AN EXISTENCE OF SIMPLE CHOREOGRAPHIES FOR N-BODY PROBLEM - A COMPUTER ASSISTED PROOF

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Abstract. We consider a question of finding a periodic solution for the planar Newtonian N-body problem with equal masses, where each body is travelling along the same closed path. We provide a computer assisted proof for the following facts: local uniqueness and convexity of Chenciner and Montgomery Eight, an existence (and local uniqueness) for Gerver's SuperEight for 4-bodies and a doubly symmetric linear chain for 6-bodies.

1. Introduction

In this paper we consider the problem of finding periodic solution to the N-body problem in which all N masses travel along a fixed curve in the plane. The N-body problem with N equal unit masses is given by a differential equation

$$\ddot{q}_i = \sum_{j \neq i} \frac{q_j - q_i}{r_{ij}^3} \tag{1.1}$$

where $q_i \in \mathbb{R}^n$, $i = 1, ..., N$, $r_{ij} = ||q_i - q_j||$. The gravitational constant is taken equal to 1.

We consider planar case (n=2) only, we set $q_i = (x_i, y_i)$, $\dot{q}_i = (v_i, u_i)$. Using this we can express (1.1) by:

$$\begin{cases} \dot{v}_i = \sum_{j \neq i} \frac{x_j - x_i}{r_{ij}^3} \\ \dot{u}_i = \sum_{j \neq i} \frac{y_j - y_i}{r_{ij}^3} \\ \dot{x}_i = v_i \\ \dot{y}_i = u_i \end{cases} \tag{1.2}$$

Recently, this problem received a lot attention in literature see [M, CM, CGMS, S1, S2, S3, MR] and papers cited there.

By a *simple choreography* [S1, S2] we mean a collision-free solution of the N-body problem in which all masses move on the same curve with a constant phase shift. This means that there exists $q : \mathbb{R} \longrightarrow \mathbb{R}^2$ a T-periodic function of time, such that the position of k-th body ($k = 0, \ldots, N-1$) is given by $q_k(t) = q(t + k\frac{T}{N})$ and $(q_0, q_1, \ldots, q_{N-1})$ is solution of the N-body problem. The simplest choreographies are Langrange solutions in which the bodies are located at the vertices of a regular N-gon and move with constant angular velocity. Another simple choreography, a figure eight curve (see Figure 1), was found numerically by C.Moore [M]. A.Chenciner and R.Montgomery [CM] gave a rigorous existence proof of the Eight

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in 2000. In December 1999, J.Gerver found orbit for $N = 4$ called 'Super-Eight' (Figure 6). After that C.Simó found a lot of simple choreographies with different shape and number of bodies ranging form 4 to several hundreds (see [S, S1, CGMS] for pictures, animations and more details).

Up to now the only choreographies whose existence has been rigorously established [MR] are Lagrange solutions and the Eight solution. While the Lagrange solution is given analytically, the existence of Eight was proven in [CM] using variational arguments and there is still a lot of open questions about it [Ch, MR]. For example uniqueness (up to obvious symmetries and rescaling) and convexity of the lobes in Eight. In Section 3 we give a computer assisted proof of existence of Eight, its local uniqueness and convexity of the lobes.

In Sections 4, 5 and 6 we concentrate on choreographies called *doubly symmetric linear chains*. They are symmetric with respect to the x and y coordinate axes and all points of the curve self-intersection are on the x axis. We give a computer assisted proof of an existence (and local uniqueness) of doubly symmetric linear chains for four (Gerver SuperEight) and six bodies.

Proofs given in this paper are computer assisted. By this we mean that we use computer to provide rigorous bounds for solutions of (1.1). The problem of proving an existence of a choreography is reduced to finding a zero for a suitable function. For this purpose we use an interval Newton method and a Krawczyk method [A, K, Mo, N], (see Section 2) which are apparently rather unknown outside the interval arithmetic community. To integrate equations (1.1) we use a C^1-Lohner algorithm [ZLo]. All computations were performed on *AMD Athlon 1700XP* with 256 MB DDRAM memory, with Windows 98SE operating system. We used CAPD package [Capd] and Borland C++ 5.02 compiler. A total computation time for 6-bodies was under 90 seconds and considerably smaller for the Eight and SuperEight solutions - see Section 7 for more details.

2. Two zero finding methods

The main technical tool used in this paper in order to establish an existence of solutions of equations of the form $f(x) = 0$ is *an interval Newton method*[A, Mo, N] and *a Krawczyk method*[A, K, N]. The interval Newton method was used to prove an existence of the Eight (Figure 1) and Gerver orbit (Figure 6). The Krawczyk method for Gerver orbit and an orbit with 6 bodies in a linear chain (Figure 8).

2.1. **Notation.** In the application of interval arithmetics to rigorous verification of theorems single valued objects, like numbers, vectors, matrices etc are in the formulas replaced by sets containing sure bounds for them. In the sequel, we will not use any special notation for single valued object and sets. For a set S by $[S]$ we denote the interval hull of S, i.e. the smallest product of intervals containing S. For a set which is an interval set (i.e. can be represented as a product of intervals) we will also use square brackets to stress its interval nature. For any interval set $[S]$ by $mid([S])$ we denote a center point of $[S]$. For any interval $[a, b]$ we define a diameter by $diam[a, b] := b - a$. For an interval vector (matrix) $S = [S]$ by $diam S$ we denote a vector (matrix) of diameters of each components.

2.2. **Interval Newton method.**

Theorem 2.1. [A, N] *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 function. Let $[X] = \Pi_{i=1}^n [a_i, b_i]$, $a_i < b_i$. Assume the interval hull of $DF([X])$, denoted here by $[DF([X])]$, is* *invertible. Let* $\bar{x} \in X$ *and we define*

$$N(\bar{x}, [X]) = -[DF([X])]^{-1}F(\bar{x}) + \bar{x} \tag{2.1}$$

Then

0. *if* $x_1, x_2 \in [X]$ *and* $F(x_1) = F(x_2)$*, then* $x_1 = x_2$
1. *if* $N(\bar{x}, [X]) \subset [X]$*, then* $\exists! x^* \in [X]$ *such that* $F(x^*) = 0$
2. *if* $x_1 \in [X]$ *and* $F(x_1) = 0$*, then* $x_1 \in N(\bar{x}, [X])$
3. *if* $N(\bar{x}, [X]) \cap [X] = \emptyset$*, then* $F(x) \neq 0$ *for all* $x \in [X]$

2.3. **Krawczyk method.** We assume that:

- $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a C^1 function,
- $[X] \subset \mathbb{R}^n$ is an interval set,
- $\bar{x} \in [X]$
- $C \in \mathbb{R}^{n \times n}$ is a linear isomorphism.

The Krawczyk [A, K, N] operator is given by

$$K(\bar{x}, [X], F) := \bar{x} - CF(\bar{x}) + (Id - C[DF([X])])([X] - \bar{x}). \tag{2.2}$$

Theorem 2.2. *1. If* $x^* \in [X]$ *and* $F(x^*) = 0$*, then* $x^* \in K(\bar{x}, [X], F)$*.*

2. If $K(\bar{x}, [X], F) \subset int[X]$*, then there exists in [X] exactly one solution of equation* $F(x) = 0$*.*

3. If $K(\bar{x}, [X], F) \cap [X] = \emptyset$*, then* $F(x) \neq 0$ *for all* $x \in [X]$

2.4. **Algorithm for Newton and Krawczyk method.** Theorems 2.1 and 2.2 can be used as a basis for an algorithm for rigorous enclosing for solution of equation $F(x) = 0$. Let $T(\bar{x}, [X]) = N(\bar{x}, [X])$ if we are using the interval Newton method and $T(\bar{x}, [X]) = K(\bar{x}, [X], F)$ in case of Krawczyk method.

