# MULTIFRACTAL ANALYSIS FOR THE CONTINUED FRACTION AND MANNEVILLE-POMEAU TRANSFORMATIONS AND APPLICATIONS TO DIOPHANTINE APPROXIMATION

MARK POLLICOTT AND HOWARD WEISS

The University of Manchester and The Pennsylvania State University

October 13, 1998 4:20pm

Abstract. In this note we extend some of the theory of multifractal analysis for conformal expanding systems to two new cases: The non-uniformly hyperbolic example of the Manneville-Pomeau equation, and the continued fraction transformation. A common point in the analysis is the use of thermodynamic formalism for transformations with infinitely many branches.

We apply the multifractal analysis to prove some new results on the precise exponential speed of convergence of the continued fraction algorithm. This gives new quantitative information on geodesic excursions up cusps on the modular surface.

## I. Introduction

In this paper we extend some aspects of the multifractal analysis which are useful in studying problems in Diophantine approximation, studying the behavior of geodesics on the modular surface, and studying an important non-uniformly hyperbolic dynamical system. In particular, we first study the continued fraction (Gauss) transformation $T_1 : [0,1] \to [0,1]$ defined by

$$T_1 x \equiv \frac{1}{x} - \left[\frac{1}{x}\right] = \left\{\frac{1}{x}\right\},$$

for $x \neq 0$ and $T_1(0) \equiv 0$. Here $[1/x]$ denotes the integer part of $1/x$.

This map (see Figure 1) is uniformly hyperbolic, but being naturally coded by an infinite alphabet and having infinite topological entropy, the usual theory of multifractal analysis for conformal maps [PW1, PW2] does not directly apply.

For the continued fraction transformation, our multifractal analysis will yield some new detailed information about the precise *rates* of Diophantine approximation to irrational

*Key words and phrases.* Continued fraction, Dimension spectrum, Diophantine approximation, Lyapunov exponent, Lyapunov spectrum, Manneville-Pomeau transformation, Modular surface, Multifractal..

The work of the second author was partially supported by a National Science Foundation grant DMS-9704913. The manuscript was completed during the second author's sabbatical visit at IPST, University of Maryland and he wishes to thank IPST for their gracious hospitality.

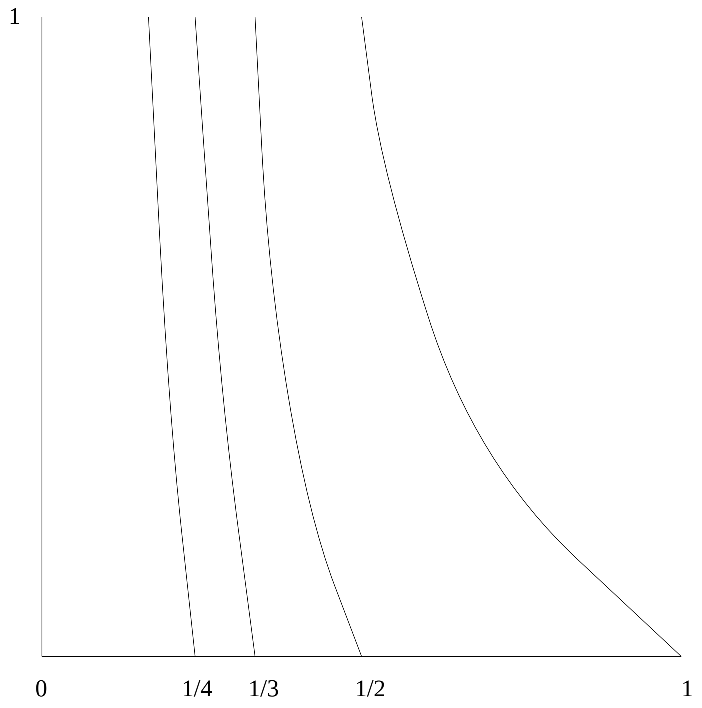


FIGURE 1: GRAPH OF THE MAP $T_1$

numbers. This has immediate implications for cuspital excursions of geodesics on the modular surface.

We also study the Manneville-Pomeau transformation [MP1] defined by

$$\begin{cases} T_2 : [0,1] \to [0,1] \\ T_2 x = x + x^{1+\alpha} \mod 1, \end{cases}$$

where $\alpha$ is a non-negative constant.

This important model (see Figure 2A) is a non-uniformly hyperbolic transformation having the most benign type of non-hyperbolicity: an indifferent fixed point at 0, i.e., $T_2(0) = 0$ and $T_2'(0) = 1$, and exhibits *intermittent* behavior [MP1]. In general one could not expect the full force of multifractal analysis to apply for general non-uniformly hyperbolic systems. However, we show that a significant part of the theory carries over to this, and similar, transformations.

A key aspect of our analysis is a reduction, via inducing, of this system to a countable state uniformly hyperbolic system (i.e., using the Schweiger jump transformation). In particular, we show that the Lyapunov exponent attains an interval of values, realized on dense sets with positive Hausdorff dimension. This quantifies the size of the set of points whose orbits spend a disproportionate amount of time near the indifferent fixed point.

A major component in the multifractal analysis is the use of symbolic dynamics and thermodynamic formalism, i.e., pressure (and its derivatives), equilibrium states, etc. In the present context of infinite state subshifts of finite type, we are fortunate in having at our disposal a theory worked out by Walters [Wa1].

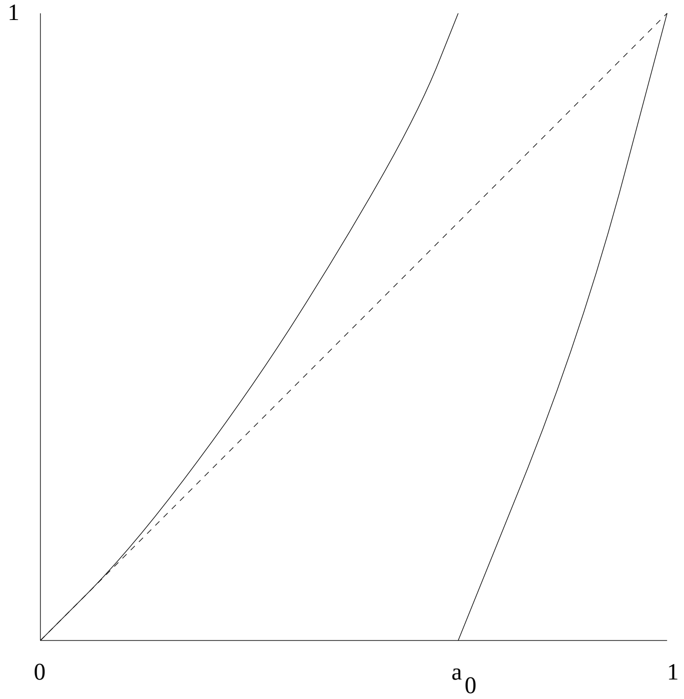


FIGURE 2A: GRAPH OF THE MAP $T_2$

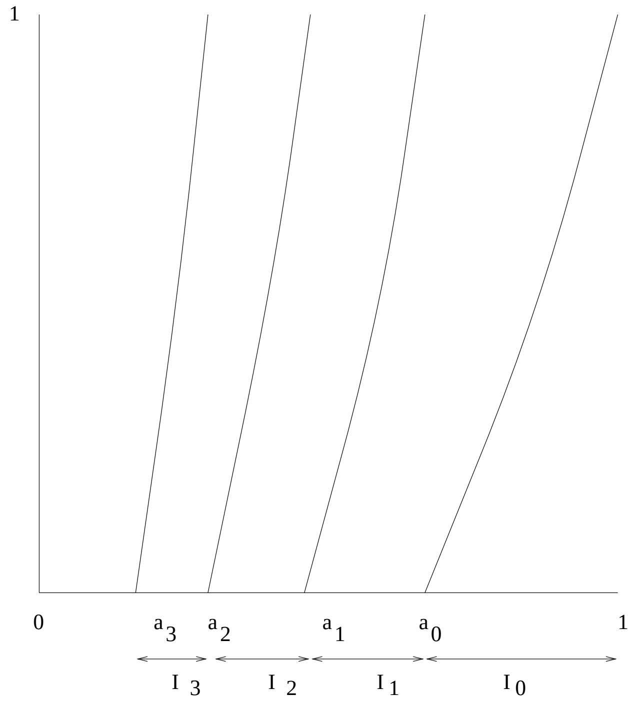


FIGURE 2B: GRAPH OF THE MAP $\widehat{T}_2$

Finally, while we state our main results for these two model transformations, all of our analysis works in greater generality. Our results are valid for interval maps which have induced maps (coded by an infinite alphabet) which satisfy the EMR (expanding-Markov-Rényi) property (see Section II) and potentials (for equilibrium states) which satisfy properties (W1) and (W2) (see Section II). In particular, our analysis works for the family of functions, which Prellberg calls $\mathcal{C}_s$ [Pr], containing certain piecewise monotone transformations of the interval with an indifferent fixed point at the boundary. For instance, the Farey map

$$F(x) = \begin{cases} x/(1-x), & 0 \leq x < 1/2 \\ (1-x)/x, & 1/2 \leq x \leq 1 \end{cases}$$

belongs to the class $C_1$.

## II. Markov Maps and Inducing

Let $I$ denote an interval of real numbers. For the maps $T_1$ and $T_2$ the interval $I$ will be $[0,1]$. The study of our two transformations can be reduced to studying transformations of the following general form:

**Definition.** We say that a transformation $T : I \to I$ is an **EMR** Transformation if we may write $I = \cup_{n=0}^{\infty} I_n$ as a countable union of closed intervals (with disjoint interiors $\overset{\circ}{I}_n$), which we call *basic subintervals*, such that

1. The map $T$ is $C^2$ on $\cup_{k=1}^{\infty} \overset{\circ}{I}_n$.
2. Some power of $T$ is uniformly expanding, i.e., there exists a positive integer $r$ and $\alpha > 0$ such that $|(T^r)'(x)| \geq \alpha > 1$ for all $x \in \cup_{n=1}^{\infty} \overset{\circ}{I}_n$.
3. The map $T$ is Markov.
4. The map $T$ satisfies Renyi's condition, i.e., there exists a positive number $K$ such that

$$\sup_n \sup_{x,y,z \in I_n} \frac{|T''(x)|}{|T'(y)||T'(z)|} \leq K < \infty.$$

It is easy to verify that $|(T_1^2)'(x)| \geq 4$ for all $x \in \cup_{n=1}^{\infty} \overset{\circ}{I}_n$.

**Proposition 1 (**[Sc], [Wa1, p.148]**).** *The continued fraction transformation $T_1$ satisfies EMR.*

For $\alpha = 0$ the Manneville-Pomeau transformation is the usual doubling map of the circle. For $0 < \alpha < 1$, Thaler [T] constructed an finite absolutely continuous invariant measure. For $\alpha > 1$, Thaler also constructed a sigma-finite but not finite absolutely continuous invariant measure. However, we shall not consider this range.

