$ can be computed using regular perturbation theory. This discrete spectrum consists of four points $\{\tilde{e}_{\pm 2}(\lambda), \tilde{e}_0^{1,2}(\lambda)\}$, where $\tilde{e}_{\pm 2}(\lambda)$ are near ± 2 and $\tilde{e}_0^{1,2}(\lambda)$ are near 0, see Figure 2 below. Since

$$(L_\lambda^*(\theta))^* \Omega = 0,$$

we have $\tilde{e}_0^1(\lambda) = 0$. Moreover, the functions $\lambda \mapsto \tilde{e}_{\pm 2}(\lambda)$ are analytic for $|\lambda| < \Lambda$,

$$\tilde{e}_{\pm 2}(\lambda) = \pm 2 + \sum_{j=1}^{\infty} \lambda^{2j} \tilde{a}_{2j}^{\pm},$$

and one finds that

$$\tilde{a}_2^{\pm} = a_2^{\pm}.$$

The resonances $\tilde{e}_0^{1,2}(\lambda)$ are the eigenvalues of a 2×2 matrix $\tilde{\Sigma}(\lambda)$ which is analytic for $|\lambda| < \Lambda$,

$$\tilde{\Sigma}(\lambda) = \sum_{j=1}^{\infty} \lambda^{2j} \tilde{\Sigma}_{2j},$$

and

$$\tilde{\Sigma}_2 = -\mathrm{i}\pi \sum_i \|\tilde{\alpha}_i(2)\|_{\mathfrak{G}}^2 \; \tilde{T}_i,$$

where

$$\tilde{T}_i = \mathrm{e}^{-\beta_i \sigma_z/2} T_i \mathrm{e}^{\beta \sigma_z/2} = \frac{1}{2\cosh \beta_i} \begin{pmatrix} \mathrm{e}^{\beta_i} & -\mathrm{e}^{-\beta_i} \\ -\mathrm{e}^{\beta_i} & \mathrm{e}^{-\beta_i} \end{pmatrix}.$$

Notice that

$$\tilde{T}_i^* \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0,$$

and so zero is always an eigenvalue of $\tilde{\Sigma}_2$. The second eigenvalue of $\tilde{\Sigma}_2$ is equal to

$$\mathrm{Tr}(\tilde{\Sigma}_2) = -\mathrm{i}\pi \sum_i \|\tilde{\alpha}_i(2)\|_{\mathfrak{G}}^2.$$

If (A3) holds, then this eigenvalue has negative imaginary part. Thus, for λ non-zero and sufficiently small, $\mathrm{Im}\tilde{e}_0^2(\lambda) < 0$.

Following the argument in the proof of Theorem 4.1, we see that

$$\mathcal{R}_D(z) = D(\theta)(z - L_\lambda^*(\theta))^{-1} D(-\theta),$$

provides the required meromorphic continuation of $\mathcal{R}_D$. By this formula, the residue P_λ^+ is related to the spectral projection $Q_\lambda^{(1)}(\theta)$ corresponding to the zero eigenvalue of $L_\lambda^*(\theta)$ by

$$P_\lambda^+ = D(\theta) Q_\lambda^{(1)}(\theta) D(-\theta). \tag{4.37}$$

This implies that $\dim \mathrm{Ran} P_\lambda^+ = 1$.

To prove the last statement of the theorem we must show that $Q_\lambda^{(1)}(\theta)$ is analytic for $|\lambda| < \Lambda$. We prove this by relating this operator to the spectral projection $\tilde{\Pi}(\lambda)$ corresponding to the zero eigenvalue of the analytic matrix $A(\lambda) = \lambda^{-2}\tilde{\Sigma}(\lambda)$. Notice that since 0 is a simple eigenvalue of $A(0)$, $\tilde{\Pi}(\lambda)$ is analytic for λ small enough.