First, we need to have a good guess for $x^* \in \mathbb{R}^n$. For this purpose we use a nonrigorous Newton method to obtain $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$. Then we choose interval set $[X]$ which contains $\bar{x}$ and perform the following algorithm:

Step 1. Compute $T(\bar{x}, [X])$.
Step 2. If $T(\bar{x}, [X]) \subset [X]$, then return **success**.
Step 3. If $X \cap T(\bar{x}, [X]) = \emptyset$, then return **fail**. There are no zeroes of F in $[X]$.
Step 4. If $[X] \subset T(\bar{x}, [X])$, then modify computation parameters (for example: a time step, the order of Taylor method, size of $[X]$). Go to Step 1.
Step 5. Define a new $[X]$ by $[X] := [X] \cap T(\bar{x}, [X])$ and a new $\bar{x}$ by $\bar{x} := mid([X])$, then go to Step 1.

In practical computation it is convenient to define a maximum number of iteration allowed and return **fail** if the actual iteration count is larger.

Observe that the third assertion in both Theorems 2.1 and 2.2 can be used to exclude an existence of zero of F. This has been used by Galias [G1, G2] to find all periodic orbits up to a given period for Hénon map and Ikeda map. It seems possible to obtain similar results for N-body problem in the future.

3. The Eight - existence, local uniqueness and convexity

Existence of the Eight has been shown in [CM] using mixture of symmetry and variational arguments. Here we give an another existence proof and in addition we obtain local uniqueness and the convexity of each lobe of the Eight. We follow [CM] in the use of the symmetry, but other component of the proof is different - we use an interval Newton method discussed in Section 2.

In notation, but only for the Eight, we follow [CM].

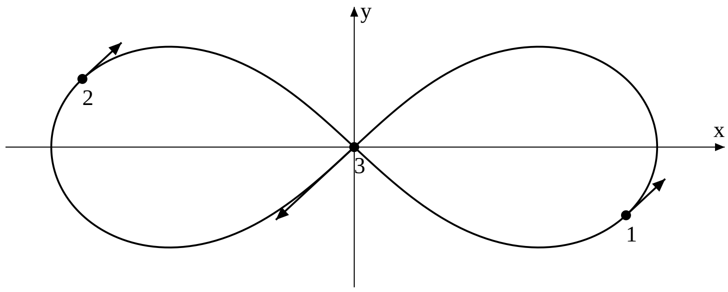

FIGURE 1. The Eight - the initial position

Let T be any positive real number. We define action of the Klein group $\mathbb{Z}_2 \times \mathbb{Z}_2$ on $\mathbb{R}/T\mathbb{Z}$ and on $\mathbb{R}^2$ as follows: if σ and τ are generators,

$$\sigma(t) = t + \frac{T}{2},\ \tau(t) = -t + \frac{T}{2},\ \sigma(x,y) = (-x,y),\ \tau(x,y) = (x,-y). \tag{3.1}$$

For loop $q : (\mathbb{R}/T\mathbb{Z}) \longrightarrow \mathbb{R}^2$ and $i = 1,2,3$ we define the position for i-th body by

$$q_i(t) = q\left(t + (3-i)\cdot\frac{T}{3}\right), \tag{3.2}$$

Here q_i is just position of i-th body.

The following theorem without the uniqueness part was proved in [CM].

Theorem 3.1. *There exists an "eight"-shaped planar loop $q : (\mathbb{R}/T\mathbb{Z}) \longrightarrow \mathbb{R}^2$ with the following properties:*

(1) *for each t,*
$$q_1(t) + q_2(t) + q_3(t) = 0;$$
(2) *q is invariant with respect to the action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ on $\mathbb{R}/T\mathbb{Z}$ and on $\mathbb{R}^2$:*
$$q\circ\sigma(t) = \sigma\circ q(t) \text{ and } q\circ\tau(t) = \tau\circ q(t);$$
(3) *the loop $x : \mathbb{R}/T\mathbb{Z} \longrightarrow \mathbb{R}^6$ defined by*
$$x(t) = (q_1(t), q_2(t), q_3(t))$$
is T-periodic solution of the planar three-body problem with equal masses.

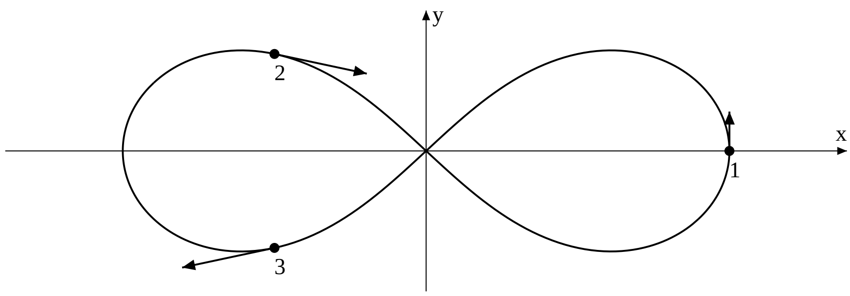

FIGURE 2. The Eight - the final position

Moreover, q is locally unique (up to obvious symmetries and rescaling).

Remark 3.2. If conditions (1),(2),(3) are satisfied then

a. $\dot{q}_1(t) + \dot{q}_2(t) + \dot{q}_3(t) = 0$ for each t,
b. $\dot{q} \circ \sigma(t) = \sigma \circ \dot{q}(t)$ and $\dot{q} \circ \tau(t) = \sigma \circ \dot{q}(t)$ for each t,
c. $q_3(0) = (0,0)$ and $\dot{q_3}(0) = -2\dot{q_1}(0)$
d. $q_1(0) = -q_2(0)$ and $\dot{q}_1(0) = \dot{q}_2(0)$,
e. $q_1(T/12)$ is on the X axis and $\dot{q}_1(T/12)$ is orthogonal to the X axis,
f. $q_3(T/12) = \tau \circ q_2(T/12)$ and $\dot{q}_3(T/12) = \sigma \circ \dot{q}_2(T/12)$,

The following lemma describes the symmetry reduction for the Eight

Lemma 3.3. *Assume that $\tilde{q} : [0, \tilde{T}] \longrightarrow \mathbb{R}^6$ is a solution of the three body problem, such that $\tilde{q} = (\tilde{q}_1, \tilde{q}_2, \tilde{q}_3)$ satisfies conditions (c),(d),(e),(f) in Remark 3.2, then exists $q : (\mathbb{R}/T\mathbb{Z}) \longrightarrow \mathbb{R}^2$ satisfying condition (1),(2),(3) in Theorem 3.1 with $T = 12\tilde{T}$.*

Proof: We define

$$\hat{q}(t) = \begin{cases} \tilde{q}_3(t) & \text{for } t \in [0, \tilde{T}] \\ \tau \circ \tilde{q}_2(2\tilde{T} - t) & \text{for } t \in [\tilde{T}, 2\tilde{T}] \\ \sigma \circ \tilde{q}_1(t - 2\tilde{T}) & \text{for } t \in [2\tilde{T}, 3\tilde{T}] \end{cases} \tag{3.3}$$

and

$$q(t) = \begin{cases} \hat{q}(t) & \text{for } t \in [0, 3\tilde{T}] \\ \tau \circ \hat{q}(\tau^{-1}(t)) & \text{for } t \in [3\tilde{T}, 6\tilde{T}] \\ \sigma \circ \hat{q}(\sigma^{-1}(t)) & \text{for } t \in [6\tilde{T}, 9\tilde{T}] \\ \sigma \circ \tau \circ \hat{q}(\tau^{-1} \circ \sigma^{-1}(t)) & \text{for } t \in [9\tilde{T}, 12\tilde{T}] \end{cases} \tag{3.4}$$

Let f_1 and f_2 be two solutions of 1.1 on intervals $[t_1, t_2]$ and $[t_2, t_3]$ respectively. If $f_1(t_2) = f_2(t_2)$ and $\dot{f}_1^-(t_2) = \dot{f}_2^+(t_2)$ then $f = \{f_1, f_2\}$ is solution on interval $[t_1, t_2]$. To show that $q(t)$ is a solution it is enough to show that "pieces fit together smoothly".