The Manneville-Pomeau map has two branches for $[0, a_0]$ and $[a_0, 1]$ where $1 = a_0 + a_0^{\alpha}$. There is a natural topological conjugacy of this map to the standard doubling map and has topological entropy equal to $\log 2$. Although this map is not hyperbolic since the derivative $T_2'(0) = 1$, the induced map on $[a_0, 1]$ is hyperbolic [Sc]. More precisely, we choose the monotone decreasing sequence $a_n \to 0$ such that $T(a_{n+1}) = a_n$ and define $I_n = [a_n, a_{n-1}]$, for $n \geq 1$, and $I_0 = [a_0, 1]$. For $a_{n+1} < x < a_n$ we define $\widehat{T}_2(x) = T_2^n(x)$ (see Figure 2B). The map $\widehat{T}_2$ is piecewise analytic on countably many intervals $\{I_n\}$ and is uniformly hyperbolic. Each point which is not an end point of an interval $I_n$ has two pre-images under $T_2$.

**Proposition 2.** [T, pp. 312-313, I §2 ] *For $0 < \alpha < 1$, the transformation $\widehat{T}_2$ satisfies ERM.*

We will first effect a multifractal analysis for the map $\widehat{T}_2$ and then *transfer* the multifractal analysis to the map $T_2$.

For an ERM transformation $T$ we define the set $\boldsymbol{\mathcal{O}} = \cup_{k=0}^{\infty} T^{-k}(\cup_{k=0}^{\infty} \partial I_k)$, where $\partial I_k$ denotes the two endpoints of the interval $I_k$. Clearly $\mathcal{O}$ is a countable subset of $I$. We denote by $\mathbf{I_n(x)}$ the element of the refinement $\vee_{i=0}^{n-1} T^{-i}\big(\{I_n\}\big)_{n=1}^{\infty}$ of the original partition containing $x$. Every $x \in I \setminus \mathcal{O}$ has a *unique* symbolic coding since for every $k \in \mathbb{N}$ there is a unique basic subinterval that contains $T^k(x)$.

A fundamental property of an ERM transformation is that it satisfies what is usually called the **Jacobian Estimate** [CFS, page 171], i.e., there exists positive $K$ such that for all $x \in I \setminus \mathcal{O}$ one has

$$0 < \frac{1}{K} \leq \sup_{n \geq 0} \sup_{y \in I_n(x)} \left| \frac{(T^n)'(x)}{(T^n)'(y)} \right| \leq K < \infty. \tag{J}$$

This property will be exploited many times in the proof of Proposition 3 and is also an essential component in the proof of Proposition 2.

For the reasons mentioned in the introduction, it is natural that for infinitely coded maps like $T_1$ and $T_2$, the class of potentials which admit unique equilibrium states is more restrictive than in the usual case. We will now discuss a class of potentials, which we denote $\boldsymbol{\mathcal{W} = \mathcal{W}(T)}$ (for Walters), for which the usual results in thermodynamic formalism are valid.

Let $T\colon I \to I$ be an ERM transformation and $\phi\colon I \to \mathbb{R}$ a function such that $\exp \phi$ is continuous and satisfies the following two properties:

**(W1)** There exits a constant $C > 0$ such that the sum $\sum_{Ty=x} \exp \phi(y) \leq C$, for all $x \in I$.

**(W2)** The function

$$C_\phi(x, x') = \sup_{n \geq 1} \sup_{y \in T^{-n}x} \sum_{i=0}^{n-1} \left| \phi(T^i y) - \phi(T^i y') \right|$$

is bounded by a constant $C_\phi$ and $C_\phi(x, x')$ tends to zero as $|x - x'| \to 0$.

Such potentials will be said to belong to the class $\boldsymbol{\mathcal{W}}$.

The thermodynamic **Pressure** $P(\phi)$ can be defined for a continuous function $\phi$ via the variational principle as

$$P(\phi) = \sup_{\substack{\mu \\ T-\text{inv}}} \left( h_\mu(T) + \int_I \phi d\mu \right),$$

where the supremum is taken over all $T-$invariant Borel probability measures.

**Example 1.** It is an important feature for the continued fraction transformation $T_1$ that $\phi(x) \to -\infty$ as $x \to 0$, in contrast to the usual boundedness of potentials for ERM maps having only finitely many intervals. Observe also that for $T_1$, the non-zero constant function never satisfies (W1).

For $T_1$ with the piecewise analytic potential $\phi = -t \log |T_1'|$ we have that

$$\sum_{T_1 y = x} \exp \phi(y) \leq \sum_{n=1}^{\infty} \frac{1}{n^{2t}} < \infty,$$

and condition (W1) holds for $t > \frac{1}{2}$.

Condition (W2) can similarly be seen to hold on the same range [Wa1]. Let $x, x' \in X$, $y \in T_1^{-n} x$ and let $y'$ be the corresponding point of $T_1^{-n} x'$. Two applications of the Mean Value Theorem yield that

$$|\phi(T^i y) - \phi(T^i y')| = \left| t \frac{T_1''(z_i)}{T_1'(z_i)(T_1^{n-i})'(w_i)} \right| |x - x'|,$$

where $w_i$ and $z_i$ lie between $T_1^i y$ and $T_1^i y'$. Using properties (2) and (4) of ERM we obtain that $|\phi(T^i y) - \phi(T^i y')| \leq Kt/(\alpha^{n-i-1}|x - x'|)$ and thus

$$\sum_{i=0}^{n-1} \left| \phi(T^i y) - \phi(T^i y') \right| \leq Kt/((\alpha - 1)|x - x'|).$$

Condition (W2) immediately follows.

On the range $t > \frac{1}{2}$, the function $t \mapsto P(-t \log |T_1'|)$ is analytic since the function $\exp(P(-t \log |T_1'|))$ is an isolated eigenvalue for the associated transfer operator, about which we shall say more in Section VI [M1]. It is also strictly convex (see Figure 3). Moreover, as in the usual theory, one can use perturbation theory to compute the second derivative of $\exp(P(-t \log |T_1'|))$ and deduce this is strictly convex. It follows from the standard Rohlin equality that $P(-\log |T_1'|) = 0$. Mayer [M1] has also shown that the function $t \mapsto P(-t \log |T_1'|)|$ has a logarithmic singularity at $t = \frac{1}{2}$. This will explain the range of values in the statement of Theorem 1.

**Example 2.** For the induced Manneville-Pomeau transformation $\widehat{T}_2$ with the piecewise analytic potential $\phi = -t \log |\widehat{T}_2'|$, we have that

$$\sum_{\widehat{T}_2 y = x} \exp \phi(y) = \sum_{\widehat{T}_2 y = x} \frac{1}{|\widehat{T}_2'(y)|^t} = \sum_{n=1}^{\infty} |G_n'(x)|^t,$$

where $G_n = F_1 F_0^{n-1}$, and $F_0, F_1$ denote the two branches of the inverse of $T_2$. Properties (W1) and (W2) for $\widehat{T}_2$ are established in [T, p.312] and [I, Lemma 2.2] for $t < 1$.

Prellburg [Pr, PS] showed that the function $P(-t \log |\widehat{T}_2'|) = 0$ for $t > 1$, and that on the range $t < 1$, the function $t \mapsto P(-t \log |\widehat{T}_2'|)$ is analytic and strictly convex (see Figure 4). It follows from the Rohlin equality that $P(-\log |\widehat{T}_2'|) = 0$. Using the variational principle, it is easy to see that for all values of $t$ we have that $P(-t \log |\widehat{T}_2'|) \geq 0$, since it we take $\mu$ to be the Dirac measure supported at 0, then $h_\mu(\widehat{T}_2) = 0$ and $\int_I \log |\widehat{T}_2'| d\mu = 0$. Lopes [L]

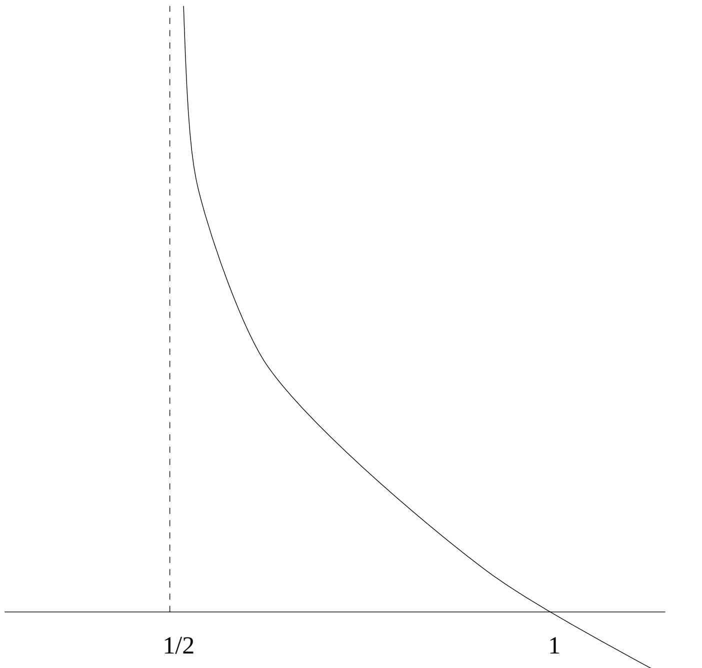


FIGURE 3: GRAPH OF $t \mapsto P(-t \log |T_1'|)$

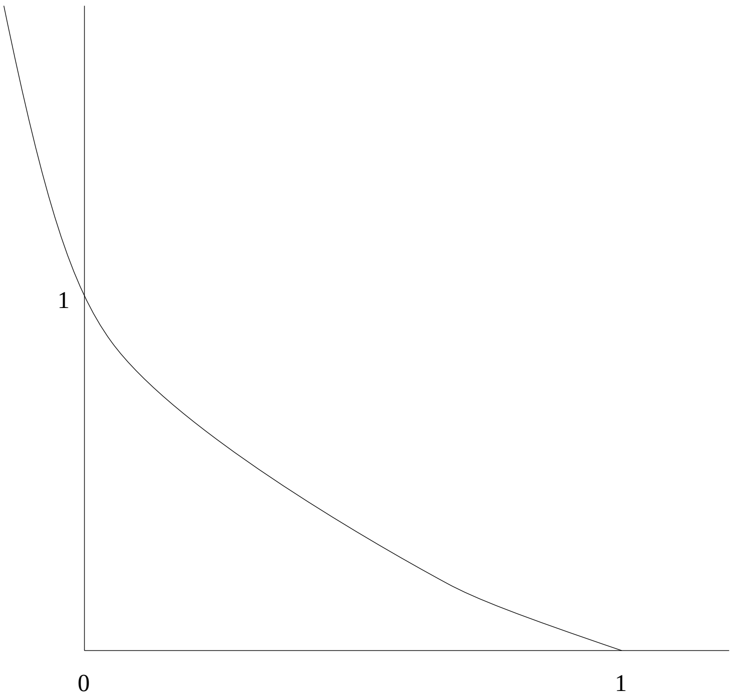


FIGURE 4: GRAPH OF $t \mapsto P(-t \log |\widehat{T}_2'|)$

has studied the precise nature of the singularity of the map $t \mapsto P(-t \log |T_1'|)|$ at $t = 1$. This explains the range of values in the statement of Theorem 2.