Let us recall the construction of the operator $\tilde{\Sigma}(\lambda)$ [JP1, HP]. By taking Λ possibly smaller, one can find a contour γ around 0 such that for θ with $\mathrm{Im}\theta$ sufficiently close to $-\kappa$ and for $|\lambda| < \Lambda$, the spectral projection corresponding to the group $\{\tilde{e}_0^1(\lambda), \tilde{e}_0^2(\lambda)\}$ is given by

$$Q_\lambda(\theta) = \frac{1}{2\pi \mathrm{i}} \oint_\gamma (z - L_\lambda^*(\theta))^{-1} \mathrm{d}z. \tag{4.38}$$

$Q_\lambda(\theta)$ is an analytic function of λ and

$$\|Q_\lambda(\theta) - Q_0(\theta)\| < 1.$$

Notice that $Q_0(\theta) = Q_0$ does not depend on θ and is the spectral projection of $\mathcal{L}$ corresponding to double degenerate eigenvalue 0. It follows that the maps

$$Q_0 : \mathrm{Ran}Q_\lambda(\theta) \to \mathrm{Ran}Q_0,$$

$$Q_\lambda(\theta) : \mathrm{Ran}Q_0 \to \mathrm{Ran}Q_\lambda(\theta),$$

are isomorphisms. Setting

$$T(\lambda) \equiv Q_0 Q_\lambda(\theta) Q_0,$$

one easily checks that the operator

$$S_\lambda(\theta) = Q_0 Q_\lambda(\theta) : \mathrm{Ran}Q_\lambda(\theta) \to \mathrm{Ran}Q_0,$$

has inverse

$$S_\lambda(\theta)^{-1} = Q_\lambda(\theta) Q_0 T(\lambda)^{-1}.$$

Using the isomorphism $S_\lambda(\theta)$, we transport the reduced operator $Q_\lambda(\theta)L_\lambda^*(\theta)Q_\lambda(\theta)$ to $\mathrm{Ran}Q_0 = \mathbb{C}^2$. A simple calculation yields:

$$\tilde{\Sigma}(\lambda) \equiv S_\lambda(\theta)Q_\lambda(\theta)L_\lambda^*(\theta)Q_\lambda(\theta)S_\lambda(\theta)^{-1} = M(\lambda)T(\lambda)^{-1}, \tag{4.39}$$

where

$$M(\lambda) \equiv Q_0 Q_\lambda(\theta) L_\lambda^*(\theta) Q_\lambda(\theta) Q_0.$$

The operators $T(\lambda)$ and $M(\lambda)$ are independent of θ as long as $|\lambda| < \Lambda$ and $\mathrm{Im}\theta$ is sufficiently close to $-\kappa$. Moreover, they are analytic functions of λ.

Formula (4.39) yields that

$$\tilde{\Pi}(\lambda) = S_\lambda(\theta)Q_\lambda^{(1)}(\theta)S_\lambda(\theta)^{-1}. \tag{4.40}$$

Inverting this formula we derive that $Q_\lambda^{(1)}(\theta)$ (and hence P_λ^+) is an analytic function for λ small enough.□

Theorem 4.3 *Assume that* (A3) *holds. Then there is* $\Lambda > 0$ *such that for* $0 < |\lambda| < \Lambda$ *all the assumptions of Theorem 2.6 hold.*

Proof. Choose $0 > \mu > -\kappa$ and Λ so that Theorem 4.2 holds. This theorem verifies assumptions (DL6), (b) and (c) of Theorem 2.6. To verify (d) it suffices to show that for some $r > 0$ large enough, all $\Psi \in \mathcal{H}$ and $j = 0, 1$

$$\sup_{y>\mu} \int_{|x|>r} |\partial_x^j(\Psi, \mathcal{R}_D(x+\mathrm{i}y)\Psi)|^{2-j} \mathrm{d}x < \infty.$$

Since

$$\mathcal{R}_D(z) = D(\theta)(z - L_\lambda^*(\theta))^{-1}D(-\theta),$$

it suffices to show that for $\mathrm{Im}\theta$ close enough to $-\kappa$, λ small enough, all $\Psi \in \mathcal{H}$ and $j = 0, 1$,

$$\sup_{y>\mu} \int_{|x|>r} |(\Psi, (x + \mathrm{i}y - L_\lambda^*(\theta))^{-1-j}\Psi)|^{2-j} \mathrm{d}x < \infty. \tag{4.41}$$