For $t = \tilde{T}$ from 3.2.f we have

$$\tilde{q}_3(\tilde{T}) = \tilde{q}_3(\frac{T}{12}) = \tau \circ \tilde{q}_2(\frac{T}{12}) = \tau \circ \tilde{q}_2(\tilde{T}) = \tau \circ \tilde{q}_2(2\tilde{T} - \tilde{T}) \tag{3.5}$$

$$\dot{\tilde{q}}_3^-(\tilde{T}) = \sigma \circ \dot{\tilde{q}}_2^+(\tilde{T}) \tag{3.6}$$

For $t = 2\tilde{T}$ from 3.2.d we have

$$\tilde{q}(2\tilde{T}) = \tau \circ \tilde{q}_2(0) = \sigma \circ \tilde{q}_1(0) \tag{3.7}$$

$$\dot{\tilde{q}}^-(2\tilde{T}) = \sigma \circ \dot{\tilde{q}}_2^-(\tilde{T}) = \sigma \circ \dot{\tilde{q}}_1^+(2\tilde{T}) = \dot{\tilde{q}}^+(2\tilde{T}) \tag{3.8}$$

From (3.5-3.8) it follows that $\hat{q}(t)$ is smooth curve for $t \in [0, 3\tilde{T}]$. We construct $q(t)$ from that curve.

For $t = 3\tilde{T}$ from 3.2.e we obtain

$$\begin{aligned} q(3\tilde{T}) &= \sigma \circ \tilde{q}_1(\tilde{T}) = \sigma \circ \tau \circ \tilde{q}_1(\tilde{T}) = \sigma \circ \tau \circ \hat{q}(\tau^{-1}(3\tilde{T})) \\ \dot{q}^-(3\tilde{T}) &= \sigma \circ \dot{\tilde{q}}_1(\tilde{T}) = \dot{\tilde{q}}_1(\tilde{T}) = \dot{q}^+(3\tilde{T}) \end{aligned}$$

Using 3.2.c for $t = 6\tilde{T}$ we infer that

$$\begin{aligned} q(6\tilde{T}) &= \tau \circ \hat{q}(\tau^{-1}(6\tilde{T})) = \tau \circ \tilde{q}_3(0) = \sigma \circ \tilde{q}_3(0) = \sigma \circ \hat{q}(\sigma^{-1}(6\tilde{T})) \\ \dot{q}^-(6\tilde{T}) &= \sigma \circ \dot{\tilde{q}}_1(\tilde{T}) = \dot{q}^+(6\tilde{T}) \end{aligned}$$

Now we show that $q(t)$ is a closed curve.

$$q(12\tilde{T}) = \sigma \circ \tau \circ \hat{q}(\tau^{-1} \circ \sigma^{-1}(12\tilde{T})) = \sigma \circ \tau \circ \hat{q}(0) = \hat{q}(0) = q(0)$$
$$\dot{q}^{-}(12\tilde{T}) = \frac{\partial}{\partial t}[\sigma \circ \tau \circ \hat{q}(\tau^{-1} \circ \sigma^{-1}(12\tilde{T}))] = \dot{\hat{q}}(0) = \dot{q}^{+}(0)$$

Hence it is easy to see that $q(t)$ can be extended to a T-periodic curve, such that $x(t) = (q_1(t), q_2(t), q_3(t))$ is a T-periodic solution of the three body problem.

Condition (2) in Theorem 3.1 follows easily from definition of $\hat{q}(t)$ and properties of σ and τ (for T-periodic orbit $\sigma^{-1}(t) = \sigma(t)$ and $\tau^{-1}(t) = \tau(t)$). For $t \in [0, 3\tilde{T}]$ we have

$$q(\sigma(t)) = \sigma \circ \hat{q}(\sigma^{-1} \circ \sigma(t)) = \sigma \circ \hat{q}(t) = \sigma \circ q(t) \tag{3.9}$$
$$q(\tau(t)) = \tau \circ \hat{q}(\tau^{-1} \circ \tau(t)) = \tau \circ \hat{q}(t) = \tau \circ q(t) \tag{3.10}$$

For $t \in [3\tilde{T}, 6\tilde{T}]$ using (3.10) we obtain

$$q(\sigma(t)) = \sigma \circ \tau \circ \hat{q}(\tau^{-1} \circ \sigma^{-1} \circ \sigma(t)) = \sigma \circ \tau \circ \hat{q}(\tau^{-1}(t)) = \sigma \circ q(t) \tag{3.11}$$
$$q(\tau(t)) = \hat{q}(\tau(t)) = \tau \circ \hat{q}(\tau \circ \tau(t)) = \tau \circ q(t) \tag{3.12}$$

We omit other two cases, because the proof is very similar.

Because there aren't any external force so the center of mass can only move with a constant velocity. But from 3.2.c and 3.2.d we have $q_1(0) + q_2(0) + q_3(0) = 0$ and $\dot{q}_1(0) + \dot{q}_2(0) + \dot{q}_3(0) = 0$. Hence we obtain condition (1) in Theorem 3.1. □

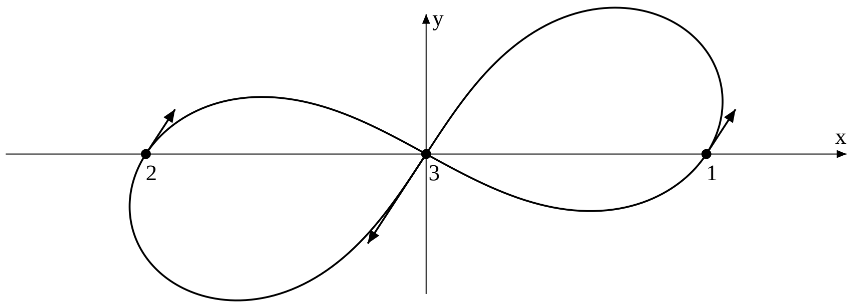

FIGURE 3. Rotated Eight - initial position

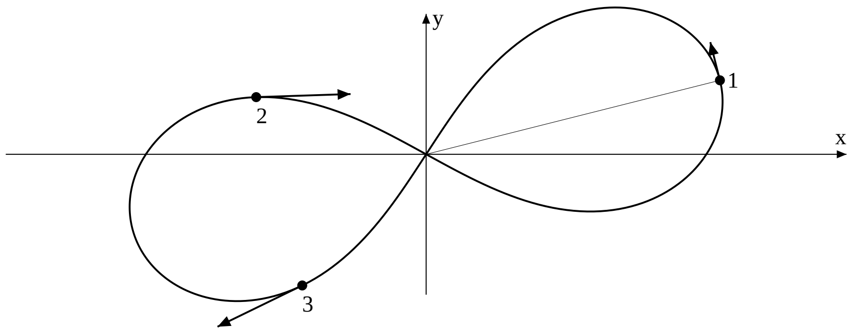

FIGURE 4. Rotated Eight - final position

Hence to prove Theorem 3.1 it is enough to show that there exists a locally unique (up to obvious degeneracies) function satisfying assumptions of Lemma 3.3. For this end we rewrite these assumptions as a zero finding problem to which we apply an interval Newton method in *the reduced space*.

Our original phase space is 12 dimensional, the state of bodies is given by $(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3)$. The center of mass is fixed at the origin. Hence one body's position and velocity is determined by other two bodies. We start from a collinear position with the third body at the origin and with equal velocities of the first and the second body (see 3). Hence it is enough to know the position and the velocity of the first body to reconstruct initial condition of other bodies. Moreover, if we have one solution we could get another solution by a suitable rotation (both have the same shape). To remove this degeneracy we place the first body on the X axis (we will make computation for rotated Eight, see Figure 3). In addition we fix the size of trajectory. This fixes the period of the solution. But if we know one periodic solution then by Kepler law we may obtain a solution of any period just by rescaling. Finally, we set $q_1(0) = (1,0)$, hence initial conditions are fixed by the velocity of the first body.

