# Continuity of the Lyapunov exponent for quasiperiodic operators with analytic potential.

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*Dedicated to David Ruelle and Yasha Sinai*

## 1 Introduction.

In this paper we study continuity of the Lyapunov exponent associated with 1D quasiperiodic operators. Assume $v$ real analytic on $\mathbb{T}$. Let $v : \mathbb{T} \to \mathbb{R}$. Consider an $SL_2(\mathbb{R})$ valued function

$$A(x,E) = \begin{pmatrix} v(x) - E & -1 \\ 1 & 0 \end{pmatrix}, \quad x \in \mathbb{T}. \tag{1.1}$$

Set

$$M_N(E,x,\omega) = \prod_{j=N}^{1} A(S^j x),\ Sx = x + \omega,$$

$$L_N(E,\omega) = \frac{1}{N}\int \log \|M_N(E,x,\omega)\| dx.$$

The Lyapunov exponent is defined by $L(E,\omega) = \lim_{N\to\infty} L_N(E) = \inf_N L_N(E,\omega)$ and exists by subadditivity.

Our main result is the following Theorem:

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**THEOREM 1** *Assume $v$ real analytic on $\mathbb{T}$. Then*

- *$L(E,\omega)$ is continuous in $E$.*
- *if $\omega_0$ is irrational, $L(E,\omega)$ is jointly continuous in $(E,\omega)$ at every $(E,\omega_0)$.*

**Remark.** $L(E,\omega)$ may be discontinuous in $\omega$ at every rational $\omega$, see below.

Matrices $M_N$ appear in the study of 1D Schrödinger operators

$$(H_x\Psi)(n) = \Psi(n+1) + \Psi(n-1) + v(S^n x)\Psi(n), \tag{1.2}$$

as $N-$step transfer-matrices, and $L(E)$ has been a subject of a considerable investigation in this context. Recently there were several results on regularity in $E$ for quasiperiodic operators (1.2) with $Sx = x+\omega, x \in T^d$. For typical $\omega$ (more precisely satisfying a strong Diophantine condition of the form

$$\|k\omega\| > C(|k|\log(1+|k|)^A)^{-1} \tag{1.3}$$

Goldstein and Schlag [10] proved Hölder regularity of $L(E)$ for $d=1$ and certain weaker regularity for $d>1$ in the regime $L>0$ (see also [5], Ch. VII). Precise estimates on Hölder regularity for the almost Mathieu operator at high coupling are contained in [4]. For $L=0$ some regularity also holds [5], Ch. VIII. For a review of results on continuity of $L$ in $E$ for strictly ergodic shifts over finite alphabet (and new result of this type for S a primitive substitution) see Lenz [17]. As far as $\omega$-dependence, there was a number of results on continuity of the spectrum (e.g., [1, 7, 3, 16, 13]), but continuity of $L$ was not addressed directly. It is however important, since various quantities, including $L(E)$ can sometimes be effectively estimated or even directly computed for periodic operators obtained from the rational approximants of $\omega$. Let $\sigma(H)$ denote the spectrum of $H$.

**COROLLARY 2** *For the almost Mathieu operator $H_{\lambda,\omega,x}$ given by (1.2) with $v(S^n x) = \lambda\cos 2\pi(x+n\omega)$, we have $L(E) = \max(0, \log\frac{|\lambda|}{2})$ for all $E \in \sigma(H)$, all $\lambda$ and all irrational $\omega$.*

**Proof:** Krasovsky [15] showed that for $E \in \sigma(H_{\lambda,\frac{p}{q}})$, $L(E, \frac{p}{q})$ converges as $q \to \infty$ to (but is not equal to) $\max(0, \log \frac{\lambda}{2})$. The result then follows from Theorem 1 and continuity of spectra [1]. ∎

We will also list another immediate corollary of Theorem 1:

**COROLLARY 3** *Suppose $v$ analytic. Then*

$$\limsup_{N\to\infty} \frac{1}{N} \log \|M_N(E, x, \omega)\| \leq L(E) \tag{1.4}$$

*uniformly in $x$ and $E$ in a compact set.*

**Proof**: Furman [8] proved uniformity of (1.4) in $x$ for any continuous cocycle on a uniquely ergodic system. The result follows then from continuity of $L$ in $E$ and compactness. ∎

This uniformity is important for various questions arising in the nonperturbative analysis of operators (1.2). For example, Corollary 3 immediately implies that the almost Mathieu operator has strong dynamical localization (see [9]) for any $\lambda > 2$ and $\omega$ satisfying $\|k\omega\| > c(\omega)(|k|^{r(\omega)})^{-1}$, that is throughout the regime of [12]. Strong dynamical localization was obtained in [9] for $\omega$ satisfying a strong Diophantine condition (1.3). The restriction on $\omega$ was needed there only for the uniformity of an upper bound, such as given in Corollary 3, which is now established for all $\omega$.