Note that $L_0^*(\theta) = \mathcal{L} + \theta N$ is a normal operator, and that the bounds

$$\begin{gathered} \sup_{y>\mu} \int_{|x|>r} \|(x + \mathrm{i}y - L_0^*(\theta))^{-1}\Psi\|^2 \mathrm{d}x < \infty, \\ \sup_{y>\mu, |x|>r} \|(x + \mathrm{i}y - L_0^*(\theta))^{-1}\| < \infty, \end{gathered} \tag{4.42}$$

follow from the spectral theorem. The second relation in (4.42) and the resolvent identity yield that for λ small enough,

$$\begin{aligned} (x + \mathrm{i}y - L_\lambda^*(\theta))^{-1} &= G(x + \mathrm{i}y - L_0^*(\theta))^{-1} \\ &= (x + \mathrm{i}y - L_0^*(\theta))^{-1}\tilde{G}, \end{aligned} \tag{4.43}$$

where the operators G and $\tilde{G}$ (which depend on θ, λ, x, y) have uniformly bounded norms. The first relations in (4.42) and (4.43) yield (4.41) for $j = 0$. The case $j = 1$ follows from the estimate

$$|(\Psi, (x + \mathrm{i}y - L_\lambda^*(\theta))^{-2}\Psi)| \leq \|G\|\|\tilde{G}\|\|(x + \mathrm{i}y - L_0^*(\theta))^{-1}\Psi\|^2.$$

It remains to verify (DL2) and (DL4). Let

$$\mathfrak{h}_{\mathrm{test}} = \left\{ f \in \mathfrak{h} \mid \tilde{f} \in \mathcal{D}(\mathrm{e}^{\kappa\langle p \rangle}) \right\}, \tag{4.44}$$

and let $\mathcal{O}_{\mathrm{f,test}}$ be the vector subspace of $\mathcal{O}_{\mathrm{f}}$ generated by $\mathbf{1}$ and

$$\left\{ a^\#(f_1) \cdots a^\#(f_n) \mid n \in \mathbb{N}, f_i \in \mathfrak{h}_{\mathrm{test}} \right\}.$$

Set

$$\mathcal{O}_{\mathrm{test}} = \mathcal{O}_{\mathrm{s}} \otimes \mathcal{O}_{\mathrm{f,test}}^{(1)} \otimes \mathcal{O}_{\mathrm{f,test}}^{(2)}.$$

Note that $\mathcal{O}_{\mathrm{test}}$ is a $*$-subalgebra of $\mathcal{O}$. Obviously, $\mathcal{O}_{\mathrm{test}} \subset \mathcal{O}_D$. Since the set $\mathfrak{h}_{\mathrm{test}}$ is dense in $\mathfrak{h}$, $\mathcal{O}_{\mathrm{test}}^{\mathrm{cl}} = \mathcal{O}$ and (DL2) follows.

To establish (DL4), note that $J\pi(\mathcal{O}_{\mathrm{test}})J \subset \mathfrak{M}_D'$. Since $\mathcal{O}_{\mathrm{test}}^{\mathrm{cl}} = \mathcal{O}$ and $\pi(\mathcal{O}_{\mathrm{test}})'' = \mathfrak{M}$, $\pi(\mathcal{O}_{\mathrm{test}})\Omega$ is dense in $\mathcal{H}$. Thus, $\mathfrak{M}_D'\Omega$ is also dense in $\mathcal{H}$.

Following the above argument one can also easily verify Hypothesis (DL3) in our model. We will not make use of this hypothesis below. □

We are now ready to finish:

Proof of Theorem 1.2. Parts (i) and (ii) follow from Theorems 2.6 and 4.3 with $\mathcal{N}_0 = \mathcal{N}_D$ and $\mathcal{O}_0 = \mathcal{O}_D$. From the construction of $\mathcal{N}_D$ and $\mathcal{O}_D$ it is immediate that $\mathcal{N}_{\mathrm{s}} \subset \mathcal{N}_D$ and $\mathcal{O}_{\mathrm{s}} \subset \mathcal{O}_D$.