The *reduced space* for Eight is two dimensional and is parameterized by velocity of second body. We define map from the reduced space to full phase space $E : \mathbb{R}^2 \longrightarrow \mathbb{R}^{12}$, which expands velocity of first body, given by (v,u), to the initial conditions $(x_1, y_1, x_2, y_2, x_3, y_3, v_1, u_1, v_2, u_2, v_3, u_3)$ for equation (1.1)

$$E(v,u) = (1, 0, -1, 0, 0, 0, v, u, v, u, -2v, -2u).$$

For each such initial condition exists a solution of the tree body problem defined on some interval. To each initial configuration, following that solution, we associate, if it exists, a configuration in which for the first body the position vector is orthogonal to its velocity vector for the first time. This defines the Poincaré map $P : \mathbb{R}^{12} \supset \Omega \longrightarrow \mathbb{R}^{12}$.

Now, we define map $R : \mathbb{R}^{12} \longrightarrow \mathbb{R}^2$, by

$$R(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3) = (\|q_2 - q_1\|^2 - \|q_3 - q_1\|^2, (\dot{q}_2 - \dot{q}_3) \times q_1),$$

where by $\times$ we denote vector product, and map $\Phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by

$$\Phi = R \circ P \circ E.$$

Remark 3.4. If $R(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3) = 0$, then $\|q_2 - q_1\| = \|q_3 - q_1\|$ and if in addition $y_1 = 0$ but $x_1 \neq 0$, then $\dot{y}_2 = \dot{y}_3$. The bodies are then located as in Fig. 4 and conditions (e) and (f) in Rem. 3.2 are satisfied in a suitably rotated coordinate frame.

The following lemma, which is crucial for the proof of Theorem 3.1 is obtained with computer assistance.

Lemma 3.5. *There exists a locally unique* $(v,u) \in \mathbb{R}^2$ *that* $\Phi(v,u) = (0,0)$.

Proof: We use an interval Newton method (Theorem 2.1). First we come close to zero, starting with some rough initial condition (for example from [S1]) using a non-rigorous Newton method. Once we have a good candidate $x_0 = (v_0, u_0)$, we set $[X] = [v_0 - \delta, v_0 + \delta] \times [u_0 - \delta, u_0 + \delta]$ and compute rigorously $\Phi(x_0)$ and $\frac{\partial \Phi([X])}{\partial x}$. For this purpose we use C^1-Lohner algorithm described in [ZLo]. In this computation we used a time step $h = 0.01$ and the order $r = 7$.

Result	
$\Phi(x_0)$	([-2.107029e-06, -2.106467e-06],[2.974991e-06, 2.976034e-06])
diam $\Phi(x_0)$	(5.625889e-10, 1.042962e-09)
$\frac{\partial\Phi([X])}{\partial x}$	$\begin{bmatrix} [17.622624, 17.643043] & [1.809772, 1.827325] \\ [-24.868548, -24.848432] & [-10.056629, -10.039221] \end{bmatrix}$
$N(x_0, [X])$	$([0.347116886243943, 0.347116889993313],$ $[0.532724941587373, 0.532724949187495])$
diam $N(x_0, [X])$	(3.749369e-09,7.600121e-09)

TABLE 1. The data from the proof of Lemma 3.5.

It turns out that the assumption of assertion 1 in Theorem 2.1 holds for $x_0 = (0.347116768716, 0.532724944657)$ and $\delta = 10^{-6}$. Numerical data from this computation are listed in Table 1.

Moreover, from Theorem 2.1 we know that this zero is unique in the set X. $\square$

Proof of Theorem 3.1: From Lemma 3.5 there exists an initial condition (v, u) in reduced space that $\Phi(v, u) = (0, 0)$. Hence there exists solution $\bar{q}(t)$ of the three body problem in some interval $[0, \tilde{T}]$ that in $t = 0$ bodies are in collinear position with third body in the origin and in $t = \tilde{T}$ they are in isosceles configuration ($r_{12} = r_{13}$). By rotation we could get solution $\tilde{q}(t)$ that in $t = \tilde{T}$ first body is on X axis. Obviously we still start in collinear position and end in isosceles. Now from Remark 3.4 $\tilde{q}(t)$ satisfy all condition in Lemma 3.3 and hence there exist $q(t)$ satisfying condition (1),(2),(3).

Local uniqueness follows from local uniqueness in Lemma 3.5. $\square$

3.1. **Convexity of the Eight.**

Theorem 3.6. *Each lobe of the Eight is convex.*

Proof: For the proof it is enough to show that the only inflection point on the curve $q(t)$ is the origin.

From Lemma 3.5 we get set $[X]$ which includes initial condition for the Eight in reduced space. We expand $[X]$ to the full space, we set $\bar{X} = E([X])$. In the coordinate frame in which the Eight looks like in Fig. 1 and symmetries σ and τ are reflections with respect to coordinate axes (see 3.1) we see immediately that the symmetry properties of the Eight imply that at the origin $\frac{\partial^2 y_i}{\partial x_i^2} = 0$ and $\frac{\partial^2 x_i}{\partial y_i^2} = 0$. To prove convexity of the Eight we follow rigorously the trajectory of set $\bar{X}$ and show that the only point in which $\frac{\partial^2 y_i}{\partial x_i^2} = 0$ and $\frac{\partial^2 x_i}{\partial y_i^2} = 0$ is the origin. The same is true if we rotate the Eight, as in Figure 3, to the coordinate system in which we performed the actual computations and in which we will work for the remainder of this proof.

Let $q_i(t) = (x_i(t), y_i(t))$ be position of i-th body . If $0 \notin \frac{\partial x_i}{\partial t}[t_{k-1}, t_k]$ then, from the implicit function theorem, we can write y_i as a function of x_i for interval between $x_i(t_{k-1})$ and $x_i(t_k)$. Otherwise we try to represent x_i as a function of y_i. For small enough time steps at least one of this representations is always possible for the Eight (this is really verified during the rigorous computations) . Now we derive the formulas for the derivatives of $y_i(t(x_i))$ with respect to x_i (for derivatives of $x_i(t(y_i))$ we just exchange x and y variables in formulas).

$$\begin{aligned}
\frac{\partial y_i(t(x_i))}{\partial x_i} &= \frac{\partial y_i}{\partial t}\left(\frac{\partial x_i}{\partial t}\right)^{-1} \\
\frac{\partial^2 y_i(t(x_i))}{\partial x_i^2} &= \left(\frac{\partial^2 y_i}{\partial t^2} - \frac{\partial^2 x_i}{\partial t^2}\frac{\partial y_i}{\partial x_i}\right)\left(\frac{\partial x_i}{\partial t}\right)^{-2} \\
\frac{\partial^3 y_i(t(x_i))}{\partial x_i^3} &= \frac{\frac{\partial^3 y_i}{\partial t^3}\frac{\partial x_i}{\partial t} - \frac{\partial^3 x_i}{\partial t^3}\frac{\partial y_i}{\partial t} + 2\left(\frac{\partial^2 x_i}{\partial t^2}\right)^2\frac{\partial y_i}{\partial x_i} - 2\frac{\partial^2 x_i}{\partial t^2}\frac{\partial^2 y_i}{\partial t^2}}{\left(\frac{\partial x_i}{\partial t}\right)^4} - \frac{\left(\frac{\partial^2 x_i}{\partial t^2}\frac{\partial^2 y_i}{\partial x_i^2}\right)}{\left(\frac{\partial x_i}{\partial t}\right)^2}
\end{aligned}$$

From this equations we need to know the derivatives x_i and y_i with respect to time. We obtain them easily by differentiation of (1.1) with respect to time. In fact this is done during each step of C^1-Lohner algorithm.