*Remark.* There are many other examples to which a similar sort of analysis extends, particularly in the realm of number theoretic analysis. For example, multi-dimensional maps with indifferent periodic points include Brun's map [Sc, §23.1] and the inhomogeneous Diophantine algorithm [Sc, §24.3]. Our analysis also applies to the family of maps $T_s(x) = \{\frac{1}{s(1-x)}\}$ for $0 < s < 4$ and $s \neq 4\cos^2(\pi/q), q = 3, 4, \cdots$, and there is a theory of $s-$backward continued fractions based on this family of maps [GH].

## III. The Multifractal Analysis of Equilibrium States

The general concept of a multifractal analysis for a dynamical system concerns a detailed study of the exceptional behavior of asymptotically defined dynamical quantities such as pointwise dimension, Lyapunov exponent, local entropy, Birkhoff average, etc. In many examples with hyperbolic structure these quantities are constant almost everywhere, with respect to an appropriate ergodic measure.

We consider the case of **pointwise dimension** of a Borel probability measure $\mu$ defined on $I$, defined by

$$d_\mu(x) = \lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}, \tag{1}$$

when the limit exists. Here $B(x,r)$ denotes the ball of radius $r$ centered at the point $x$. Assume that $d_\mu(x) = \alpha_0$ for almost every $x$. There is a natural decomposition of the interval $I$ by level sets $K_\alpha = \{x\colon d_\mu(x) = \alpha\}$

$$I = K_{\alpha_0} \cup \bigcup_{\substack{-\infty<\alpha<\infty \\ \alpha\neq\alpha_0}} K_\alpha \cup \{x \in I \mid d_\nu(x) \text{ does not exist}\}.$$

To study this decomposition we define the **dimension spectrum** $f_\mu(\alpha) = \dim_H(K_\alpha)$, where $\dim_H(K_\alpha)$ denotes the Hausdorff dimension of the set $K_\alpha$. The parts of the multifractal analysis which we will establish are that under suitable conditions, the function $f_\mu(\alpha)$ is real analytic and strictly convex (on a suitable interval), and is given in terms of thermodynamic formalism.

More precisely, let $T\colon I \to I$ be an ERM transformation, and let $\phi$ be a Hölder continuous function on $I$. The **equilibrium state** $\mu$ is a $T$-invariant probability measure $\mu$ such that there exists a positive $C$ such that

$$\frac{1}{C} \leq \frac{\mu(I_n(x))}{\exp(-nP(\phi) + \sum_{j=0}^{n-1} \phi(T^j y))} \leq C, \tag{2}$$

for all $x \in I$ and $y \in I_n(x)$.

We assume that $\phi$ is not cohomologous to $\log|T'|$ [PW1]. Let $\psi$ be the positive function such that $\log\psi = \phi - P(\phi)$, where $P(\phi)$ denotes the thermodynamic pressure of $\phi$ [Wa2]. Clearly $\psi$ is a Hölder continuous function on $X$ such that $P(\log\psi) = 0$ and $\mu$ is also the equilibrium state for $\log\psi$. We define the two parameter family of Hölder continuous functions $\phi_{q,t} = -t\log|T'| + q\log\psi$. Define the function $t(q)$ by requiring that $P(\phi_{q,t(q)}) = 0$ and let $\mu_q$ be the equilibrium state for $\phi_{q,t(q)}$

**Definition.** We say that a triple $(T, \phi, \mu)$ satisfies a **multifractal analysis** if

(1) The pointwise dimension $d_\mu(x)$ exists for $\mu$-almost every $x \in I$. Moreover, $d_\mu(x) = d_\mu \equiv h_\mu(T)/\int_X \log|T'|\, d\mu$ for $\mu$-almost every $x \in I$.

(2) The function $t(q)$ is the Legendre transform of the dimension spectrum, i.e., we have that $f_\mu(\alpha(q)) = t(q) + q\alpha(q)$, where $\alpha(q) = -t'(q) = -\int_I \log\psi d\mu_q / \int_I \log|T'|d\mu_q$. In particular, $t(q)$ is smooth and strictly convex on some interval $(q_{\min}, q_{\max})$.

An immediate consequence is that if $(T, \phi, \mu)$ satisfies a multifractal analysis, then the dimension spectrum $f_\mu(\alpha)$ is smooth and strictly convex on an interval, and hence the pointwise dimension $d_\mu(x)$ attains the interval of values $(\alpha(q_{\max}), \alpha(q_{\min}))$, where each value is attained on an uncountable dense set which supports an equilibrium state.

We note that this is only a partial multifractal analysis in the sense of [PW1, PW2], in that a complete multifractal analysis also establishes analogous results for the Rényi spectrum of dimensions for $\mu$ and then establishes a Legendre transform relation between the dimension spectrum and the Rényi spectrum. Here we only extend those aspects of the theory which we use in our applications. One technical complication in extending the entire theory is that the equilibrium state $\mu$ may not be included in the family of equilibrium states $\mu_q$, where in the usual theory $\mu_1 = \mu$. In other words, the interval $(\alpha(q_{\max}), \alpha(q_{\min}))$ may not contain $d_\mu$. This happens for the continued fraction traFnsformation $T_1$ with potential $\phi = -s\log|T_1'|$, $s > \frac{1}{2}$ (see Corollary 5).

Henceforth for $T_1$ we shall only consider potentials $\phi$ which are elements of $\mathcal{W}(T_1)$, and for $T_2$ we shall only consider potentials $\phi$ which are elements of $\mathcal{W}(\widehat{T}_2)$. For these classes of potentials we establish a multifractal analysis for the pointwise dimension of the associated equilibrium state. For applications to Diophantine approximation, we will only require a multifractal analysis for the very special class of potentials of the form $-s\log|T'|$.

**Theorem 1.** *A multifractal analysis holds for the continued fraction transformation $T_1$ in the range of $q$ such that $t(q) > \frac{1}{2}$ (see Figure 5).*

**Theorem 2.** *A multifractal analysis holds for holds for the Manneville-Pomeau transformation $T_2$ for $q > 0$ (or equivalently, $t(q) < 1$) (see Figure 6).*

Lyapunov exponents measure the exponential rate of divergence of *infinitesimally close* orbits of a smooth dynamical system. These exponents are intimately related with the global stochastic behavior of the system and are fundamental invariants of a smooth dynamical system. For a transformation $T\colon I \to I$ we define the **Lyapunov exponent** of $T$ by

$$\lambda(x) \equiv \lim_{n\to\infty} \frac{1}{n} \log|(T^n)'(x)| = \lim_{n\to\infty} \frac{1}{n} \log \prod_{i=0}^{n-1} |T'(T^i x)|, \tag{3}$$

when the limit exists. The function $\lambda(x)$ is clearly $T-$invariant.

The multifractal analysis of the Lyapunov exponent follows easily from the multifractal analysis for pointwise dimension via the following proposition.

**Proposition 3.** *Let $T : I \to I$ be an ERM transformation and let $\mu$ be the equilibrium state corresponding to the Hölder continuous potential $\phi$. Define $\psi$ by $\log\psi = \phi - P(\phi)$.*

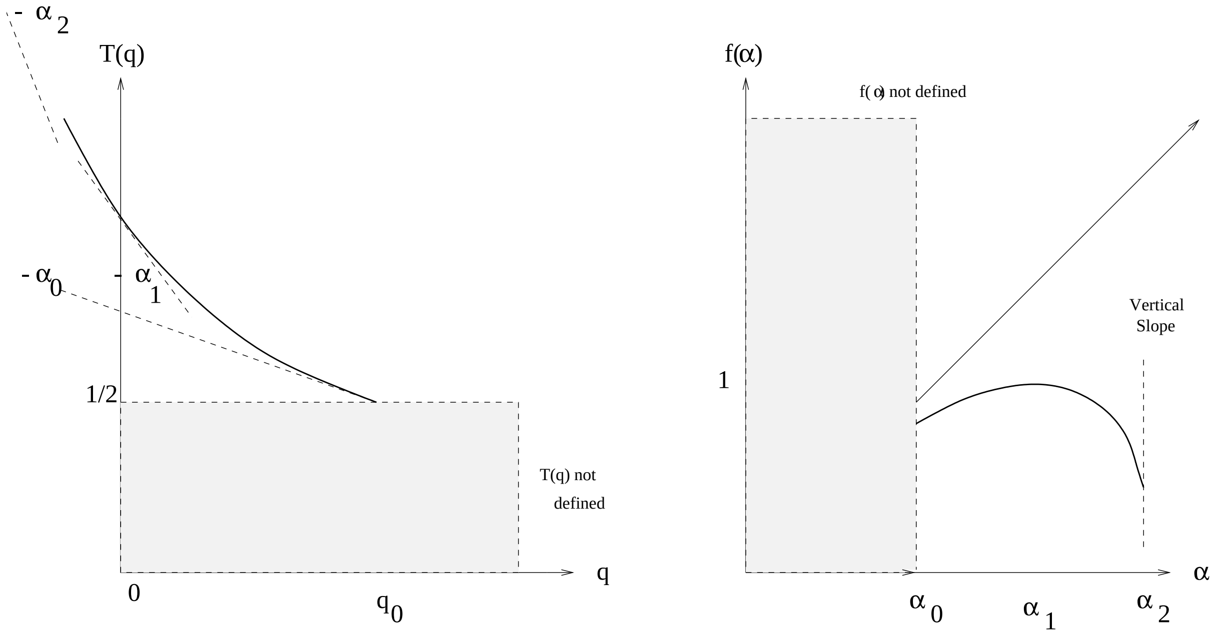


FIGURE 5: MULTIFRACTAL ANALYSIS FOR THE CONTINUED FRACTION TRANSFORMATION

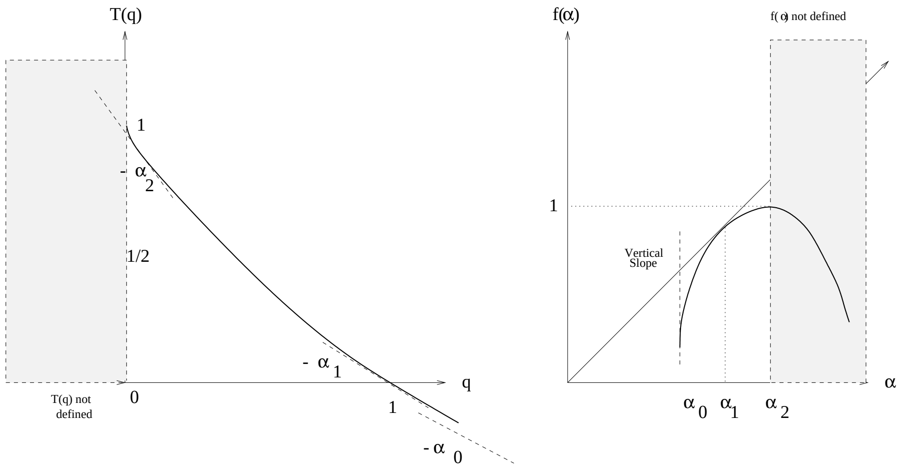


FIGURE 6: MULTIFRACTAL ANALYSIS FOR THE MANNEVILLE-POMEAU TRANSFORMATION

*For $x \in I \setminus \mathcal{O}$, consider the Lyapunov exponent $\lambda(x)$ and the Birkhoff average $\overline{\phi}(x) \equiv \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \phi(T^k(x))$, where it exists.*