Since for $A \in \mathcal{O}_D$,

$$\omega_\lambda^+(A) = (D^{-1}A^*\Omega, \Omega_\lambda^+) = (D^{-1}A^*\Omega, P_\lambda^+\Omega),$$

Part (iii) follows from the last statement of Theorem 4.2. □

As we have pointed in the Remark 3 after Theorem 1.2, Part (iii) of Theorem 1.2 yields that for $A \in \mathcal{O}_D$ we have an expansion

$$\omega_\lambda^+(A) = \sum_{k=0}^{\infty} \lambda^k \omega_k^+(A). \tag{4.45}$$

It is an important question whether the functionals ω_k^+ can be (at least in principle) computed. If

$$\Omega_\lambda^+ = \sum_{k=0}^{\infty} \lambda^k \Omega_k^+,$$

then $\omega_k^+(A) = (D^{-1}A^*, \Omega_k^+)$, so ω_k^+ is determined Ω_k^+ ((DL3) implies that the opposite is also true). To compute the expansion of Ω_λ^+, one uses that $P_\lambda^+\Omega = \Omega_\lambda^+$ and the identity (4.37). First, using (4.38), one expands $Q_\lambda(\theta)$ in powers of λ. Using this result, one expands $T(\lambda)$, $S_\lambda(\theta)$, $M(\lambda)$ and $\tilde{\Sigma}(\lambda)$. The expansion of $\tilde{\Sigma}(\lambda)$ and regular perturbation theory yield the expansion of $\tilde{\Pi}(\lambda)$. The formulas (4.40) and (4.37) then yield the expansions of $Q_\lambda^{(1)}(\theta)$ and $P_\lambda^+(\theta)$. Although clearly the resulting formulas are complicated, at least in principle it is possible to compute any term in the expansion (4.45). In particular, the first term ω_0 is determined by the vector

$$\Omega_0^+ = P_0^+\Omega = (\tilde{\Pi}(0)\Omega_{\mathrm{s}}) \otimes \Omega_{\mathrm{f}}^{(1)} \otimes \Omega_{\mathrm{f}}^{(2)}.$$

5 Entropy production

Proof of Theorem 1.1. We assume that the reader is familiar with basic properties of relative entropy (a particularly clear review is given in [Don]). Let $\mathfrak{M} = \pi_\omega(\mathcal{O})''$ and let $\mathfrak{M}_*$ be the predual of $\mathfrak{M}$.

Assume that (a) and (b) hold, and that $\mathrm{Ep}(\omega_V^+) = \omega_V^+(\sigma_V) = 0$. Then, by the formula (1.2) and (b),

$$\mathrm{Ent}(\omega \circ \tau_V^t \,|\, \omega) = -\int_0^t \left(\omega(\tau_V^s(\sigma_V)) - \omega_V^+(\sigma_V)\right) \mathrm{d}s \geq -C,$$

for all $t > 0$ and some $C > 0$. Set

$$\omega_T \equiv \frac{1}{T}\int_0^T \omega \circ \tau_V^t \mathrm{d}t.$$

The convexity and the upper semicontinuity of the relative entropy yield that

$$\mathrm{Ent}(\omega_T \,|\, \omega) \geq \frac{1}{T}\int_0^T \mathrm{Ent}(\omega \circ \tau_V^t \,|\, \omega)\mathrm{d}t \geq -C.$$

Since the set of all states $\eta \in \mathcal{N}_\omega$ such that $\mathrm{Ent}(\eta \,|\, \omega) \geq -C$ is $\sigma(\mathfrak{M}_*, \mathfrak{M})$-compact, the set of weak-* limit points of $\{\omega_T \,|\, T > 0\}$ is contained in $\mathcal{N}_\omega$. It follows that $\omega_V^+ \in \mathcal{N}_\omega$, and this contradicts (a). □