By $\varphi(t, x_0)$ we denote the state of bodies (the position and the velocity) at time t with the initial condition x_0 at $t = 0$. Let h_k be length of k-th time step, $t_k = h_1 + \cdots + h_k$ - the time after k steps, $[q^k] = [q_1^k] \times [q_2^k] \times [q_3^k] \times [\dot{q}_1^k] \times [\dot{q}_2^k] \times [\dot{q}_3^k] \subset \mathbb{R}^{12}$ be product of intervals, such that $\varphi(t_k, \bar{X}) \subset [q^k]$ and $[Q^k] = [Q_1^k] \times [Q_2^k] \times [Q_3^k] \times [\dot{Q}_1^k] \times [\dot{Q}_2^k] \times [\dot{Q}_3^k] \subset \mathbb{R}^{12}$, where $[Q_i^k] = [X_i^k] \times [Y_i^k]$, be product of intervals, such that $\varphi([t_{k-1}, t_k], \bar{X}) \subset [Q^k]$ Note that $[Q^k]$ can be seen as an interval enclosure for whole trajectory between $[q^{k-1}]$ and $[q^k]$, which is computed during each step of C^1-Lohner algorithm.

For k-th step and i-th body (except first step for third body) we check if at least one of the following conditions is true

$$0 \notin \frac{\partial x_i}{\partial t}[X_i^k] \text{ and } 0 \notin \frac{\partial^2 x_i}{\partial y_i^2}[X_i^k] \tag{3.13}$$

$$0 \notin \frac{\partial y_i}{\partial t}[Y_i^k] \text{ and } 0 \notin \frac{\partial^2 y_i}{\partial x_i^2}[Y_i^k]. \tag{3.14}$$

For first step for third body (starting in the origin) we check if one of following conditions is satisfied

$$0 \notin \frac{\partial x_3}{\partial t}[X_3^1] \text{ and } 0 \in \frac{\partial^2 x_3}{\partial y_3^2}[X_3^1] \text{ and } 0 \notin \frac{\partial^3 x_3}{\partial y_3^3}[X_3^1] \tag{3.15}$$

$$0 \notin \frac{\partial y_3}{\partial t}[Y_3^1] \text{ and } 0 \in \frac{\partial^2 y_3}{\partial x_3^2}[Y_3^1] \text{ and } 0 \notin \frac{\partial^3 y_3}{\partial x_3^3}[Y_3^1]. \tag{3.16}$$

To verify above conditions we follow rigorously using C^1-Lohner algorithm the trajectory of set $\bar{X}$ until reaching the section described in proof of Lemma 3.5 and for each time step and each body we check suitable condition. It turns out that for each time step and each body these conditions were satisfied. This finishes the proof. □

3.2. **Some numerical data from the convexity of Eight computation.** The parameters of the methods were: a time step $h = 0.01$, order $r = 7$. We needed 53 steps to cross the section, below we show data for characteristic cases in order to show that it was really quite easy to verify, with algorithms we used.

i	$\frac{\partial x_i}{\partial t}[X_i^1]$	$\frac{\partial^2 x_i}{\partial y_i^2}[X_i^1]$	$\frac{\partial^3 x_i}{\partial y_i^3}[X_i^1]$
1	[0.334402,0.347118]	[15.3592,17.9897]	[136.616,219.114]
2	[0.347116,0.360049]	[-16.5013,-14.111]	[119.951,192.562]
3	[-0.695034,-0.69385]	[-0.0682713,0.269952]	[-30.969,-26.3718]

Table 2. The data from the proof of Theorem 3.6, for step 1, neighborhood of deflection point.

i	$\frac{\partial x_i}{\partial t}[X_i^2]$	$\frac{\partial^2 x_i}{\partial y_i^2}[X_i^2]$	$\frac{\partial^3 x_i}{\partial y_i^3}[X_i^2]$
1	[0.32222,0.334722]	[16.7085,19.6225]	[155.007,250.021]
2	[0.35972,0.372882]	[-15.2203,-13.0177]	[105.778,171.191]
3	[-0.695669,-0.69444]	[0.126359,0.472046]	[-30.9533,-26.1203]

Table 3. The data from the proof of Theorem 3.6, for step 2.

i	$\frac{\partial x_i}{\partial t}[X_i^2]$	$\frac{\partial^2 x_i}{\partial y_i^2}[X_i^2]$	$\frac{\partial^3 x_i}{\partial y_i^3}[X_i^2]$
1	[-0.00287209,0.00468403]	-	-
2	[0.904939,0.922079]	[-2.55715,-2.16831]	[4.35197,10.6201]
3	[-0.919428,-0.909617]	[2.56371,3.03259]	[-3.51203,3.48564]
i	$\frac{\partial y_i}{\partial t}[Y_i^2]$	$\frac{\partial^2 y_i}{\partial x_i^2}[Y_i^2]$	$\frac{\partial^3 y_i}{\partial x_i^3}[Y_i^2]$
1	[0.480975,0.48288]	[-3.24824,-3.11737]	[-2.98453,-0.860616]

Table 4. The data from the proof of Theorem 3.6, for step 37.

Step 1. We start in collinear position with third body in the origin. So we are in an *inflection point*. Numerical data for this case are given in Table 2. We see that for third body we have $0 \in \frac{\partial^2 x_i}{\partial y_i^2}[X_i^1]$, but this derivative is monotonic ($\frac{\partial^3 x_i}{\partial y_i^3}[X_i^1] < 0$), hence there can be only one zero of it in interval $[X_3^1]$. But we know that one zero is at $(0,0) \in [X_3^1]$.
Steps 2-36 and 38-53 In this case all second derivatives do not contain 0. Table 3 contains data obtained in second step.
Step 37. In this case we cannot represent y_1 as function of x_1, hence we interchange variables and represent x_1 as function of y_1 and check condition 3.14 instead of 3.13. The first body is in the rightmost position. Table 4 contains the derivatives for this case.

4. Doubly symmetric choreographies with even number of bodies

4.1. **Symmetries.** Many of the choreographies found by Simo [S1] have at least one symmetry. Just as in case of the Eight this is not only a symmetry of trajectory image, but also of how the curve describing the trajectory is parameterized with time.

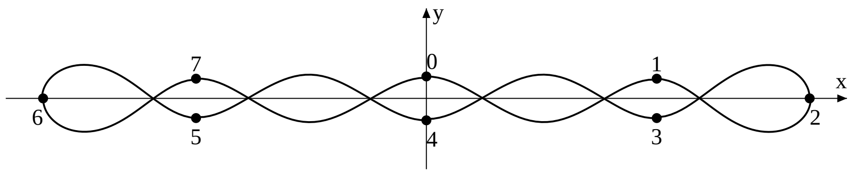

FIGURE 5. Linear chain with 8 bodies.

In this section we introduce a notation for symmetries which will be used till the end of this paper. By S_x, S_y, S_0 we want to denote a symmetry with respect to the x axis, the y axis and to the origin. These spatial symmetries act also on time variable parameterizing curves.
Let T be any positive real number. We define actions of S_x, S_y and S_0 on $\mathbb{R}/T\mathbb{Z}$ and on $\mathbb{R}^2$ as follows:

$$\begin{array}{ll} S_x(t) = -t + \frac{T}{2}, & S_x(x,y) = (-x,y), \\ S_y(t) = -t, & S_y(x,y) = (x,-y), \\ S_0(t) = t + \frac{T}{2}, & S_0(x,y) = (-x,-y). \end{array}$$

It follows from this definition that $S_0 = S_x \circ S_y = S_y \circ S_x$.