We note that the continuity issue, even in $E$ alone, is nontrivial, as $L$ considered as a function on $C(\mathbb{T}, SL_2(\mathbb{R}))$, with $\omega$ fixed, is discontinuous at $A(\cdot, E)$ for a dense set of $E$ in the spectrum of corresponding $H$ provided $L(E)$ is positive and either $\omega$ is Liouville or $v$ even (follows from a theorem of Furman [8] and a combination of [2, 14], see also a discussion in [17] and a related result in [11]). Moreover, the restriction of $L$ to $C(\mathbb{T}, M)$ where $M$ is any locally closed submanifold of $SL_2(\mathbb{R})$ such that $A$ takes values in $M$, is also discontinuous at all such $A$ [8].

The rest of the paper is devoted to the proof of Theorem 1. Section 2 contains a large deviation theorem, which is applied in Section 3 together with avalanche principle to obtain estimates on convergence. Those estimates allow to approximate $L$ with $2L_{2N} - L_N$, uniformly in $(\omega, E)$ sufficiently close to $(\omega_0, E_0)$, where $N$ is also uniform in $(\omega, E)$, provided $\omega_0$ is irrational and

$L(\omega_0, E_0)$ is positive. This is done in Section 4, and the proof of Theorem 1 is completed there.

Our proof builds on many of the same ideas and techniques as the proof of the regularity of $L(E)$ in [5], Ch. VII. A few arguments taken directly from [5] are given here without explanation, and the reader is therefore referred to [5] for more detail.

All constants $c, C$ in what follows will depend, unless otherwise noted, only on $v$ and $E$, being uniform for $E$ in a bounded range. Same notations will be sometimes used for different such constants.

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# 2 Large deviations.

**LEMMA 4** *Let*

$$|\omega - \frac{a}{q}| < \frac{1}{q^2} \qquad (a, q) = 1. \tag{2.1}$$

*Let* $0 < \kappa < 1$. *Then, for appropriate* $c > 0$ *and* $C < \infty$, *for* $N > C\kappa^{-2}q$,

$$\text{mes}\left[x \middle| \left|\frac{1}{N}\log\|M_N(E,x)\| - L_N(E)\right| > \kappa\right] < e^{-c\kappa q}. \tag{2.2}$$

**Proof.** Put

$$u(x) = \frac{1}{N}\log\|M_N(E;x)\| = \sum_{k\in\mathbb{Z}} \hat{u}(k)e^{2\pi ikx}$$

where

$$\hat{u}(0) = L_N(E)$$

As shown in [6] (see also [5], Ch. IV)

$$|\hat{u}(k)| < \frac{C}{|k|}. \tag{2.3}$$

Function $u$ also satisfies

$$|u(x)| < C \ \text{ and } \ |u(x) - u(x+\omega)| < \frac{C}{N} \tag{2.4}$$

(see [5] for details.) Take

$$R \sim \kappa^{-1} q \tag{2.5}$$

and, using (2.4), estimate for $N > C\kappa^{-2} q$

$$|u(x) - \sum_{|j|<R} \frac{R-|j|}{R^2} u(x+j\omega)| < C\frac{R}{N} < \frac{\kappa}{10}. \tag{2.6}$$

for an appropriate $C$. Considering the Féjèr average, we obtain therefore

$$|u(x) - \hat{u}(0)| < \frac{\kappa}{10} + \sum_{0<|k|\le K} |\hat{u}(k)|\ (1 + (R\|k\omega\|)^2)^{-1} + \alpha(x) \tag{2.7}$$

where

$$\|\alpha\|_2^2 \le \sum_{|k|>K} |\hat{u}(k)|^2 < C \sum_{k>K} \frac{1}{k^2} \sim K^{-1}. \tag{2.8}$$

We estimate the second term in (2.7) as

$$\sum_{0<|k|<\frac{q}{4}} |k|^{-1}(2R\|k\omega\|)^{-1} + \sum_{\ell=1}^{4Kq^{-1}} \frac{1}{\ell q} \sum_{k\in I_\ell} (1+(R\|k\omega\|)^2)^{-1} = (I) + (II)$$

where $I_\ell = [\ell\frac{q}{4}, (\ell+1)\frac{q}{4})$.