(1) *The* **Markov pointwise dimension** *is defined by*

$$\delta_\mu(x) \equiv \lim_{n\to\infty} \frac{\log \mu(I_n(x))}{-\log \ell(I_n(x))} = \lim_{n\to\infty} \frac{\prod_{i=0}^{n-1} \log \psi(T^i x)}{-\log |(T^n)'(x)|},$$

*when the limits exist.*

(2) *The pointwise dimension* $d_\mu(x)$ *exists if and only if the Markov pointwise dimension,* $\delta_\mu(x)$*, exists.*
(3) *If* $\overline{\phi}(x)$ *and* $\lambda(x)$ *both exist, then*

$$d_\mu(x) = \delta_\mu(x) = \frac{P(\phi) - \overline{\phi}(x)}{\lambda(x)} = \frac{-\overline{\log\psi}(x)}{\lambda(x)}.$$

The Markov pointwise dimension will be a useful tool to compute the pointwise dimension. To show that the pointwise dimension attains a particular value on a set of positive Hausdorff dimension, it is sufficient to verify the same for the Markov pointwise dimension, since the two notions of dimension agree except on a countable set. Thus to compute the Hausdorff dimension of a level set for the Lyapunov exponent attains a particular value, one can work with either notion of pointwise dimension. The essential difference between the two notions of dimension is that while the definition of pointwise dimension uses arbitrary intervals, the Markov pointwise dimension uses only those intervals of the form $I_n(x)$. For arbitrary invariant measures, there are subtleties in trying to relate the quantities [PW1, PW3], but Proposition 3 says that for equilibrium states, the two quantities, as long as they are well defined, coincide.

*Proof of Proposition 3.* The proof of (1) follows immediately from the Jacobian Estimate (J) and (2).

The proof of (2) is modeled on the proof of Theorem 7 in [PW3]. The Jacobian estimate (J) allows us to estimate the lengths ($\ell$) of the intervals $I_n(x)$ using the derivative of $T$ [PW1], i.e., there exist positive constants $C_1$ and $C_2$ such that for all $x \in I \setminus \mathcal{O}$ and all $n \in \mathbb{N}$, we have that

$$C_1 \le \frac{\ell(I_n(x))}{|(T^n)'(x)|^{-1}} \le C_2. \tag{4}$$

Suppose that the Markov pointwise dimension $\delta_\mu(x)$ exists at a point $x$. Given $r > 0$ there exists a unique $n = n(r)$ such that $C_1|(T^n)'(x)|^{-1} < r \le C_1|(T^{n-1})'(x)|^{-1}$. It immediately follows from (4) that

$$\frac{\log\mu(B(x,r))}{\log r} \ge \frac{\log\mu(B(x, C_1|(T^{n-1})'(x)|^{-1})}{\log r} \ge \frac{\log\mu(I_{n-1}(x))}{\log r} \tag{5}$$

$$\ge \frac{\log\mu(I_{n-1}(x))}{\log(C_1|(T^n)'(x)|^{-1})} = \frac{\log\mu(I_{n-1}(x))}{\log\mu(I_n(x))}\frac{\log\mu(I_n(x))}{\log(C_1|(T^n)'(x)|^{-1})}. \tag{6}$$

Since $\mu$ is an equilibrium state for a potential whose exponential is smooth on $I$, it is easy to verify that there exist positive constants $D_1$ and $D_2$ such that $D_1 \le \mu(I_{n-1}(x))/\mu(I_n(x)) \le D_2$ for all $x \in I \setminus \mathcal{O}$ and all $n \in \mathbb{N}$, and thus $\lim_{n\to\infty} \log\mu(I_{n-1}(x))/\log\mu(I_n(x)) = 1$ for all $x \in I \setminus \mathcal{O}$. We immediately obtain that $\underline{d}_\mu(x) \ge \delta_\mu(x)$.

Next given $r > 0$ there exists a unique $n = n(r)$ such that $C_2|(T^n)'(x)|^{-1} < r \le C_2|(T^{n-1})'(x)|^{-1}$. It immediately follows from (4) that

$$\frac{\log\mu(B(x,r))}{\log r} \le \frac{\log\mu(B(x, C_2|(T^n)'(x)|^{-1})}{\log r} \le \frac{\log\mu(I_n(x))}{\log r} \le \frac{\log\mu(I_n(x))}{\log(C_2|(T^n)'(x)|^{-1})}.$$

It immediately follows that $\overline{d}_\mu(x) \le \delta_\mu(x)$ and hence $d_\mu(x) = \delta_\mu(x)$.

Now suppose that the Markov pointwise dimension $d_\mu(x)$ exists at a point $x$. Equation (4) immediately implies that

$$\frac{\log \mu(I_n(x))}{\log |(T^n)'(x)|^{-1}} \ge \frac{\log \mu(B(x, C_2|(T^n)'(x)|}{\log(|(T^n)'(x)|^{-1})},$$

and we immediately obtain that $\underline{\delta}_\mu(x) \ge d_\mu(x)$.

Finally, choose an increasing sequence of positive integers $\{n_k\}$ such that

$$\lim_{n\to\infty} \frac{\log \mu(I_{n_k}(x))}{\log(C_1|(T^{n_k})'(x)|^{-1})} = \overline{\delta}_\mu(x),$$

and for each $n_k$ choose some $r_k > 0$ such that $C_1|(T^{n_k})'(x)|^{-1} < r_k \le C_1|(T^{n_k-1})'(x)|^{-1}$. From (5) and (6) we have that

$$\frac{\log \mu(B(x, r_k))}{\log r_k} \ge \frac{\log \mu(I_{n_k-1}(x))}{\log \mu(I_{n_k}(x))} \frac{\log \mu(I_{n_k}(x))}{\log(C_1|(T^{n_k})'(x)|^{-1})}.$$

Since $\lim_{n\to\infty} \log \mu(I_{n_k-1}(x))/\log \mu(I_{n_k}(x)) = 1$ for all $x \in I \setminus \mathcal{O}$, we obtain that $d_\mu(x) \ge \overline{\delta}_\mu(x)$ and hence $\delta_\mu(x) = d_\mu(x)$.

Finally, the proof of (3) follows from (1) and (2).

In a similar spirit as for pointwise dimension, we define the **Lyapunov spectrum** for a smooth map $T$ by considering the level sets of the Lyapunov exponent $L_\beta = \{x \in I \colon \lambda(x) = \beta\}$ and by defining

$$g(\beta) = \dim_H(L_\beta).$$

The following is an immediate consequence of Proposition 3 and Theorems 1 and 2, and allows us to relate the Lyapunov spectrum to the dimension spectrum for a special class of equilibrium states (see [We]).

**Corollary 1.** *Let $T : I \to I$ be an ERM transformation. If $\phi(x) = -s \log |T'|$, then $\lambda(x) = P(-s\log|T'|)/(\delta_{\mu_s}(x) - s)$, where $\mu_s$ is the equilibrium state for $-s\log|T'|$. Hence, $\lambda(x) = P(-s\log|T'|)/(d_{\mu_s}(x) - s)$ if $x \in I \setminus \mathcal{O}$. In the case $s = 0$ we obtain, that except on a countable set, $\lambda(x) = h_{TOP}(T)/d_{\mu_{\mathrm{MAX}}}(x)$, where $h_{TOP}(T)$ denotes the topological entropy of the map $T$ and $\mu_{\mathrm{MAX}}$ denotes the measure of maximal entropy[1]. Since countable sets have zero Hausdorff dimension, we have that*

$$f_{\mu_s}(\alpha) = g\left(\frac{P(-s\log|T'|)}{\alpha - s}\right).$$

*The Lyapunov spectrum $g(\beta)$ is smooth and strictly convex on an interval.*

We will see in Section IV that the Lyapunov exponent for the continued fraction transformation measures the precise exponential rate of rational approximation for the continued fraction algorithm.

---

[1]We remind the reader that $h_{TOP}(T_1) = \infty$ and thus care must be taken in the allowable range of $s$.

## IV. Continued Fractions, Diophantine Approximation, and Cuspital Excursions on the Modular Surface

For a wealth of classical results about continued fractions we recommend the superb books [C], [HW] and [K]. The books [B, CFS] contain an excellent introduction to the dynamics of the continued fraction transformation and the connection with Diophantine approximation.

Every irrational number $0 < x < 1$ has a continued fraction expansion of the form

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}} \quad = \quad [a_1, a_2, a_3, \dots],$$

where $a_1, a_2, \cdots$ are positive integers. For every positive integer $n$ define the n-th approximant $p_n/q_n$ to be the rational number

$$\frac{p_n}{q_n} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\dots + \cfrac{1}{a_n}}}}.$$

There is a simple recursive relationship between $p_n, q_n$ and $a_n$:

$$p_0 = 0, \quad p_{-1} = 1, \quad p_n = a_n p_{n-1} + p_{n-2}, \quad k = 1, 2, \cdots$$

$$q_0 = 1, \quad q_{-1} = 0, \quad q_n = a_n q_{n-1} + q_{n-2}, \quad k = 1, 2, \cdots. \tag{7}$$

The continued fraction transformation can be considered as a simple *algorithm* for associating to irrational numbers $0 < x < 1$ a sequence of rational numbers $\frac{p_n}{q_n}$. It is well known that the approximants $p_n/q_n$ satisfy

$$\frac{1}{2q_{n+1}^2} < \frac{1}{q_n(q_n + q_{n+1})} \leq \left| x - \frac{p_n}{q_n} \right| \leq \frac{1}{q_n(a_{n+1}q_n + q_{n-1})} = \frac{1}{q_n q_{n+1}} < \frac{1}{q_n^2}. \tag{8}$$

There is an intimate connection between the numbers $a_1, a_2, \cdots$ and the continued fractions transformation $T_1$. Given $0 < x < 1$ we can write

$$x = \frac{1}{\frac{1}{x}} = \frac{1}{\left[\frac{1}{x}\right] + \left\{\frac{1}{x}\right\}} = \frac{1}{a_1 + T_1 x} = \frac{1}{a_1 + \frac{1}{\frac{1}{T_1 x}}} = \frac{1}{a_1 + \frac{1}{\left[\frac{1}{T_1 x}\right] + \left\{\frac{1}{T_1 x}\right\}}}$$

$$= \cfrac{1}{a_1 + \cfrac{1}{a_2 + \left\{\cfrac{1}{T_1 x}\right\}}} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + T_1^2 x}} = \cdots.$$

Thus $a_1 = [1/x], a_2 = [1/T_1 x], \cdots, a_k = \left[1/T_1^{k-1}x\right]$. Alternatively, if $x = [a_1, a_2, \cdots]$, then $T_1^n(x) = [a_{n+1}, a_{n+2}, a_{n+3}, \cdots]$. From this relation one immediately sees a close connection between the distribution of the values of $a_k$ and the ergodic properties of the map $T_1$. It also easily follows from the recursion that $p_n(x) = q_{n-1}(T_1 x)$. To see this

$$\frac{p_n(x)}{q_n(x)} = [a_1, \cdots, a_n] = \frac{1}{a_1 + [a_2, \cdots, a_n]} = \frac{1}{a_1 + [p_{n-1}(T_1 x)/q_{n-1}(T_1 x)]}$$

$$= \frac{q_{n-1}(T_1 x)}{a_1 q_{n-1}(T_1 x) + p_{n-1}(T_1 x)}. \tag{9}$$

Since all fractions are irreducible, the result immediately follows.

