Theory of Non-Equilibrium Stationary States as a Theory of Resonances. Existence and Properties of NESS

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Abstract

We study a small quantum system (e.g. a simplified model for an atom or molecule) interacting with two bosonic or fermionic reservoirs (say, photon or phonon fields) at different temperatures T_1 and T_2. We show that if T_1 is not equal to T_2 then the combined system has a stationary, non-equilibrium state (NESS). We show that this state has nonvanishing heat fluxes and positive entropy production and that it is dynamically asymptotically stable. The latter means that the evolution with an initial condition, normal with respect to any state where the reservoirs are in equilibria at temperatures T_1 and T_2, converges to this NESS. Our results are valid for the temperatures satisfying the bound $\min(T_1, T_2) > g^{2+\alpha}$, where g is the coupling constant and $0 < \alpha < 1$ is a power related to the infra-red behaviour of the coupling functions. This restriction is introduced in order to present the setup and techniques without extra hard and lengthy technical estimates. In a subsequent work we combine the present setup and techniques with the spectral renormalization group method to obtain the results as above for all temperatures.

1 Introduction

The present paper is a contribution to rigorous quantum statistical mechanics. Key problems here are dynamical stability of equilibrium states, and characterization (if not the definition) and stability of non-equilibrium stationary states (NESS).

While our understanding of the quantum equilibrium states, the subject of equilibrium statistical mechanics (see [9, 17, 16, 29]), and the recent progress in proving their dynamical stability ([18, 8, 11, 10, 22, 14, 23]) are fairly satisfactory, results on non-equilibrium stationary states are just beginning to emerge. The problem is that we do not have a simple stationary characterization of NESS similar to the principle of maximum of entropy or the KMS characterization for equilibrium states. Thus it is remarkable that certain characterizations of NESS and their stability were recently shown for (idealized) particle systems coupled to Fermi reservoirs at high temperatures, $\min(T_1, T_2) >$

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$C\max(g, \log\frac{1}{g})$, in [21], for XY-chains [4] and for coupled Fermi reservoirs in [12, 15, 2, 1]. Here, $T_{1,2}$ are the temperatures of the reservoirs, and g is the coupling constant.

In this paper we establish a spectral characterization of the NESS and prove their dynamical stability for (idealized) particle systems coupled to two Bose reservoirs (e.g. photons or phonons) for reservoir temperatures satisfying $\min(T_1, T_2) > Cg^{2+\alpha}$, where C is a constant and $0 < \alpha < 1$ ($\alpha = \frac{\mu-1/2}{\mu+1/2}$, where $\mu > 1/2$ is given in Condition (A) below). Our approach applies to an arbitrary finite number of bosonic or fermionic reservoirs; in the latter case, it gives an extension of the results of [21] to the temperature range mentioned above. Moreover, we develop a perturbation theory for NESS and use it to prove that the entropy production is strictly positive.

An iteration of our estimates in the spirit of the spectral renormalization group of [5, 6, 7, 8] gives the above results for *all temperatures*. For reasons of space this extension will be presented in the subsequent paper [26].

Similarly to [21], we construct a NESS from a zero (non-degenerate) resonance eigenvector of a certain non-self-adjoint Liouville operator, K, acting on a positive temperature Hilbert space. The operator K is the analogue of a C-Liouvillean in the terminology for C^*-dynamical systems [21].

To show dynamical stability of a NESS we have to establish certain long-time (ergodic) properties of the evolution, $U(t)$, generated by K. The operator K does not belong to a class for which the evolution is a priori known to exist (e.g. classes of normal or accretive operators). To overcome this problem we establish a direct connection between desired ergodic properties of $U(t)$ and certain spectral properties of a complex deformation, K_θ, $\theta \in \mathbb{C}^2$, of K. For technical reasons we can use neither the complex deformations introduced in [18] nor those introduced in [8] but we combine both types, hence θ is in $\mathbb{C}^2$ rather than in $\mathbb{C}$. (Such a combination was already mentioned in [8]). In order to establish the desired spectral characteristics of the operator family K_θ, we use the method of the Feshbach map, including the RG-approach (renormalization group), as developed in [5, 6, 7].

Already a single application of the Feshbach map considered in this paper yields the results mentioned above under the additional restriction that $\min_j(T_j) \gg g^{2+\alpha}$ (but with a relaxed infrared condition). Using results of [5, 7, 8] on the iteration of the Feshbach map (the RG approach), this restriction on the temperatures can be removed. We present in [26] a detailed analysis of the RG to the specific model at hand. It relies on [6, 7, 8] and features some simplifications due to the specificity of our problem and some recent developments [5].

A more detailed outline of our approach and of the organization of the paper is given in Section 2.

2 Model and Approach

We consider a system consisting of a particle system, described by a self-adjoint Hamiltonian H_p on a Hilbert space $\mathcal{H}_p$, and two bosonic reservoirs, at inverse temperatures β_1 and β_2, described by the Hamiltonians H_{r1} and H_{r2} acting on Hilbert spaces $\mathcal{H}_{r1}$ and $\mathcal{H}_{r2}$, respectively. The full Hamiltonian is

$$H := H_0 + gv\ , \tag{2.1}$$

acting on the tensor product space $\mathcal{H}_0 := \mathcal{H}_p \otimes \mathcal{H}_{r1} \otimes \mathcal{H}_{r2}$. Here

$$H_0 := H_p \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes H_{r1} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes H_{r2} \tag{2.2}$$

is the unperturbed Hamiltonian, v is an operator on $\mathcal{H}_0$ describing the interaction and $g \in \mathbb{R}$ is a coupling constant.

The operators H_{rj} describe free scalar (or vector, if wished) quantum fields on $\mathcal{H}_{rj}$, the bosonic Fock spaces over the one-particle space $L^2(\mathbb{R}^3, d^3k)$,

$$H_{rj} = \int \omega(k) a_j^*(k) a_j(k)\, d^3k, \tag{2.3}$$

where $a_j^*(k)$ and $a_j(k)$ are creation and annihilation operators on $\mathcal{H}_{rj}$ and $\omega(k) = |k|$ is the dispersion relation for relativistic massless bosons. The interaction operator is given by

$$v = \sum_{j=1}^{2} v_j \quad \text{with} \quad v_j = a_j(G_j) + a_j^*(G_j). \tag{2.4}$$

Its choice is motivated by standard models of particles interacting with the quantized electromagnetic field or with phonons.

In (2.4), $G_j : k \mapsto G_j(k)$ is a map from $\mathbb{R}^3$ into $\mathcal{B}(\mathcal{H}_p)$, the algebra of bounded operators on $\mathcal{H}_p$, and

$$a_j(G_j) := \int G_j(k)^* \otimes a_j(k)\, d^3k \qquad \text{and} \qquad a_j^*(G_j) := a_j(G_j)^*. \tag{2.5}$$

If the coupling operators G_j are such that

$$g^2 \int\limits_{\mathbb{R}^3} \left(1 + |k|^{-1}\right) \|G_j(k)\|^2 \, dk \qquad \text{is sufficiently small,} \tag{2.6}$$

then the operator H is self-adjoint (see e.g. [8]).

Now we set up a mathematical framework for non-equilibrium statistical mechanics. Operators on the Hilbert space $\mathcal{H}_0$ will be called observables. (Only certain self-adjoint operators on $\mathcal{H}_0$ are actually *physical* observables.) As an algebra of observables describing the system we take the C^*-algebra

$$\mathcal{A} = \mathcal{B}(\mathcal{H}_p) \otimes \mathfrak{W}(L_0^2) \otimes \mathfrak{W}(L_0^2), \tag{2.7}$$

where $\mathfrak{W}(L_0^2)$ denotes the Weyl CCR algebra over $L_0^2 := L^2(\mathbb{R}^3, (1+|k|^{-1})d^3k)$. States of the system are positive linear ('expectation') functionals ψ on the algebra $\mathcal{A}$, normalized as $\psi(\mathbf{1}) = 1$.

The reason we chose $\mathcal{A}$ rather than $\mathcal{B}(\mathcal{H}_0)$ is that the algebra $\mathcal{A}$ supports states in which each reservoir is at a thermal equilibrium at its own temperature. More precisely, consider the evolution for the j-th reservoir given by

$$\alpha_{rj}^t(A) := e^{iH_{rj}t} A e^{-iH_{rj}t}. \tag{2.8}$$

Then there are stationary states on the j-th reservoir algebra of observables, $\mathfrak{W}(L_0^2)$, which describe (single-phase) thermal equilibria. These states are parametrized by the inverse temperature $\beta_j = 1/T_j$ and their generating functional is given by

$$\omega_{rj}^{(\beta_j)}(W_j(f)) = \exp\left\{-\frac{1}{4}\int_{\mathbb{R}^3} \frac{e^{\beta_j|k|}+1}{e^{\beta_j|k|}-1}|f(k)|^2 d^3k\right\}, \tag{2.9}$$

where $W_j(f) := e^{i\phi_j(f)}$, with $\phi_j(f) := \frac{1}{2}\left(a_j^*(f) + a_j(f)\right)$, is the Weyl operator, see e.g. [9]. The choice of the space L_0^2 above is dictated by the need to have the r.h.s. of this functional finite. These states are characterized by the KMS condition and are called the (α_{rj}^t, β_j)-KMS states.

Remark. It is convenient to define states ψ on products $a^\#(f_1)\dots a^\#(f_n)$ of the creation and annihilation operators, where $a^\#$ denotes either a or a^*. This is done using derivatives ∂_{s_k} of its values on the Weyl operators $W(s_1 f_1)\dots W(s_n f_n)$ (see [9], Section 5.2.3 and (2.15)).

Consider states (on $\mathcal{A}$) of the form

$$\omega_0 := \omega_p \otimes \omega_{r1}^{(\beta_1)} \otimes \omega_{r2}^{(\beta_2)}, \tag{2.10}$$

where ω_p is a state of the particle system and $\omega_{ri}^{(\beta)}$ is the (α_{ri}^t, β_i)-KMS state of the i-th reservoir. The set of states which are normal w.r.t. ω_0 is the same for any choice of ω_p. A state normal w.r.t. ω_0 will be called a $\beta_1\beta_2$-normal state.

In the particular case $\omega_p(\cdot) = \mathrm{Tr}(e^{-\beta_p H_p}\,\cdot)/\mathrm{Tr}(e^{-\beta_p H_p})$ we call ω_0 a *reference state.*

The Hamiltonian H generates the dynamics of observables $A \in \mathcal{B}(\mathcal{H}_0)$ according to the rule

$$A \mapsto \alpha^t(A) := e^{iHt} A e^{-iHt} . \tag{2.11}$$

Eqn (2.11) defines a group of *-automorphisms of $\mathcal{B}(\mathcal{H}_0)$. However, α^t does not map the subalgebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H}_0)$ into itself, so (2.11) does not define a dynamics on $\mathcal{A}$. To circumvent this problem we define the interacting evolution of a class of *states on* $\mathcal{A}$ by using the Araki-Dyson expansion. Namely, we define the evolution of a state ψ on $\mathcal{A}$ which is normal w.r.t. ω_0 by

$$\psi^t(A) := \lim_{n\to\infty} \sum_{m=0}^{\infty} (ig)^m \int_0^t dt_1 \cdots \int_0^{t_{m-1}} dt_m \; \psi_n^{t,t_1,\ldots,t_m}(A), \tag{2.12}$$

where the term with $m = 0$ is $\psi(\alpha_0^t(A))$, and, for $m \geq 1$,

$$\psi_n^{t,t_1,\ldots,t_m}(A) := \psi\left([\alpha_0^{t_m}(v_n). \cdots [\alpha_0^{t_1}(v_n), \alpha_0^t(A)] \cdots]\right).$$

Here, $v_n \in \mathcal{A}$ is an approximating sequence for the operator v, satisfying the relation

$$\lim_{n\to\infty} \omega_0(A^*(v_n^* - v^*)(v_n - v)A) = 0, \tag{2.13}$$

for all $A = W(f) \in \mathcal{A}$ with $f \in L_0^2$. Such a sequence is constructed as follows. Let $\{e_m\}$ be an orthonormal basis in L_0^2. We define the approximate creation operators

$$a_{j,n}^*(G_j) = \sum_{m=1}^{\mu} \langle e_m, G_j \rangle b_{j,\lambda}^*(e_m), \tag{2.14}$$

where $n = (\lambda, \mu)$, and, for any $f \in L^2(\mathbb{R}^3)$ and $\lambda > 0$,

$$b_{j,\lambda}^*(f) := \frac{\lambda}{\sqrt{2}i} \left\{ W_j(f/\lambda) - \mathbf{1} - iW_j(if/\lambda) + i\mathbf{1} \right\}. \tag{2.15}$$

Similarly we define the approximate annihilation operators $a_{j,n}(G_j)$. Via the above construction we obtain the family of interactions v_n which belongs to $\mathcal{A}$ and, as can be easily shown using (2.9), satisfies (2.13).

We show in Appendix A that under condition (2.13) the integrands on the r.h.s. of (2.12) are continuous functions in $t_1, \ldots, t_m$, that the series is absolutely convergent and that the limit exists and is independent of the approximating sequence v_n.

Our goal is to understand stationary states of the interacting system originating from $\beta_1\beta_2$-normal states either by a perturbation theory or through an ergodic limit of the full evolution α^t. These states are not equilibrium (KMS) states states. They will be called *non-equilibrium stationary states* or NESS for short.

Assuming certain smoothness and smallness conditions on the coupling operators $gG_j(k)$ and assuming that the particle system is effectively coupled to the reservoirs, we show that, starting initially in any $\beta_1\beta_2$-normal state ψ, the system converges, under the evolution α^t, to a state η:

$$\psi^t \longrightarrow \eta \quad \text{as } t \to \infty. \tag{2.16}$$

The convergence (2.16) is understood in the weak$*$ sense on the sub-C^*-algebra of "analytic observables"

$$\mathcal{A}_1 = \mathcal{B}(\mathcal{H}_p) \otimes \mathfrak{W}(D^{\rm anal}) \otimes \mathfrak{W}(D^{\rm anal}). \tag{2.17}$$

Here, $\mathfrak{W}(D^{\rm anal})$ is the Weyl CCR algebra over the dense set $D^{\rm anal} \subset L_0^2$ which we define in Appendix C. Roughly speaking, $D^{\rm anal}$ consists of vectors from the space $\bigcap_{b\geq 0} e^{-b|k|} L_0^2$ which have some analyticity properties in $|k|$ and a certain behaviour at $k = 0$. The density of $D^{\rm anal} \subset L^2_0$ implies that $\mathcal{A}_1$ is strongly dense in $\mathcal{A}$.

As mentioned in the introduction, so far, we do not have a simple characterization of NESS. However, there is a key physical quantity which differentiates between equilibrium and non-equilibrium stationary states – the entropy production. Motivated by the second law of thermodynamics ($dS = \sum dQ_j/T_j$) we introduce the observable of entropy production (rate) as

$$s := \sum_{j=0}^{2} \beta_j \phi_j \ , \tag{2.18}$$

where ϕ_j, $j = 1, 2$, is the heat flow rate (i.e. the energy flow rate due to thermal contact) into the j-th reservoir:

$$\phi_j := \frac{\partial}{\partial t}\Big|_{t=0} \alpha^t(H_{rj}), \tag{2.19}$$

and ϕ_0 is the energy flow into the particle system,

$$\phi_0 := \frac{\partial}{\partial t}\Big|_{t=0} \ \alpha^t(H_p); \tag{2.20}$$

for notational convenience, we write $\beta_0 := \beta_p$.

The entropy production, $EP(\omega)$, in a state ω is defined as (see [34, 30, 31, 32, 33, 19, 20, 21])

$$EP(\omega) = \omega(s) \ . \tag{2.21}$$

Since s is not a bounded operator, we have to use an approximation procedure similar to the one mentioned in the remark after (2.9) in order to define the r.h.s. of (2.21) for sufficiently regular states.

The entropy production $EP(\eta)$ of the NESS is independent of the particle state ω^p entering Definition (2.10) of the state of the decoupled system, since η is independent of ω_p. Notice that $\eta(\phi_0) = \partial_t\big|_{t=0}\eta\big(\alpha^t(H_p)\big)$, since $H_p \in \mathcal{B}(\mathcal{H}_p)$ is a proper observable and η is a continuous and stationary state. Hence

$$\eta(\phi_0) = 0 \ . \tag{2.22}$$

Therefore, writing $\sum_{j=1}^{2} \eta(\phi_j) = \sum_{j=0}^{2} \eta(\phi_j) = \eta(\partial_t|_{t=0}\alpha^t(H_0)) = -\partial_t\big|_{t=0}\eta(\alpha^t(v)) = 0$, we obtain

$$\sum_{j=1}^{2} \eta(\phi_j) = 0. \tag{2.23}$$

Observe that the zero total flow relation (2.23) and Definition 3.1 for the entropy production rate imply that

$$EP(\eta) = (\beta_1 - \beta_2)\eta(\phi_1) \ . \tag{2.24}$$

Thus, the relation $EP(\eta) > 0$ is equivalent to

$$\eta(\phi_1) > 0 \text{ whenever } T_2 > T_1 \ , \tag{2.25}$$

where $T_j = \beta_j^{-1}$ is the temperature of the j-th reservoir. In other words, in the state η the energy flows from the hotter to the colder reservoir.

A general result due to [19] shows that $EP(\omega) \geq 0$ for any NESS ω. We show that for the NESS η,

$$EP(\eta) > 0 \quad \text{iff} \quad \beta_1 \neq \beta_2,$$

see Theorem 3.2 and Section 12 for a precise statement of this result. Moreover, we develop a perturbation theory for the NESS and compute $EP(\eta)$ in leading order in the coupling constant g.

Let us outline the main steps of our proof of the convergence (2.16) (c.f. [21]). We pass to the Araki-Woods GNS representation of $(\mathcal{A}, \omega_0)$, with ω_0 of the form (2.10) and $\omega_p(A) := \mathrm{Tr}(e^{-\beta_p H_p} A)/\mathrm{Tr}(e^{-\beta_p H_p})$;

$$(\mathcal{A}, \omega_0) \to (\mathcal{H}, \pi, \Omega_0),$$

where $\mathcal{H}$, π and Ω_0 are a Hilbert space, a representation of the algebra $\mathcal{A}$ by bounded operators on $\mathcal{H}$, and a cyclic element in $\mathcal{H}$ (meaning that $\overline{\pi(\mathcal{A})\Omega_0} = \mathcal{H}$) s.t.

$$\omega_0(A) = \langle \Omega_0, \pi(A)\Omega_0 \rangle .$$

The GNS representation provides us with a Hilbert space framework which we use to convert the dynamical problem described above into a spectral problem for a certain non-self-adjoint operator K on the Hilbert space $\mathcal{H}$. With the free evolution $\alpha_0^t(A) := e^{itH_0} A e^{-itH_0}$ one associates the unitary one-parameter group $U_0(t) = e^{itL_0}$ on $\mathcal{H}$ s.t.

$$\pi(\alpha_0^t(A)) = U_0(t)\pi(A)U_0(t)^{-1} \tag{2.26}$$

and $U_0(t)\Omega_0 = \Omega_0$. Define the operator $L^{(\ell)} := L_0 + g\pi(v)$ on the dense domain $\mathrm{Dom}(L_0) \cap \mathrm{Dom}(\pi(v))$. Here $\pi(v)$ can be defined either using explicit formulae for π in the Araki-Woods representation given below or by using the approximation, $v_n \in \mathcal{A}$, for the operator v constructed above. By the Glimm-Jaffe-Nelson commutator theorem the operator $L^{(\ell)}$ is essentially self-adjoint; we denote its self-adjoint closure again by the same symbol $L^{(\ell)}$. The operator $L^{(\ell)}$ induces the one-parameter group σ^t on $\pi(\mathcal{A})''$, the weak closure of $\pi(\mathcal{A})$,

$$\sigma^t(B) := e^{itL^{(\ell)}} B e^{-itL^{(\ell)}} \tag{2.27}$$

for any $B \in \pi(\mathcal{A})''$. Let ψ be a state on the algebra $\mathcal{A}$ normal w.r.t. ω_0, i.e.

$$\psi(A) = \mathrm{Tr}(\rho\pi(A)) \tag{2.28}$$

for some positive trace class operator ρ on $\mathcal{H}$ of trace one. It is shown in Appendix A that for ψ as above the limit on the r.h.s. of (2.12) exists and equals

$$\psi^t(A) = \mathrm{Tr}(\rho\sigma^t(\pi(A))). \tag{2.29}$$

In particular, the limit is independent of the choice of the approximating family v_n.

Due to (2.29) the dynamics on normal states, defined in (2.12), gives rise to the dynamics on the Hilbert space $\mathcal{H}$, determined by a one-parameter group $U(t)$, satisfying

$$\mathrm{Tr}(\rho U(t)\pi(A)U(t)^{-1}) = \psi^t(A), \quad \forall A \in \mathcal{A}. \tag{2.30}$$

Due to fact that $\pi(\mathcal{A})''$ has a large commutant (isomorphic to itself, as is known from Tomita-Takesaki theory), relation (2.30) does not define $U(t)$ uniquely; however, if we impose in addition to (2.30) the invariance condition

$$U(t)\Omega = \Omega, \tag{2.31}$$

where Ω is a fixed cyclic and separating vector, then $U(t)$ is uniquely determined. (The vector Ω is called cyclic if $\pi(\mathfrak{A})\Omega$ is dense in $\mathcal{H}$ and separating if $\pi(\mathfrak{A})'\Omega$ is dense in $\mathcal{H}$, the prime denoting the commuant.) If Ω were the vector representing an equilibrium state then $U(t)$ satisfying (2.30) and (2.31) would be a unitary group. In the non-equilibrium case $\beta_1 \neq \beta_2$, one can see that (2.31) cannot be satisfied for a unitary $U(t)$ implementing the dynamics as in (2.30). For technical reasons, we choose $U(t)$ to satisfy (2.31) for a convenient vector Ω, rather than to be unitary.

We will show that $U(t)$ is strongly differentiable on a dense set of vectors and we will calculate explicitly its generator, $K := -i\frac{\partial}{\partial t}U(t)|_{t=0}$. In the non-equilibrium situation $K^* \neq K$ ($U(t)$ is not unitary!) and (2.31) implies that $K\Omega = 0$. The main effort of our analysis is to derive enough spectral information on the operator K to enable us to show (2.16) and to identify the NESS with

$$\eta(A) = \langle \Omega^*, \pi(A)\Omega \rangle \,, \tag{2.32}$$

where Ω^* is a zero resonance of the operator K^*: $K^*\Omega^* = 0$ and $\Omega^* \in \mathcal{D}'_{\text{anal}}$, for an appropriate dense set $\mathcal{D}_{\text{anal}} \subset \mathcal{H}$, and A are such that $\pi(A)\Omega \in \mathcal{D}_{\text{anal}}$.

In order to obtain rather subtle spectral information on the operator K, and to give a precise meaning to expression (2.32), we develop a new type of spectral deformation, $K \mapsto K_\theta$, with a spectral deformation parameter $\theta \in \mathbb{C}^2$, in combination with an (iterative) application of a Feshbach map acting on K_θ.