Let $q(t) : \mathbb{R}/T\mathbb{Z} \longrightarrow \mathbb{R}^2$. We say that $q(t)$ is invariant (equivariant) with respect to the action of S if $S(q(t)) = q(S(t))$ for all t. Moreover we assume that $q(t)$ is C^2 function. If $q(t)$ is invariant with respect to S_x (resp. S_y) then $\dot{q}(S_x(t)) = S_y(\dot{q}(t))$ (resp. $\dot{q}(S_y(t)) = S_x(\dot{q}(t))$). Hence S_0 invariance implies that $\dot{q}(S_0(t)) = S_0(\dot{q}(t))$.

From now on we will enumerate bodies starting from 0. We set $q_i(t) = q\left(t + \frac{T}{N} \cdot i\right)$ for $i = 0, \ldots, N-1$.

4.2. **Doubly symmetric choreographies with even number of bodies.** We will consider only cases with even number of bodies.

Let $T = N \cdot \bar{T} > 0$. We search for a function $q(t) : \mathbb{R}/T\mathbb{Z} \longrightarrow \mathbb{R}^2$ which has following properties:

P1. for each t the origin is center of mass,

$$\sum_{i=0}^{N-1} q\left(t + \bar{T} \cdot i\right) = 0, \tag{4.1}$$

P2. $q(t)$ is invariant with respect to
  (a) S_x i.e. $q(S_x(t)) = S_x(q(t))$,
  (b) S_0 i.e. $q(S_0(t)) = S_0(q(t))$,

P3. function $x = (q_0(t), q_2(t), \ldots, q_{N-1})$ where $q_i(t) = q\left(t + \bar{T} \cdot i\right)$ for $i = 0, \ldots, N-1$, is a T-periodic solution of the Newtonian N-body problem (1.1).

Observe that condition **P3** says that bodies trace each other with a constant phase shift.

The following Lemma, which is analogous to Lemma 3.3, gives necessary and sufficient conditions for an existence of a choreography satisfying P1,P2 and P3.

Lemma 4.1. *Let $N = 2n$ be a number of bodies. There exists a function q(t) with properties P1, P2, P3 if and only if there are functions $q_i : [0, \bar{T}/2] \longrightarrow R^2$ for $i = 0, 1, \ldots, N-1$ such as:*

(1) *$q_0(0) = (0, y_0)$ for some $y_0 \neq 0$ ($q_0(0)$ is on Y axis),*

(2) *At time* $t_0 = 0$ *for* $i = 0, 1, \dots, N/2 - 1$ *we have*
   (a) $q_i(t_0) = S_x(q_{\frac{N}{2}-i}(t_0))$
   (b) $\dot{q}_i(t_0) = S_y(\dot{q}_{\frac{N}{2}-i}(t_0))$
   (c) $q_i(t_0) = S_0(q_{\frac{N}{2}+i}(t_0))$
   (d) $\dot{q}_i(t_0) = S_0(\dot{q}_{\frac{N}{2}+i}(t_0))$
(3) *At time* $t_1 = \bar{T}/2$ *for* $i = 0, 1, \dots, N/2 - 1$ *we have*
   (a) $q_i(t_1) = S_x(q_{\frac{N}{2}-i-1}(t_1))$
   (b) $\dot{q}_i(t_1) = S_y(\dot{q}_{\frac{N}{2}-i-1}(t_1))$
   (c) $q_i(t_1) = S_0(q_{\frac{N}{2}+i}(t_1))$
   (d) $\dot{q}_i(t_1) = S_0(\dot{q}_{\frac{N}{2}+i}(t_1))$
(4) $x(t) = (q_0(t), q_1(t), \dots, q_{N-1}(t))$ *is a solution of the Newton N-body problem for* $t \in [0, \bar{T}/2]$.

Proof: If we have a trajectory with properties P1, P2, P3 then it is easy to show that functions $q_i(t)$ defined in P3 satisfy conditions (1)-(3). On the other hand if we have a functions $q_i(t)$ which have properties (1)-(3) then we define $q(t)$ by (4.2)

$$q(t) = \begin{cases} q_i(t - i\bar{T}) & \text{for } t \in [i\bar{T}, (i+\frac{1}{2})\bar{T}] \text{ and } i = 0, 1, \dots, N-1 \\ S_x(q_{\frac{N}{2}-i}(i\bar{T} - t)) & \text{for } t \in [(i-\frac{1}{2})\bar{T}, i\bar{T}] \text{ and } i = 1, 2, \dots, \frac{N}{2} \\ S_x(q_{\frac{3N}{2}-i}(i\bar{T} - t)) & \text{for } t \in [(i-\frac{1}{2})\bar{T}, i\bar{T}] \text{ and } i = \frac{N}{2}+1, \dots, N \end{cases}$$

□

From Lemma 4.1 it follows that the proof of an existence of a doubly symmetric choreography is equivalent to some boundary value problem for the N-body equation. We will now formulate this problem as a zero finding problem for a suitable map.

Original phase space for the planar N-body problem has $4N$ dimensions. It turns out that, if initial conditions satisfy all conditions in point 2 of Lemma 4.1 it is enough to know values of only N variables to recover rest of them. We still may obtain solutions of any period, to determine this we can fix the size of curve by fixing one variable. Hence our *reduced space* be $(N-1)$-dimensional. In the next paragraph we will be more specific.

We define map $E : \mathbb{R}^{N-1} \longrightarrow \mathbb{R}^{4N}$, which expands, using symmetries from (2), initial conditions from reduced space to the full phase space: $(x_0, y_0, \dot{x}_0, \dot{y}_0, x_1, y_1, \dot{x}_1, \dot{y}_1, \dots, x_{N-1}, y_{N-1}, \dot{x}_{N-1}, \dot{y}_{N-1})$. We consider two cases: $N = 4k$ and $N = 4k + 2$.
For $N = 4k$ we set

$$\begin{aligned} E(\dot{x}_0 \times \prod_{i=1}^{k-1}(x_i, y_i, \dot{x}_i, \dot{y}_i) \times (x_k, \dot{y}_k)) &= (0, a, \dot{x}_0, 0) \times \prod_{i=1}^{k-1}(x_i, y_i, \dot{x}_i, \dot{y}_i) \\ &\times (x_k, 0, 0, \dot{y}_k) \times \prod_{i=1}^{k-1}(x_{k-i}, -y_{k-i}, -\dot{x}_{k-i}, \dot{y}_{k-i}) \\ &\times (0, -a, -\dot{x}_0, 0) \times \prod_{i=1}^{k-1}(-x_i, -y_i, -\dot{x}_i, -\dot{y}_i) \\ &\times (-x_k, 0, 0, -\dot{y}_k) \times \prod_{i=1}^{k-1}(-x_{k-i}, y_{k-i}, \dot{x}_{k-i}, -\dot{y}_{k-i}) \end{aligned}$$

For $N = 4k + 2$ we set

$$E(\dot{x}_0 \times \prod_{i=1}^{k}(x_i, y_i, \dot{x}_i, \dot{y}_i)) = (0, a, \dot{x}_0, 0) \times \prod_{i=1}^{k}(x_i, y_i, \dot{x}_i, \dot{y}_i)$$
$$\times \prod_{i=0}^{k-1}(x_{k-i}, -y_{k-i}, -\dot{x}_{k-i}, \dot{y}_{k-i}) \times (0, -a, -\dot{x}_0, 0)$$
$$\times \prod_{i=1}^{k}(-x_i, -y_i, -\dot{x}_i, -\dot{y}_i) \times \prod_{i=0}^{k-1}(-x_{k-i}, y_{k-i}, \dot{x}_{k-i}, -\dot{y}_{k-i})$$

In both cases a is a parameter fixing a size of an orbit.

Again, like for the Eight, for every such initial condition exists a solution of the N-body problem (1.1) defined on some interval. To every initial configuration, following that solution, we associate, if it exists, a configuration in which for the first time

- for $N = 4k$: bodies k and $k - 1$ have equal x coordinate ($x_k = x_{k-1}$),
- for $N = 4k + 2$: k-th body is on the X axis ($y_k = 0$).

This defines the Poincaré map $P : \mathbb{R}^{4N} \supset \Omega \longrightarrow \mathbb{R}^{4N}$.