Proof of Theorem 1.4. Theorem 1.3 yields that the assumption (a) of Theorem 1.1 holds. Let us verify (b) for the initial state $\omega = \omega_{\mathrm{s}} \otimes \omega_{\beta_1} \otimes \omega_{\beta_2}$, where ω_{s} is given by (3.32). By Takesaki's theorem [BR1],

$$\delta_\omega = -\beta_1 \delta_1 - \beta_2 \delta_2, \tag{5.46}$$

and

$$\delta_\omega(V) = -\beta_1 \sigma_x \otimes \varphi(\mathrm{i}s\alpha_1) \otimes \mathbf{1} - \beta_2 \sigma_x \otimes \mathbf{1} \otimes \varphi(\mathrm{i}s\alpha_2).$$

Since $\mathrm{i}s\alpha_i \in \mathfrak{h}_{\mathrm{test}}$, ($\mathfrak{h}_{\mathrm{test}}$ is given by (4.44)), $\delta_\omega(V) \in \mathcal{O}_0$. Hence, by Part (ii) of Theorem 1.2, the assumption (b) of Theorem 1.1 holds, and $\mathrm{Ep}(\omega_\lambda^+) > 0$.

It remains to show that the entropy production does not depend on the choice of the initial state in $\widehat{\mathcal{N}}_{\mathrm{s}}$. Let $\eta = \eta_{\mathrm{s}} \otimes \omega_{\beta_1} \otimes \omega_{\beta_2} \in \widehat{\mathcal{N}}_{\mathrm{s}}$. Then, by Theorem 1.1 in [JP3],

$$\mathrm{Ent}(\omega \circ \tau_V^t \,|\, \eta) = \mathrm{Ent}(\omega \,|\, \eta) - \int_0^t \eta(\tau_V^s(\delta_\eta(\lambda V))\mathrm{d}s.$$

By the proof of Proposition 1.3 in [JP3],

$$\mathrm{Ent}(\omega \circ \tau_V^t \,|\, \omega) = \mathrm{Ent}(\omega \circ \tau_V^t \,|\, \eta) + O(1),$$

uniformly for $t > 0$. This implies that

$$\omega_\lambda^+(\delta_\omega(\lambda V)) = \omega_\lambda^+(\delta_\eta(\lambda V)). \tag{5.47}$$

□

Relation (5.47) has one important consequence. Let ω and η be as in the above proof and

$$\eta_{\mathrm{s}}(A) = \mathrm{Tr}(A\mathrm{e}^{H_{\mathrm{s}}})/\mathrm{Tr}(\mathrm{e}^{H_{\mathrm{s}}}).$$

Then,

$$\delta_\eta(\,\cdot\,) = \mathrm{i}[H_{\mathrm{s}}, \,\cdot\,] + \delta_\omega(\,\cdot\,),$$

and (5.47) yield that

$$\omega_\lambda^+([H_{\mathrm{s}}, V]) = 0. \tag{5.48}$$

Proof of Theorem 1.5. The second relation in (1.11) follows from the definition of entropy production and Relation (5.46). To prove the first, note that

$$\delta(\,\cdot\,) = \mathrm{i}[H_{\mathrm{s}}, \,\cdot\,] + \delta_1(\,\cdot\,) + \delta_2(\,\cdot\,),$$

and

$$\delta_\lambda(\,\cdot\,) = \delta(\,\cdot\,) + \mathrm{i}\lambda[V,\,\cdot\,],$$

are the generators of the free and the perturbed dynamics. Since ω_λ^+ is τ_λ-invariant and $V \in \mathcal{D}(\delta) = \mathcal{D}(\delta_\lambda)$,

$$\begin{aligned} 0 &= \omega_\lambda^+(\delta_\lambda(\lambda V)) = \omega_\lambda^+(\delta(\lambda V)) \\ &= \mathrm{i}\lambda\omega_\lambda^+([H_{\mathrm{s}}, V]) + \omega_\lambda^+(\Phi_1) + \omega_\lambda^+(\Phi_2) \\ &= \omega_\lambda^+(\Phi_1) + \omega_\lambda^+(\Phi_2), \end{aligned}$$

where we used (5.48). □

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