In conclusion of this outline we present here the GNS triple provided by the Araki-Woods construction, which forms a mathematical framework for our analysis (see [8, 18, 9] for details and [3, 17] for original papers). In the Araki-Woods GNS representation the (positive temperature) Hilbert space is given by

$$\mathcal{H} = \mathcal{H}^p \otimes \mathcal{H}^r, \tag{2.33}$$

where $\mathcal{H}^p = \mathcal{H}_p \otimes \mathcal{H}_p$ and $\mathcal{H}^r = \mathcal{H}^{r1} \otimes \mathcal{H}^{r2}$ with

$$\mathcal{H}^{rj} = \mathcal{H}_{rj} \otimes \mathcal{H}_{rj}. \tag{2.34}$$

We denote by $a^{\#}_{\ell,j}(f)$ (resp., $a^{\#}_{r,j}(f)$) the creation and annihilation operators which act on the left (resp., right) factor of (2.34). They are related to the zero temperature creation and annihilation operators $a^{\#}_j(f)$ by

$$\pi(a_j(f)) = a_{\ell j}(\sqrt{1+\rho_j}\, f) + a^*_{rj}(\sqrt{\rho_j}\, \bar{f}) \tag{2.35}$$

and

$$\pi'(a_j(f)) = a^*_{\ell j}(\sqrt{\rho_j}\, f) + a_{rj}(\sqrt{1+\rho_j}\, \bar{f}) \tag{2.36}$$

where $\rho_j \equiv \rho_j(k) = (e^{\beta_j\omega(k)} - 1)^{-1}$ with $\omega(k) = |k|$. Finally, we denote $\Omega_r := \Omega_{r1} \otimes \Omega_{r2}$, where $\Omega_{rj} := \Omega_{rj,\ell} \otimes \Omega_{rj,r}$ are the vacua in $\mathcal{H}^{rj}$. Thus, Ω_r is the vacuum in $\mathcal{H}^r$.

Definition (2.10) and our choice of ω_p made at the beginning of this section imply that

$$\Omega_0 = \Omega_p \otimes \Omega_r \quad \text{with} \quad \Omega_p \equiv \Omega_p^{(\beta_p)} = \frac{\sum_j e^{-\beta_p E_j/2}\varphi_j \otimes \varphi_j}{[\sum_j e^{-\beta_p E_j}]^{1/2}}, \tag{2.37}$$

where E_j and φ_j are the eigenvalues and normalized eigenvectors of H_p.

The self-adjoint operator L_0 generating the free evolution $U_0(t)$ defined in (2.26) is of the form $L_0 = L_p \otimes \mathbf{1}^r + \mathbf{1}^p \otimes L_r$ with $L_r = \sum_{j=1}^2 L_{rj}$. The operator L_p has the standard form

$$L_p = H_p \otimes \mathbf{1}_p - \mathbf{1}_p \otimes H_p,$$

and

$$L_{rj} = \int \omega(k) \left(a^*_{\ell,j}(k)a_{\ell,j}(k) - a^*_{r,j}(k)a_{r,j}(k) \right) d^3k.$$

The operator K can be written as $K = L_0 + g(V - W)$ with $V = \pi(v)$ and $W = \pi'(w)$ with w a non-self-adjoint operator obtained by a simple transformation of v.

A standard argument shows that the spectrum of the operator L_0 fills the axis $\mathbb{R}$ with the thresholds and eigenvalues located at $\sigma(L_p)$ and with 0 an eigenvalue of multiplicity at least $\dim H_p$ and at most $(\dim H_p)^2$ (depending on the degeneracy of the spectrum of L_p). A priori we do not know anything about the spectrum of the non-self-adjoint operator K besides the fact that it has an eigenvalue 0. For all we know its spectrum might fill in the entire complex plane! Thus understanding the evolution generated by the operator K is a subtle matter.

This paper is organized as follows. In Section 3 we give a precise formulation of our assumptions, state the results and discuss assumptions and results. In Section 4 we present the Hilbert space framework and the evolution $U(t)$ and in Section 5 we describe the generator K. In Section 6 we introduce the complex deformation K_θ of K and establish the connection between the resolvents of K and K_θ. In Section 7 we establish the spectral properties of K_θ which we then use in Section 8 to express the dynamics in terms of an integral over the resolvent of K_θ. In Sections 9 and 10 we prove our first main result, the existence and explicit form of the NESS, and its dynamical stability. In Section 11 we develop a perturbation theory for NESS and in Section 12 we prove the positivity of the entropy production. Finally, in Appendices A–C we collect some technical results.

3 Assumptions and Results

In order to state our assumptions (A) - (C) below, it is practical to define the map $\gamma : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}\times S^2)$,

$$(\gamma f)(u,\sigma) = \sqrt{|u|}\left\{\begin{array}{ll} f(u\sigma), & u\geq 0,\\ -\overline{f}(-u\sigma), & u<0.\end{array}\right. \tag{3.1}$$

Let $j_\theta(u) = e^{\delta\,\mathrm{sgn}(u)}u+\tau$ for $\theta=(\delta,\tau)\in\mathbb{C}^2$ and $u\in\mathbb{R}$ (see (B.2.2)) and define $(\gamma_\theta f)(u,\sigma) = (\gamma f)(j_\theta(u),\sigma)$, for $f\in L^2(\mathbb{R}\times S^2)$, $\theta\in\mathbb{R}^2$.

The maps γ and γ_θ have obvious extensions to operator valued functions.

(A) *Analyticity.* For $j=1,2$ and every fixed $(u,\sigma)\in\mathbb{R}\times S^2$, the maps

$$\theta\mapsto(\gamma_\theta G_j)(u,\sigma) \tag{3.2}$$

from $\mathbb{R}^2$ to the bounded operators on $\mathcal{H}_p$ have analytic continuations to

$$\left\{(\delta,\tau)\in\mathbb{C}^2\,\middle|\,|\mathrm{Im}\,\delta|<\delta_0, |\tau|<\tau_0\right\}, \tag{3.3}$$

for some δ_0, $\tau_0>0$, $\frac{\tau_0}{\cos\delta_0}\leq\frac{2\pi}{\beta}$, where $\beta=\max(\beta_1,\beta_2)$. Moreover,

$$\|G_j\|_{\delta\beta_j,\mu,\theta} := \sum_{\nu=1/2,\mu}\left[\int_{\mathbb{R}\times S^2}\left\|\gamma_\theta\left(\frac{\sqrt{|u|+1}}{|u|^\nu}e^{\delta\beta_j|u|/2}G_j\right)(u,\sigma)\right\|^2 dud\sigma\right]^{1/2}<\infty, \tag{3.4}$$

for some fixed $\mu>1/2$ and where $\delta\beta_j=\beta-\beta_j$.

(B) *Non-Degeneracy of the Particle System.* We have $\dim\mathcal{H}_p=N<\infty$ and the Hamiltonian H_p has non-degenerate spectrum $\{E_n\}_{n=0}^{N-1}$.

(C) *Fermi Golden Rule Condition.* We have, for $j=1,2$,

$$\gamma_0 := \min_{0\leq n<m\leq N-1}\int_{\mathbb{R}^3}\delta(|k|-E_{mn})|G_j(k)_{nm}|^2d^3k>0, \tag{3.5}$$

where $G_j(k)_{mn}:=\langle\varphi_m,G_j(k)\varphi_n\rangle$, the φ_n are normalized eigenvectors of H_p corresponding to the eigenvalues E_n, and δ is the Dirac delta distribution.

(D) Either $\dim\mathcal{H}_p=2$, or, if $\dim\mathcal{H}_p\geq 3$, then the inverse temperatures satisfy

$$|\beta_1-\beta_2|\leq c,\quad\text{or}\quad|\beta_1-\beta_2|\geq C \text{ and } \min_j\beta_j\geq C\ , \tag{3.6}$$

where $0<c,C<\infty$ are constants depending only on the interaction $G_{1,2}$.

Remarks. 1) The map (3.1) has the following origin. In the positive-temperature representation of the CCR (the Araki-Woods representation on a suitable Hilbert space, see Appendix A), the interaction term v_j is represented by $a_j(\widetilde{\gamma}_{\beta_j} G_j) + a_j^*(\widetilde{\gamma}_{\beta_j} G_j)$, where

$$\widetilde{\gamma}_\beta := \sqrt{\frac{u}{1-e^{-\beta u}}}\ \gamma. \tag{3.7}$$

2) A class of interactions satisfying Condition (A) is given by $G_j(k) = g(|k|)G$, where $g(u) = u^\alpha e^{-u^2}$, with $u \geq 0$, $\alpha = n + 1/2$, $n = 0, 1, 2, \ldots$, and $G = G^* \in \mathcal{B}(\mathcal{H}_p)$. A straightforward estimate gives that the norms (3.4) have the bound

$$\|G_j\|_{\delta\beta_j,\mu,\theta} \leq C\left(1 + e^{(\delta\beta_j)^2/4}\right)\|G\|, \tag{3.8}$$

provided $\mu < \alpha + 1$, where the constant C does not depend on the inverse temperatures, nor on θ varying in any compact set (compare this with the bound (4.13) of [21]). The restriction $\alpha = n+1/2$ with $n = 0, 1, 2, \ldots$ comes from the requirement of translation analyticity (the τ–component of θ), which appears also in [21].

3) The condition $\tau_0/\cos\delta_0 < 2\pi/\beta$ after (3.3) guarantees that the square root in (3.7) is analytic in translations $u \mapsto u + \tau$.

4) What we need in our analysis is that the level shift operator Λ_0, the $N \times N$ matrix defined in (7.1), has a spectral gap at zero which is bounded below by a strictly positive constant independent of the temperatures. Condition (D) ensures this property. If one can show the desired property on the gap by other means then Condition (D) can be dropped.

Let

$$\sigma := \{|\lambda - \mu| \mid \lambda, \mu \in \sigma(H_p), \lambda \neq \mu\} \tag{3.9}$$

and define

$$g_0 := C\sigma^{1/2}\sin(\delta_0)\left[(1 + \beta_1^{-1/2} + \beta_2^{-1/2})\max_j \sup_{|\theta|\leq\theta_0} \|G_j\|_{\delta\beta_j,1/2,\theta}\right]^{-1},$$

where C is a constant depending only on $\tan\delta_0$, and $\delta\beta_j = \beta - \beta_j$, $\delta\beta_p = |\beta - \beta_p|$, and set

$$g_1 := \min\left((g_0)^{1/\alpha}, [\min(T_1, T_2)]^{\frac{1}{2+\alpha}}\right), \tag{3.10}$$

where $\alpha = \frac{\mu-1/2}{\mu+1/2}$, and $\mu > 1/2$ is given in (3.4).

The main results of this paper are given in the following theorems, where by a "state" on a subalgebra (which is not necessarily a C^*-subalgebra), we mean a positive normalized linear functional.

Theorem 3.1 *Assume conditions (A) – (D) are obeyed for some $0 < \beta_1, \beta_2 < \infty$, $\mu > 1$, and let $\beta = \max(\beta_1, \beta_2)$.*

If $0 < |g| < g_1$ then there is a stationary state $\eta = \eta_{\beta_1\beta_2}$, defined on a strongly dense subalgebra $\mathcal{A}_1$ of $\mathcal{A}$ (see (2.7)), satisfying

$$\psi^t \to \eta, \quad t \to \infty \tag{3.11}$$

for any $\beta_1\beta_2$-normal initial state ψ. η is continuous in the norm of $\mathcal{A}$. The convergence is in the weak sense (i.e., pointwise for each $A \in \mathcal{A}_1$). For $A \in \mathcal{A}_1$, $\eta(A)$ is analytic in g, and $\eta|_{g=0} = \omega_0$ (the reference state (2.10) where ω_p is the particle Gibbs state at temperature β_p).*

Remark. 5) Our analysis shows that the NESS is actually defined on a bigger (but somewhat less explicit) Banach space of operators $\mathcal{A}_0 \supseteq \mathcal{A}_1$ (see (9.12)), and the convergence to the NESS, (3.11), holds on $\mathcal{A}_0$. On $\mathcal{A}_1$ one can introduce a "deformation norm" $\|\cdot\|_\sim \leq \|\cdot\|$, see (9.10), such that in this norm, the convergence in (3.11) is uniform, $\sup_{A\in\mathcal{A}_1}(\psi^t(A)-\eta(A))/\|A\|_\sim \to 0$. Moreover, on $\mathcal{A}_1$, the convergence is exponentially fast for initial conditions ψ in a dense set (in the topology of bounded linear functionals on $\mathcal{A}$) – this set is the convex hull of vector states with deformation analytic vectors.

Theorem 3.2 *Assume that the conditions of Theorem 3.1 are satisfied and let $\eta_{\beta_1\beta_2}$ and g be as in Theorem 3.1. If $\beta_1 \neq \beta_2$ and if either g and $|\beta_1-\beta_2|$ are sufficiently small (independently of each other) and $G_1 = G_2$, or if $\dim\mathcal{H}_p = 2$, then $EP(\eta_{\beta_1\beta_2}) > 0$.*

Our analysis gives a stronger result than the one presented in Theorem 3.2. Namely, for $\mu > 3/2$, we show that $EP(\eta_{\beta_1\beta_2}) > 0$, provided $o(g^0)O(\delta\beta) \le \eta'$, where $\delta\beta = |\beta_1-\beta_2|$ (see Theorem 12.1). Here, η' depends on the inverse temperatures and the coupling functions and is given by

$$\eta' = 2\pi\sum_{j>i}(\gamma_j e^{\beta_1 E_{ji}} - \gamma_i)\frac{E_{ji}\ g_{ji}(E_{ji})^2}{e^{\beta_1 E_{ji}}-1},$$

where $E_{ji} = E_j - E_i$, $g_{ji}(E)^2 = \int_{\mathbb{R}^3} d^3k|\langle\varphi_i, G_1(k)\varphi_j\rangle|^2\delta(E_{ji}-\omega)$ (see Condition (C)). The numbers $\gamma_j \ge 0$ are the coordinates (in the basis $\{\varphi_j\otimes\varphi_j\}$ of $\mathrm{Null}(L_p)$) of the unique vector ζ^* in the kernel of the adjoint level shift operator Λ_0^*, at the value $\beta_p = 0$ (and normalized as $\sum_j\gamma_j = 1$). (The operator Λ_0 is defined in Section 10.)

By general arguments one can prove that $\eta' \ge 0$ for $\beta_1 > \beta_2$, $\eta' \le 0$ for $\beta_1 < \beta_2$, and $\eta' = 0$ if $\beta_1 = \beta_2$. We also show that $\eta' > 0$ for $\beta_1 > \beta_2$ and $\eta' < 0$ for $\beta_1 < \beta_2$, for all β_1, β_2, except possibly for finitely many values in any compact set, see after (12.38).

The dependence of η' on $\delta\beta$ is determined by the coordinates γ_j. We compute those in the cases when $G_1 = G_2$ and $\dim\mathcal{H}_p = 2$ (see the proof of Theorem 12.1, and equation (11.20), respectively).

Remarks. 6) Using Araki's theory of perturbation of KMS states, one shows that if the temperatures of both reservoirs are equal then the limit state is an equilibrium state and has zero entropy production. Non-existence of equilibrium states for $\beta_1 \neq \beta_2$ has been shown in [25].

7) For a model with fermionic reservoirs, using a sufficiently fast convergence rate in (3.11) (e.g. $O(t^{-\alpha})$ with $\alpha > 1$ suffices) and the fact that η is not a normal state for $\beta_1 \neq \beta_1$, it has been shown by an abstract argument that $EP(\eta_{\beta_1\beta_2}) > 0$, provided $|\beta_1-\beta_2| \ge Cg$ for some $C > 0$ (see [21]). Instead of this indirect derivation we compute $EP(\eta_{\beta_1\beta_2})$ to the leading order in g and derive the results stated in the theorem.

8) The condition $G_1 = G_2$ can be relaxed to $G_1 - G_2$ being small in a suitable sense

9) In this paper we prove Theorems 3.1 and 3.2 under the additional condition that $\min(T_1, T_2) > Cg^{2+\alpha}$, see (3.10), with $\alpha = \frac{\mu-1/2}{\mu+1/2}$, where $\mu > 1/2$ is given in Condition (A). We will remove this restriction in [26].

4 Spectral Theory of NESS

In this section we outline a spectral theory of NESS applicable to Bose and Fermi reservoirs. Our approach follows the one developed for the Fermi reservoirs in [21]. Fix a state, ω_0, of the form (2.10) with $\omega_p(A) := \mathrm{Tr}(e^{-\beta_p H_p}A)/\mathrm{Tr}(e^{-\beta_p H_p})$.

In this and the next section we use the following conditions on the coupling functions and the particle Hamiltonian

$$\int \|G_j(k)\|^2 e^{(\beta-\beta_j)|k|}[1+(\beta_j|k|)^{-1}]dk < \infty, \tag{4.1}$$

where, recall, $\beta = \max(\beta_1, \beta_2)$, and

$$(H_p + i)^{-1} \text{ is a compact operator.} \tag{4.2}$$

(The exponential weight in (4.1) is used in Section 5 and could be omitted in the present section as well.) These conditions are considerably weaker than Conditions (A) and (B). In particular, the θ-analyticity of the coupling functions and the finiteness of the dimension of the particle space are not required.

We pass to the Araki-Woods GNS representation for the unperturbed system:

$$(\mathcal{A}, \omega_0) \to (\mathcal{H}, \pi, \Omega_0)$$

where $\mathcal{H}$, π and Ω_0 are a Hilbert space, a representation of the algebra $\mathcal{A}$ by bounded operators on $\mathcal{H}$, and a cyclic element in $\mathcal{H}$ s.t.

$$\omega_0(A) = \langle \Omega_0, \pi(A)\Omega_0 \rangle\,.$$

There is also an anti-linear representation, π', of the algebra $\mathcal{A}$ in bounded operators on the space $\mathcal{H}$, s.t. π' commutes with π (i.e. $[\pi'(A), \pi(B)] = 0\ \forall A, B \in \mathcal{A}$), $\overline{\pi'(\mathcal{A})\Omega_0} = \mathcal{H}$.

The full dynamics is implemented by a one-parameter group $U(t)$ satisfying

$$U(t)B\Omega = \sigma^t(B)\Omega, \quad \forall B \in \pi(\mathcal{A}), \tag{4.3}$$

where Ω is a cyclic and separating vector for $\pi(\mathcal{A})$ to be specified below and where

$$\sigma^t(B) := e^{itL^{(\ell)}} B e^{-itL^{(\ell)}} \tag{4.4}$$

for any $B \in \pi(\mathcal{A})$ with $L^{(\ell)} := L_0 + g\pi(v)$. Observe that (4.3) implies that

$$U(t)\Omega = \Omega. \tag{4.5}$$

If the state ω corresponding to Ω is stationary (i.e. $\omega^t = \omega$) then $U(t)$ comes out to be unitary. In our situation we expect that there is no ω_0-normal stationary state and $U(t)$ will be a non-unitary group.

We pick the vector Ω as follows. Let $\beta = \max_{j=1,2} \beta_j$. We define

$$\Omega := e^{-\beta L^{(\ell)}/2}\Omega_0 / \| e^{-\beta L^{(\ell)}/2}\Omega_0 \|. \tag{4.6}$$

The facts that Ω is well defined, i.e., that $\Omega_0 \in \mathrm{Dom}(e^{-\beta L^{(\ell)}/2})$, and that Ω is cyclic and separating, are established in Proposition 4.1 at the end of this section.

The family $U(t)$ is not unitary since $\omega := \langle \Omega, \pi(\,\cdot\,)\Omega \rangle$ is not stationary:

$$\begin{aligned} \langle U(t)\pi(A)\Omega, U(t)\pi(B)\Omega \rangle &= \left\langle \Omega, \sigma^t(\pi(A^*B))\Omega \right\rangle = \omega^t(A^*B) \\ &\neq \omega(A^*B) = \langle \pi(A)\Omega, \pi(B)\Omega \rangle\,, \end{aligned} \tag{4.7}$$

for some A, B, t. Let now ψ be an ω_0-normal state corresponding to the vector $Q\Omega \in \mathcal{H}$ (i.e. $\psi(A) = \langle Q\Omega, \pi(A)Q\Omega \rangle$), where $Q \in \pi'(\mathcal{A})$ then

$$\psi^t(A) = \left\langle Q\Omega, \sigma^t(\pi(A))Q\Omega \right\rangle\ . \tag{4.8}$$

Due to Eqns (2.27) and $\sigma_0^t(\pi(A)) := \pi(\alpha_0^t(A))$ we see that the operator Q commutes with $\sigma_n^t(\pi(A))$. Using this together with (4.3) and (4.5) we arrive at

$$\psi^t(A) = \langle Q^*Q\Omega, U(t)\pi(A)\Omega \rangle\ . \tag{4.9}$$

This key formula, due to [21], connects the long time behaviour of $\psi^t(A)$ with spectral properties of $U(t)$ or its generator. We explain what this means.

Assume we can show that, for a certain class of ϕ and Ψ, and as $t \to \infty$,

$$\langle \phi, U(t)\Psi \rangle \to \langle \phi, P\Psi \rangle \tag{4.10}$$

where P is the eigenprojection on the fixed point subspace of $U(t)$ (i.e. $U(t)P = PU(t) = P$), which we assume for a moment to exist. Relations (4.9) and (4.10) imply

$$\psi^t \to \eta, \quad t \to \infty, \tag{4.11}$$

where the state η is defined (on an appropriate set of observables) by

$$\eta(A) := \langle Q^* Q\Omega, P\pi(A)\Omega\rangle . \tag{4.12}$$

We will show below that (4.10) holds for some unbounded projection operator P. To understand the structure of this operator, we proceed as follows.

We will show that $U(t)$ is strongly differentiable on a dense set and we will compute its non-self-adjoint generator, $K := -i\frac{\partial}{\partial t}|_{t=0}U(t)$, which satisfies (see (4.5))

$$K\Omega = 0. \tag{4.13}$$

The operator P is the eigenprojection onto the eigenspace of K associated with the eigenvalue 0 (i.e. $KP = PK = 0$). We show that $\dim P = 1$ and

$$P = |\Omega\rangle\langle\Omega^*| \tag{4.14}$$

for some $\Omega^* \notin \mathcal{H}$, satisfying $K^*\Omega^* = 0$ in a weak sense ($\Omega^* \in \mathcal{D}'_{\text{anal}}$, where $\mathcal{D}_{\text{anal}} = \cup_{\text{Im}\theta>0}\text{Dom}(U_\theta)$ with the family U_θ defined in Section 6). Understanding the nature of the vector Ω^*, which we call the *NESS vector*, is a goal of our analysis.

Substituting (4.14) into (4.12) and using that $\langle Q^*Q\Omega, \Omega\rangle = \|Q\Omega\|^2 = \psi(\mathbf{1}) = 1$, we obtain

$$\eta(A) = \langle\Omega^*, \pi(A)\Omega\rangle . \tag{4.15}$$

Since $\Omega^* \notin \mathcal{H}$ the state η is not normal but it is well defined for a dense set of observables. The question now is what is Ω^*? The answer, provided in subsequent sections, is that Ω^* is a resonance of K^*.

In the following sections we construct a mathematical framework which provides meaningful expressions replacing formal ones, (4.10)–(4.15) and with the help of which we can prove the convergence (4.11).