Now, we define the reduction map $R : \mathbb{R}^{4N} \longrightarrow \mathbb{R}^{N-1}$ in such way that R has zeroes in points satisfying conditions (3a) and (3b) in Lemma 4.1 and only in such points. Observe that we don't have to worry about conditions (3c) and (3d) in Lemma 4.1, because from the properties of (1.1) if follows that if (2c) and (2d) holds, then (3c) and (3d) are satisfied for *any* t_1.

For $N = 4k$ we set

$$R\left(\prod_{i=0}^{N-1}(x_i, y_i, \dot{x}_i, \dot{y}_i)\right) = (y_k + y_{k-1}, \dot{x}_k + \dot{x}_{k-1}, \dot{y}_k - \dot{y}_{k-1})$$
$$\times \prod_{i=0}^{k-2}(x_i - x_{2k-i-1}, y_i + y_{2k-i-1}, \dot{x}_i + \dot{x}_{2k-i-1}, \dot{y}_i - \dot{y}_{2k-i-1})$$

For $N = 4k + 2$ we set

$$R\left(\prod_{i=0}^{N-1}(x_i, y_i, \dot{x}_i, \dot{y}_i)\right) = \{v_k\} \times \prod_{i=0}^{k-1}(x_i - x_{2k-i}, y_i + y_{2k-i}, \dot{x}_i + \dot{x}_{2k-i}, \dot{y}_i - \dot{y}_{2k-i})$$

We define map $\Phi : \mathbb{R}^{N-1} \supset E^{-1}(\Omega) \longrightarrow \mathbb{R}^{N-1}$ by

$$\Phi = R \circ P \circ E.$$

Theorem 4.2. *If for some $x \in \mathbb{R}^{N-1}$ $\Phi(x) = 0$, then there exists a trajectory with properties P1, P2, P3.*

Proof: Let x be point in which we have $\Phi(x) = 0$. Then $E(x)$ satisfies condition (1),(2) in Lemma 4.1 and from construction of Φ we obtain that there exist trajectories $(q_i(t))$ starting with initial condition $E(x)$ and ending with $\hat{x}$ such that $R(\hat{x}) = 0$, hence condition (3) in Lemma 4.1 is satisfied. Now the assertion follows from Lemma 4.1. □

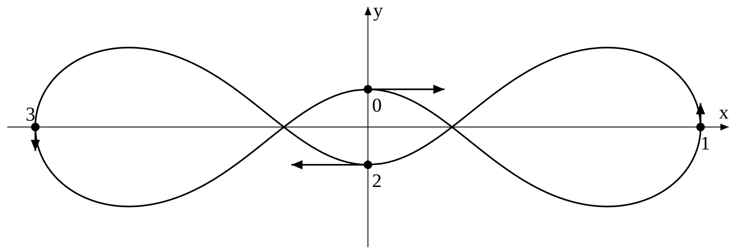

FIGURE 6. Gerver orbit - initial position

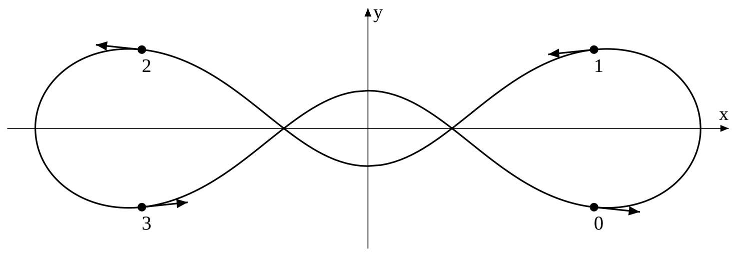

FIGURE 7. Gerver orbit - final position

5. Existence of the SuperEight - Gerver orbit

The Gerver orbit (Figure 6, 7) is a choreography with 4 bodies forming a linear chain. It's the simplest trajectory after the Eight.

With a computer assistance we proved the following

Theorem 5.1. *The Gerver SuperEight exists and is locally unique (up to obvious symmetries and rescaling).*

We show the existence of SuperEight using an approach described in Section 4, i.e. we verify assumptions of Theorem 4.2. Below we give some details.

We set

$$E(x_1, \dot{x}_0, \dot{y}_1) = (0, a, \dot{x}_0, 0, x_1, 0, 0, \dot{y}_1, 0, -a, -\dot{x}_0, 0, -x_1, 0, 0, -\dot{y}_1), \tag{5.1}$$

where a is a parameter fixing the size of the orbit.

$$R(x_0, y_0, \dot{x}_0, \dot{y}_0, x_1, y_1, \dot{x}_1, \dot{y}_1, x_2, y_2, \dot{x}_2, \dot{y}_2, x_3, y_3, \dot{x}_3, \dot{y}_3) = (y_1 + y_0, \dot{x}_1 + \dot{x}_0, \dot{y}_1 - \dot{y}_0), \tag{5.2}$$

Poincáre section S is defined by

$$S(x_0, y_0, \dot{x}_0, \dot{y}_0, x_1, y_1, \dot{x}_1, \dot{y}_1, x_2, y_2, \dot{x}_2, \dot{y}_2, x_3, y_3, \dot{x}_3, \dot{y}_3) = x_1 - x_0 = 0. \tag{5.3}$$

At first we have proved an existence of a zero of $\Phi(x)$ using an interval Newton method, but in the paper we present data from proof based on Krawczyk method. In case of Gerver orbit the choice of the method isn't so important, because if we take a time step small enough or smaller set $[X]$, then the computed matrix $\left[\frac{\partial \Phi}{\partial x}([X])\right]$ becomes invertible and the proof usually goes through. But in general the main problem of application of an interval Newton method is that of an invertibility of $\left[\frac{\partial \Phi}{\partial x}([X])\right]$. We avoid this using the Krawczyk method.

Initial values	
$\bar{x}$	(1.382857, 1.87193510824,0.584872579881)
a	0.157029944461
$[X]$	$\bar{x} + [-10^{-7}, 10^{-7}]^3$

TABLE 5. Data from the proof of existence of Gerver SuperEight. Initial values.

Computed values	
C	$\begin{bmatrix} -2.15400 & 0.257911 & 0.786925 \\ -0.08163 & 0.293713 & 0.043565 \\ 0.939059 & -0.10027 & 0.158399 \end{bmatrix}$
$\Phi(\bar{x})$	$\begin{bmatrix} [-2.87020e-09, -2.26613e-09] \\ [-1.21155e-08, -1.06812e-08] \\ [-5.45542e-08, -5.10016e-08] \end{bmatrix}$
diam $\Phi(\bar{x})$	$\begin{bmatrix} 6.04064e-10 \\ 1.43432e-09 \\ 3.55268e-09 \end{bmatrix}$
$\frac{\partial \Phi}{\partial x}([X])$	$\begin{bmatrix} [-0.1664, -0.1657] & [0.39070, 0.39119] & [0.71771, 0.71790] \\ [-0.1764, -0.1750] & [3.52548, 3.52654] & [-0.0968, -0.0964] \\ [0.87189, 0.87534] & [-0.0867, -0.0842] & [1.99599, 1.99697] \end{bmatrix}$
$K(\bar{x}, [X], \Phi)$	$\begin{bmatrix} [1.382857036247056692, 1.382857041633411832] \\ [1.871935113301492981, 1.871935114053588922] \\ [0.5848725887384301769, 0.5848725902808686872] \end{bmatrix}$
diam $K(\bar{x}, [X], \Phi)$	$\begin{bmatrix} 5.386355139691545446e-09 \\ 7.520959410811656198e-10 \\ 1.54243851024915557e-09 \end{bmatrix}$

TABLE 6. Data from the proof of existence of Gerver SuperEight. Matrix C and results of computation.