There is an absolutely continuous $T_1$-invariant probability measure $\mu_G$ on $[0, 1]$, usually called the **Gauss measure**, defined by

$$\mu_G(B) = \frac{1}{\log 2} \int_B \frac{1}{(1+x)} dx.$$

Clearly

$$\frac{1}{2 \log 2} \ell(B) \leq \mu_G(B) \leq \frac{1}{\log 2} \ell(B),$$

where $\ell$ denotes Lebesgue measure. The map $T_1$ is ergodic with respect to $\mu_G$ [K].

For $x \in I \setminus \mathcal{O}$ the set $I_n(x)$ consists of all points $0 \leq y \leq 1$ whose n-th approximant is the same as for $x$. Dynamically this is equivalent to $T^k x, T^k y \in \left[\frac{1}{a_k(x)+1}, \frac{1}{a_k(x)}\right]$ for $1 \leq k \leq n$. An easy calculation shows that $\ell(I_n(x)) = \frac{1}{q_n(x)(q_n(x)+q_{n-1}(x))}$. Applying (4) and (8) we obtain

$$\lambda(x) = -\lim_{n\to\infty} \frac{1}{n} \log \ell(I_n(x)) = 2 \lim_{n\to\infty} \frac{1}{n} \log q_n(x) \equiv 2q(x), \tag{10}$$

when the limits exist, where $q(x)$ denotes the exponential growth rate of the sequence $\{q_n(x)\}$. Moreover, when one or other limit exists, the other limit must also exist.

An immediate consequence of (8) is that

$$\lambda(x) = -\lim_{n\to\infty} \frac{1}{n} \log \left| x - \frac{p_n(x)}{q_n(x)} \right|. \tag{11}$$

when one or other limit exists.

It also follows from the simple estimate on the density of the Gauss measure $\mu_G$ that

$$\lambda(x) = -\lim_{n\to\infty} \frac{1}{n} \log \mu_G(I_n(x)) = 2q(x) = -\lim_{n\to\infty} \frac{1}{n} \log \left| x - \frac{p_n(x)}{q_n(x)} \right|.$$

when the limit exists. Again, when one or other limit exists the other limit must also exist.

Since $T_1$ is ergodic with respect to $\mu_G$, if follows from the Birkhoff ergodic theorem applied to the function $\log |T_1 x|$ and a simple calculation [B] that for $\mu_G$-almost all $x \in I$

$$\lambda(x) = \lambda_0 \equiv \frac{\pi^2}{6\log 2} = 2.37314\cdots \qquad \text{and} \qquad q(x) = q_0 \equiv \frac{\pi^2}{12\log 2} = 1.18657\cdots. \quad (12)$$

Thus the Lyapunov exponent of the continued fraction transformation measures both the precise exponential speed of approximation of a number by its approximants and the exponential growth rate of the sequence $\{q_n(x)\}$. For $\mu_G$-almost all $x \in I$, $|x - p_n(x)/q_n(x)| \asymp \exp(-n\pi^2/6\log 2)$ and $q_n(x) \asymp \exp(n\pi^2/12\log 2)$.

Our aim now is to understand all the possible values which the Lyapunov exponent attains on the exceptional set of zero measure, as well understanding the distribution and structure of the sets of points where the exceptional values are realized.

By studying periodic points of $T_1$ one can easily find points $x$ such that $\lambda(x) \neq \lambda_0$. The fixed points of $T_1$ correspond to numbers of the form $x = [a, a, a, \cdots]$. The Lyapunov exponent at a fixed point $x$ is $-2\log x$, and since fixed points exist arbitrarily close to 0 ($[2a, 2a, 2a, \cdots] = \sqrt{a^2+1} - a$), it follows that $\lambda(x)$ attains arbitrarily large values. On the other hand, it is easy to see that the minimum value that $\lambda(x)$ can attain is $2\log(\gamma) = 0.962424\cdots$, where $\gamma = (1+\sqrt{5})/2$ is the Golden Mean. This follows from the fundamental recursion (7) since $q_n = a_n q_{n-1} + q_{n-2} \geq q_{n-1} + q_{n-2}$, and thus $q_n(x) \geq c\gamma^n$ for all $x$ and all $n$, where $c$ is a constant which is determined by initial conditions. This value of $\lambda(x)$ is attained, for example, at the fixed point $x = \gamma - 1 = [1, 1, 1, \cdots]$. This value of $\lambda(x)$ is also attained at any number $x$ whose continued fraction expansion consists of all 1's from some point on, i.e., $x = [a_1, \cdots, a_n, 1, 1, 1, \cdots]$. Such numbers are sometimes called *noble* numbers. More generally, this value will be realized by precisely those numbers whose continued fraction expansions have a proportion of 1s which increases to 100 percent. The set of such numbers is dense and uncountable.

We know that $\lambda(x)$ can realize the values $\gamma$ and $\lambda_0$. In the case of $\gamma$ we can find a periodic orbit $x_\infty$ such that $\lambda(x_\infty) = \gamma$. We claim that the Lyapunov exponent for $T_1$ possesses an intermediate value property: any intermediate value can also be realized as a Lyapunov exponent.

**Lemma 1.** *Given any value $2\log\gamma < \xi < \lambda_0$, there exists a point $x \in I$ such that $\lambda(x) = \xi$.*

*Proof.* Fix any value $2\log\gamma < \xi < \lambda_0$. We first note that since the $T_1$-invariant measures are convex, then

$$[2\log\gamma, \lambda_0] \cap \left\{ \int \log |T_1'| d\mu : \mu\, T_1 - \text{invariant} \right\} = [2\log\gamma, \lambda_0].$$

Moreover, since the periodic point measures are weak star dense in the $T_1$-invariant measures, we can choose a sequence of periodic orbits $T_1^{N_n} x_n = x_n$, $n \geq 1$, such that the associated Lyapunov exponent $\lambda(x_n)$ satisfies $|\lambda(x_n) - \xi| < \frac{1}{n}$.

We can write each periodic orbit in terms of its continued fraction expansion i.e. $x_n = [a_0(x_n), a_1(x_n), \cdots, a_{N_n - 1}(x_n)]$. We can choose an increasing sequence $n_k$ inductively such that $\frac{1}{n_k}\sum_{i=1}^{k-1} n_i N_i \to 0$.

Finally, we define the point $x \in I$ having continued fraction expansion

$$x = [\underbrace{a_0(x_1), \cdots, a_{N_1-1}(x_1)}_{n_1 \text{ times}}, \underbrace{a_0(x_2), \cdots, a_{N_2-1}(x_2)}_{n_2 \text{ times}}, \cdots],$$

i.e., we concatenate the repeated block in the continued fraction expansion of $x_1$ ($n_1$ times), followed by the repeated block in the continued fraction expansion of $x_2$ ($n_2$ times), etc. By construction we have that $\lambda(x) = \xi$. ■

To study the distribution of values of $\lambda(x)$ more precisely we define the ($T_1-$invariant) level sets of the Lyapunov exponent

$$\Lambda_\alpha = \{x \in I \colon \lambda(x) = \alpha\}.$$

These sets (along with the set on which $\lambda(x)$ does not exist) provide a decomposition of the interval. From (10) and (11) we see that for $x \in \Lambda_\alpha$, $|x - p_n(x)/q_n(x)| \asymp \exp(-n\alpha)$ and $q_n(x) \asymp \exp(n\alpha/2)$.

The following proposition on the distribution of values of $\lambda$ is an easy consequence of Lemma 1, Theorem 1, and Proposition 3 applied to the function $\phi = -s \log |T_1'|$ for $s > \frac{1}{2}$:

**Proposition 4.** *Let $T_1 : I \to I$ be the continued fraction transformation.*

(1) *The Lyapunov exponent $\lambda(x)$ attains the interval of values*

$$[2 \log \gamma, \infty) = [2 \log \gamma, \lambda_0) \cup [\lambda_0, \infty).$$

(2) *For $\alpha \in [\lambda_0, \infty)$ the value $\alpha = \int_I \log |T_1'| d\mu_s$ is attained by the Lyapunov exponent on a set of (positive) Hausdorff dimension $h_{\mu_s}(T_1)/\int_I \log |T_1'| d\mu_s = h_{\mu_s}(T_1)/\alpha$. This level set is uncountable and also dense in $I$.*

*Proof.* Lemma 1 implies that $\lambda(x)$ attains the interval of values $[\gamma, \lambda_0]$.

Consider the family of potentials $\phi^s = -s \log |T_1'|$, $s > \frac{1}{2}$ and let $\mu_s$ be the corresponding family of equilibrium states. To prove Proposition 4, we shall only require (1) in our definition of multifractal analysis, that

$$d_{\mu_s}(x) = d_{\mu_s} \equiv \frac{h_{\mu_s}(T_1)}{\int_I \log |T_1'| d\mu_s} = s + \frac{P(-s \log |T_1'|)}{\int_I \log |T_1'| d\mu_s} > 0$$

for $\mu_s$-almost all $x \in I$. Since $d_{\mu_s}(x) = d_{\mu_s}$ for all $x \in K_{d_{\mu_s}} \equiv \{x \in I \colon d_{\mu_s}(x) = d_{\mu_s}\}$, this gives $\dim_{\mathrm{H}} K_{d_{\mu_s}} = d_{\mu_s}$ [PW1, pp 253–254]. Proposition 1 and the Variational Principle immediately imply that if $d_{\mu_s}(x) = d_{\mu_s}$ then $\lambda(x) = \int_I \log |T_1'| d\mu_s$, and thus $\lambda(x) = \int_I \log |T_1'| d\mu_s$ for $\mu_s$-almost all $x \in I$. Thus $\dim_{\mathrm{H}} \Lambda_{\int_I \log |T_1'| d\mu_s} = d_{\mu_s} > 0$.

Since $\mu_s(K_{d_{\mu_s}}) = \mu_s(\Lambda_{\int_I \log |T_1'| d\mu_s}) = 1$, and equilibrium states are positive on open sets, we have that each of the sets $K_{d_{\mu_s}}$ and $\Lambda_{\int_I \log |T_1'| d\mu_s}$ are dense in $I$.

Recall that $\mu_1 = \mu_G$ and thus $\int_I \log |T_1'| d\mu_1 = \lambda_0$. It immediately follows from the first derivative formula for pressure [R] that

$$\int_I \log |T_1'| d\mu_s = -\frac{d}{ds} P(-s \log |T_1'|).$$

Since $\lim_{s\searrow\frac{1}{2}} P(-s\log|T_1'|) = \infty$ and $P(-s\log|T_1'|)$ is smooth and strictly convex on $(\frac{1}{2},\infty)$, it follows that $\lim_{s\searrow\frac{1}{2}} \frac{d}{ds}P(-s\log|T_1'|) = -\infty$. Hence $\lim_{s\searrow\frac{1}{2}} \int_I \log|T_1'|d\mu_s = \infty$. The map $s \mapsto \int_I \log|T_1'|d\mu_s$ is smooth on $(\frac{1}{2},\infty)$, by analytic perturbation theory since $\mu_s$ corresponds to an isolated maximal eigenvalue for a transfer operator [M1], and this implies that the Lyapunov exponent $\lambda$ attains all values between $\lambda_0$ and $\infty$, each on an uncountable dense set of positive Hausdorff dimension. ■

Proposition 4 immediately implies the following two number theoretic corollaries.