Remark. Evolution groups and their generators given by conditions of the type of (4.5) (or (4.13)) were introduced in [21], where the group $U(t)$ is specified by the condition $U(t)\Omega_0 = \Omega_0$. where Ω_0 is the unperturbed vector ("vacuum") introduced in (2.37) above. However, an analysis of the operator K (see Sections 6 and 7) defined this way requires, instead of Condition (A), the condition obtained from Condition (A) by replacing the weight $e^{\delta\beta_j|u|/2}$ by $e^{\beta_j|u|/2}$ (and correpondingly, $e^{(\beta-\beta_j)|k|}$ has to be replaced by $e^{\beta_j|k|}$ in (4.1)). This leads to an additional restriction on the temperatures of the form

$$\min_j T_j \geq c\ln(1/g). \tag{4.16}$$

Using in [21] the vector Ω instead of Ω_0 would improve this bound to $\min_j T_j \geq cg$.

Now we proceed to the main technical result of this section - the proof of the existence of the vector Ω and establishing its properties mentioned and used above.

Proposition 4.1 *$\Omega_0 \in \text{Dom}(e^{-\beta L^{(\ell)}/2})$ and the vector $\Omega := e^{-\beta L^{(\ell)}/2}\Omega_0/\|e^{-\beta L^{(\ell)}/2}\Omega_0\|$ is cyclic and separating for the von Neumann algebra $\pi(\mathcal{A})''$.*

We begin with with some preliminary technical results. To do manipulations with unbounded operators we use the dense subset, $\mathcal{D}$, of our Hilbert space, $\mathcal{H}$, defined by

$$\mathcal{D} := \pi(\tilde{\mathcal{A}})\Omega_0, \tag{4.17}$$

where $\tilde{\mathcal{A}} = \mathcal{B}(\mathcal{H}_p) \otimes \mathcal{P}_1 \otimes \mathcal{P}_2$. Here $\mathcal{P}_j$ is the polynomial algebra generated by the annihilation and creation operators, $a_{rj}(f)$ and $a^*_{rj}(f)$, of the j-th reservoir acting on $\mathcal{H}_{rj}$ with $f \in C_0^\infty(\mathbb{R}^3)$ satisfying $f(0) = 0$ and $\nabla f(0) = 0$.

Let Δ and J be the modular operator and modular conjugation associated with the pair $\{\pi(\mathcal{A})'',\ \Omega_0\}$ and let κ^t be defined by

$$\kappa^t = \alpha_p^{\beta_p t} \otimes \alpha_{r1}^{\beta_1 t} \otimes \alpha_{r2}^{\beta_2 t}. \tag{4.18}$$

The vector Ω_0 defines a $(\kappa, 1)$-KMS state and one can show that

$$\Delta^{-it} A \Delta^{it} = \pi\left(\kappa^t(A)\right) \quad \forall\ A \in \mathcal{A}. \tag{4.19}$$

Using this one computes that $\Delta = e^{-\tilde{L}}$, where $\tilde{L} = \beta_p L_p + \beta_{r1} L_{r1} + \beta_{r2} L_{r2}$, and, in particular, Δ^{it} commutes with e^{itL_0}.

Observe that

$$e^{\tau L_0}\mathcal{D} = \mathcal{D} \ \text{ and } \ \Delta^\tau \mathcal{D} = \mathcal{D} \quad \forall \tau \in \mathbb{R}. \tag{4.20}$$

In the sequel, an important role is played by the operator $\Gamma := e^{-\beta L^{(\ell)}/2} e^{\beta L_0/2}$ which is defined and closable on $\mathcal{D}$ (see the proof of Lemma 4.2 below). We denote the closure of Γ by the same symbol. The following results, which we prove at the end of this section, are useful.

Lemma 4.2 *Let $\Gamma^\#$ stand for either Γ^* or Γ. The following statements hold:*

$$\pi(\mathcal{A})'\mathcal{D} \subset \mathrm{Dom}(\Gamma^\#), \tag{4.21}$$

$$\Gamma B \Omega_0 = B \Gamma \Omega_0,\ \forall B \in \pi(\mathcal{A})'\ , \tag{4.22}$$

$$J A \Gamma \Omega_0 = \Delta^{1/2} \Gamma^* A^* \Omega_0,\ \forall A \in \pi(\mathcal{A})''\ , \tag{4.23}$$

$$\mathrm{Null}(\Gamma^\#) = \{0\}. \tag{4.24}$$

Remark. $\pi(\mathcal{A})'$ is the von Neumann algebra $\pi'(\mathcal{A})''$, the weak closure of $\pi'(\mathcal{A})$.

Proof of Proposition 4.1. Since $e^{\tau L_0}\Omega_0 = \Omega_0$, the property $\Omega_0 \in \mathrm{Dom}(e^{-\beta L^{(\ell)}/2})$ is equivalent to $\Omega_0 \in \mathrm{Dom}(\Gamma)$, which is proven in Lemma 4.2, relation (4.21). Hence the vector Ω exists.

For the cyclicity it is enough to show that if $B \in \pi(\mathcal{A})'$ and $B\Omega = 0$ then $B = 0$. Let $\Omega' := e^{-\beta L^{(\ell)}/2}\Omega_0$. By (4.23) and (4.24), $B\Omega_0 \in \mathrm{Dom}(\Gamma)$ and $0 = B\Omega' = B\Gamma\Omega_0 = \Gamma B \Omega_0$. Since $\mathrm{Null}(\Gamma) = \{0\}$, this implies $B\Omega_0 = 0$. Since Ω_0 is separating for $\pi(\mathcal{A})'$ we have that $B = 0$.

Now we show that Ω is separating for $\pi(\mathcal{A})''$. Let $A \in \pi(\mathcal{A})''$ be such that $A\Omega = 0$. The relation $\Gamma\Omega_0 = \Omega'$ and equation (4.23) imply that

$$0 = \langle J A \Gamma \Omega_0, \psi \rangle = \left\langle A^*\Omega_0\ , \Gamma \Delta^{1/2} \psi \right\rangle \tag{4.25}$$

for any $\psi \in \mathcal{D}$. Since $\Delta^{1/2}\mathcal{D} = \mathcal{D}$ and $\Gamma\mathcal{D}$ is dense (see at the end of the proof of Lemma 4.2 below) we have that $\Gamma\Delta^{1/2}\mathcal{D}$ is dense and it follows from (4.25) that $A^*\Omega_0 = 0$. Since Ω_0 is separating this implies that $A = 0$ and therefore Ω is separating. ■

Proof of Lemma 4.2. We consider only the operator Γ; the operator Γ^* is treated similarly. We approximate the operator v by UV-cut-off operators $v_n \in \mathcal{A}$ by replacing the coupling functions G_j by the coupling functions $e^{-|k|^2/n} G_j$ in (2.14). Let $L_n^{(\ell)} := L_0 + V_n$, where $V_n := \pi(v_n)$, and let $\Gamma_n := e^{-\beta L_n^{(\ell)}/2} e^{\beta L_0/2}$. Note that the V_n are σ_0-entire. We claim that

$$\mathcal{D} \subset \mathrm{Dom}(\Gamma_n), \tag{4.26}$$

$$\Gamma_n \varphi \ \text{ converges as } n \to \infty,\ \forall \varphi \in \mathcal{D}. \tag{4.27}$$

Indeed, due to the UV-cut-off, the transformation $\sigma_0^{i\tau}(V_n) = e^{-\tau L_0} V_n e^{\tau L_0}$ is well defined. Hence, the Dyson series for $\Gamma_n \varphi$ is well defined:

$$\Gamma_n \varphi = \sum_{m \geq 0} (-g)^m \int_0^{\beta/2} d\tau_1 \cdots \int_0^{\tau_{m-1}} d\tau_m \sigma_0^{i\tau_m}(V_n) \cdots \sigma_0^{i\tau_1}(V_n) \varphi, \tag{4.28}$$

where the integrands of the r.h.s. are in $\mathcal{H}$ and are strongly continuous in $\tau_1, \ldots, \tau_m$ and the integrals are understood as strong limits of Riemann sums.

Since the V_n are bounded, the series above converges. Moreover, it converges uniformly in $n > 0$. This can be seen using a standard argument to estimate the norms of the integrands (to keep notation simple we set $\varphi = \Omega_0$),

$$\| \sigma_0^{i\tau_m}(V_n) \cdots \sigma_0^{i\tau_1}(V_n) \Omega_0 \| = \omega_0(\alpha_0^{-i\tau_1}(v_n) \cdots \alpha_0^{-i\tau_m}(v_n) \alpha_0^{i\tau_m}(v_n) \cdots \alpha_0^{i\tau_1}(v_n))^{1/2}, \tag{4.29}$$

by using Wick's theorem, the expression for the imaginary-time two-point functions and condition (4.1), in the same way as it is done, for instance, in [8], Thm IV.3.

A similar argument shows that $\forall \varphi \in \mathcal{D}$, $\{\Gamma_n \varphi\}$ is a Cauchy sequence as $n \to 0$ and therefore $\Gamma_n \varphi$ converges as $n \to \infty$, which completes the proof of (4.26) and (4.27).

Now we show that $\Omega_0 \in \mathrm{Dom}(\Gamma)$ (and $\Omega_0 \in \mathrm{Dom}(e^{-\beta L^{(\ell)}/2})$). The set $\mathcal{D}$ given in (4.17) is a core for the selfadjoint operators $L_n^{(\ell)}$ and $L^{(\ell)}$. Since $L_n^{(\ell)} \to L^{(\ell)}$ on $\mathcal{D}$, the convergence, in fact, is in the strong resolvent sense (see Theorem VIII.25 of [28]). Take $\varphi \in \mathcal{H}$ s.t. $\varphi = f(L^{(\ell)})\varphi$ for some $f \in C_0^\infty(\mathbb{R})$. Such φ's form a core for the operator $e^{-\beta L^{(\ell)}/2}$. Since $L_n^{(\ell)} \to L^{(\ell)}$ in the strong resolvent sense, we have

$$\left\langle e^{-\beta L^{(\ell)}/2} \varphi, \Omega_0 \right\rangle = \lim_{n \to 0} \left\langle e^{-\beta L_n^{(\ell)}/2} f(L_n^{(\ell)}) \varphi, \Omega_0 \right\rangle = \lim_{n \to 0} \left\langle f(L_n^{(\ell)}) \varphi, \Gamma_n \Omega_0 \right\rangle$$

and therefore, by (4.27), $\left\langle e^{-\beta L^{(\ell)}/2} \varphi, \Omega_0 \right\rangle = \langle \varphi, \chi \rangle$, where $\chi := \lim_{n \to 0} \Gamma_n \Omega_0$. Since φ varies through a core for the operator $e^{-\beta L^{(\ell)}/2}$ we have that $\Omega_0 \in \mathrm{Dom}(e^{-\beta L^{(\ell)}/2})$ (and $e^{-\beta L^{(\ell)}/2} \Omega_0 = \chi$). Since $e^{\tau L_0} \Omega_0 = \Omega_0$, we conclude that $\Omega_0 \in \mathrm{Dom}(\Gamma)$. Similarly one shows that $\varphi \in \mathrm{Dom}(\Gamma)$ for any $\varphi \in \mathcal{D}$, i.e. $\mathcal{D} \subset \mathrm{Dom}(\Gamma)$.

Note that $\mathcal{D} \subset \mathrm{Dom}(\Gamma)$, (4.27), and similar statements for Γ_n^* and Γ^*, imply that

$$\Gamma_n^\# \varphi \to \Gamma^\# \varphi, \text{ as } n \to \infty, \forall \varphi \in \mathcal{D}. \tag{4.30}$$

Since $\sigma_0^{i\tau}(V_n) = \pi(\alpha_0^{i\tau}(v_n))$ we have $\sigma_0^{i\tau_m}(V_n) \cdots \sigma_0^{i\tau_1}(V_n) = \pi(\alpha_0^{i\tau_m}(v_n) \cdots \alpha_0^{i\tau_1}(v_n))$. Truncating the sum on the r.h.s. of (4.28), and approximating the integrals by Riemann sums, one shows that

$$\Gamma_n \varphi = \lim_{k \to \infty} \Gamma_{n,k} \varphi \ \text{ with } \ \Gamma_{n,k} \in \pi(\mathcal{A}), \ \forall \varphi \in \mathcal{D}. \tag{4.31}$$

Equations (4.30) and (4.31) imply that for any $\varphi, \psi \in \mathcal{D}$ and for any $B \in \pi(\mathcal{A})'$

$$\langle B\psi, \Gamma^* \varphi \rangle = \lim_{n,k \to \infty} \langle B\psi, \Gamma_{n,k}^* \varphi \rangle = \lim_{n,k \to \infty} \langle B \Gamma_{n,k} \psi, \varphi \rangle = \langle B \Gamma \psi, \varphi \rangle .$$

Hence by an elementary abstract argument $B\psi \in \mathrm{Dom}(\Gamma)$ and $B\Gamma\psi = \Gamma B \psi$, $\forall\ B \in \pi(\mathcal{A})'$. This completes the proof of (4.21) and shows (4.22) as well.

To prove Eqn (4.23) we use Eqns (4.30) and (4.31) and the defining property of the operator $S = \Delta^{-1/2} J$ to obtain $\forall A \in \pi(\mathcal{A})''$ and $\forall \psi \in \mathcal{D} = \Delta^{-1/2} \mathcal{D}$

$$\langle JA\Gamma\Omega_0, \psi \rangle = \lim_{n,k \to \infty} \langle JA\Gamma_{n,k} \Omega_0, \psi \rangle = \lim_{n,k \to \infty} \left\langle \Gamma_{n,k}^* A^* \Omega_0, \Delta^{1/2} \psi \right\rangle = \left\langle A^* \Omega_0, \Gamma \Delta^{1/2} \psi \right\rangle .$$

Finally, to prove (4.24), it suffices to show that $\Gamma^{\#}\mathcal{D}$ is dense. Recall that $e^{\tau L_0}\mathcal{D} = \mathcal{D}$, $\forall\tau \in \mathbb{R}$. Let ψ be such that $0 = \langle \psi, \Gamma e^{-\beta L_0/2}\varphi\rangle = \left\langle \psi, e^{-\beta L^{(\ell)}/2}\varphi\right\rangle$, $\forall\varphi \in \mathcal{D}$. Since $\mathcal{D}$ is dense, this implies that $\psi \in \mathrm{Dom}(e^{-\beta L^{(\ell)}/2})$ which, in turn, yields that $\psi = 0$. Hence $\Gamma\mathcal{D}$ is dense. Similarly, one shows that $\Gamma^*\mathcal{D}$ is dense as well. ■

Remark. Another way to demonstrate the properties of the operators $\Gamma^{\#}$ listed in Lemma 4.2 is to use the analyticity of the operator family $\Gamma(z) := e^{-zL^{(\ell)}}e^{zL_0}$, for $|\mathrm{Re} z| < \beta$. The advantage of this approach is that $\Gamma(it)$, $t \in \mathbb{R}$, are bounded operators and many of the relations above can be easily verified by using e.g. the Kato-Trotter formula.

5 Generator K and interpolating family $K_{(s)}$

In this section we find an explicit form and some properties of the generator K of the one-parameter group $U(t)$ introduced in the preceding section (cf. [21]), and of the family $K_{(s)}$ which interpolates K to a selfadjoint operator.

Let ω_0 be the state of the algebra $\mathcal{A}$ fixed at the beginning of the Section 4 and let J and Δ be the Tomita-Takesaki modular conjugation and modular operator associated with the couple $(\mathcal{A}, \omega_0)$. We have the following standard relations:

$$J\pi(A)J = \pi'(A), \tag{5.1}$$

$J\Delta^{1/2}\pi(A)\Omega_0 = \pi(A^*)\Omega_0$, $\Delta^{1/2}\Omega_0 = \Omega_0$ and $\Delta^{-it}\pi(A)\Delta^{it} = \pi(\kappa^t(A))$, where κ is the automorphism of the algebra $\mathcal{A}$ defined in (4.18). The last three equations imply

$$J\pi(A)\Omega_0 = \pi(\kappa^{i/2}(A^*))\Omega_0. \tag{5.2}$$

Finally, we recall that $\beta = \max(\beta_1, \beta_2)$ and that $L^{(\ell)}$ is the self-adjoint operator defined as $L^{(\ell)} := L_0 + gV$ where $V := \pi(v)$.

Theorem 5.1 *Assume that (4.1) and (4.2) hold. The semigroup $U(t)$, defined in (4.3)–(4.5), is differentiable on the domain $\mathrm{Dom}(L^{(\ell)}) \cap \pi(\mathcal{A})\Omega$, and the generator $K = -i\frac{\partial}{\partial t}\Big|_{t=0} U(t)$ is given on this domain by the expression*

$$K = L_0 + g(V - V'_{-i/2}), \tag{5.3}$$

where $V = \pi(v)$ and $V'_s = \pi'(\gamma^{\overline{s}}(v))$ with $\gamma^s := \alpha_0^{-\beta s} \circ \kappa^s$ (due to condition (4.1) the operator $V'_{-i/2}$ is well defined). Furthermore, the domain $\mathrm{Dom}(L^{(\ell)}) \cap \pi(\mathcal{A})\Omega$ is dense in $\mathcal{H}$.

Remark. The imaginary part of the generator K is not semi-bounded. Therefore, the group $U(t)$, densely defined on $\pi(\mathcal{A})\Omega$, does not extend to a group of bounded operators.

Proof of Theorem 5.1. The argument which was used to prove that $\Omega_0 \in \mathrm{Dom}(e^{-\beta L^{(\ell)}/2})$ (see the proof of Proposition 4.1 and Lemma 4.2) also shows that $L^{(\ell)}\Omega_0 = g\pi(v)\Omega_0 \in \mathrm{Dom}(e^{-\beta L^{(\ell)}/2})$. The latter fact implies that $e^{itL^{(\ell)}/2}\Omega$ is differentiable at $t = 0$ and therefore $\Omega \in \mathrm{Dom}(L^{(\ell)})$. Now, let $B \in \pi(\mathcal{A})$ be such that $B\Omega \in \mathrm{Dom}(L^{(\ell)})$. Taking into account equation (4.4) we see that $\sigma^t(B)$ and therefore, due to (4.3), also $U(t)B\Omega$, is differentiable in t. Differentiating equation (4.3) we find

$$KB\Omega = L^{(\ell)}B\Omega - BL^{(\ell)}\Omega. \tag{5.4}$$

Now we compute the last term on the r.h.s. of this expression. To this end we use the following relations:

$$\pi(A)\Omega_0 = \pi'(\kappa^{i/2}(A^*))\Omega_0 \tag{5.5}$$

and, for $z = it$,

$$e^{zL^{(\ell)}}\pi'(A)e^{-zL^{(\ell)}} = \pi'(\alpha_0^{i\overline{z}}(A)). \tag{5.6}$$

Eqn (5.5) follows from relations (5.1) and (5.2). Equality (5.6) is proven by using the Kato-Trotter product formula.

Now, we claim that $\forall \psi \in \mathcal{D}$

$$\langle \psi, Be^{-\beta L^{(\ell)}/2}\pi(v)\Omega_0\rangle = \langle [\pi'(\alpha_0^{-i\beta/2} \circ \kappa^{i/2}(v))]^* \psi, Be^{-\beta L^{(\ell)}/2}\Omega_0\rangle. \tag{5.7}$$

Indeed, approximate the operator v as in the proof of Lemma 4.2 by UV-cut-off operators v_n. Since both sides of Eqn (5.7) change continuously under this approximation, it suffices to prove (5.7) for v_n replacing v. We prove the more general relation

$$\langle \psi, Be^{zL^{(\ell)}}\pi(v_n)\Omega_0\rangle = \langle [\pi'(\alpha_0^{i\overline{z}} \circ \kappa^{i/2}(v_n))]^* \psi, Be^{zL^{(\ell)}}\Omega_0\rangle \tag{5.8}$$

for $|\mathrm{Re}\, z| < \beta$. Observe that the vectors $e^{zL^{(\ell)}}\varphi$, $\varphi \in \mathcal{D}$, are analytic in z for $|z| < \beta$ as can be shown by using the Araki-Dyson expansion as in the proof of Lemma 4.2, with $\beta = 2$ and with $L^{(\ell)}$ replaced by $zL^{(\ell)}$. Hence both sides of Eqn (5.8) are analytic in z for $|z| < \beta$. Therefore it suffices to prove (5.8) for $z = it$. Eqn (5.8) with $z = it$ follows from the relations (5.5) and (5.6) proven above. Thus (5.7) is demonstrated.

Now, let $\Omega' := e^{-\beta L^{(\ell)}/2}\Omega_0$. Recall that $L^{(\ell)}\Omega' = ge^{-\beta L^{(\ell)}/2}\pi(v)\Omega_0$. Then (5.7) and the definition of the transformation γ^s imply that

$$\begin{aligned} \langle \psi, BL^{(\ell)}\Omega'\rangle &= g\langle \psi, Be^{-\beta L^{(\ell)}/2}\pi(v)\Omega_0\rangle \\ &= g\langle [\pi'(\gamma^{i/2}(v))]^*\psi, B\Omega'\rangle. \end{aligned} \tag{5.9}$$

This relation shows that $B\Omega' \in \mathrm{Dom}(\pi'(\gamma^{i/2}(v))$ and

$$BL^{(\ell)}\Omega = \pi'(\gamma^{i/2}(v))B\Omega. \tag{5.10}$$

Since the r.h.s. of (5.10) is $V'_{-i/2}B\Omega$, this equation together with (5.4) implies (5.3). Finally, $\mathrm{Dom}(L^{(\ell)}) \cap \pi(\mathcal{A})\Omega$ contains the set $\mathcal{D}$ and is therefore dense in $\mathcal{H}$. ■

Observe that

$$\kappa^t|_{\beta_j=\beta_p=\beta} = \alpha_0^{\beta t} \qquad \text{and} \qquad \gamma^t|_{\beta_j=\beta_p=\beta} = \mathrm{id}. \tag{5.11}$$

This implies that

$$K|_{\beta_j=\beta_p=\beta} = L, \tag{5.12}$$

where $L := L_0 + gV - gV'$, with $V' := \pi'(v)$, is the standard self-adjoint Liouville operator. In what follows we write $K = L_0 + gI$, where

$$I = V - V'_{-i/2}.$$

The operator K is non-self-adjoint for $\delta\beta \neq 0$, and the perturbation I not relatively bounded w.r.t. the unperturbed operator L_0. To study the evolution generated by K we use the family of operators

$$K_{(s)} := L_0 + g(V - V'_s) \ , \tag{5.13}$$

where, recall, $V = \pi(v)$, and $V'_s := \pi'(\gamma^{\overline{s}}(v))$. This family interpolates between the operator K,

$$K = K_{(-i/2)} \ , \tag{5.14}$$

(see Eqn (5.3)) and the self-adjoint operators $K_{(s)}$ with real s. Under condition (4.1) on v, $V'_{(s)}$ and $K_{(s)}$ are well defined on the dense domain $\mathrm{Dom}(L^{(\ell)}) \cap \pi(\mathcal{A})\Omega$ for all s in the strip

$$S_\varepsilon := \left\{ t \in \mathbb{C} \middle| |\mathrm{Im}\, t| < \tfrac{1}{2} + \varepsilon \right\}, \tag{5.15}$$

for $\varepsilon > 0$, and are strongly analytic there (recall that π' is anti-linear).

6 Spectral Deformation of K and $K_{(s)}$

Since the operator K is not self-adjoint it is not a simple matter to derive long-time properties of the dynamics e^{iKt} from spectral properties of K. As a result we bypass establishing the connection of the dynamics to the spectrum of K and instead connect it to certain spectral properties of a complex deformation, K_θ, of this operator. To do this we use the interpolating family $K_{(s)\theta}$, which is the complex deformation of the family $K_{(s)}$, (5.13). In this section we define complex deformations K_θ and $K_{(s)\theta}$ and in the next section we establish their spectral characteristics which are relevant for us.