To check the assertion (2) in Theorem 2.2 we used the C^1-Lohner algorithm with order $r = 6$. The time step was set to $h = 0.002$. As matrix C we used an inverse of the monodromy matrix computed nonrigorously in a point $\bar{x}$, i.e. $C = \frac{\partial \Phi(\bar{x})}{\partial x}^{-1}$.

Tables 5 and 6 contain numerical data from this proof.

6. EXISTENCE OF THE 'LINEAR CHAIN' ORBIT FOR THE 6 BODIES

Figure 8 displays a linear chain choreography with 6 bodies.

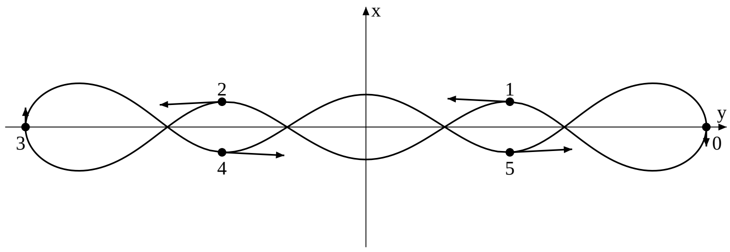

FIGURE 8. 'Linear chain' orbit for the 6 bodies - initial position

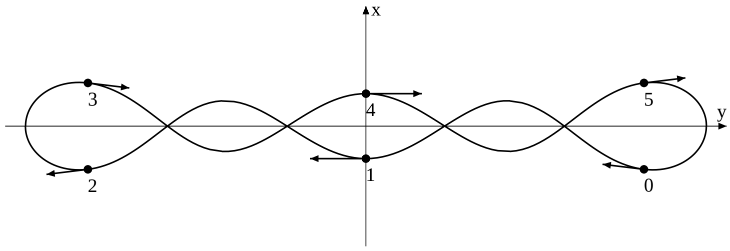

FIGURE 9. 'Linear chain' orbit for the 6 bodies - final position

In this section we report about the computer assisted proof of the following

Theorem 6.1. *The linear chain for* 6 *bodies exists and is locally unique (up to obvious symmetries and rescaling).*

Again, we show an existence of this orbit using an approach described in Section 4 with some minor changes. To speed up calculation and to increase accuracy we take into account in computation that all time $q_3(t) = -q_0(t)$, $q_4(t) = -q_1(t)$ and $q_5(t) = -q_2(t)$ (we change equation (1.1) doing a suitable substitution). So, our full space for ODE has now 12 dimensions. We use also a different time parameterization (we do time shift of $\frac{1}{4}$ of the period). To use exactly approach described in Section 4 we interchange axes (see Fig. 8 and 9). From Lemma 4.1 we will get, in this coordinate frame, a doubly symmetric periodic solution $q(t)$. It is easy to see that $\bar{q} = q(t - \frac{T}{4})$ is then a solution sharing needed symmetries in the original coordinate frame. All data are given in frame with interchanged axes.

We set

$$E(\dot{x}_0, x_1, y_1, \dot{x}_1, \dot{y}_1) = (0, a, \dot{x}_0, 0, x_1, y_1, \dot{x}_1, \dot{y}_1, x_1, -y_1, -\dot{x}_1, \dot{y}_1), \tag{6.1}$$

where a is a parameter fixing the size of the orbit.

(6.2)

$$R(x_0, y_0, \dot{x}_0, \dot{y}_0, x_1, y_1, \dot{x}_1, \dot{y}_1, x_2, y_2, \dot{x}_2, \dot{y}_2) = (\dot{x}_1, x_0 - x_2, y_0 + y_2, \dot{x}_0 + \dot{x}_2, \dot{y}_0 - \dot{y}_2).$$

Poincáre section S is defined by

$$S(x_0, y_0, \dot{x}_0, \dot{y}_0, x_1, y_1, \dot{x}_1, \dot{y}_1, x_2, y_2, \dot{x}_2, \dot{y}_2) = y_1 = 0. \tag{6.3}$$

To find a zero of $\Phi(x)$ we use Krawczyk method. We check the assertion (2) in Theorem 2.2 using C^1-Lohner algorithm [ZLo] of order $r = 9$ and a time step $h = 0.0025$ for a computation in point $\bar{x}$ and $h = 0.001$ for a computation on set $[X]$.

	Initial value
$\bar{x}$	$\begin{bmatrix} -0.635277524319 \\ 0.140342838651 \\ 0.797833002006 \\ 0.100637737317 \\ -2.03152227864 \end{bmatrix}$
a	1.887041548253914
$[X]$	$\begin{bmatrix} [-0.635277525319, -0.635277523319] \\ [0.140342837651, 0.140342839651] \\ [0.797833001006, 0.797833003006] \\ [0.100637736317, 0.100637738317] \\ [-2.03152227964, -2.03152227764] \end{bmatrix}$
diam $[X]$	$\begin{bmatrix} 2.0e-09 \\ 2.0e-09 \\ 2.0e-09 \\ 2.0e-09 \\ 2.0e-09 \end{bmatrix}$

TABLE 7. Data from the proof of existence of linear chain for 6 bodies. Initial values for Krawczyk method.

As matrix C we used an inverse of the monodromy matrix computed nonrigorously in a point $\bar{x}$, i.e. $C = \frac{\partial \Phi(\bar{x})}{\partial x}^{-1}$.

In Tables 7 and 8 we present data from computation of Krawczyk method.

7. Some technical data

All computations were performed on *AMD Athlon 1700XP* with 256 MB DDRAM memory, with Windows 98SE operating system. We used CAPD package[Capd] and Borland C++ 5.02 compiler.

In the listing below r is an order and h is a time step used in the C^1-Lohner algorithm [ZLo].

The computation times for The Eight, $h = 0.01$, $r = 7$

- in point $\bar{x}$: 1.417 sec
- for set $[X]$: 2.66 sec
- convexity : 1.15501 sec

For the proof of an existence of Gerver solution in 4-body problem we used $r = 6$, $h = 0.002$. The computation times for both $\bar{x}$ and set $[X]$ were approximately equal to 30.5 seconds.

For the proof of linear chain for 6-body problem

- computation of Poincaré Map for set $[X]$ took 57.5 seconds with $h = 0.001$ and $r = 9$
- computation for $\bar{x}$ took 23.8 seconds with $h = 0.0025$ and $r = 9$

	Computed value
$\Phi(\bar{x})$	$\begin{bmatrix} [-3.1311957909658e-11, 3.156277062700585e-11] \\ [-4.528821762050939e-12, 4.574757239694804e-12] \\ [-1.063704679893362e-11, 1.051470022161993e-11] \\ [-3.084105193451592e-11, 3.117495150917193e-11] \\ [-1.203726007759087e-11, 1.193112275643671e-11] \end{bmatrix}$
$\operatorname{diam} \Phi(\bar{x})$	$\begin{bmatrix} 6.287472853666386e-11 \\ 9.103579001745743e-12 \\ 2.115174702055356e-11 \\ 6.201600344368785e-11 \\ 2.396838283402758e-11 \end{bmatrix}$
$K(\bar{x},[X],\Phi)$	$\begin{bmatrix} [-0.6352775243616679557, -0.6352775242763283314] \\ [0.1403428386430521646, 0.1403428386590999943] \\ [0.797833001999263769, 0.797833002012834469] \\ [0.10063773728817425324, 0.1006377373457752189] \\ [-2.031522278710178764, -2.031522278575771612] \end{bmatrix}$
$\operatorname{diam}\ K(\bar{x},[X],\Phi)$	$\begin{bmatrix} 8.53396242561643703e-11 \\ 1.604782973174678773e-11 \\ 1.357070011920313846e-11 \\ 5.760096566387318262e-11 \\ 1.344071520748002513e-10 \end{bmatrix}$

TABLE 8. Data from the proof of an existence of linear chain for 6 bodies. Results of computation of Krawczyk method.

The programm performing the proofs is available on *http://www.ap.krakow.pl/~tkapela*

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