**Corollary 2.** *The quantity* $\lim_{n\to\infty} \frac{1}{n}\log\left|x - \frac{p_n(x)}{q_n(x)}\right|$ *attains the interval of values* $[\lambda_0,\infty)$ *and each value in this interval is attained on an uncountable dense set of positive Hausdorff dimension. There is an explicit formula for the Hausdorff dimension of each level set, which in particular shows that the Hausdorff dimension of the level sets vary smoothly.*

**Corollary 3.** *The quantity* $\lim_{n\to\infty} \frac{1}{n}\log q_n(x)$ *attains the interval of values* $[\frac{1}{2}\lambda_0,\infty)$ *and each value in this interval is attained on an uncountable dense set of positive Hausdorff dimension. There is an explicit formula for the Hausdorff dimension of each level set, which in particular shows that the Hausdorff dimension of the level sets vary smoothly.*

We have seen that $p_n(x) = q_{n-1}(T_1x)$ and this easily implies that

$$p(x) \equiv \lim_{n\to\infty} \frac{1}{n}\log p_n(x) = \lim_{n\to\infty} \frac{1}{n}\log q_{n-1}(T_1x) = q(T_1x).$$

It follows from (12) that $p(x) = q_0 = \pi^2/12\log 2$ for $\mu_G-$almost all $x \in I$. The next corollary, which analyzes the exceptional set, follows immediately.

**Corollary 4.** *The quantity* $\lim_{n\to\infty} \frac{1}{n}\log p_n(x)$ *attains the interval of values* $[\frac{1}{2}\lambda_0,\infty)$ *and each value in this interval is attained on an uncountable dense set of positive Hausdorff dimension. There is an explicit formula for the Hausdorff dimension of each level set, which in particular shows that the Hausdorff dimension of the level sets vary smoothly.*

*Remark.* Consider the function

$$\tau(x) = -\lim_{n\to\infty} \frac{\log\left|x - \frac{p_n(x)}{q_n(x)}\right|}{\log q_n(x)} = -\lim_{n\to\infty} \frac{1}{n}\log\left|x - \frac{p_n(x)}{q_n(x)}\right| \frac{1}{\frac{1}{n}\log q_n(x)},$$

when the limits exist. It immediately follows from (10) and (11) that for $\mu_G$-almost all $x$ the function $\tau(x) = \tau_0 \equiv \lambda_0/q_0 = 2$. More precisely, if $q(x) = \lim_{n\to\infty}\frac{1}{n}\log q_n(x)$ exists at a point $x \in I\setminus\mathcal{O}$ then $\tau(x) = 2$.

We now consider the set of points for which $\lambda(x)$ does not exist. An example is the Liouville number $x = \sum_{k=1}^{\infty} 10^{-k!}$. It is easy to see that $|x - \frac{p_n(x)}{q_n(x)}| = \sum_{k=n+1}^{\infty} 10^{-k!}$ and thus $10^{-(n+1)!} < |x - \frac{p_n(x)}{q_n(x)}| < 2\times 10^{-(n+1)!}$. It follows that $\lambda(x) = \infty$.

It is also easy to construct numbers for which the limit defining $\lambda(x)$ does not exist and is not infinite. The construction uses the trick in Lemma 2. Consider the number $x$ with continued fraction expansion

$$x = [\underbrace{1, \cdots, 1}_{n_1 \text{ times}}, \underbrace{2, \cdots, 2}_{m_1 \text{ times}}, \underbrace{1, \cdots, 1}_{n_2 \text{ times}}, \underbrace{2, \cdots, 2}_{m_2 \text{ times}}, \cdots],$$

with each $n_i$ and $m_i$ being much much larger than the sum of all the proceeding choices. A routine argument shows that for suitable choices of $n_i$ and $m_i$, the Lyapunov exponent $\lambda(x)$ does not exist.

A straightforward extension of an argument by Shereshevsky [Sh] gives that the set of points for which $\lambda(x)$ does not exist has positive Hausdorff dimension. A natural problem is to compute the *precise* Hausdorff dimension. In [BaS] the authors show that for a conformal expanding maps (they assume that their map is a local homeomorphism), the Hausdorff dimension of the set of points where the Lyapunov exponent does not exist is *maximal*, i.e., the same dimension as the limit set (repeller). The essential hypothesis for their result is a smooth map which possesses a sequence $\mu_k$ of ergodic invariant measures such that $\lim_{k\to\infty} \dim_{\mathrm{H}}(\mu_k) = \dim_{\mathrm{H}}(J)$, where $J$ is the limit set and $\dim_{\mathrm{H}}(\mu) \equiv \inf\{\dim_{\mathrm{H}}(U), \mu(U) = 1\}$ [Pe, p.42] is the Hausdorff dimension of the measure $\mu$. As will be noted after the proof of Theorems 1 and 2, the existence of such sequences of measures for the maps $T_1$ and $T_2$ follow immediately from the proof of Theorems 1 and 2. Thus a straightforward extension of the proof of Barreira and Schmeling proves the following result.

**Theorem 3.** *The set of points $x \in I$ for which $\lambda(x)$ does not exist has Hausdorff dimension equal to 1.*

There are classical results on Hausdorff dimension and Diophantine approximation, due to Jarnik, to which our results can be viewed as complimentary. Recall that the continued fraction approximants $p_n/q_n$ of a number $x$ all satisfy the Diophantine condition

$$\left| x - \frac{p}{q} \right| \leq \frac{1}{q^2}.$$

Let us now consider the set of numbers which admit a *faster approximation* by rational numbers. For $\tau > 2$ let $\mathcal{F}_\tau$ denote the set of $\tau-$well approximable numbers, i.e., those that satisfy

$$\mathcal{F}_\tau = \left\{ x \in I\colon \left| x - \frac{p}{q} \right| \leq \frac{1}{q^\tau} \quad \text{infinitely often} \right\}.$$

Here *infinitely often* means that there are infinitely many distinct rational $p/q$ which satisfy the relation. Legendre showed that if $p/q$ is any rational approximation to an irrational number $x$ satisfying $\left| x - \frac{p}{q} \right| \leq \frac{1}{2q^2}$, then $p/q$ must be an approximant for $x$. Thus the rationals $p/q$ in $\mathcal{F}_\tau$ are all approximants. It is easy to show that this set has zero measure for each $\tau > 2$. Jarnik [J] computed the Hausdorff dimension of $\mathcal{F}_\tau$ and showed that

$\dim_H(\mathcal{F}_\tau) = \frac{2}{\tau}$. While Jarnik explicitly computes the Hausdorff dimension of the set of numbers $x$ such the approximation error $|x - \frac{p}{q}|$ is bounded above by $q^{-\tau}$ for infinitely many approximants $p_n/q_n$, Corollary 2 is a statement about the Hausdorff dimension of sets of numbers $x$ such that the error $|x - \frac{p_n}{q_n}|$ admits precise asymptotic limiting behavior. Corollary 2 also quantifies the precise speed of convergence of the continued fraction algorithm on exceptional sets. Although our results are related, e.g., the sets $\Lambda_\alpha \subset \mathcal{F}_\tau$ for some $\tau = \tau(\alpha)$, they do not seem to be obtainable from each other.

**Application to Geodesics on the Modular Surface.** The map $T_1$ is also closely related to the symbolic description of the geodesic flow $\phi_t : PSL(2,\mathbb{R})/PSL(2,\mathbb{Z}) \to PSL(2,\mathbb{R})/PSL(2,\mathbb{Z})$ defined on the the unit tangent bundle of the modular surface $M = \mathbb{H}^2/PSL(2,\mathbb{Z})$ by $\phi_t(g)PSL(2,\mathbb{Z}) = gg_tPSL(2,\mathbb{Z})$ with $g_t = \mathrm{diag}(\exp(t), \exp(-t))$ and $\mathbb{H}^2 = \{x + iy \colon y > 0\}$. There is an interesting connection between Corollary 1 and such geodesic flows. The Lyapunov exponent quantifies the proportion of time that a geodesic spends in excursions up the cusp [St, Su].

Given a geodesic $\gamma$ on the modular surface we can consider a lift $\hat{\gamma}$ to the Poincaré upper half-plane $\mathbb{H}^2$. Such geodesics on $\mathbb{H}^2$ correspond to Euclidean semi-circles which meet the real line perpendicularly. In particular, we can choose our lift so that the endpoints satisfy and $1 < x \equiv \hat{\gamma}(\infty) < \infty$ and $-1 < y \equiv \hat{\gamma}(-\infty) < 0$. Let us consider the continued fraction expansion $x = [a_0(x), a_1(x), a_2(x), \ldots]$ then we see that the Euclidean height of the arc is equal to $(x - y)/2$ and lies between $a_0/2$ and $a_0/2 + 1$. In particular, the hyperbolic distance of the excursion into the cusp can be estimated by $\log a_0(x)$ (see Figure 7).

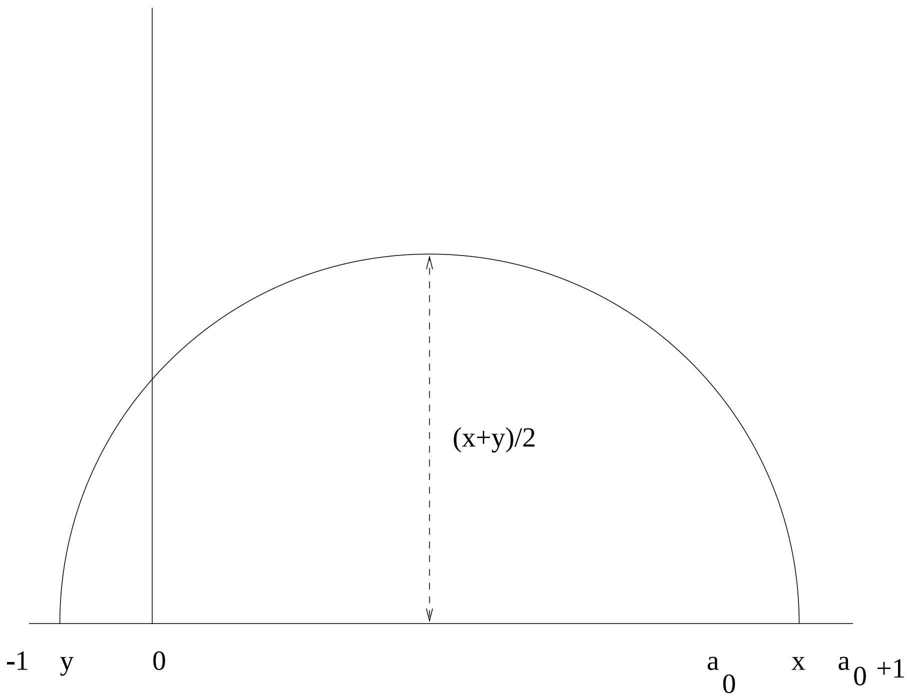


Figure 7: Continued fractions and geodesic excursions

Fix $p \in M$. Given a unit tangent vector $v \in T_1M = PSL(2,\mathbb{R})/PSL(2,\mathbb{Z})$ at $p$, we let

$\gamma : \mathbb{R} \to M$ be the unique unit speed geodesic with $\dot{\gamma}(0) = v$.