In order to carry out the spectral analysis of the operator K, which we begin in this section, we use the specifics of the Araki-Woods representation in an essential way. They were not used in an essential way for the developments up to this section.

As a complex deformation we choose a combination of the complex dilation used in [8] and complex translation due to [18] (see [8], Section V.2 for a sketch of the relevant ideas). This complex deformation was used in [25] in the spectral analysis for a general class of Liouville type operators.

First we define the group of dilations. Let $\hat{U}_{d,\delta}$ be the second quantization of the one-parameter group

$$u_{d,\delta} : f(k) \to e^{3\delta/2} f(e^\delta k)$$

of dilations on $L^2(\mathbb{R}^n)$. This group acts on creation and annihilation operators $a_r^\#(f)$ on the Fock space, $\mathcal{H}_r$, according to the rule

$$\hat{U}_{d,\delta} a_r^\#(f) \hat{U}_{d,\delta}^{-1} = a_r^\#(u_{d,\delta} f), \qquad \hat{U}_{d,\delta}\Omega_{rj} = \Omega_{rj}. \tag{6.1}$$

We lift this group to the positive-temperature Hilbert space, (2.33), according to the formula

$$U_{d,\delta} = \mathbf{1}_p \otimes \mathbf{1}_p \otimes \hat{U}_{d,\delta} \otimes \hat{U}_{d,-\delta} \otimes \hat{U}_{d,\delta} \otimes \hat{U}_{d,-\delta}. \tag{6.2}$$

Note that we could dilate each reservoir by a different amount. However, this does not give us any advantage, so to keep notation simple we use one dilation parameter for both reservoirs. We record for future reference how the group $U_{d,\delta}$ acts on the Liouville operator L_0 and the positive-temperature photon number operator $N := \sum_{j=1}^2 N_j$, where

$$N_j := \int \left[a_{\ell,j}^*(k) a_{\ell,j}(k) + a_{r,j}^*(k) a_{r,j}(k) \right] d^3k, \tag{6.3}$$

and the operators $a_{\{\ell,r\},j}^\#(k)$ were introduced after (2.34). We have (below we do not display the identity operators):

$$U_{d,\delta} L_{rj} U_{d,\delta}^{-1} = \cosh(\delta) L_{rj} + \sinh(\delta)\Lambda_j, \tag{6.4}$$

where Λ_j is the positive operator on the jth reservoir Hilbert space given by

$$\Lambda_j = \int \omega(k) \left(a_{\ell,j}^*(k) a_{\ell,j}(k) + a_{r,j}^*(k) a_{r,j}(k) \right) d^3k, \tag{6.5}$$

and

$$U_{d,\delta} N_j U_{d,\delta}^{-1} = N_j. \tag{6.6}$$

Now we define a one-parameter group of translations. It can be defined as one-parameter group arising from transformations of the underlying physical space similarly to the dilation group. This is done in Appendix B. Here we define the translation group by means of the selfadjoint generator $T := \sum_{j=1}^2 T_j$, where

$$T_j = \int \left[a_{\ell,j}^*(k) \gamma a_{\ell,j}(k) + a_{r,j}^*(k) \gamma a_{r,j}(k) \right] d^3k. \tag{6.7}$$

Here, $\gamma = i(\hat{k}\cdot\nabla + \nabla\cdot\hat{k})$ with $\hat{k} = k/|k|$. Since γ is a self-adjoint operator on $L^2(\mathbb{R}^3)$, the operators $T_j, j = 1,2$, and therefore the operator T, are self-adjoint as well. We define the one-parameter group of translations as

$$U_{t,\tau} := \mathbf{1}_p \otimes \mathbf{1}_p \otimes e^{i\tau T}. \tag{6.8}$$

Eqns. (6.7) - (6.8) imply the following expressions for the action of this group on the Liouville operators:

$$U_{t,\tau} L_{rj} U_{t,\tau}^{-1} = L_{rj} + \tau N_j. \tag{6.9}$$

Observe that neither the dilation nor the translation group affects the particle vectors, and that $U_{t,\tau} N_j U_{t,\tau}^{-1} = N_j$.

Now we want to apply the product of these transformations to the full operator $K = L_0 + gI$. Since the dilation and translation transformations do not commute we have to choose the order in which we apply them. Since the operator $\Lambda = \sum_j \Lambda_j$ is not analytic under the translations while the operator N is analytic under dilations we apply first the translation and then the dilation transformation. We define the combined translation-dilation transformation as

$$U_\theta = U_{d,\delta} U_{t,\tau} \tag{6.10}$$

where $\theta = (\delta, \tau)$. In what follows we will use the notation $|\theta| = (|\delta|, |\tau|)$, $\mathrm{Im}\theta = (\mathrm{Im}\delta, \mathrm{Im}\tau)$, and similarly for $\mathrm{Re}\theta$, and

$$\mathrm{Im}\theta > 0 \quad \Longleftrightarrow \quad \mathrm{Im}\delta > 0 \wedge \mathrm{Im}\tau > 0. \tag{6.11}$$

Now we are ready to define a complex deformation of the operator K. On the set $\mathrm{Dom}(\Lambda) \cap \mathrm{Dom}(N)$ we define for $\theta \in \mathbb{R}^2$

$$K_\theta := U_\theta K U_\theta^{-1}. \tag{6.12}$$

Recalling the decomposition $K = L_0 + gI$, where $L_0 := L_p + L_r$, $L_r := \sum_{j=1}^2 L_{rj}$ and $I = V - V'_{-i/2}$, we have

$$K_\theta = L_{0,\theta} + gI_\theta, \tag{6.13}$$

where the families $L_{0,\theta}$ and I_θ are defined accordingly. Due to Eqns. (6.4), (6.6) and (6.9) we have:

$$L_{0,\theta} = L_p + \cosh(\delta) L_r + \sinh(\delta)\Lambda + \tau N, \tag{6.14}$$

where $\theta = (\delta, \tau)$, and $\Lambda = \sum_{j=1}^2 \Lambda_j$. An explicit expression for the family I_θ is given in Appendix B.2 (see Eqns (B.2.5) and (B.2.7)).

Similarly, we define the family $K_{(s)\theta} := U_\theta K_{(s)} U_\theta^{-1}$ (recall (5.13)).

Of course the operator families above are well defined for real θ. Our task is to define them as analytic families on the strips

$$S_{\theta_0}^{\pm} = \left\{\theta \in \mathbb{C}^2 \,|\, 0 < \pm\mathrm{Im}\theta < \theta_0\right\} \tag{6.15}$$

where $\theta_0 = (\delta_0, \tau_0) > 0$ is the same as in Condition (A). Recall that the inequality $\pm\mathrm{Im}\theta < \theta_0$ is equivalent to the following inequalities: $\pm\mathrm{Im}\delta < \delta_0$ and $\pm\mathrm{Im}\tau < \tau_0$. (The fact that analyticity in a neighbourhood of a fixed $\theta \in S_{\theta_0}^{\pm}$ implies analyticity in the corresponding strip in which $\mathrm{Re}\theta$ is not constraint follows from the explicit formulas (6.14), (B.2.5) and (B.2.7).) The analytic continuations (if they exist) are denoted by the same symbols.

We define the family K_θ for $\theta \in \{\theta \in \mathbb{C}^2 \,\big|\, |\mathrm{Im}\,\theta| < \theta_0\}$ by the explicit expressions (6.13), (6.14), (B.2.5) and (B.2.7). Clearly, $\mathrm{Dom}(\Lambda) \cap \mathrm{Dom}(N) \subset \mathrm{Dom}(L_{0\theta})$ and on this domain the family $L_{0\theta}$ is manifestly strongly analytic in $\theta \in \{\theta \in \mathbb{C}^2 \,\big|\, |\mathrm{Im}\theta| < \theta_0\}$. It is shown in Appendix B that for $|\mathrm{Im}\theta| < \theta_0$ we have $\mathrm{Dom}(\Lambda^{1/2}) \subset \mathrm{Dom}(I_\theta)$ and $I_\theta f$ is analytic $\forall f \in \mathrm{Dom}(\Lambda^{1/2})$. Here Condition (A) of Section 3 is used. Hence the family K_θ for $\theta \in \{\theta \in \mathbb{C}^2 \,\big|\, |\mathrm{Im}\,\theta| < \theta_0\}$ is bounded from $\mathrm{Dom}(\Lambda) \cap \mathrm{Dom}(N)$ to $\mathcal{H}$ (and $K_\theta f$ is analytic in $\theta \in \{\theta \in \mathbb{C}^2 \,\big|\, |\mathrm{Im}\,\theta| < \theta_0\}, \forall f \in \mathrm{Dom}(\Lambda) \cap \mathrm{Dom}(N)$). Moreover, for $|\mathrm{Im}\ \theta| > 0$ the operators K_θ are closed on the domain $\mathrm{Dom}(\Lambda) \cap \mathrm{Dom}(N)$.

However, $\{K_\theta \mid \mathrm{Im}\theta < \theta_0\}$ is not an analytic family in the sense of Kato. The problem here is the lack of coercivity – the perturbation I is not bounded relatively to the unperturbed operator L_0. To compensate for this we have chosen the deformation U_θ in such a way that the operator $M_\theta := \mathrm{Im}L_{0,\theta}$ is coercive for $\mathrm{Im}\theta > 0$, i.e., the perturbation I_θ as well as $\mathrm{Re}L_{0,\theta}$ are bounded relative to this operator. The problem here is that $M_\theta \to 0$ as $\mathrm{Im}\theta \to 0$ so we have to proceed carefully.

Everything said about K_θ applies also to the family $K_{(s)\theta}$.

The next result is gives some analyticity properties and some global spectral properties of $K_{(s)\theta}$.

Theorem 6.1 ([25]) *Assume that Condition (A) holds and let $\theta_0 = (\delta_0, \tau_0)$ be as in that condition. Take an*

$$a > \frac{g^2}{\sin(\mathrm{Im}\delta)} C_0^2 \left(\sum_{j=1}^{2} \|G_j\|_{\delta\beta_j,1/2,\theta} \right)^2 , \tag{6.16}$$

where $C_0 := C(1 + \beta_1^{-1/2} + \beta_2^{-1/2})$, and where C is a constant depending only on $\tan\delta_0$. Then we have:

(i) $\{z \in \mathbb{C} | \mathrm{Im}\, z \le -a\} \subseteq \rho(K_{(s)\theta})$ (resolvent set) if either $s \in \mathbb{R}$ and $\theta \in \overline{S^+_{\theta_0}}$, or if $s \in S_\epsilon$ and $\theta \in S^+_{\theta_0}$;

(ii) Let $C_{a,b}$ be the truncated cone

$$C_{a,b} := \left\{ z \in \mathbb{C} \mid \mathrm{Im}\, z > -\tfrac{a}{2},\ |\mathrm{Re}\, z| < 2[(\sin b)^{-1} + a/4](\mathrm{Im}\, z + a) + \|L_p\| + 1 \right\}.$$

Take $s \in S_\varepsilon$, $\theta \in S^+_{\theta_0}$, and take a as in (6.16). Then $\sigma(K_\theta) \subset C_{a,\mathrm{Im}\,\delta}$, and for $z \in \mathbb{C}\backslash C_{a,\mathrm{Im}\,\delta}$ we have

$$\|(K_\theta - z)^{-1}\| \le (\mathrm{dist}\,(z, C_{a,\mathrm{Im}\,\delta}))^{-1}. \tag{6.17}$$

(iii) The family $K_{(s)\theta}$, $s \in S_\epsilon$, $\theta \in \overline{S^+_{\theta_0}}$, is analytic of type A (in the sense of Kato) in $\theta \in S^+_{\theta_0}$, for all $s \in S_\epsilon$, and in $s \in S_\epsilon$, for all $\theta \in S^+_{\theta_0}$;

(iv) Let $s \in \mathbb{R}$. For any u and v which are U_θ-analytic in a strip $\left\{\theta \in \mathbb{C}^2 | 0 \le \mathrm{Im}\theta < \theta_1\right\}$, for some $\theta_1 = (\delta_1, \tau_0)$, $\delta_1 \in [0, \min\{\pi/3, \theta_0\})$, the following relation holds:

$$\left\langle u, (K_{(s)} - z)^{-1} v \right\rangle = \left\langle u_{\overline{\theta}}, (K_{(s)\theta} - z)^{-1} v_\theta \right\rangle, \tag{6.18}$$

where $u_\theta = U_\theta u$, etc., $\mathrm{Im}\, z \le -a$ and $0 < \mathrm{Im}\,\theta < \theta_1/2$.

Proof. Statements (i), (iii) and (iv) are special cases of Theorem 5.1 in [25], with the exception of the assertion about analyticity of $s \mapsto K_{(s)\theta}$ in (iii). This assertion is easily proven by noticing that $\partial_s (K_{(s)\theta} - z)^{-1} = -(K_{(s)\theta} - z)^{-1} (\partial_s V_s')(K_{(s)\theta} - z)^{-1}$. Statement (ii) is the content of Proposition 5.2 in [25]. ■

7 Spectral Analysis of K_θ

In what follows, for a self-adjoint operator A and $a \in \mathbb{R}$ we use the notation $\chi_{A \le a}$ for the spectral projection of A associated to the set $\{\lambda \in \mathbb{R} \,| \lambda \le a\}$ and similarly for $\chi_{A \ge a}$ and $\chi_{A=a}$, etc. Fix a θ satisfying $0 < \mathrm{Im}\,\theta < \theta_0$.

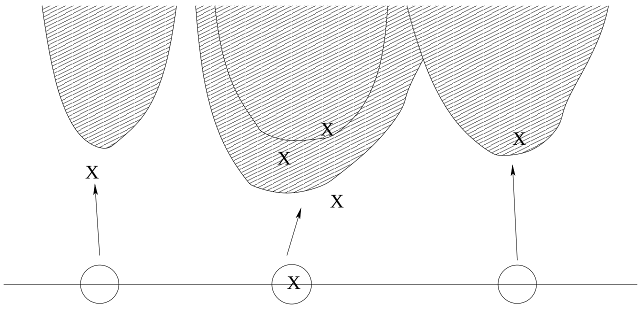
Theorem 7.1 *Assume Conditions (A) – (C). Let $\theta_0 \ge \mathrm{Im}\theta = (\delta', \tau') > 0$ and let g_1 be as in Theorem 3.1. If $0 < |g| < g_1$ then*

(a) 0 is an isolated and simple eigenvalue of K_θ;

(b) $\sigma(K_\theta)\backslash\{0\} \subset \{z \in \mathbb{C}^+ \,|\, \mathrm{Im} z \geq \min(cg^2, \frac{1}{2}\tau')\}$,

for some $c > 0$, *independent of* θ.

Theorem 7.1 is proven at the end of this section. Together with Theorem 6.1 it gives the following picture for the spectrum of K_θ.



Spectrum of K_θ for $\mathrm{Im}\theta > 0$

The motion of resonances bifurcating out of the eigenvalues of L_0 is governed, to second order in the coupling constant g, by *level shift operators*, see [25, 24] for a discussion closest to the situation at hand. Let e be an eigenvalue of L_p and let Λ_e be the level shift operator acting on $\mathrm{Ran}\chi_{L_p=e}$, defined by

$$\Lambda_e := -P_e I (L_0 - e + i0)^{-1} I P_e, \tag{7.1}$$

where $P_e = \chi_{L_p=e} \otimes \chi_{L_r=0}$. The notation $+i0$ in (7.1) stands for the limit of $i\epsilon$ as $\epsilon \downarrow 0$. The following result summarizes properties of the level shift operators which are essential for the proof of Theorem 7.1.

Theorem 7.2 *The level shift operators* Λ_e, *(7.1), satisfy*

$$\sigma(\Lambda_e) \subset \overline{\mathbb{C}^+} \tag{7.2}$$

$$\sigma(\Lambda_e) \cap \mathbb{R} = \begin{cases} \emptyset & \text{if } e \neq 0 \\ \{0\} & \text{if } e = 0 \end{cases} \tag{7.3}$$

$$\dim \mathrm{Ker}\,(\Lambda_{e=0}) = 1\ . \tag{7.4}$$

Furthermore, there is a $\gamma_0 > 0$ *which does not depend on the inverse temperatures, s.t.* $\mathrm{Im}\Lambda_e \geq \gamma_0$, *for all* $e \neq 0$. *Moreover, if Condition (D) is satisfied, then there is a* $\delta_0 > 0$ *which does not depend on the inverse temperatures, s.t.* $\mathrm{Im}(\sigma(\Lambda_0)\backslash\{0\}) \geq \delta_0$.

We prove this theorem is in Section 10.

Proof of Theorem 7.1. That 0 is an eigenvalue of K_θ follows readily from the equations $K\Omega = 0$ and the fact that Ω is U_θ-analytic in the strip (6.15), as we show in Lemma 7.3 below. So we have

$$K_\theta \Omega_\theta = 0, \tag{7.5}$$

where $\Omega_\theta := U_\theta \Omega$.

Lemma 7.3 Ω *is* U_θ*-analytic, for* $\theta = (\delta, \tau) \in S^{\pm}_{\theta_0}$, *see (6.15).*

Proof of Lemma 7.3. As in the proof of Theorem 4.1 we consider the regularized vector $\Omega_n = e^{-\beta L_n^{(\ell)}/2}\Omega_0$. The vectors $U_{t,\tau}\Omega_n$ exist and have limits as $n \to \infty$. By the closedness of $U_{t,\tau}$ we have that $\Omega \in \mathrm{Dom}(U_{t,\tau})$ and $U_{t,\tau}\Omega_n \to U_{t,\tau}\Omega$, i.e. Ω is $U_{t,\tau}$-analytic. In the same way one sees that $U_{d,\delta}U_{t,\tau}\Omega_n$ exists and converges, and from the closedness of $U_{d,\delta}$ it follows that $U_{t,\tau}\Omega$ is in the domain of $U_{d,\delta}$. Consequently, Ω is U_θ-analytic. This completes the proof of Lemma 7.3. ■

Let $\rho_0 \in (0, \sigma/2)$, where σ is given in (3.9), and consider the half-space

$$S = \left\{ z \in \mathbb{C} \middle| \operatorname{Im} z < \tfrac{1}{4}\rho_0 \sin(\operatorname{Im}\delta)\rho_0 \right\}. \tag{7.6}$$

We decompose this region into the strips $S_e = \{z \in S | |\operatorname{Re} z - e| \le \rho_0\}$, where $e \in \sigma(L_p)$. The following result is a special case of Theorem 6.1 of [25].

Theorem 7.4 *Assume that condition (A) holds and that there are constants $\gamma_0 > 0$ and $\delta_0 > 0$ satisfying $\operatorname{Im}\Lambda_e \ge \gamma_0$ for all $0 \ne e \in \sigma(L_p)$ and $\operatorname{Im}(\sigma(\Lambda_0)\backslash\{0\}) \ge \delta_0$.*

Take $0 < |g| < \sqrt{\rho_0}g_0$ and let $\alpha = (\mu - 1/2)/(\mu + 1/2)$, where $\mu > 1/2$ is given in Condition (A). Then

1. We have $\sigma(K_\theta) \cap S \subset \bigcup_{e\in\sigma(L_p)} S_e$.

2. Choose $\rho_0 = |g|^{2-2\alpha}$. There is a $C > 0$ s.t. if $0 < |g| < C(\gamma_0)^{1/\alpha}$, where γ_0 is given in Theorem 7.2 above, then

$$\sigma(K_\theta) \cap S_e \subset \{z \in \mathbb{C} \mid \operatorname{Im} z \ge \tfrac{1}{2}g^2\gamma_0\}, \tag{7.7}$$

for all $e \ne 0$.

3. Choose $\rho_0 = |g|^{2-2\alpha}$. There is a $C > 0$ s.t. if $0 < |g| < C\min[(\delta_0)^{1/\alpha}, (\tau')^{\frac{1}{2+\alpha}}]$ then

$$\sigma(K_\theta) \cap S_0 \subset \{z_0\} \cup \{z \in \mathbb{C} \mid \operatorname{Im} z \ge \tfrac{1}{2}\min(g^2\delta_0, \tau')\}, \tag{7.8}$$

where z_0 is a simple isolated eigenvalue of K_θ, satisfying $|z_0| = O(|g|^{2+\alpha})$.

For a coupling constant satisfying

$$0 < |g| < \min\left[(g_0)^{1/\alpha}, (\gamma_0)^{1/\alpha}, (\delta_0)^{1/\alpha}, (\tau')^{\frac{1}{2+\alpha}}\right] \tag{7.9}$$

all three parts of Theorem 7.4 apply. Thus K_θ has a simple isolated eigenvalue $\{z_0\}$ in a neighbourhood $O(|g|^{2+\alpha})$ of the origin, and $\sigma(K_\theta)\backslash\{z_0\} \subset \{z \in \mathbb{C}_+ \mid \operatorname{Im} z \ge \min(cg^2, \frac{1}{2}\tau')\}$, where $c = \frac{1}{2}\min(\gamma_0, \delta_0)$. In order to complete the proof of Theorem 7.1 we only need to remark that $z_0 = 0$ since zero is an eigenvalue of K_θ, see (7.5). (One can also give a dynamical argument to prove that $z_0 = 0$, see the remark after (8.12).) ■

8 Resolvent representation and pole approximation

In order to study the long-time behaviour of the evolution $U(t) = e^{iKt}$ (defined on the domain $\pi(\mathcal{A})\Omega$ by (4.3)) we relate it to an object which we understand relatively well, namely the resolvent $(K_\theta - z)^{-1}$ of the deformation K_θ, defined in Section 6. The main result of this section is

Theorem 8.1 *Let ϕ and Ψ be U_θ-analytic vectors, and let $\Psi = \pi(A)\Omega$ for some $A \in \mathcal{A}$. We have the following representation*

$$\langle \phi, e^{iKt}\Psi \rangle = -\frac{1}{2\pi i}\oint_\Gamma \langle \phi_{\overline{\theta}}, (K_\theta - z)^{-1}\Psi_\theta \rangle e^{izt} dz\ , \tag{8.1}$$

where $\Psi_\theta = U_\theta\Psi$ (similarly for ϕ or any other vector), $\operatorname{Im}\theta > 0$, and Γ is the path

$$\Gamma := \{z = \lambda - i\tau'/3, |\lambda| \le C\} \cup \{z = \lambda - i2\tau'/3 + i\lambda\tau'/3C, \lambda \ge C\}$$
$$\cup \{z = \lambda - i2\tau'/3 - i\lambda\tau'/3C, \lambda \le -C\} \tag{8.2}$$

for a sufficiently large constant C. The integral on the r.h.s. of (8.1) is well defined in virtue of Theorem 6.1, (6.17), and the estimate $|e^{izt}| \le e^{-\lambda\tau'/3C}$ on the infinite branches of Γ.

Proof. In the proof below, the vectors ϕ and Ψ are as in the theorem. To prove the equality in (8.1) we use the family $K_{(s)}$ of operators defined in (5.13)-(5.14). First we show

Lemma 8.2 $\Omega \in \mathrm{Dom}(e^{\nu N})$ *for any* $\nu > 0$.

Proof of Lemma 8.2. The result follows from the relation $\Omega_0 \in \mathrm{Dom}(e^{\nu N} e^{-\beta L^{(\ell)}/2})$ which is shown in the same way as the relation $\Omega_0 \in \mathrm{Dom}(e^{-\beta L^{(\ell)}/2})$, see Proposition 4.1. ■

Next, we define the operator $e^{iK_{(s)}t}$ as follows: $e^{iK_{(s)}t}\Omega$ is given by a Dyson expansion, where the part $-gV_s'$ of $K_{(s)}$ is treated as a perturbation. The fact that the Dyson series converges is easily seen from the relation $\Omega \in \mathrm{Dom}(e^{\nu N})$, $\nu > 0$, shown above. Moreover, it is clear that this series defines a vector which is analytic in $s \in S_\varepsilon$, i.e., $s \mapsto e^{iK_{(s)}t}\Omega$ is analytic for $s \in S_\varepsilon$. We define the action of $e^{iK_{(s)}t}$ on vectors $\pi(A)\Omega$, $A \in \mathcal{A}$ (which form a dense set), by

$$e^{iK_{(s)}t}\pi(A)\Omega = \sigma^t(\pi(A))e^{iK_{(s)}t}\Omega. \tag{8.3}$$

Consequently, the map $s \mapsto e^{iK_{(s)}t}\pi(A)\Omega$ is analytic for $s \in S_\varepsilon$. For $s = \frac{-i}{2}$ this definition gives e^{iKt}.

Since $K_{(s)}$ is self-adjoint for s real we derive from Stone's formula

$$\langle \phi, e^{iK_{(s)}t}\Psi \rangle = -\frac{1}{2\pi i} \oint_{\mathbb{R}-i} \langle \phi, (K_{(s)} - z)^{-1}\Psi \rangle e^{izt} dz, \quad s \in \mathbb{R}. \tag{8.4}$$

Next, using $e^{izt} = \frac{1}{it}\frac{\partial}{\partial(\mathrm{Re}\, z)}e^{izt}$ and integrating by parts we can represent the r.h.s. of (8.4) as

$$RHS(8.4) = -\frac{1}{2\pi i} \int_{\mathbb{R}-i} \langle \phi, (K_{(s)} - z)^{-2}\Psi \rangle e^{izt} dz.$$

Now we perform the spectral deformation, Theorem 6.1, (iv), to obtain for $\mathrm{Im}\theta > 0$

$$RHS(8.4) = -\frac{1}{2\pi i}\frac{1}{it} \int_{\mathbb{R}-i} \langle \phi_{\overline{\theta}}, (K_{(s)\theta} - z)^{-2}\Psi_\theta \rangle e^{izt} dz. \tag{8.5}$$

The integral converges since due to (6.17) we have $\|(K_{(s)\theta} - z)^{-n}\| \le C_n \langle \mathrm{Re}\, z\rangle^{-n}$ for $z \in \mathbb{R} - i\varepsilon$, where $\langle x \rangle := \sqrt{1+x^2}$.

In (8.5) we deform the contour of integration from $\mathbb{R} - i$ to Γ which is fine due to (6.17), and we integrate by parts in the opposite direction of above, to obtain

$$\langle \phi, e^{iK_{(s)}t}\Psi \rangle = -\frac{1}{2\pi i} \oint_{\Gamma} \langle \phi_{\overline{\theta}}, (K_{(s)\theta} - z)^{-1}\Psi_\theta \rangle e^{izt} dz\ . \tag{8.6}$$

Both sides of this expression are well defined and analytic for $s \in S_\varepsilon$ (see Theorem 6.1, (6.17), and after equation (8.3)). Since they are equal for real s they are equal for all s in their domain of analyticity. Taking $s = -\frac{i}{2}$ in this formula gives (8.1). ■

It is shown in Section 7 that the operator family K_θ has a simple isolated eigenvalue at 0 and the rest of its spectrum is located in a truncated cone in $\{z \in \mathbb{C}^+ \,|\mathrm{Im}\, z > \frac{1}{3}\tau'\}$, where $\tau' = \mathrm{Im}\,\tau$. In the integral on the r.h.s. of formula (8.1) we deform the contour of integration to

$$\Gamma' := \{z = \lambda + \tfrac{1}{3}i\tau', |\lambda| \le C\} \cup \{z = \lambda + i\lambda\tau'/3C, \lambda \ge C\} \cup \{z = \lambda - i\lambda\tau'/3C, \lambda \le -C\} \tag{8.7}$$

where C is sufficiently large. Picking up the residue from the simple eigenvalue 0 of K_θ we derive from (8.1)

$$\langle \phi, e^{iKt}\Psi \rangle = \langle \phi_{\overline{\theta}}, P_\theta \Psi_\theta \rangle + O(e^{-\tau' t/3})\ , \tag{8.8}$$

where $P_\theta = \frac{-1}{2\pi i}\oint (K_\theta - z)^{-1}dz$ is the eigenprojection of K_θ corresponding to the simple and isolated eigenvalue 0. The contour integral is over a small circle around the origin. This is the only place where we use that 0 is an isolated eigenvalue of K_θ – the fact we show using complex translation in addition to complex dilation.

P_θ is a rank one projection which is analytic in $\theta \in S^+_{\theta_0}$. One proves using a standard argument that it satisfies $K_\theta P_\theta = P_\theta K_\theta = 0$. Hence, P_θ can be written as

$$P_\theta = |\Omega_\theta\rangle \langle \Omega^*_{\overline{\theta}}| , \tag{8.9}$$

where Ω_θ and $\Omega^*_{\overline{\theta}}$ are zero eigenvectors of K_θ and its adjoint operator

$$(K_\theta)^* = (K^*)_{\overline{\theta}} =: K^*_{\overline{\theta}}, \tag{8.10}$$

i.e. we have

$$K_\theta\Omega_\theta = 0 \ \text{ and } \ K^*_{\overline{\theta}}\Omega^*_{\overline{\theta}} = 0, \tag{8.11}$$

with the normalization $\langle \Omega^*_{\overline{\theta}}, \Omega_\theta\rangle = 1$. Since P_θ and Ω_θ are analytic in $\theta \in S^+_{\theta_0}$, then so is $\Omega^*_{\overline{\theta}}$ in the variable $\overline{\theta} \in S^-_{\theta_0}$. (The possibility of the normalization, $\left\langle \Omega^*_{\overline{\theta}}, \Omega_\theta \right\rangle \neq 0$, follows also from results of Section 11. Analyticity of $\Omega^*_{\overline{\theta}}$ in $\overline{\theta} \in S \in S^\pm_{\theta_0}$ can also be shown directly by using the analyticity and spectral properties of $K^*_{\overline{\theta}}$, see Section 7). Equation (8.9) implies

$$\left\langle \phi_{\overline{\theta}}, P_\theta \Psi_\theta \right\rangle = \langle \phi, \Omega\rangle \left\langle \Omega^*_{\overline{\theta}}, \Psi_\theta \right\rangle \ . \tag{8.12}$$

Remark. We present here another proof of the relation $z_0 = 0$, where z_0 is the simple isolated eigenvalue of K_θ given in Theorem 7.4. Starting with the information on the spectrum of K_θ given in Theorem 7.4 and proceeding with a contour deformation as above we find that $\left\langle \phi, e^{iKt}\psi \right\rangle = e^{iz_0t}\left\langle \phi_{\overline{\theta}}, P_\theta\psi_\theta \right\rangle + O(e^{(|\mathrm{Im}\, z_0| - \tau'/3)t})$ instead of (8.8). Applying this formula to $\phi = \psi = \Omega$ and using (4.3) we obtain $\langle \Omega, \Omega\rangle = e^{iz_0t}\langle \Omega, \Omega\rangle + O(e^{(|\mathrm{Im}\, z_0| - \tau'/3)t})$. Since $|z_0| = O(|g|^{2+\alpha}) \ll \tau'$ the error term tends to zero as $t \to \infty$, so by taking $\lim_{T\to\infty}\int_0^T dt$ on both sides we see that z_0 must be zero.

9 Proof of Theorem 3.1

Let ψ be any ω_0-normal state on $\mathcal{A}$. For the following reasoning, we may assume without loss of generality that $\psi(A) = \langle \Omega_\psi, \pi(A)\Omega_\psi\rangle$, for some $\Omega_\psi \in \mathcal{H}$. Since Ω is cyclic for $\pi'(\mathcal{A})$, ψ can be approximated as follows. For any $\epsilon > 0$ there is a $Q \in \pi'(\mathcal{A})$ s.t., for all $A \in \mathcal{A}$,

$$|\psi(A) - \langle Q\Omega, \pi(A)Q\Omega\rangle\,| < \epsilon\|A\|. \tag{9.1}$$

Taking into account (2.30), $U(t) = e^{itK}$ and $U(t)\Omega = 0$, we obtain

$$|\psi^t(A) - \left\langle Q^*Q\Omega, e^{itK}\pi(A)\Omega\right\rangle| < \epsilon\|A\|, \tag{9.2}$$

uniformly in $t \in \mathbb{R}$. In order to examine the long time behaviour of $\left\langle Q^*Q\Omega, e^{itK}\pi(A)\Omega\right\rangle$, via (8.8), we first approximate the vector $Q^*Q\Omega \in \mathcal{H}$ by a family of U_θ-analytic vectors χ_ϵ, s.t. $\|\chi_\epsilon - Q^*Q\Omega\| < \epsilon$. We have

$$|\left\langle Q^*Q\Omega - \chi_\epsilon, e^{itK}\pi(A)\Omega\right\rangle| < \epsilon\|e^{itK}\pi(A)\Omega\| = \epsilon\|\sigma^t(\pi(A))\Omega\| < \epsilon\|A\|. \tag{9.3}$$

It follows from (9.2), (9.3), (8.8) and (8.12) that

$$\left|\psi^t(A) - \langle \chi_\epsilon, \Omega\rangle \left\langle \Omega^*_{\overline{\theta}}, (\pi(A)\Omega)_\theta\right\rangle\right| \le 2\epsilon\|A\| + |R_t(A)|, \tag{9.4}$$

where

$$|R_t(A)| = \frac{1}{2\pi}\left|\oint_{\Gamma'} \left\langle \chi_{\epsilon,\overline{\theta}}, (K_\theta - z)^{-1}(\pi(A)\Omega)_\theta\right\rangle e^{izt}dz\right| \le C\|\chi_{\epsilon,\overline{\theta}}\|\ \|(\pi(A)\Omega)_\theta\|e^{-\tau' t/3}. \tag{9.5}$$

The path Γ' in (9.5) is defined in (8.7).

Since $\langle Q^*Q\Omega, \Omega\rangle \to 1$ as $\epsilon \to 0$, we have $\langle \chi_\epsilon, \Omega\rangle = 1 + o(\epsilon^0)$, where $o(\epsilon^0)$ denotes a quantity that vanishes in the limit $\epsilon \to 0$. Thus

$$|\psi^t(A) - \langle \chi_\epsilon, \Omega\rangle \left\langle \Omega^*_{\overline{\theta}}, (\pi(A)\Omega)_\theta \right\rangle| \geq |\psi^t(A) - \left\langle \Omega^*_{\overline{\theta}}, (\pi(A)\Omega)_\theta \right\rangle| - o(\epsilon^0)\ |\left\langle \Omega^*_{\overline{\theta}}, (\pi(A)\Omega)_\theta \right\rangle|.$$

Combining this estimate with (9.4) and (9.5) we arrive at

$$\begin{aligned} &|\psi^t(A) - \left\langle \Omega^*_{\overline{\theta}}, (\pi(A)\Omega)_\theta \right\rangle| \\ &\quad \leq o(\epsilon^0)\left(\|A\| + |\left\langle \Omega^*_{\overline{\theta}}, (\pi(A)\Omega)_\theta \right\rangle|\right) + C\|\chi_{\epsilon,\overline{\theta}}\|\ \|(\pi(A)\Omega)_\theta\| e^{-\tau' t/3}, \end{aligned} \tag{9.6}$$

where $o(\epsilon^0)$ is independent of A and θ. In particular, taking first $t \to \infty$ and then $\epsilon \to 0$ yields

$$\lim_{t\to\infty} \psi^t(A) = \left\langle \Omega^*_{\overline{\theta}}, (\pi(A)\Omega)_\theta \right\rangle. \tag{9.7}$$

Equation (9.7) shows that $|\left\langle \Omega^*_{\overline{\theta}}, (\pi(A)\Omega)_\theta \right\rangle| \leq \|A\|$. We re-inject this inequality into r.h.s. of (9.6) to arrive at

$$\lim_{t\to\infty} \sup_{A\in\widetilde{\mathcal{A}}} \frac{|\psi^t(A) - \eta(A)|}{\|A\|_\sim} = 0, \tag{9.8}$$

where

$$\widetilde{\mathcal{A}} = \{A \in \mathcal{A} \mid \pi(A)\Omega \text{ is } U_\theta\text{-analytic for } |\theta| < \theta_0\}, \tag{9.9}$$

and where $\|\cdot\|_\sim$ is the norm on $\widetilde{\mathcal{A}}$ defined by

$$\|A\|_\sim = \|A\| + \sup_{|\theta|<\theta_0} \|(\pi(A)\Omega)_\theta\|, \quad \text{for } A \in \widetilde{\mathcal{A}}. \tag{9.10}$$

The state η on $\widetilde{\mathcal{A}}$ in (9.8) is given by

$$\eta(A) := \left\langle \Omega^*_{\overline{\theta}}, (\pi(A)\Omega)_\theta \right\rangle, \tag{9.11}$$

it is independent of the deformation parameter θ, if $\operatorname{Im}\theta > 0$, and $0 < g < |g_0|$ (the upper bound depending on $\sin(\operatorname{Im}\delta)$, see Theorem 3.1).

Observe that $\mathbf{1} \in \widetilde{\mathcal{A}}$, and that the normalization $\left\langle \Omega^*_{\overline{\theta}}, \Omega_\theta \right\rangle = 1$ implies $\eta(\mathbf{1}) = 1$. $\widetilde{\mathcal{A}}$ is a linear subspace of $\mathcal{A}$, but not an algebra.

We show in Appendix C, Proposition C.1, that $\mathcal{A}_1$ (defined in (2.17)) is strongly dense in $\mathcal{A}$ (defined in (2.7)), and that any $A \in \mathcal{A}_1$ has the property that $\pi(A)\Omega$ is U_θ-analytic, for θ in a neighbourhood of $\theta = 0$. Hence $\mathcal{A}_1 \subseteq \widetilde{\mathcal{A}} \subseteq \mathcal{A}$, and therefore $\widetilde{\mathcal{A}}$ is strongly dense in $\mathcal{A}$.

We have thus shown that for any ω_0-normal state ψ, $\psi^t \to \eta$ as $t \to \infty$, where the convergence is understood in the $\|\cdot\|_\sim$-topology of linear functionals on $\widetilde{\mathcal{A}}$.

It is clear from (9.8) that $|\eta(A)| \leq \|A\|$, for $A \in \widetilde{\mathcal{A}}$, hence η extends to a bounded positive linear functional on the Banach space of observables

$$\mathcal{A}_0 := \|\cdot\|\text{-closure of } \widetilde{\mathcal{A}}, \tag{9.12}$$

normalized as $\eta(\mathbf{1}) = 1$. Standard perturbation theory shows that $\eta(A)$, $A \in \widetilde{\mathcal{A}}$, is analytic in the coupling constant g.

Observe that we can rewrite the state $\eta(A)$ also in the form

$$\eta(A) = \operatorname{Tr}\left(\pi(A)_\theta P_\theta\right) \tag{9.13}$$

where $\pi(A)_\theta := U_\theta \pi(A) U_\theta^{-1}$. Formally one can undo the rotation in (9.11) to obtain (4.15) with $\Omega^* := U_{\overline{\theta}}^{-1}\Omega^*_{\overline{\theta}}$. However, in the non-equilibrium situation $\Omega^* \notin \mathcal{H}$! The set $\mathcal{A}_1$ is exactly the set on which (4.15) makes sense. Thus we gave a rigorous meaning to (4.15) and the NESS vector Ω^*.

10 Proof of Theorem 7.2

Our task is to show that the spectrum of (7.1) lies in the upper complex half plane $\{\mathrm{Im}\, z > 0\}$ if $e \neq 0$; and that it has a simple eigenvalue at zero and all the other eigenvalues lie in the upper complex half plane if $e = 0$. While this analysis is standard in the case when I is a selfadjoint operator (then the imaginary part of (7.1) is just $P_e I \delta(L_0 - e) I P_e$, manifestly a non-negative operator; see e.g. [8]), it needs some more thought in our case, where the interaction is non-selfadjoint. Let

$$V_j = \pi(v_j), \quad \text{and} \quad V_j' = \pi'(v_j).$$

The main ingredient of the proof is

Proposition 10.1 *We have*

$$\Lambda_e = \left(e^{-\beta_p H_p/2} \otimes \mathbb{1}_p\right) \left[\sum_{j=1,2} \left(e^{\beta_j H_p/2} \otimes \mathbb{1}_p\right) \Lambda_{je} \left(e^{-\beta_j H_p/2} \otimes \mathbb{1}_p\right)\right] \left(e^{\beta_p H_p/2} \otimes \mathbb{1}_p\right), \tag{10.1}$$

where, setting $\mathcal{R} := (L_0 - e + i0)^{-1}$,

$$\Lambda_{je} = -P_e (V_j - V_j') \mathcal{R} (V_j - V_j') P_e. \tag{10.2}$$

Notice that (10.1) shows that the spectrum of Λ_e *is independent of* β_p.

The importance of (10.1) is that it relates Λ_e to the operators Λ_{je} whose spectral characteristics are well known. Indeed, Λ_{je} are the level shift operators corresponding to the reservoir j coupled to the particle system, studied in [8, 24].

Before proceeding to the proof we examine some consequences of this proposition.

The case $e \neq 0$. Let us assume that the nonzero eigenvalues of L_p are simple, i.e. $E_i - E_j = E_m - E_n \Leftrightarrow i = m, j = n$. For a treatment of the more general case where $E_i - E_j = E_m - E_n$, for $(i,j) \neq (m,n)$, with simple E_j, we refer to [27]. Since P_e is of rank one Λ_e is just a complex number, namely the sum of $\Lambda_{1e} + \Lambda_{2e}$ (the dependence on β_1, β_2 disappears). Under condition (3.5), one has $\mathrm{Im}\, \Lambda_e \geq \gamma_0$, where γ_0 is a strictly positive constant which is independent of the inverse temperatures, see [8]). This shows (7.2) and (7.3) for $e \neq 0$.

The case $e = 0$. Zero is necessarily a degenerate eigenvalue of L_0, so the above reasoning does not apply. In particular, Λ_{10} and Λ_{20} do not commute. It is shown in [24, 27] that

$$\Lambda_{j0} = i \mathrm{Im}\, \Lambda_{j0} =: i \Gamma_{j0}, \tag{10.3}$$

where $\mathrm{Im}\, \Lambda_{j0} := \frac{1}{2i}(\Lambda_{j0} - \Lambda_{j0}^*)$. We use here implicitly Condition (B) on the non-degeneracy of H_p; if the small system has degenerate energy levels then Λ_{j0} are not purely imaginary [24]. One shown as in [8, 27, 24] that Γ_{j0} are real matrices having strictly negative off-diagonal entries, $(\Gamma_{j0})_{mn} < 0$, for $m \neq n$, and satisfying

$$\Gamma_{j0} \Omega_p^{(\beta_j)} = 0, \tag{10.4}$$

where $\Omega_p^{(\beta_j)}$ is the particle Gibbs state at temperature β_j. Hence, since

$$\left(e^{(\beta_p - \beta_j) H_p/2} \otimes \mathbb{1}_p\right) \Omega_p^{(\beta_p)} = \frac{\mathrm{tr} e^{-\beta_j H_p/2}}{\mathrm{tr} e^{-\beta_p H_p/2}}\, \Omega_p^{(\beta_j)},$$

we see that

$$\Gamma_0 \Omega_p^{(\beta_p)} = \sum_{j=1,2} \left(e^{(-\beta_p + \beta_j) H_p/2} \otimes \mathbb{1}_p\right) \frac{\mathrm{tr} e^{-\beta_j H_p/2}}{\mathrm{tr} e^{-\beta_p H_p/2}} \Gamma_{j0} \Omega_p^{(\beta_j)} = 0,$$

where $\Gamma_0 := -i\Lambda_0$. Thus, $\Omega_p^{(\beta_p)}$ is an eigenvector of the real matrix Γ_0 with eigenvalue zero. Notice that the vector $\Omega_p^{(\beta_p)}$ has strictly positive components, $[\mathrm{tr} e^{-\beta_p H_p/2}]^{-1} e^{-\beta_p E_n/2}$, in the orthonormal basis $\{\varphi_n \otimes \varphi_n\}$ of $\mathrm{Ran}\,\chi_{L_p=0}$ (where $H_p\varphi_n = E_n\varphi_n$, $\|\varphi_n\| = 1$). Moreover, the off-diagonal elements of the real matrix (which is not symmetric for $\beta_1 \neq \beta_2$) Γ_0 are given by

$$
\begin{aligned}
(\Gamma_0)_{m,n} &= \sum_{j=1,2} \left[\left(e^{(-\beta_p+\beta_j)H_p/2} \otimes \mathbb{1}_p \right) \Gamma_{j0} \left(e^{-(-\beta_p+\beta_j)H_p/2} \otimes \mathbb{1}_p \right) \right]_{mn} \\
&= -\pi \sum_{j=1,2} \frac{E_{mn}^2 e^{-(\beta_p-\beta_j)E_{mn}/2}}{|\sinh(\beta_j E_{mn}/2)|} \int_{S^2} d\sigma \; |[G_j(E_{mn},\sigma)]_{nm}|^2 \qquad (10.5)
\end{aligned}
$$

for $m > n$ (and similarly for $m < n$, see also [8, 27]). Hence Condition (C) implies that $(\Gamma_0)_{m,n} < 0$. A standard *Perron-Frobenius argument* shows that zero is a simple eigenvalue of Γ_0, and that $\sigma(\Gamma_0)\backslash\{0\} \subset \mathbb{C}^+$. This shows equations (7.3) – (7.4) for $e = 0$. It is shown in [8] that the gap at the bottom of the spectrum of Γ_{j0}, $j = 1, 2$ has a lower bound which is independent of the inverse temperatures.

We now prove the existence of δ_0, assuming that Condition (D) is satisfied. If $\dim \mathcal{H}_p = 2$ then the two eigenvalues of Λ_0 are zero and the trace of Λ_0. Expressions (10.1) and (10.3) show that $\mathrm{Tr}(\Lambda_0) = i[\mathrm{Tr}(\Gamma_{10}) + \mathrm{Tr}(\Gamma_{20})]$. Thus the spectral gap of Λ_0 is the sum of the gaps of Γ_{01} and Γ_{02}, which have lower bounds uniform in the inverse temperatures.

Next take $\dim \mathcal{H}_p \geq 3$. For $\delta\beta = |\beta_1 - \beta_2| = 0$ the matrix Λ_0 has the same spectrum as $\sum_j \Lambda_{j0}$, see (10.1). An application of the minimax principle demonstrates that the spectral gap of the latter operator has to be at least as large as the maximum of the gaps of Γ_{j0}, $j = 1, 2$. For small values of $\delta\beta$ (c.f. (3.6)), the existence of δ_0 follows by perturbation theory.