We denote by $s_n$, $n \geq 1$, the times $s > 0$ at which $\gamma_s(v)$ maximizes $d(\gamma_s(v), p)$ on each successive excursion into the cusp (i.e., $s_1 < s_2 < s_3 < \dots$ are local maxima for $d(\gamma_s(v), p)$), where $d$ denotes the hyperbolic distance on $M$. By the above observations we see that these heights can be estimated with the function $-\log a_n(x)$. Unfortunately, this function does *not* satisfy condition (W1), so that we need to consider instead functions of the form $\phi(x) \equiv -\beta \log a_n(x)$, for $\beta > 1$. Moreover, $s_n$ can be estimated by $\log |(T_1^n)'(x)|$. We can therefore interpret the quantity $A_n(v) \equiv \sum_{i=0}^{n-1} \log a_i(x)/\log |(T_1^n)'(x)| \sim \sum_{i=0}^{n-1} d(\gamma_{s_i}(v), p)/s_n$ as an estimate on the average height of the first $n$ geodesic excursions, compared with the time required for the excursions. Let us denote

$$\Lambda_\alpha \equiv \left\{ v : \lim_{n\to\infty} \frac{1}{s_n} \sum_{i=0}^{n-1} d(\gamma_{s_i}(v), p) = \frac{\alpha}{\beta} \right\} = \left\{ v : \lim_{n\to\infty} A_n(v) = \frac{\alpha}{\beta} \right\}.$$

Theorem 1 implies the following result.

**Theorem 4.** *There exists an interval of values $(\alpha_{min}, \alpha_{max})$ such that for $\alpha$ in this interval the set $\Lambda_\alpha$ is an uncountable dense set of positive Hausdorff dimension, and the Hausdorff dimension $dim_H(\Lambda_\alpha)$ varies smoothly (analytically).*

This can be compared with the results of Sullivan [Su], Melián-Pestana [MP2], and Stratmann [St], which are of a somewhat complementary nature. In our notation these authors' results relate to the subsequence $t_m = s_{n_m}$ of successive *farthest* geodesic excursions (i.e., $d(\gamma_{t_1}(v), p) \leq d(\gamma_{t_2}(v), p) \leq \dots$ ). Sullivan [Su] showed that a typical geodesic extends a distance at most $\log t$ into a geodesic at time $t$. More precisely, Sullivan shows that for almost every unit tangent vector $v$ at $p$,

$$\lim_{m\to\infty} \frac{d(\gamma_{t_m}(v), p)}{\log t_m} = 1.$$

Stratmann and Melián-Pestana computed the Hausdorff dimension of the sets $\Pi_\alpha = \{v : \lim_{m\to\infty} \frac{d(\gamma_{t_m}(v), p)}{t_m} = \alpha\}$.

*Remark.* We do not yet have a multifractal analysis of Birkhoff sums for $T_1$. With such machinery one could make similar statements about $\sum_{i=0}^{n-1} d(\gamma_{s_i}(v), p)/n$ as we can for $\sum_{i=0}^{n-1} d(\gamma_{s_i}(v), p)/s_n$.

*Remark.* As one would imagine, a similar study can be made of the behaviour of geodesics on other surfaces with cusps, generalizing those for the modular surface. In this case, one needs to use the general analysis described in [BoS]. There is also an analogous notion of Diophantine for more general Fuchsian groups [Pa], to which our results would naturally apply.

## VI. Thermodynamic Formalism for Infinite State Systems

The following proposition contains useful formulas for the derivatives of pressure.

**Proposition 5.** *Let $T\colon I \to I$ be an ERM transformation. Let $f, g$ and $h$ be Hölder continuous functions on $I$ such that for sufficiently small $\epsilon_1, \epsilon_2$ the family of functions $f + \epsilon_1 g + \epsilon_2 h$ satisfy (W1), (W2), and $P(f + \epsilon_1 g + \epsilon_2 h) > -\infty$. Then the function $(\epsilon_1, \epsilon_2) \mapsto P(f + \epsilon_1 g + \epsilon_2 h)$ is analytic, convex (in each variable), strictly convex if $f$ is not cohomologous to a constant, and satisfies the following derivative formulas:*

$$\frac{d}{d\epsilon_1}\bigg|_{\epsilon=0} P(f + \epsilon_1 g) = \int_I g \, d\mu_f,$$

*and*

$$\frac{\partial^2 P(f + \epsilon_1 g + \epsilon_2 h)}{\partial \epsilon_1 \partial \epsilon_2}\bigg|_{\epsilon_1 = \epsilon_2 = 0} \equiv Q_f(g, h),$$

*where $Q_f$ is the bilinear form on $C^\alpha(I, \mathbb{R})$ defined by*

$$Q_f(g, h) = \sum_{k=0}^{\infty} \left( \int_I g \cdot \ (h \circ T^k) \, d\mu_f - \int_I g \, d\mu_f \int_I h \, d\mu_f \right),$$

*and $\mu_f$ is the equilibrium state for $f$. Also $Q_f(g, g) \geq 0$ for all $g$ and $Q_f(g, g) > 0$ if and only if $f$ is not cohomologous to a constant function.*

*Proof.* The proof if very similar to the proof in the *usual setting* where the interval map is piecewise smooth and expanding on finitely many intervals [R]. There is additional potential complication which needs to be addressed in that there may be an infinite sum, rather than just a finite sum, for the transfer operator. This occurs for $T_1$ and $\widehat{T}_2$. This complication is *handled* with conditions (W1) and (W2) which ensure that the infinite sum is well defined and can be treated in the usual ways.

The key to the proof is the of study is the transfer operator $L_{\phi_{q,t}} : C^0(I) \to C^0(I)$ given by

$$L_{\phi_{q,t}} k(x) = \sum_{Ty = x} \exp(\phi_{q,t}(y)) k(y),$$

whose maximal eigenvalue is $\exp(P(\phi_{q,t}))$ Moreover, to obtain an isolated eigenvalue in the appropriate range, we study $L_{\phi_{q,t}} : BV(I) \to BV(I)$ acting on the space of functions of bounded variation. Prellburg [Pr] has shown that for $\widehat{T}_2$, the spectrum of this operator consists of the closed unit ball, plus at most a countable number of isolated eigenvalues of modulus strictly greater than 1. Thus when the quantity $P(\phi_{q,t}) > -\infty$, its exponential is the maximal positive isolated eigenvalue for $L_{\phi_{q,t}}$. The result follows by analytic perturbation theory. The convexity is a direct consequence of the second derivative formula. ■

The next proposition is an immediate consequence of Proposition 5 and the implicit function theorem [PW1].

**Proposition 6.** *Let $T\colon I \to I$ be an ERM transformation. Assume that for a range of values $(q, t)$ the family of functions $\phi_{q,t}$, satisfy (W1), (W2), and $P(\phi_{q,t}) > -\infty$. Then the function $(q, t) \mapsto P(\phi_{q,t})$ is analytic and convex (in each variable). Furthermore, for* *the one parameter family of potentials* $\phi_q = \phi_{q,t(q)}$*, where* $t = t(q)$ *is defined by requiring that* $P(\phi_{q,t(q)}) = 0$*, we have that*

(1) *The function* $t(q)$ *is real analytic and convex. It is strictly convex if* $\log \psi$ *is not cohomologous to* $-\log |T'|$.

(2) *The derivative*

$$t'(q) = -\frac{\int_I \log \psi \, d\mu_q}{\int_I \log |T'| \, d\mu_q},$$

*where* $\mu_q$ *is the equilibrium state for* $\phi_q$.

(3) *The second derivative satisfies*

$$t''(q) = \frac{t'(q)^2 \left(\frac{\partial^2 P(\phi_{q,r})}{\partial r^2}\right) - 2t'(q)\left(\frac{\partial^2 P(\phi_{q,r})}{\partial q \, \partial r}\right) + \left(\frac{\partial^2 P(\phi_{q,r})}{\partial q^2}\right)}{\left(\frac{\partial P(\phi_{q,r})}{\partial r}\right)},$$

*evaluated at* $(q, r) = (q, -t(q))$.

The following useful corollary follows easily from Proposition 2. Here we collect many results which we will require in Section VII .

**Corollary 5.** *In the special case* $\phi = -s \log |T'|$ *we have that the function* $\phi_q = -(t(q) + qs)\log |T'| - qP(s \log |T'|)$ *and that*

(1) *The function* $t(q)$ *is defined implicitly by*

$$P(-(t(q) + qs)\log |T'|) = qP(-s \log |T'|).$$

(2) *The derivative*

$$t'(q) = -\frac{h_{\nu_q}(T)}{\int_I \log |T'| d\nu_q},$$

*where* $\nu_q$ *is the equilibrium state for* $\phi_q$.

(3) *We have the special values* $t(0) = 1, t(1) = 0$*, and* $t'(0) = -1$.

(4) *For the continued fraction transformation* $T_1$ *the expressions in (1)-(3) are well defined (satisfy W1, W2, and* $P \neq -\infty$*) provided that* $t(q) > \frac{1}{2}$ *and* $s > \frac{1}{2}$.

(5) *For the Manville-Pomeau transformation* $T_2$ *the expressions in (1)-(3) are well defined (satisfy W1, W2, and* $P \neq -\infty$*) provided that* $t(q) < 1$ [Pr].

**Example 1.** The transfer operator for the continued fraction transformation $T_1$ for the family $\phi_{q,t}$, can be written explicitly as

$$L_{\phi_{q,t}} k(x) = \sum_{n=1}^{\infty} \psi\left(\frac{1}{x+n}\right)^q \left(\frac{1}{x+n}\right)^{2t} k\left(\frac{1}{x+n}\right).$$

For the potentials $\phi_{q,t}$ (for $t > \frac{1}{2}$), conditions (W1) and (W2) apply.

In the special case that $\phi$ is real analytic, the operator $L_{\phi_{q,t}}$ preserves the smaller space of analytic functions and is compact. In particular, we can waive the assumption that $P(\phi_{q,t}) > -\infty$.

**Example 2.** The transfer operator for the induced Manville-Pomeau transformation $\widehat{T}_2$ for the family $\phi_{q,t}$, can be written explicitly as

$$L_{\phi_{q,t}} k(x) = \sum_{\widehat{T}_2 y = x} \frac{\psi^q(x)}{|\widehat{T}_2'(x)|^t} k(x) = \sum_{n=1}^{\infty} \psi(G_n(x))^q |G_n'(x)|^t k(G_n(x)),$$

where where $G_n = F_1 F_0^{n-1}$, and $F_0, F_1$ denote the two branches of the inverse of $\widehat{T}_2$.