Finally we consider the case where $\delta\beta$ and β_1, β_2 are large (see (3.6)). Let us take $\beta_2 = \beta_1 + \delta\beta$. As is easily seen from (10.1) we have

$$
\sigma(\Lambda_0) = i\sigma\left(\left[\Gamma_{10} + (e^{\delta\beta H_p/2} \otimes \mathbb{1})\Gamma_{20}(e^{-\delta\beta H_p/2} \otimes \mathbb{1}) \right] \right). \qquad (10.6)
$$

Using the explicit expression for the matrix elements of Γ_{20} in the basis $\varphi_j \otimes \varphi_j$ (which can be read off of relation (10.5) for off-diagonal terms, and is easy to obtain for the diagonal ones), one verifies that the matrix $(e^{\delta\beta H_p/2} \otimes \mathbb{1})\Gamma_{20}(e^{-\delta\beta H_p/2} \otimes \mathbb{1})$ converges to a lower-triangular matrix $Q(\beta_1)$, in the limit $\delta\beta \to \infty$ (uniformly in β_1), and furthermore, that $Q(\beta_1) \to D$ as $\beta_1 \to \infty$, where D is a diagonal matrix with non-negative entries. The minimax principle implies that all but one eigenvalues of $\Gamma_{10} + D$ are greater than, or equal to the gap of Γ_{10}. From perturbation theory we know that for $\delta\beta$ and β_1 sufficiently large (independently of each other), all but one eigenvalues of the operator $\Gamma_{10} + (e^{\delta\beta H_p/2} \otimes \mathbb{1})\Gamma_{20}(e^{-\delta\beta H_p/2} \otimes \mathbb{1})$ must have real part greater than, or equal to half of the gap of Γ_{10}. The existence of δ_0 now follows from (10.6). ■

Proof of Proposition 10.1. Let $V = \sum_{j=1}^2 V_j$ and $V' = \sum_{j=1}^2 V_j'$. By the definition of the operator I, $I := V - V'_{-i/2}$, and the relation $V'_{-i/2} = e^{\tilde{L}/2} V' e^{-\tilde{L}/2}$ we have

$$
I = V - e^{\tilde{L}/2} V' e^{-\tilde{L}/2},
$$

where $\tilde{L} := \delta\beta_p L_p + \delta\beta_1 L_{r1} + \delta\beta_2 L_{r2}$ with $\delta\beta_p = \beta_p - \beta$ and $\delta\beta_j = \beta_j - \beta$. Using that $P_e e^{-\delta\beta_p L_p/2} = P_e e^{-\delta\beta_p e/2}$ we decompose

$$
\begin{aligned}
\Lambda_e \;=\; & P_e V \mathcal{R} V P_e + P_e V' \mathcal{R} V' P_e \\
& - P_e V \mathcal{R} e^{\tilde{L}/2} V' P_e e^{-\delta\beta_p e/2} - P_e V' e^{-\tilde{L}/2} \mathcal{R} V P_e e^{\delta\beta_p e/2}. \qquad (10.7)
\end{aligned}
$$

Notice that V and $\mathcal{R}$ commute with $\mathbb{1}_p \otimes e^{\delta\beta_p H_p/2}$. Using this and the relation

$$
(\mathbf{1}_p \otimes e^{\delta\beta_p H_p/2}) P_e = e^{-\delta\beta_p e} (e^{\delta\beta_p H_p/2} \otimes \mathbf{1}_p) P_e, \qquad (10.8)
$$

we obtain

$$\begin{aligned} P_eV\mathcal{R}VP_e &= e^{\delta\beta_p e/2}P_e\left(e^{-\delta\beta_pH_p/2}\otimes e^{\delta\beta_pH_p/2}\right)V\mathcal{R}VP_e \\ &= e^{\delta\beta_p e/2}P_e\left(e^{-\delta\beta_pH_p/2}\otimes \mathbb{1}_p\right)V\mathcal{R}V\left(\mathbb{1}_p\otimes e^{\delta\beta_pH_p/2}\right)P_e \\ &= \left(e^{-\delta\beta_pH_p/2}\otimes \mathbb{1}_p\right)P_eV\mathcal{R}VP_e\left(e^{\delta\beta_pH_p/2}\otimes \mathbb{1}_p\right). \end{aligned}$$

Next, using (10.8) again, we find

$$\begin{aligned} & P_eV\mathcal{R}e^{\overline{L}/2}V'P_ee^{-\delta\beta_pe/2} \\ &= P_eV\mathcal{R}\left(e^{\delta\beta_pH_p/2}\otimes e^{-\delta\beta_pH_p/2}\right)e^{(\delta\beta_1L_{r1}+\delta\beta_2L_{r2})/2}V'P_ee^{-\delta\beta_pe/2} \\ &= P_e\left(\mathbb{1}_p\otimes e^{-\delta\beta_pH_p/2}\right)V\mathcal{R}e^{(\delta\beta_1L_{r1}+\delta\beta_2L_{r2})/2}V'\left(e^{\delta\beta_pH_p/2}\otimes\mathbb{1}_p\right)P_ee^{-\delta\beta_pe/2} \\ &= \left(e^{-\delta\beta_pH_p/2}\otimes\mathbb{1}_p\right)P_eV\mathcal{R}e^{(\delta\beta_1L_{r1}+\delta\beta_2L_{r2})/2}V'P_e\left(e^{\delta\beta_pH_p/2}\otimes\mathbb{1}_p\right). \end{aligned}$$

Treating the other two terms in (10.7) in a similar way, we arrive at

$$\begin{aligned} -\Lambda_e = \left(e^{-\delta\beta_pH_p/2}\otimes\mathbb{1}_p\right)P_e[V\mathcal{R}V+V'\mathcal{R}V' \\ -V\mathcal{R}e^{(\delta\beta_1L_{r1}+\delta\beta_2L_{r2})/2}V'-V'\mathcal{R}e^{(-\delta\beta_1L_{r1}-\delta\beta_2L_{r2})/2}V]P_e\left(e^{\delta\beta_pH_p/2}\otimes\mathbb{1}_p\right). \end{aligned} \tag{10.9}$$

We now examine the term in $[\cdots]$. Write $V=V_1+V_2$, $V'=V_1'+V_2'$, where, recall, $V_j=\pi(v_j)$, $V_j'=\pi'(v_j)$. Notice that we have $(\mathbb{1}_p\otimes\mathbb{1}_p\otimes P_\Omega\otimes\mathbb{1}_1)V_1(\mathbb{1}_p\otimes\mathbb{1}_p\otimes P_\Omega\otimes\mathbb{1}_2)=0$, and similarly for V_2, from which it follows that the expression $P_e[\cdots]P_e$ in (10.9) splits into a sum

$$P_e\sum_{j=1,2}\left[V_j\mathcal{R}V_j+V'\mathcal{R}V_j'-V_j\mathcal{R}e^{(\delta\beta_1L_{r1}+\delta\beta_2L_{r2})/2}V_j'-V_j'\mathcal{R}e^{(-\delta\beta_1L_{r1}-\delta\beta_2L_{r2})/2}V_j\right]P_e.$$

We consider the $j=1$ term. Using that L_{r2} commutes with V_1,V_1' and that $L_{r2}P_e=0$, we see that

$$\begin{aligned} P_eV_1\mathcal{R}e^{\delta\beta_1L_{r1}+\delta\beta_2L_{r2}}V_1'P_e &= P_eV_1\mathcal{R}e^{\delta\beta_1L_{r1}/2}V_1'P_e=P_eV_1\mathcal{R}e^{\delta\beta_1L_0/2}e^{-\delta\beta_1L_p/2}V_1'P_e \\ &= P_e\left(\mathbb{1}_p\otimes e^{\delta\beta_1H_p/2}\right)V_1\mathcal{R}e^{\delta\beta_1L_0/2}V_1'\left(e^{-\delta\beta_pH_p/2}\otimes\mathbb{1}_p\right)P_e \\ &= \left(e^{\delta\beta_1H_p/2}\otimes\mathbb{1}_p\right)P_eV_1\mathcal{R}e^{\delta\beta_1(L_0-e)/2}V_1'P_e\left(e^{-\delta\beta_1H_p/2}\otimes\mathbb{1}_p\right). \end{aligned}$$

All other terms in (10.9) for $j=1$, as well as the terms for $j=2$, are treated similarly and one arrives at

$$\begin{aligned} -\Lambda_e = \left(e^{-\beta_pH_p/2}\otimes\mathbb{1}_p\right)\sum_{j=1,2}\left(e^{\beta_jH_p/2}\otimes\mathbb{1}\right)P_e[V_j\mathcal{R}V_j+V_j'\mathcal{R}V_j' \\ -V_j\mathcal{R}e^{\delta\beta_j(L_0-e)/2}V_j'-V_j'\mathcal{R}e^{-\delta\beta_j(L_0-e)/2}V_j]P_e\left(e^{-\beta_jH_p/2}\otimes\mathbb{1}_p\right)\left(e^{\beta_pH_p/2}\otimes\mathbb{1}_p\right), \end{aligned}$$

where we used $\delta\beta_p-\delta\beta_j=\beta_p-\beta_j$. Hence, $-\Lambda_e=RHS(10.1)+(e^{-\delta_pH_p}\otimes\mathbb{1}_p)R(e^{\delta_pH_p}\otimes\mathbb{1}_p)$, where

$$\begin{aligned} R=\sum_{j=1,2}\left(e^{\beta_jH_p/2}\otimes\mathbb{1}\right)P_e[V_j\mathcal{R}(1-e^{\delta\beta_j(L_0-e)/2})V_j' \\ +V_j'\mathcal{R}(1-e^{-\delta\beta_j(L_0-e)/2})V_j]P_e\left(e^{-\beta_jH_p/2}\otimes\mathbb{1}_p\right). \end{aligned}$$

Since L_0 implements the free dynamics, we have that $e^{izL_0}V_je^{-izL_0}$ commutes with V_j', for $z \in \mathbb{C}$. Using this, writing $\frac{\mathbf{1}-e^{\delta\beta_j(L_0-e)/2}}{L_0-e\pm i0} = -\int_0^{\delta\beta_j/2} ds\, e^{s(L_0-e)}$ and using that $P_eL_0 = eP_e$, we see that

$$P_eV_j\frac{\mathbf{1}-e^{\delta\beta_j(L_0-e)/2}}{L_0-e-i0}V_j'P_e = -P_eV_j'\frac{\mathbf{1}-e^{-\delta\beta_j(L_0-e)/2}}{L_0-e+i0}V_jP_e.$$

Consequently, $R = 0$. This concludes the proof of Proposition 10.1. ■

11 Perturbation Theory for NESS

In this section we develop a perturbation theory for the NESS $\eta(A) := \left\langle \Omega_{\overline{\theta}}^*, \pi(A)_\theta\Omega_\theta \right\rangle$. The vectors $\Omega_{\overline{\theta}}^*$ and Ω_θ are the zero eigenvectors of the operators $K_{\overline{\theta}}^*$ and K_θ respectively, see (8.10) and (8.11). We derive perturbation expansions for $\Omega_{\overline{\theta}}^*$ and Ω_θ, see (11.10) and (11.15) below using the Feshbach maps introduced in [6, 7], and extended in [5]. We review the definitions and some properties of these maps referring the reader to [7, 5] for more detail. For simplicity we present here the original version, [6, 7], though the refined one, [5], the smooth Feshbach map, is easier to use from a technical point of view.

Let X be a Banach space and let P be a projection on X. Define $\overline{P} := \mathbf{1}-P$ and let $H_{\overline{P}} := \overline{P}H\overline{P}$ and $R_{\overline{P}}(H) := \overline{P}H_{\overline{P}}^{-1}\overline{P}$ if $H_{\overline{P}}$ is invertible on $\mathrm{Ran}\overline{P}$. We define the Feshbach map F_P by the relation $F_P(H) := P\,(H - HR_{\overline{P}}(H)H)\,P$ on the domain

$$\begin{aligned}&\mathrm{Dom}(F_P) = \\ &\quad\{H : X \to X | H_{\overline{P}} \text{ is invertible}, \mathrm{Ran}P \subseteq \mathrm{Dom}(H), \mathrm{Ran}R_{\overline{P}}(H) \subseteq \mathrm{Dom}(PH\overline{P})\}.\end{aligned}$$

A key property of the maps F_P is given in the following statement proven in [7]:

Theorem 11.1 (Isospectrality Theorem) *(i)* $0 \in \sigma(H) \iff 0 \in \sigma(F_P(H))$,

(ii) $H\psi = 0 \iff F_P(H)\varphi = 0$ *with* $\varphi = P\psi$ *("⇒") and* $\psi = (\mathbf{1} - R_{\overline{P}}(H)H)\varphi$ *("⇐").*

Let $P_{e\rho}$ be defined as

$$P_{e\rho} := \chi_{L_p=e} \otimes \chi_{M_\theta\le\rho}, \tag{11.1}$$

where $\chi_{L_p=e}$ is the eigenprojection for the operator L_p corresponding to an eigenvalue $e \in \sigma(L_p)$ and $\chi_{M_\theta\le\rho}$ is the spectral projection for the self-adjoint operator M_θ corresponding to the spectral interval $[0, \rho]$ (remember that M_θ is a positive operator).

The following result is proven in [25], Lemma 6.3.

Lemma 11.2 *Take* $\rho_0 \in (0, \sigma/2)$ *and let* $|g| < \sqrt{\rho_0}\, g_0$. *If* $z \in S_e$ *then* $K_{\theta z} := K_\theta - z \in \mathrm{Dom}(F_{P_{e\rho_0}})$, *and the operator* $K_{\theta z}^{(1)} := F_{P_{e\rho_0}}(K_{\theta z})$ *acting on* $\mathrm{Ran}P_{e\rho_0}$ *is of the form*

$$K_{\theta z}^{(1)} = (e - z)\mathbf{1} + L_{r\theta} + g^2\Lambda_e + O(\epsilon(g, \rho_0)). \tag{11.2}$$

The remainder is estimated in operator norm, $\|O(\epsilon(g, \rho_0))\| \le C\epsilon(g_{\delta\beta}, \rho_0)$, *where* $g_{\delta\beta} = |g|\max_j \sup_{|\theta|<\theta_0} \|G_j\|_{\delta\beta_j,1/2,\theta}$, *with a* C *independent of* $\delta\beta_j$, θ, *and where we have set*

$$\epsilon(g, \rho) := |g|\rho^\mu + |g|^3\rho^{-1/2} + |g|^2\rho^{2\mu-1}. \tag{11.3}$$

To unify the following analysis we write $\Omega_{\theta\#}^\#$ for either Ω_θ or $\Omega_{\overline{\theta}}^*$. Correspondingly, $I_{\theta\#}^\#$, $L_{0\theta\#}$ and $K_{\theta\#}^\#$ stand for either I_θ or $I_{\overline{\theta}}^*$, for either $L_{0\theta}$ or $L_{0\overline{\theta}}$, and for either K_θ or $K_{\overline{\theta}}^*$, respectively. We use the shorthand $P_0 \equiv P_{0\rho_0}$ and $\overline{R}_0(A) := \overline{P}_0A_{\overline{P}_0}^{-1}\overline{P}_0$, where $A_P := PAP$.

Theorem 11.1 and Lemma 11.2 imply that $K^{\#}_{\theta\#} \in \mathrm{Dom}(F_{P_0})$, that

$$F_{P_0}(K^{\#}_{\theta\#})P_0\Omega^{\#}_{\theta\#} = 0, \tag{11.4}$$

and that the original eigenvector $\Omega^{\#}_{\theta\#}$ can be reconstructed as

$$\Omega^{\#}_{\theta\#} = \left[\mathbf{1} - g\overline{R}_0(K^{\#}_{\theta\#})I^{\#}_{\theta\#}\right]P_0\Omega^{\#}_{\theta\#}. \tag{11.5}$$

We expand $\overline{R}_0(K^{\#}_{\theta\#})$ in this expression into a Neumann series,

$$\Omega^{\#}_{\theta\#} = \sum_{n=0}^{N-1} g^n(-\overline{R}_0(L_{0\theta\#})I^{\#}_{\theta\#})^n P_0\Omega^{\#}_{\theta\#} + O\left((g_{\delta\beta}\rho_0^{-1/2})^N\right), \tag{11.6}$$

for any $N \geq 1$, provided that $O((g_{\delta\beta}\rho_0^{-1/2})) = o_g(1)$. The remainder term in (11.6) is obtained by using a standard estimate on the Nth term of the convergent Neumann series. Indeed, writing

$$\begin{aligned} &\left[\overline{R}_0(L_{0\theta\#})I^{\#}_{\theta\#}\right]^N \\ = \quad &(M_\theta+\rho_0)^{-1/2}\left[\frac{M_\theta+\rho_0}{L_{0\theta\#}\overline{P}_0}\overline{P}_0(M_\theta+\rho_0)^{-1/2}I^{\#}_{\theta\#}(M_\theta+\rho_0)^{-1/2}\right]^N(M_\theta+\rho_0)^{1/2} \end{aligned} \tag{11.7}$$

and using the estimates

$$|g|\left\|(M_\theta+\rho_0)^{-1/2}I^{\#}_{\theta\#}(M_\theta+\rho_0)^{-1/2}\right\| \leq g_{\delta\beta}\rho_0^{-1/2} \tag{11.8}$$

(see also Lemma 5.3 of [25]) and $\left\|\frac{M_0+\rho_0}{L_{0\theta\#}\overline{P}_0}\overline{P}_0\right\| \leq C$, we obtain

$$|g|^N\left\|\left[\overline{R}_0(L_{0\theta\#})I^{\#}_{\theta\#}\right]^N P_0\Omega^{\#}_{\theta\#}\right\| \leq \rho_0^{-1/2}(g_{\delta\beta}\rho_0^{-1/2})^N\rho_0^{1/2}\ . \tag{11.9}$$

Observe that since $g_{\delta\beta}\rho_0^{-1} = o_g(1)$ we have

$$\Omega^{\#}_{\theta\#} = \sum_{n=0,1} g^n(-\overline{R}_0(L_{0\theta\#})I^{\#}_{\theta\#})^n P_0\Omega^{\#}_{\theta\#} + o(g). \tag{11.10}$$

Let $K^{\#(1)} := F_{\rho_0}(K^{\#}_{\theta\#})$. As in (11.2) it can be written as

$$K^{\#(1)} = K_0^{\#(1)} + W^{\#},$$

where $K_0^{\#(1)} := L_{r\theta\#} + g^2\Lambda^{\#}$ with $\Lambda^{\#}$ either Λ_0 or Λ_0^*, where Λ_0 is given by (7.1) with $e=0$. We assume that $\delta\beta$ varies in the set $|\delta\beta| \leq c$, for some $c>0$, so that $g_{\delta\beta}$ can be replaced by g. Take $\rho_0 = g^{2-2\alpha}$ with $\alpha = \frac{\mu-1/2}{\mu+1/2}$, then Lemma 11.2 gives

$$W^{\#} = O(g^{2+\alpha}). \tag{11.11}$$

By Theorem 11.1,

$$K^{\#(1)}P_0\Omega^{\#}_{\theta\#} = 0. \tag{11.12}$$

Let $Q_0^{\#} := \chi_{K_0^{\#(1)}=0} = \chi_{\Lambda^{\#}=0} \otimes \chi_{L_r=0}$, where $\chi_{\Lambda^{\#}=0}$ is the Riesz projection onto the kernel of $\Lambda^{\#}$. Since the eigenvalue zero of the operator $K^{\#(1)}$ (and of the operator $K_0^{\#(1)}$) is separated from the rest of its spectrum by a gap of order

$$\min(\tau', g^2) \gg g^{2+\alpha}, \tag{11.13}$$

as is seen by a standard argument, we conclude that $K^{\#(1)} \in \mathrm{Dom}(F_{Q_0^\#})$. Therefore, by Theorem 11.1,

$$P_0 \Omega^\#_{\theta\#} = \left(\mathbf{1} - R_{\overline{Q_0^\#}}(K^{\#(1)})W^\#\right) Q_0^\# \Omega^\#_{\theta\#}. \tag{11.14}$$

Since $\|R_{\overline{Q_0^\#}}(K^{\#(1)})\| \leq C[\min(\tau', g^2)]^{-1}$ we have the absolutely convergent perturbation expansion

$$P_0 \Omega^\#_{\theta\#} = C^\# \sum_{n=0}^{\infty} \left(-R_{\overline{Q_0^\#}}(K_0^{\#(1)})W^\#\right)^n \zeta^\# \otimes \Omega_r. \tag{11.15}$$

Here, $\zeta^\# \in \mathrm{Ran}\, \chi_{L_p=0}$ is the unique vector in the kernel of $\Lambda^\#$, normalized as

$$\Lambda^\# \zeta^\# = 0,\ \|\zeta\| = 1,\ \langle \zeta^*, \zeta\rangle = 1. \tag{11.16}$$

Letting $\zeta^{\#\#}$ equal ζ if $\# = *$ and letting it equal ζ^* otherwise, the constant in (11.15) takes the form $C^\# = \left\langle \zeta^{\#\#} \otimes \Omega_r, \Omega^\#_{\theta\#}\right\rangle$.

The overlap $\langle \zeta^*, \zeta\rangle$ can be chosen strictly positive since the $\zeta^\#$ are the Perron-Frobenius eigenvectors of $\Lambda^\#$ (i.e., their components can be chosen non-negative), and every component of ζ is strictly positive (see below). The last relation in (11.16) is then achieved by scaling ζ^* properly. The normalization $\langle \Omega^*_\theta, \Omega_\theta\rangle = 1$ together with (11.6), (11.11) (11.15) and (11.16) implies that

$$\overline{C^*} C = 1 + o(g). \tag{11.17}$$

If the condition (11.13), $\tau' \gg g^{2+\alpha}$, does not hold then we have to apply the Feshbach map iteratively and use a corresponding novel perturbation theory for eigenvectors. We omit here this analysis and refer the reader to [6, 7, 5] for general references on such a RG perturbation theory and [26, 27] for the RG perturbation theory in our specific case.

In Section 10 we have shown that $\mathrm{Null}\Lambda = \mathbb{C}\Omega_p$ and consequently $\zeta = \Omega_p$ and the vector $\Omega_0 = \zeta \otimes \Omega_r = \Omega_p \otimes \Omega_r$ is our unperturbed state introduced in Section 6. Recall that $\Omega_p = \Omega_p^{\beta_p}$ is the particle Gibbs state at temperature β_p.

In general there is no simple expression for the zero eigenvector ζ^*. However, there are three cases where such an expression does exist:

(i) If $\beta_1 = \beta_2 = \beta$ then

$$\zeta^* = \sum_j \gamma_j \varphi_j \otimes \varphi_j, \quad \text{with} \quad \gamma_j = \frac{\mathrm{Tr} e^{-\beta_p H_p}}{\mathrm{Tr} e^{-\beta H_p}} e^{(\beta_p/2 - \beta)E_j} \tag{11.18}$$

(ii) If β_1 is fixed and $\beta_2 \to 0$ then by perturbation theory

$$\zeta^* = \sum_j \gamma_j \varphi_j \otimes \varphi_j \quad \text{with} \quad \gamma_j = \frac{\mathrm{Tr} e^{-\beta_p H_p}}{\mathrm{Tr} e^{-\beta H_p/2}} (1 + O(\beta_2)) \tag{11.19}$$

(iii) If $N = \dim \mathcal{H}_p = 2$, then

$$\zeta^* = \alpha \varphi_1 \otimes \varphi_1 + \varphi_2 \otimes \varphi_2 \tag{11.20}$$

where

$$\alpha = \left[1 + \frac{\sum_{j=1,2} g_j(E)}{\sum_{j=1,2} g_j(E)\rho_j(E)}\right] e^{-\beta_p E/2}. \tag{11.21}$$

Here, we use the notation $E := E_2 - E_1$, $\rho_j(E) := \frac{1}{e^{\beta_j E} - 1}$ and

$$g_j(E) := \int_{|k|=|E|} \left[|G_j(k)_{12}|^2 + |G_j(k)_{21}|^2\right] dk\ .$$

Equation (11.18) follows from (10.1), (10.3) and (10.4). The expressions (11.20) – (11.21) come simply from solving a two-dimensional eigenvalue problem. Equation (11.19) follows from a straightforward perturbation theory in β_2.

Moreover, expressions (10.1)–(10.2) for Λ_0 implies the following relations among vectors $\zeta^* \equiv \zeta^*(\beta_p)$ corresponding to different particle temperatures

$$\zeta^*(\beta_p) = (e^{\beta_p H_p/2} \otimes \mathbb{1}_p)\zeta^*(0) \ . \tag{11.22}$$

12 Entropy production rate for η

In this section we prove Theorem 3.2. Recall that the stationary state η mentioned in this theorem is, in fact, given in (9.11). To analyze the entropy production, $EP(\eta)$, in this state η we use expression (2.24),

$$EP(\eta) = (\beta_1 - \beta_2)\eta(\phi_1) = ig(\beta_1 - \beta_2)\eta\big(a_1(\omega G_1) - a_1^*(\omega G_1)\big), \tag{12.1}$$

which relates it to the heat flow, $\eta(\phi_1)$, in the state η. Recall that

$$\phi_1 = ig[v_1, H_{r1}]. \tag{12.2}$$

If $\beta_1 = \beta_2$, then $\eta(\phi_1) = 0$, c.f. [19]. We want to show here that

$$\eta(\phi_1) > 0 \quad \text{if} \quad \beta_1 > \beta_2. \tag{12.3}$$

Our proof is based on

Theorem 12.1 *Set $\beta = \max(\beta_1, \beta_2)$ and let $\Omega_0^* = \zeta^* \otimes \Omega_r$ with $\Lambda^*\zeta^* = 0$ (see (11.16)) and with the vector Ω_0 defined in (2.37). Under the conditions of Theorem 3.1, with $\mu > 3/2$, we have*

$$\eta(\phi_1) = g^2\eta' + o(g^2)O\,(\delta\beta)\,, \tag{12.4}$$

where

$$\eta' = 2\mathrm{Re}\left\langle \Omega_0^*, \pi(v_1)L_{r1}i(L_0 + i0)^{-1}e^{-\beta L_0/2}\pi(v_1)\Omega_0 \right\rangle. \tag{12.5}$$

Moreover, we have the explicit expression

$$\eta' = 2\pi \sum_{j>i} (\gamma_j e^{\beta_1 E_{ji}} - \gamma_i)\frac{E_{ji}\ g_{ji}(E_{ji})^2}{e^{\beta_1 E_{ji}} - 1}, \tag{12.6}$$

where $E_{ji} = E_j - E_i$, $g_{ji}(E)^2 = \int_{\mathbb{R}^3} d^3k |\langle\varphi_j, G_1(k)\varphi_i\rangle|^2\delta(E_{ji} - \omega)$ (see Condition (C)). The numbers $\gamma_j \geq 0$ are the components of the vector ζ^*, normalized as in (11.16), at $\beta_p = 0$:

$$\zeta^*|_{\beta_p=0} = \sum_j \gamma_j\varphi_j \otimes \varphi_j. \tag{12.7}$$

Observe that by (11.18), $\eta' = 0$ for $\beta_1 = \beta_2$.

The following result shows that η' is strictly positive for small nonzero temperature differences.

Theorem 12.2 *If $\delta\beta = \beta_1 - \beta_2 > 0$ is small and the coupling functions (2.4) satisfy $G = G_1 = G_2$ then the linear term of η' in $\delta\beta$ is*

$$\frac{\delta\beta}{2}\frac{Z_p(\beta_p)}{Z_p(\beta_1 + \beta_p/2)}\sum_{j>k}\frac{E_{jk}^2\ g_{jk}(E_{jk})^2}{e^{\beta_1 E_j} - e^{\beta_1 E_k}} > 0, \tag{12.8}$$

where $Z_p(\beta)=\operatorname{tr} e^{-\beta H_p}$ is the particle partition function.

Remark. Using (11.20) for the two-dimensional case it is easy to verify that the linear term of η' in $\delta\beta$ is strictly positive (without the assumption $G_1=G_2$). Also, in the general case, if G_1 is close to G_2 one deduces strict positivity of η' in the linear term in $\delta\beta$ by a perturbation argument.

Proof of Theorem 12.1. We first show 1. To simplify the exposition we restrict ourselves to the case $\tau'\gg g^{2+\alpha}$ for $\alpha=\frac{\mu-1/2}{\mu+1/2}$, and $\mu>3/2$. Pick $\rho_0=g^{2-2\alpha}$. First we prove an estimate on $\eta(\phi_1)$ which is rougher than (12.4) and then we explain how to obtain (12.4). Recall that

$$\eta(\phi_1)=\left\langle\Omega_{\overline{\theta}}^*,\pi(\phi_1)_\theta\Omega_\theta\right\rangle. \tag{12.9}$$

Let $\Omega_0^\#=\zeta^\#\otimes\Omega_r$ and take a number M so large that $(g^{2+\alpha}[\min(\tau',g^2)]^{-1})^M=o(g)$. Substituting expansions (11.6) and (11.15) into the r.h.s. of this expression, using that $(g_{\delta\beta}\rho_0^{-1/2})^2=o(g)$, for $\rho_0=g^{2-2\alpha}$, $\alpha>\frac{1}{2}$, and using that ϕ_1 is proportional to g, we find

$$\eta(\phi_1)=\eta_0+\eta_1+\mathrm{Rem}+o(g^2), \tag{12.10}$$

where

$$\begin{aligned}\eta_0 &:= \langle\Omega_0^*,\pi(\phi_1)\Omega_0\rangle, &(12.11)\\ \eta_1 &:= -g\left\langle\Omega_0^*,\left[I_\theta\overline{R}_0(L_{0\theta})\pi(\phi_1)_\theta+\pi(\phi_1)_\theta\overline{R}_0(L_{0\theta})I_\theta)\right]\Omega_0\right\rangle &(12.12)\end{aligned}$$

and

$$\begin{aligned}\mathrm{Rem} &= \sum_{m+n\geq 1}^{M}\sum_{k,l=0}^{1} g^{k+l}\Big\langle(-\overline{R}_0(L_{0\overline{\theta}})I_{\overline{\theta}}^*)^k(-R_{\overline{Q_0^*}}(K_0^{\#(1)})W^*)^m\Omega_0^*,\\ &\quad \pi(\phi_1)_\theta(-\overline{R}_0(L_{0\theta})I_\theta)^l(-R_{\overline{Q}_0}(K_0^{(1)})W)^n\Omega_0\Big\rangle. \qquad (12.13)\end{aligned}$$

Here we replaced the factor $\overline{C^*}C=1+o(g)$ by 1 (see (11.15) and (11.17)) in the r.h.s. Using the pull-through procedure and elementary estimates of the resulting integrals we obtain that

$$\mathrm{Rem}=o(g^2). \tag{12.14}$$

Since ϕ_1 is linear in creation and annihilation operators, see (12.1), we have

$$\eta_0=0. \tag{12.15}$$

It remains to compute η_1. Using

$$(\overline{P}_{\rho_0}-\mathbf{1})I_{\theta\#}^{\#}\Omega_0^\#=O(\rho_0) \tag{12.16}$$

([25] Lemma 5.3), removing the spectral deformation and using $\pi(\phi_1)=g[\pi(v_1),iL_{r1}]$, we obtain $\eta_1=g^2\eta'+o(g^2)$, where

$$\eta'=-\left\langle\Omega_0^*,\left[\pi(v_1)iL_{r1}(L_0+i0)^{-1}I-I(L_0+i0)^{-1}iL_{r1}\pi(v_1)\right]\Omega_0\right\rangle. \tag{12.17}$$

Next, note that the contribution of the v_2-part of I to η' is zero since the resulting expression is linear in creation and annihilation operators for the first and second reservoirs separately. The contribution of the $\pi(v_1)$-part of I is also zero by the symmetry of (12.17). Hence we have $\eta'=A-B$, where

$$\begin{aligned}A &= \langle\Omega_0^*,\pi(v_1)iL_{r1}(L_0+i0)^{-1}\pi'\big(\gamma^{i/2}(v_1)\big)\Omega_0\rangle,\\ B &= \langle\Omega_0^*,\pi'\big(\gamma^{i/2}(v_1)\big)iL_{r1}(L_0+i0)^{-1}\pi(v_1)\Omega_0\rangle.\end{aligned}$$

Using that $\pi'\big(\gamma^{i/2}(v_1)\big)\Omega_0 = Je^{(\beta L_0-\tilde{L})/2}\pi(v_1)\Omega_0$, where $\tilde{L}$ is given after (4.19), and the fact that $Je^{-\tilde{L}/2}\pi(v_1)\Omega_0 = \pi(v_1^*)\Omega_0 = \pi(v_1)\Omega_0$, we transform

$$A = \langle\Omega_0^*, \pi(v_1)iL_{r1}(L_0+i0)^{-1}e^{-\beta L_0/2}\pi(v_1)\Omega_0\rangle \ .$$

We use the relations $\pi'\big(\gamma^{i/2}(v_1)\big) = Je^{(\beta L_0-\tilde{L})/2}\pi(v_1)e^{-(\beta L_0-\tilde{L})/2}J$ and

$$JiL_{r1}(L_0+i0)^{-1}\pi(v_1)\Omega_0 = -iL_{r1}(L_0+i0)^{-1}e^{-\tilde{L}/2}\pi(v_1)\Omega_0 \ ,$$

see also (5.1), to find that

$$B = -\langle\Omega_0^*, Je^{(\beta L_0+\tilde{L})/2}\pi(v_1)iL_{r1}(L_0+i0)^{-1}e^{-\beta L_0/2}\pi(v_1)\Omega_0\rangle \ .$$

Fingally, since $\langle Ju, Jv\rangle = \overline{\langle u,v\rangle}$, $J\Omega_0^* = \Omega_0^*$, $L_p\Omega_0^* = 0$ and $(\beta L_0 - \overline{L})\Omega_0^* = 0$ we obtain

$$\overline{B} = -\langle\Omega_0^*, \pi(v_1)iL_{r1}(L_0+i0)^{-1}e^{-\beta L_0/2}\pi(v_1)\Omega_0\rangle = -A.$$

Since $\eta' = A - B$ this gives

$$\eta' = 2\mathrm{Re}\,\langle\Omega_0^*, \pi(v_1)iL_{r1}(L_0+i0)^{-1}e^{-\beta L_0/2}\pi(v_1)\Omega_0\rangle \ . \tag{12.18}$$

Collecting estimates (12.10), (12.14), (12.15) and $\eta_1 = g^2\eta' + o(g^2)$ we find that

$$\eta(\phi_1) = g^2\eta' + o(g^2) \tag{12.19}$$

where η' is given by (12.5). This proves the rough version of (12.4)–(12.5).

Before we refine estimate (12.19) let us show (12.6). We expand the vectors Ω_0^* and Ω_0 in the basis $\varphi_{jj}\otimes\Omega_r$, $\varphi_{jj} = \varphi_j\otimes\varphi_j$,

$$\Omega_0 = \sum_{j=1}^n \alpha_j\varphi_{jj}\otimes\Omega_r, \quad \Omega_0^* = \sum_{j=1}^n \tilde{\gamma}_j\varphi_{jj}\otimes\Omega_r \tag{12.20}$$

with $\alpha_j \geq 0$, $\sum_j \alpha_j = 1$ and $\tilde{\gamma}_j \geq 0$, $\sum_j\tilde{\gamma}_j = 1$. Plugging the expressions in (12.20) into the r.h.s. of (12.5), using (2.35) in order to express $\pi(v_1)$ in terms of creation and annihilation operators, $a_{\ell 1}^\#$ and $a_{r1}^\#$,

$$\pi(v_1) = a_{\ell 1}(\sqrt{1+\rho_1}\,G_{1\ell}) + a_{r1}(\sqrt{\rho_1}\,G_{1\ell}) + h.c.,$$

where $\rho_1 = (e^{\beta_1\omega}-1)^{-1}$, pulling through the annihilation operators to the right and using that $\pi(v_1)$ (or $G_{1\ell}$) acts only on the first (left) factor in $\varphi_{jj} = \varphi_j\otimes\varphi_j$, we obtain

$$\begin{aligned}\eta' \quad &= \quad -2\sum_j \alpha_j\tilde{\gamma}_j\mathrm{Im}\int\Big\{(1+\rho_1)\omega\left\langle\varphi_j, G_1^*(H_p - E_j + \omega + i0)^{-1}e^{-\beta(H_p-E_j+\omega)/2}G_1\varphi_j\right\rangle \\ &\qquad -\rho_1\omega\left\langle\varphi_j, G_1^*(H_p - E_j - \omega + i0)^{-1}e^{-\beta(H_p-E_j-\omega)/2}G_1^*\varphi_j\right\rangle\Big\}d^3k.\end{aligned}$$

Inserting the partition of unity $\mathbf{1} = \sum_j|\varphi_j\rangle\langle\varphi_j|$ into the inner products on the r.h.s. we obtain furthermore

$$\eta' = 2\pi\sum_{i,j}\alpha_i\tilde{\gamma}_i\int\omega|\langle\varphi_i, G_1\varphi_j\rangle|^2\left\{(1+\rho_1)\delta(E_{ji}+\omega) - \rho_1\delta(E_{ji}-\omega)\right\}d^3k.$$

Interchanging the labels in the sum of the first term and noticing that in the resulting expression the integrals vanish unless $E_{ji} > 0$, i.e. $E_j > E_i$, or $j > i$, we arrive at

$$\eta' = 2\pi\sum_{j>i}(\alpha_j\tilde{\gamma}_je^{\beta_1E_{ji}} - \alpha_i\tilde{\gamma}_i)E_{ji}\ g_{ji}(E_{ji})^2\ (e^{\beta_1E_{ji}}-1)^{-1},$$

where $g_{ji}(E)^2 := \int_{\mathbb{R}^3} d^3k \,|\langle\varphi_j, G_1(k)\varphi_i\rangle|^2\delta(E_{ji}-\omega)$. Observing that, due to (11.22), $\alpha_i\tilde{\gamma}_i = \tilde{\gamma}_i|_{\beta_p=0} \equiv \gamma_i$, we arrive at (12.6).

Since $\eta' = O(\delta\beta)$ (the energy flow vanishes if $\beta_1=\beta_2$) estimate (12.19) is ineffective if $\delta\beta$ is so small that $\delta\beta\, g^2 = o(g^2)$. However, with a little bit more work (12.19) can be upgraded to estimate (12.4). We sketch a proof of this estimate without going into much detail. We begin with some notation.

Consider the *self-adjoint* Liouville operator for equal reservoir temperatures $\beta_2=\beta_1=\beta$,

$$\tilde{L} = L_0 + g\tilde{I}, \tag{12.21}$$

where

$$\tilde{I} = \tilde{\pi}(v) - \tilde{\pi'}(v). \tag{12.22}$$

Define $\widetilde{\Omega} = \frac{e^{-\beta\widetilde{L}^{(l)}/2}\Omega_0}{\|e^{-\beta\widetilde{L}^{(l)}/2}\Omega_0\|}$, where $\widetilde{L}^{(l)} = L_0+\widetilde{\pi}(v)$ and $\widetilde{\pi} = \pi|_{\beta_1=\beta_2=\beta}$. Since the entropy production does not depend on β_p we put from now on $\beta_p=\beta_1$. The operator $K|_{\beta_p=\beta_1=\beta_2=\beta} = L$ is selfadjoint, (5.12), and hence $\widetilde{\Omega}^*_{\overline{\theta}} = \widetilde{\Omega}_{\overline{\theta}}$. From (12.9) we obtain

$$\begin{aligned}\eta(\phi_1) &= \left\langle(\Omega^*_{\overline{\theta}}-\widetilde{\Omega}_{\overline{\theta}}), \pi(\phi_1)_\theta\Omega_\theta\right\rangle + \left\langle\widetilde{\Omega}_{\overline{\theta}}, \pi(\phi_1)_\theta(\Omega_\theta-\widetilde{\Omega}_\theta)\right\rangle \\ &\quad + \left\langle\widetilde{\Omega}_{\overline{\theta}}, \pi(\phi_1)_\theta\widetilde{\Omega}_\theta\right\rangle.\end{aligned} \tag{12.23}$$

We consider the last term first. Recall that $\phi_1 = g[v, iH_{r1}] = g[v_1, iH_{r1}]$ and therefore $\pi(\phi_1) = \widetilde{\pi}(\phi_1)$. It follows that

$$\langle\widetilde{\Omega}_{\overline{\theta}}, \pi(\phi_1)_\theta\widetilde{\Omega}_\theta\rangle = \langle\widetilde{\Omega}_{\overline{\theta}}, \widetilde{\pi}(\phi_1)_\theta\widetilde{\Omega}_\theta\rangle = \langle\widetilde{\Omega}, \widetilde{\pi}(\phi_1)\widetilde{\Omega}\rangle. \tag{12.24}$$

Note that the r.h.s. of (12.24) describes the heat flow into reservoir $r1$ for the equal temperature system. Since the heat flows vanish individually in the equal temperature case we have shown that the last term in (12.23) vanishes.

To estimate the first two terms on the r.h.s. we use as before expansions (11.6) and (11.15) for $\Omega^{\#}_{\theta\#} = \Omega_\theta, \Omega^*_{\overline{\theta}}$ and similar expansions (obtained by setting $\beta_2=\beta_1=\beta$ in (11.6) and (11.15)) for $\widetilde{\Omega}_\theta$. As a result we obtain an expression for (12.23) of the type (12.10) – (12.13) but with some of the powers in Rem, (12.13), replaced by the differences, e.g. $(-\overline{R}_0(L_{0\theta})I_\theta)^l - (-\overline{R}_0(L_{0\theta})\widetilde{I}_\theta)^l$ or $(R_{\overline{Q}_0}(K_0^{(1)})W)^n - (R_{\overline{Q}_0}(K_0^{(1)})W)^n|_{\beta_2=\beta_1=\beta}$. These differences are estimated by using a telescopic expansion, e.g.,

$$\begin{aligned}&(-\overline{R}_0(L_{0\theta})I_\theta)^l - (-\overline{R}_0(L_{0\theta})\widetilde{I}_\theta)^l \\ &= \sum_{j=1}^{l}(-\overline{R}_0(L_{0\theta})I_\theta)^{j-1}(-\overline{R}_0(L_{0\theta}))(I_\theta-\widetilde{I}_\theta)(-\overline{R}_0(L_{0\theta})\widetilde{I}_\theta)^{l-j},\end{aligned} \tag{12.25}$$

and then estimating the first type of the differences in norm while for the second type we do first the pull-through and contraction procedure and then estimate the resulting integrals. As a result we have

$$\begin{aligned}\eta(\phi_1) &= -g\left\langle\Omega_0^*-\Omega_0, \pi(\phi_1)_\theta\overline{R}_0(L_{0,\theta})I_\theta\Omega_0\right\rangle \\ &\quad -g\left\langle\overline{R}_0(L_{0,\overline{\theta}})\left[I^*_{\overline{\theta}}\Omega_0^* - \widetilde{I}_{\overline{\theta}}\Omega_0\right], \pi(\phi_1)_\theta\Omega_0\right\rangle \\ &\quad -g\left\langle\Omega_0, \pi(\phi_1)_\theta\overline{R}_0(L_{0,\theta})\left[I_\theta-\widetilde{I}_\theta\right]\Omega_0\right\rangle \\ &\quad +o(g^2)O(\delta\beta) \\ &= \eta_1+\eta_2+o(g^2)O(\delta\beta)\end{aligned} \tag{12.26}$$

where η_1 is given in (12.12) and

$$\eta_2 = g\left\langle \Omega_0, \left[\widetilde{I}_\theta \overline{R}_0(L_{0,\theta})\pi(\phi_1)_\theta + \pi(\phi_1)_\theta \overline{R}_0(L_{0,\theta})\widetilde{I}_\theta\right]\Omega_0\right\rangle. \tag{12.27}$$

Since the contribution of the v_2-component of $\widetilde{I}_\theta$ is zero we can omit the tilde ($\sim$) in (12.27). Thus the expression for η_2 coincides up to the sign and the substitution $\Omega_0^* \to \Omega_0$ with the expression (12.12) for η_1, i.e.

$$\eta_1 + \eta_2 = -g\left\langle \Omega_0^* - \Omega_0, \left[I_\theta \overline{R}_0(L_{0,\theta})\pi(\phi_1)_\theta + \pi(\phi_1)_\theta \overline{R}_0(L_{0,\theta})I_\theta\right]\Omega_0\right\rangle. \tag{12.28}$$

We proceed with the r.h.s. of (12.28) exactly as we did above with η_1 alone in equations (12.16) – (12.18), and we arrive at

$$\eta(\phi_1) = 2g^2 \mathrm{Re}\left\langle \Omega_0^* - \Omega_0, \pi(v_1) i L_{r1}(L_0 + i0)^{-1} e^{-\beta L_0/2}\pi(v_1)\Omega_0\right\rangle + o(g^2)O(\delta\beta).$$

(12.4) follows by noticing that

$$\mathrm{Re}\left\langle \Omega_0, \pi(v_1) i L_{r1}(L_0 + i0)^{-1} e^{-\beta L_0/2}\pi(v_1)\Omega_0\right\rangle = 0. \quad \blacksquare$$

Proof of Theorem 12.2. We want to control the components γ_j, (12.7), appearing in the expression (12.6) for η'. To this end we employ basic analytic perturbation theory (in $\delta\beta$) for the matrix family $M(\delta\beta) := \Lambda_0^*(\delta\beta)|_{\beta_p=0}$, where we consider β_1 to be fixed. Write Γ_j instead of Γ_{j0}, see (10.3). According to Proposition 10.1 we have $M(\delta\beta) = M_0 + \delta\beta M_1 + O(\delta\beta^2)$, where

$$\begin{aligned} M_0 &= -i(e^{-\beta_1 H_p/2}\otimes \mathbb{1})\left\{\Gamma_1(\beta_1) + \Gamma_2(\beta_1)\right\}(e^{\beta_1 H_p/2}\otimes \mathbb{1}), && (12.29)\\ M_1 &= -i(e^{-\beta_1 H_p/2}\otimes \mathbb{1})\left\{\tfrac{1}{2}[H_p\otimes \mathbb{1}, \Gamma_2(\beta_1)] - (\partial_\beta\Gamma_2)(\beta_1)\right\}(e^{\beta_1 H_p/2}\otimes \mathbb{1}). && (12.30)\end{aligned}$$

Let $\zeta^* = \zeta_0^* + \delta\beta\zeta_1^* + O(\delta\beta^2)$. The normalization $\langle\zeta^*, \zeta\rangle = 1$ (where $\zeta = \Omega_p(\beta_p)$ is the particle Gibbs state) implies that $\langle\zeta_0^*, \zeta\rangle = 1$ and $\left\langle\zeta_j^*, \zeta\right\rangle = 0$, for $j \geq 1$. Solving the zero-oder eigenvalue equation $M_0\zeta_0^* = 0$ gives

$$\zeta_0^* = \sum_j \gamma_j^{(0)}\varphi_j\otimes\varphi_j, \quad \gamma_j^{(0)} = \frac{Z_p(\beta_p)}{Z_p(\beta_1+\beta_p/2)}e^{-\beta_1 E_j}. \tag{12.31}$$

The first-order eigenvalue equation reads $M_1\zeta_0^* + M_0\zeta_1^* = 0$, which implies

$$[\Gamma_1(\beta_1) + \Gamma_2(\beta_1)](e^{\beta_1 H_p/2}\otimes \mathbb{1})\zeta_1^* = \partial_\beta|_{\beta_1}\Gamma_2(\beta)(e^{\beta H_p/2}\otimes \mathbb{1})\zeta_0^*. \tag{12.32}$$

We use here that $(e^{\beta_1 H_p/2}\otimes \mathbb{1})\zeta_0^*$ is in the kernel of H_p. Let $\Psi(\beta) := \sum_j e^{-\beta E_j/2}\varphi_j\otimes\varphi_j$. Since $\Gamma_2(\beta)\Psi(\beta) = 0$ we have $(\partial_\beta\Gamma_2)(\beta)\Psi(\beta) = -\Gamma_2(\beta)(\partial_\beta\Psi)(\beta)$, so

$$\begin{aligned} \partial_\beta|_{\beta_1}\Gamma_2(\beta)(e^{\beta H_p/2}\otimes \mathbb{1})\zeta_0^* &= C(\beta_1)\partial_\beta|_{\beta_1}\Gamma_2(\beta)\Psi(2\beta_1 - \beta)\\ &= C(\beta_1)\Gamma_2(\beta_1)(H_p\otimes \mathbb{1})\Psi(\beta_1), && (12.33)\end{aligned}$$

where $C(\beta_1) = Z_p(\beta_p)/Z_p(\beta_1+\beta_p/2)$. The r.h.s. of (12.33) is a vector in the orthogonal complement of $\ker\Gamma_2(\beta_1) = \mathbb{C}\Psi(\beta_1)$. Using this fact and (12.33) we solve (12.32) for ζ_1^*:

$$\begin{aligned} \zeta_1^* &= C(\beta_1)(e^{-\beta_1 H_p/2}\otimes \mathbb{1})[\Gamma_1(\beta_1) + \Gamma_2(\beta_1)]^{-1}\Gamma_2(\beta_1)(H_p\otimes \mathbb{1})\Psi(\beta_1) && (12.34)\\ &\quad + C'(e^{-\beta_1 H_p/2}\otimes \mathbb{1})\Psi(\beta_1), && (12.35)\end{aligned}$$

where the constant C' is determined by the normalization condition $\langle\zeta_1^*, \zeta\rangle = 0$. From expression (12.6) it is clear that the term (12.35) does not contribute to the value of η' (this is the same as saying that $\eta' = 0$ for $\delta\beta = 0$).