In the case of the Manneville-Pomeau transformation, the induced transformation $\widehat{T}_2 : I \to I$ is a smooth map on each of the intervals $[a_n, a_{n+1}]$ [Pr].

For the potentials $\phi_{q,t}$ (for $t > 1$), conditions (W1) and (W2) apply.

The next result describes the construction of equilibrium states using transfer operators.

**Proposition 7.** *Let $T\colon I \to I$ be an ERM transformation. Assume that $\phi$ satisfies (W1) and (W2). Then there exists a unique equilibrium state.*

*Sketch of Proof.* The associated transfer operator satisfies the following [Wa1, p. 128]

(1) There exists $\lambda > 0$ and $h \in C^0(I)$ with $L_{\psi_q} h = \lambda h$;
(2) There exists a $T$-invariant probability measure $\mu$ such that for $f \in C^0(I)$ we have that $\lambda^{-n} L_{\psi_q}^n f(x) \to h \int f d\mu$.

If we denote $g = \frac{\exp(\psi) h}{\lambda h \circ T}$ we have that $\lambda^{-n} L_{\log g}^n f(x) \to \int f d\mu$ [Wa1, p. 128]. Moreover, $\frac{dT^*\mu}{d\mu} = \frac{1}{g}$ [Wa1, p. 124], where $T^*\mu$ denotes the pull-back of the measure $\mu$ by $T$. The proof that $\mu$ is the required equilibrium state comes from (2). A useful property of $g$ is that it satisfies:

$$\exp(-2C(x,y)) \le \prod_{i=0}^{n-1} \frac{|g(T^i x)|}{|g(T^i y')|} \le \exp(2C(x,y)),$$

[Wa1, p.130]. In particular, this implies that there exists $C > 0$ such that

$$\frac{1}{C} \le \frac{\mu(I_n(x))}{\prod_{i=0}^{n-1} g(T^i x)} \le C,$$

where $\exp(2C(x,y)) \le C$. Uniqueness comes by the ergodicity of $\mu$ and the fact that any two solutions are absolutely continuous. ■

Finally, we observe that for all but a finite set of points, the Lyapunov exponents of $T_2$ and $\widehat{T}_2$ are the same. Similarly, other corresponding averages along orbits agree.

## VII. Proofs of Theorems 1 and 2

Consider the transformation $T = T_1$ or $\widehat{T}_2$. Statement (1) in our definition of multifractal analysis easily follows from the Birkhoff ergodic theorem. Given $x \in I \setminus \mathcal{O}$, it follows from Proposition 3 that $d_\mu(x) = -\overline{\log\psi}(x)/\lambda(x)$. Since the equilibrium state $\mu$ is an ergodic measure we have that $\lambda(x) = \int_I \log|T'| d\mu$ and $\overline{\log\psi}(x) = \int_I \log\psi d\mu$ for $\mu$-almost every $x \in I$. Since $P(\log\psi) = 0$, it follows from the Variational Principle that $h_\mu(T) = -\int_I \log\psi d\mu$. The first statement follows.

Recall that $t(q)$ is the unique solution to $P(\phi_{(q,t(q))}) = P(-t(q)\log|T'| + q\log\psi) = 0$ and $\mu_q$ is the equilibrium state for $\phi_q = \phi_{(q,t(q))}$. For each map $T$ and each potential $\phi$, the function $t(q)$ is defined for a certain range $(q_{\min}, q_{\max})$ of $q$ which we have discussed in Section VI. By Propositon 6 the function $t(q)$ is smooth and strictly convex on this interval. Using the derivative formula in Proposition 6, we define for each $q$ the function

$$\alpha(q) \equiv -t'(q) = \frac{\int_I \log\psi(x) d\mu_q}{-\int_I \log|T'| d\mu_q},$$

and consider the level sets

$$M_{\alpha(q)} = \{x\colon \delta_\mu(x) = \alpha(q)\} \qquad \text{and} \qquad K_{\alpha(q)} = \{x\colon d_\mu(x) = \alpha(q)\}.$$

An immediate consequence of the Birkhoff ergodic theorem is that the pointwise dimensions satisfy $\mu_q(M_{\alpha(q)}) = \mu_q(K_{\alpha(q)}) = 1$. Proposition 3 immediately implies that for all $x \in M_{\alpha(q)}$

$$\delta_{\mu_q}(x) = -\frac{\overline{\phi}_q(x)}{\lambda(x)} = \frac{t(q)\overline{\log|T'|} - q\overline{\log\psi}}{\lambda(x)} = t(q) + q\alpha(q),$$

and thus for *all* $x \in K_{\alpha(q)} \cap (I \setminus \mathcal{O})$ the pointwise dimension $d_{\mu_q}(x) = \delta_{\mu_q}(x) = t(q) + q\alpha(q)$. By a standard argument in dimension theory [PW1, pp 253–254] this implies that $\dim_H(K_{\alpha(q)} \cap (I \setminus \mathcal{O})) = t(q) + q\alpha(q)$. Since the Hausdorff dimension of a countable set is zero, we obtain that $f_\mu(\alpha(q)) = \dim_H(K_{\alpha(q)}) = t(q) + q\alpha(q)$.

The smoothness and convexity properties of $t(q)$ follow from Proposition 6. ■

*Remark.* As mentioned before the statement of Theorem 3, the above proof of Theorems 1 and 2 provides a sequence of ergodic invariant measures for $T_1$ or $T_2$ such that

$$\lim_{k\to\infty} \dim_{\mathrm{H}}(\mu_k) = \dim_{\mathrm{H}}(I) = 1.$$

Since $d_{\mu_q}(x) = t(q) + q\alpha(q)$ for $\mu_q$ almost-all $x \in K_{\alpha(q)}$ and $\mu_q(K_{\alpha(q)}) = 1$, a weaker conclusion than used in the theorem above is that for the (ergodic) equilibrium state $\mu_q$ we have that $\dim_{\mathrm{H}}(\mu_q) = t(q) + q\alpha(q)$ [Pe, p. 42]. Since $t(0) = 1$ we have that $\lim_{q\to 0}\dim_{\mathrm{H}}(\mu_q) = 1$. Thus any allowable potential $\phi \in \mathcal{W}$ gives rise to such a one parameter family (and hence such a sequence) of measures.

## References


[B] P. Billingsly, *Ergodic Theory and Information*, Krieger, 1978.

[BaS] L. Bareirra and J. Schmelling, *Sets of "non-typical" points have full topological entropy and full Hausdorff dimension*, preprint.

[BoS] R. Bowen and C. Series, *Markov maps associated with Fuchsian groups*, Publ. Math.(IHES) **50** ((1979)), 153-170.

[C] J. Cassels, *An Introduction to Diopphantine Approximation*, Cambridge University Press, 1957.

[CFS] I. Cornfeld, S. Fomin, Ya. Sinai, *Ergodic theory*, Springer-Verlag, 1982.

[GH] K. Gröchenig and A. Hass, *Backwards Continued Fractions, Hecke Groups and Invariant Measures for Transformations of the Interval*, preprint.

[HW] G. Hardy and E. Wright, *An Introduction to the Theory of Numbers*, fifth edition, Oxford University Press, 1979.

[I] S. Isola, *Dynamical Zeta Functions and Correlation Functions for Non-uniformly Hyperbolic Systems*, preprint.

[J] V. Jarnik, *Über die simultanen diophantischen Approximationen*, Math. Zeit. **33** (1931), 505–543.

[K] A. Khinchin, *Continued fractions*, University of Chicago Press, 1964.

[L] A. Lopes, *The Zeta Function, Nondifferentiability of Pressure, and the Critical Exponent of Transition*, Adv. Math. **101** (1993), no. 2, 133–165.

[M1] D. Mayer, *On the Thermodynamics Formalism for the Gauss Map*, Comm. Math. Phys. **130** (1990), 311–333.

[MP1] Y. Pomeau and P. Manneville, *Intermittent Transition to Turbulence in Dissipative Dynamical Systems*, Comm. Math. Phys. **74** (1980), 189–197.

[MP2] M. Melián and D. Pestana, *Geodesic Excursions into Cusps in Finite-Volume Hyperbolic Manifolds*, Michigan Math. J. **40** (1993), 77–93.

[Pa] S. Patterson, *Diophantine approximation in Fuchsian groups*, Philos. Trans. roy. Soc. London, Ser. A **282** (1976), 527-563.

[Pe] Y. Pesin, *Dimension Theory in Dynamical Systems*, CUP, 1997.

[Pr] T. Prellberg, *Maps of Intervals with Indifferent Fixed Points: Thermodynamic Formalism and Phase Transitions*, Va. Polytechnique Institute Theses, 1991.

[PS] T. Prellberg and J. Slawny, *Maps of Intervals with Indifferent Fixed Points: Thermodynamic Formalism and Phase Transitions*, J. Stat. Physics **66** (1992), 503–514.

[PW1] Y. Pesin and H. Weiss, *A Multifractal Analysis of Equilibrium Measures for Conformal Expanding Maps and Moran-like Geometric Construction*, J. of Stat. Physics **86** (1997), 233–275.

[PW2] Y. Pesin and H. Weiss, *The Multifractal Analysis of Gibbs Measures: Motivation, Mathematical Foundation and Examples*, Chaos **7** (1997), 89–106.

[PW3] Y. Pesin and H. Weiss, *On the Dimension of Deterministic and Random Cantor-like sets, Symbolic Dynamics, and the Eckmann-Ruelle Conjecture*, Comm. Math. Physics **182** (1996), 105–153..

[Sc] P. Schweiger, *Ergodic Theory of Fibred Systems and Metric Number Theory*, Oxford University Press, 1995.

[Sh] M. Shereshevsky, *A Complement to Young's Theorem on Measure Dimension: The Difference Between Lower and Upper Pointwise Dimension*, Nonlinearity **4** (1991), 15–25.

[St] B. Stratmann, *Fractal Dimensions for Jarnik Limit Sets of Geometrically Finite Kleinian Groups; The Semi-Classical APproach*, Ark. Mat. **33** (1995), 385–403.

[Su] D. Sullivan, *Disjoint Spheres, Approximation by Imaginary Quadratic Numbers and the Logarithmic Law for Geodesics*, Acta. Math. **149** (1982), 215–237.

[R] D. Ruelle, *Thermodynamic Formalism*, Addison-Wesley, 1978.

[T] M. Thaler, *Estimates on the invariant densities of endomorphisms with indifferent fixed points*, Israel J. Math. **37** (1980), 303-314.

[Wa1] P. Walters, *Invariant Measures and Equilibrium States for Some Mappings Which Expand Distances*, Transactions of the AMS **236** (1996), 121–153.

[Wa2] P. Walters, *Introduction to Ergodic Theory*, Springer Verlag, 1982.

[We] H. Weiss, *The Lyapunov Spectrum of Equilibrium Measures for Conformal Expanding Maps and Axiom-A Surface Diffeomorphisms*, Preprint (1997).



Mark Pollicott
Department of Mathematics
The University of Manchester
Oxford Road M13 9PL Manchester
England
Email: mp@ma.man.ac.uk

Howard Weiss
Department of Mathematics
The Pennsylvania State University
University Park, PA 16802
U.S.A.
Email: weiss@math.psu.edu