Under the assumption $G_1 = G_2 = G$ we have $\Gamma_1(\beta_1) = \Gamma_2(\beta_1)$ and the r.h.s. of (12.34) simplifies to an easy expression, which, when used in (12.6), yields (12.8). ■

Proof of Theorem 3.2. Now we are ready to prove (12.3). By an abstract result of [19], $EP(\eta) \geq 0$. Therefore, due to (12.1), $\eta(\phi_1) \geq 0$ for $\beta_1 \geq \beta_2$. Hence, due to (12.4), for g sufficiently small (depending on $\delta\beta$ in general),

$$\eta' \geq 0 \quad \text{for} \quad \beta_1 \geq \beta_2. \tag{12.36}$$

Next, expression (12.6) shows that

$$\eta' \text{ is analytic in } \beta_1 \text{ and } \beta_2 \tag{12.37}$$

(the γ_j are analytic in β_1, β_2). Equations (11.19) and (12.6) imply that

$$\eta' > 0 \quad \text{if } \beta_1 \text{ is fixed and } \beta_2 \text{ is sufficiently small.} \tag{12.38}$$

Relations (12.36) – (12.38) imply (12.3) for almost all values of $(\beta_1, \beta_2) \in (0,\infty) \times (0,\infty)$, in the sense that for fixed $\beta_1 \in (0,\infty)$, $EP(\eta_{\beta_1\beta_2})$ can vanish only for finitely many values of β_2 in any bounded subset of $(0,\infty)$. (The same holds for β_1 and β_2 interchanged).

If $G_1 = G_2$ or if the dimension of the particle system is 2 then we have $\eta' = \overline{\eta}\delta\beta + O((\delta\beta)^2)$ with $\overline{\eta} > 0$ independent of $\delta\beta$. Hence for g and $\delta\beta$ both small, but independent of each other, we have $\eta' > 0$. This follows from Theorem 12.2 and for the two-dimensional case from (11.20), (11.21).

The above results together with (12.4) imply the statement in Theorem 3.2 for $\beta_1 \neq \beta_2$. ■

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A Proof of existence of dynamics

In this appendix we prove the existence of dynamics (2.12). Recall the definition of the operator $L^{(\ell)} := L_0 + g\pi(v)$ and the one parameter group $\sigma^t(B) := e^{itL^{(\ell)}} B e^{-itL^{(\ell)}}$, $B \in \pi(\mathcal{A})''$.

Proposition A.1 *Assume the operators $v_n \in \mathcal{A}$ satisfy (2.13). Then the integrands on the r.h.s. of (2.12) are continuous functions, the series is absolutely convergent, the limit exists and equals*

$$\psi^t(A) = \mathrm{Tr}(\rho\sigma^t(\pi(A))) \tag{A.1}$$

and, consequently, is independent of the approximating operators.

Proof. Let $v_n \in \mathcal{A}$ be an approximating sequence for the operator v satisfying (2.13). We define the selfadjoint operators $L_n^{(\ell)} := L_0 + g\pi(v_n)$ on the dense domain $\mathcal{D}(L_0)$. Let the one parameter group $\sigma_{(n)}^t$ on $\pi(\mathcal{A})$ be given by

$$\sigma_{(n)}^t(B) := e^{itL_n^{(\ell)}} B e^{-itL_n^{(\ell)}}.$$

Set $\sigma_0^t(\pi(A)) := \pi(\alpha_0^t(A))$ and let ψ be an ω_0-normal state on $\mathcal{A}$, i.e.

$$\psi(A) = \mathrm{Tr}(\rho\pi(A)) \tag{A.2}$$

for some positive, trace class operator ρ on $\mathcal{H}$ of trace 1. Then using the definition $V_n = \pi(v_n)$ we find

$$\psi([\alpha_0^{t_m}(v_n), \cdots [\alpha_0^{t_1}(v_n), \alpha_0^t(A)] \cdots]) = \mathrm{Tr}(\rho[\sigma_0^{t_m}(V_n), \cdots [\sigma_0^{t_1}(V_n), \sigma_0^t(A)] \cdots]). \qquad \text{(A.3)}$$

Clearly the r.h.s. is continuous in $t_1, \cdots, t_m$ and therefore the integrals in (2.12) are well defined and, by a standard estimate, the series on the r.h.s. of (2.12) converges absolutely. In fact, using the Araki-Dyson series

$$\begin{aligned}\sigma_{(n)}^t(\pi(A)) \quad &= \quad \sum_{m=0}^{\infty} (ig)^m \int_0^t dt_1 \cdots \int_0^{t_{m-1}} dt_m \; [\sigma_0^{t_m}(\pi(v_n)), \cdots \\ &\qquad [\sigma_0^{t_1}(\pi(v_n)), \sigma_0^t(\pi(A))] \cdots],\end{aligned} \qquad \text{(A.4)}$$

one can easily see that this series is nothing but the Araki-Dyson expansion of the function $\mathrm{Tr}(\rho\sigma_{(n)}^t(\pi(A)))$. Thus we have shown that the r.h.s. of (2.12) is equal to $\lim_{n\to\infty} \mathrm{Tr}(\rho\sigma_{(n)}^t(\pi(A)))$.

Now, V_n converges to V strongly on the dense set $\mathrm{Span}\{\pi(W(f))\Omega_0 | f \in L_0^2\}$ as follows from (2.13) and the relation

$$\|(V_n - V)\pi(A)\Omega_0\|^2 = \omega_0(A^*(v_n^* - v^*)(v_n - v)A). \qquad \text{(A.5)}$$

Hence $L_n^{(\ell)}$ converges to $L^{(\ell)}$ strongly on the same set. Since this set is a core for $L_n^{(\ell)}$ and $L^{(\ell)}$ we conclude that $L_n^{(\ell)}$ converge to $L^{(\ell)}$ in the strong resolvent sense as $n \longrightarrow \infty$. Hence the functions $\mathrm{Tr}(\rho\sigma_n^t(\pi(A))$ converge to $\mathrm{Tr}(\rho\sigma^t(\pi(A)))$ which, in particular, shows (A.1). ■

B Positive Temperature Representation

B.1 Jakšić-Pillet Gluing

In this appendix, we represent the Hilbert space $\mathcal{H}$ in a form which is well suited for a definition of the translation transformation. This representation is due to [18].

Consider the Fock space

$$\mathcal{F} := \mathcal{F}(L^2(X \times \{1,2\})), \quad X = \mathbb{R} \times S^2 \qquad \text{(B.1.1)}$$

and denote $x = (u, \sigma) \in X$. The vacuum in $\mathcal{F}$ is denoted by $\tilde{\Omega}_r$. The smeared-out creation operator $a^*(F)$, $F \in L^2(X \times \{1,2\})$ is given by

$$a^*(F) = \sum_\alpha \int_X F(x,\alpha) a^*(x,\alpha)$$

and analogously for annihilation operators. The CCR read

$$[a(x,\alpha), a^*(x',\alpha')] = \delta_{\alpha,\alpha'}\delta(x - x').$$

Following [18], we introduce the unitary map

$$U : \left[\mathcal{F}(L^2(\mathbb{R}^3)) \otimes \mathcal{F}(L^2(\mathbb{R}^3))\right] \otimes \left[\mathcal{F}(L^2(\mathbb{R}^3)) \otimes \mathcal{F}(L^2(\mathbb{R}^3))\right] \to \mathcal{F}(L^2(X \times \{1,2\})) \qquad \text{(B.1.2)}$$

defined by

$$U\left([\Omega_{r1} \otimes \Omega_{r1}] \otimes [\Omega_{r2} \otimes \Omega_{r2}]\right) := \tilde{\Omega}_r \qquad \text{(B.1.3)}$$

and

$$\begin{aligned}U\Big(&[a^*(f_1) \otimes \mathbf{1} + \mathbf{1} \otimes a^*(g_1)] \otimes \mathbf{1} \otimes \mathbf{1} \\ &+ \mathbf{1} \otimes \mathbf{1} \otimes [a^*(f_2) \otimes \mathbf{1} + \mathbf{1} \otimes a^*(g_2)] \Big) U^{-1} \quad := \quad a^*(f \oplus g),\end{aligned} \qquad \text{(B.1.4)}$$

where, for $x = (u, \sigma) \in X$,

$$[f \oplus g](u, \sigma, \alpha) := \begin{cases} u\, f_\alpha(u\sigma), & u \geq 0, \\ u\, g_\alpha(-u\sigma), & u < 0. \end{cases} \tag{B.1.5}$$

This map is extended to the Hilbert space $\mathcal{H} = \mathcal{H}^p \otimes \mathcal{F}$ in the obvious way. We keep the same notation for its extension.

The operators $L_{r1} \otimes \mathbf{1}_{r2} + \mathbf{1}_{r1} \otimes L_{r2}$ and $N_{r1} \otimes \mathbf{1}_{r2} + \mathbf{1}_{r1} \otimes N_{r2}$ are mapped under U to the (total) free field Liouvillian and number operator given by

$$\begin{aligned} L_f &= \mathrm{d}\Gamma(u) = \sum_\alpha \int_X a^*(x, \alpha) u a(x, \alpha), \\ N &= \mathrm{d}\Gamma(\mathbb{1}) = \sum_\alpha \int_X a^*(x, \alpha) a(x, \alpha). \end{aligned}$$

Moreover, the interaction takes the form

$$UIU^{-1} = a^*(F_1) + a(F_2), \tag{B.1.6}$$

where $F_{1,2} \in L^2(X \times \{1, 2\}, \mathcal{B}(\mathcal{H}_p \otimes \mathcal{H}_p))$ are explicitly given by ($x = (u, \sigma) \in X = \mathbb{R} \times S^2$)

$$\begin{aligned} F_1(u, \sigma, \alpha) &= \sqrt{\frac{u}{1 - e^{-\beta_\alpha u}}} \\ &\times \begin{cases} \sqrt{u}\left(G_\alpha(u\sigma) \otimes \mathbb{1}_p - e^{-\beta_\alpha u/2} e^{\delta\beta_\alpha u/2}\ \mathbb{1}_p \otimes \alpha_p^{i\,\delta\beta_p/2}(\overline{G_\alpha}^*(u\sigma))\right), & u > 0 \\ -\sqrt{-u}\left(G^*_\alpha(-u\sigma) \otimes \mathbb{1}_p - e^{-\beta_\alpha u/2} e^{\delta\beta_\alpha u/2}\ \mathbb{1}_p \otimes \alpha_p^{i\,\delta\beta_p/2}(\overline{G_\alpha}(-u\sigma))\right), & u < 0 \end{cases} \end{aligned} \tag{B.1.7}$$

and

$$\begin{aligned} F_2(u, \sigma, \alpha) &= \sqrt{\frac{u}{1 - e^{-\beta_\alpha u}}} \\ &\times \begin{cases} \sqrt{u}\left(G_\alpha(u\sigma) \otimes \mathbb{1}_p - e^{-\beta_\alpha u/2} e^{-\delta\beta_\alpha u/2}\ \mathbb{1}_p \otimes \alpha_p^{-i\,\delta\beta_p/2}(\overline{G_\alpha}^*(u\sigma))\right), & u > 0 \\ -\sqrt{-u}\left(G^*_\alpha(-u\sigma) \otimes \mathbb{1}_p - e^{-\beta_\alpha u/2} e^{-\delta\beta_\alpha u/2}\ \mathbb{1}_p \otimes \alpha_p^{-i\,\delta\beta_p/2}(\overline{G_\alpha}(-u\sigma))\right), & u < 0 \end{cases} \end{aligned} \tag{B.1.8}$$

where $\delta\beta_\alpha = \beta_\alpha - \beta$, $\delta\beta_p = \beta_p - \beta$, and $\beta = \max(\beta_1, \beta_2)$.

Thus the operator $\tilde{K} := UKU^{-1}$ can be written as

$$\tilde{K} = \tilde{L}_0 + g\tilde{I}$$

where $\tilde{I} = UIU^{-1}$ is given in (B.1.6) and $\tilde{L}_0 := UL_0U^{-1}$ is of the form

$$\tilde{L}_0 = L_p \otimes \mathbf{1}_f + \mathbf{1}_p \otimes L_f.$$

B.2 Complex Deformation

Now we express the complex deformation operators U_θ introduced in Section 6 in the Jakšić-Pillet glued Hilbert space. For a function $F \in L^2(X \times \{1, 2\})$ and $\theta = (\delta, \tau)$, $x = (u, \sigma) \in X$, define

$$[\tilde{u}_\theta F](u, \sigma, \alpha) = e^{\frac{1}{2}\delta \mathrm{sgn}(u)} F(j_\theta(u), \sigma, \alpha), \tag{B.2.1}$$

where

$$j_\theta(u) = e^{\delta \mathrm{sgn}(u)} u + \tau, \tag{B.2.2}$$

and sgn is the sign function, $\mathrm{sgn}(u) = 1$ if $u \geq 0$, $\mathrm{sgn}(-u) = -\mathrm{sgn}(u)$. Next, we lift the operator family $\tilde{u}_\theta$ from $L^2(X \times \{1,2\})$ to the operator family, $\tilde{U}_\theta$, on $\mathcal{H}^p \otimes \mathcal{F}(L^2(X \times \{1,2\}))$ in a standard way (cf. (6.1)). The family $\tilde{U}_\theta$ is related to the family U_θ introduced in Section 6 as

$$U_\theta = U\tilde{U}_\theta U^{-1}.$$

The operator $\tilde{K}$ becomes after spectral deformation

$$\tilde{K}_\theta := \tilde{U}_\theta K \tilde{U}_\theta^{-1} = \tilde{L}_{0,\theta} + g\tilde{I}_\theta \tag{B.2.3}$$

where

$$\begin{aligned} \tilde{L}_{0,\theta} &= L_p + \cosh\delta\ L_f + \sinh\delta\ \Lambda_f + \tau N, & \text{(B.2.4)} \\ \Lambda_f &= \mathrm{d}\Gamma(|u|) = \sum_\alpha \int_X a^*(x,\alpha)|u|a(x,\alpha), & \\ \tilde{I}_\theta &= a^*(F_{1,\theta}) + a(F_{2,\theta}) \quad \text{with} \quad F_{j,\theta} = \tilde{u}_\theta F_j. & \text{(B.2.5)} \end{aligned}$$

This spectral deformation can be translated to the original space $\mathcal{H}$ as

$$K_\theta := U^{-1}\tilde{K}_\theta U^{-1} = L_{0,\theta} + gI_\theta \tag{B.2.6}$$

where $L_{0,\theta} := U^{-1}\tilde{L}_{0,\theta}U$ is given by (6.14) and

$$I_\theta = U^{-1}\tilde{I}_\theta U. \tag{B.2.7}$$

C The C^*-algebra $\mathcal{A}_1$

Proposition C.1 *Let the algebras $\mathcal{A}_1$ and $\mathcal{A}$ be defined by (2.17), and (2.7), respectively. Then $\mathcal{A}_1$ is strongly dense in $\mathcal{A}$.*

Proof. Let $L^2_1 = \{f \text{ a.e. continuous}, \ ||f||^2_{L^2_1} := \int |f(k)|^2(|k|^{-1}+1)d^3k < \infty\}$. Recall the real linear map $\widetilde{\gamma}_\beta$ defined in (3.7). Using that for $x \geq 0$, $\max(1/x, 1) \leq \frac{1+e^{-x}}{1-e^{-x}} \leq 4\max(1/x,1)$, we obtain

$$c||f||^2_{L^2_1} \leq \|\widetilde{\gamma}_\beta f\|^2_{L^2} = \int_{S^2} d\sigma \int_0^\infty du\ \frac{1+e^{-\beta u}}{1-e^{-\beta u}}\ u^2\ |f(u\sigma)|^2 \leq C\|f\|^2_{L^2_1}, \tag{C.1}$$

so $\widetilde{\gamma}_\beta$ is bounded and invertible (on $R := \widetilde{\gamma}_\beta L^2_1 \subset L^2$), and $\widetilde{\gamma}_\beta^{-1}$ is a (real linear) bounded map. We have the equivalence

$$\begin{aligned} g \in R \quad \Leftrightarrow \quad & g(u,\sigma) = -e^{\beta u/2}\overline{g}(-u,\sigma) \ \text{ for a.e. } u \in \mathbb{R} \\ & \text{and} \quad \int_{\mathbb{R}} du \int_{S^2} d\sigma |g(u,\sigma)|^2 < \infty. \end{aligned} \tag{C.2}$$

Let $R_0 := \{g \in R,\ e^{bu^2}g \in L^2 \text{ for some } b \geq 0\} \subset R$. The set R_0 is dense in R and (C.2) implies that

$$R_0 = \{e^{\beta u/4}h| : e^{bu^2}h \in L^2 \text{ for some } b > 0 \text{ and } h(u,\sigma) = -\overline{h}(-u,\sigma)\}. \tag{C.3}$$

Given $g = e^{\beta u/4}h \in R_0$, define $h_\epsilon := G_\epsilon * h$, the convolution in the variable u of h with the Gaussian $G_\epsilon(u) := \epsilon^{-1}G(u/\epsilon)$, where $G(u) = \pi^{-1/2}e^{-u^2}$ and $\epsilon > 0$. h_ϵ is continuous (actually analytic), satisfies $e^{bu^2}h_\epsilon \in L^2$, and since $G_\epsilon(\cdot)$ is real valued and odd:

$$h_\epsilon(u,\sigma) = -\overline{h}_\epsilon(-u,\sigma).$$

Therefore, $g_\epsilon := e^{\beta u/4} h_\epsilon \in R_0$. Since $h_\epsilon \to h$ in L^2, we conclude that $g_\epsilon \to g$ in L^2 as $\epsilon \to 0$. Clearly, g_ϵ extends to an entire function $z \mapsto g_\epsilon(z, \sigma)$. Define the set

$$R^{\mathrm{anal}} := \{e^{\beta u/4} h_\epsilon | \ h \text{ satisfies the conditions on r.h.s. of (C.3)}, \ \epsilon > 0\}.$$

R^{anal} is a subset of R that is dense in R. Since $\widetilde{\gamma}_\beta^{-1}$ is bounded, then $D^{\mathrm{anal}} := \widetilde{\gamma}_\beta^{-1}(R^{\mathrm{anal}})$ is dense in L_1^2. Since L_1^2 is dense in L^2, we conclude that D^{anal} is also dense in L^2.

Define $\mathcal{A}_1$ as in (2.17) with $\mathcal{D}^{\mathrm{anal}}$ given above. Since $\mathcal{D}^{\mathrm{anal}}$ is dense in L^2, $\mathcal{A}_1$ is strongly dense in $\mathcal{A}$ (defined by (2.7)). Next we have for real θ

$$\begin{aligned} & U_\theta \pi\big(A_p \otimes W(f_1) \otimes W(f_2)\big)\Omega \\ = \quad & \pi_p(A_p) \otimes W\big((\widetilde{\gamma}_\beta f_1)_\theta\big) \otimes W\big((\widetilde{\gamma}_\beta f_2)_\theta\big)\Omega_\theta \end{aligned} \tag{C.4}$$

where the map $g \to g_\theta$ is defined by $g_\theta(u, \sigma) := g(j_\theta(u), \sigma)$ with the function j_θ defined in (A2.2) and where we understand the Weyl operators on the r.h.s. as acting on the (Jakšić-Pillet glued) GNS space and Ω_θ given in the same representation. Using that $\theta \to (\widetilde{\gamma}_\beta f_j)_\theta$ are analytic for $f_j \in \mathcal{D}^{\mathrm{anal}}$ and $(\widetilde{\gamma}_\beta f_j)_\theta \in L^2(\mathbb{R} \times S^2)$ as long as $|\tan(\operatorname{Im}\delta)| < \frac{b}{1+b}$ and expanding $W\big((\widetilde{\gamma}_\beta f_j)_\theta\big)$ into the Taylor series and Ω_θ into the Dayson-Araki one, one can show the r.h.s. of (C.4) has an analytic continuation in θ into the neighbourhood

$$\Big\{\theta \in \mathbb{C}^2 \,\Big|\, |\tan(\operatorname{Im}\delta)| < \frac{b}{1+b} \quad \text{and} \quad |\operatorname{Im}\tau| < \beta^{-1}\Big\}$$

of $\mathbb{R}$. ■

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