It follows from (2.1) that for $|k| \le \frac{q}{2}$, $|k\omega - \frac{ka}{q}| < \frac{1}{2q}$ and hence $\|k\omega\| > \frac{1}{2q}$. Let $\alpha_1, \dots, \alpha_{q/4}$ be the decreasing rearrangement of $(\|k\omega\|^{-1})_{0<k\le\frac{q}{4}}$. Then we have $\alpha_i \le \frac{2q}{i}$. Moreover, if $I$ is any interval of length $q/4$, same is true for $(\|k\omega\|^{-1}),\ k \in I$, if we exclude at most one value of $k$.

Hence, for an appropriate choice of $R$ in (2.5),

$$(I) \le CR^{-1} \sum \frac{1}{k}\frac{q}{k} < CqR^{-1} < \frac{\kappa}{10},$$

and, for each $\ell$

$$\sum_{k\in I_\ell} (1+(R\|k\omega\|)^2)^{-1} \le 1 + \sum_{s=1}^{q} \left(R\frac{s}{q}\right)^{-2} \le 1 + C\left(\frac{q}{R}\right)^2 \le C$$

and

$$(II) < C \sum_{\ell=1}^{4Kq^{-1}} \frac{1}{\ell q} < Cq^{-1}\log K.$$

Letting $\log K \sim \kappa q$, $(II) < \frac{\kappa}{10}$, and (2.8) implies (2.2). ∎

## 3 Applications of Avalanche principle

Avalanche principle [10] (full details is also given in [5], Ch. VI) is the following:

Let $A_1, \cdots, A_n$ be a sequence in $SL_2(\mathbb{R})$ such that

$$\|A_j\| \geq \mu \tag{3.1}$$

$$\mu > n \tag{3.2}$$

and

$$|\log\|A_j\| + \log\|A_{j+1}\| - \log\|A_{j+1}A_j\|| < \frac{1}{2}\log\mu,\ j = 1, \ldots, n. \tag{3.3}$$

Then

$$\left|\log\left\|\prod_{j=n}^{1} A_j\right\| + \sum_{j=2}^{n-1}\log\|A_j\| - \sum_{j=1}^{n-1}\log\|A_{j+1}A_j\|\right| < C\frac{n}{\mu} \tag{3.4}$$

where $C$ is an absolute constant. We will also need the following extension, relaxing condition (3.2):

**LEMMA 5** *Assume $A_1, \cdots, A_N$ satisfy (3.1),(3.3) with $\mu$ sufficiently large and $N = \prod_{i=1}^{s} n_i$ where $3 \leq n_i < \frac{\mu}{2}$, $i = 1, \ldots, s-1$ and $n_s < \mu$. Then*

$$\left|\log\left\|\prod_{j=N}^{1} A_j\right\| + \sum_{j=2}^{N-1}\log\|A_j\| - \sum_{j=1}^{N-1}\log\|A_{j+1}A_j\|\right| < C_1\frac{N}{\mu} \tag{3.5}$$

**Remarks:**

1. We show (3.5) with $C_1 = 5C$, $C$ from the avalanche principle. As will be seen from the proof, $C_1 = (3+\epsilon)C$ will also work for large $\mu$.

2. The largeness condition on $\mu$ is explicit. For example, it is sufficient to have $\mu\log\mu > 27C$ with $C$ from (3.4).

**Proof:** For the sake of less cumbersome notations our proof will assume $N = 3^s$. The proof for the general case is exactly the same with obvious changes.

We use induction in $s$ with the beginning provided by (3.4) with $n = n_1, 2n_1$. Set $N_1 = 3^{s-1}$; $B_i = A_{3i}A_{3i-1}A_{3i-2}$, $i = 1, \dots, N_1$. Then, by (3.4), for all $j$,

$$\Big| \log \|B_j\| + \log \|A_{3j-1}\| - \log \|A_{3j-1}A_{3j-2}\| - \log \|A_{3j}A_{3j-1}\| \Big| < \frac{3C}{\mu} \quad (3.6)$$

and similarly for $\log \|B_{i+1}B_i\|$.

(3.6) and (3.3) imply

$$\log \|B_j\| > \sum_{k=0}^{2} \log \|A_{3j-k}\| - \frac{3C}{\mu} - \log \mu > 2\log \mu - \frac{3C}{\mu} = \log \mu_1 \quad (3.7)$$

where $\mu_1 > \mu$. Also,

$$\begin{aligned} |\log \|B_j\| \ &+ \ \log \|B_{j+1}\| - \log \|B_{j+1}B_j\|| < \\ &\frac{12C}{\mu} + |\log \|A_{3j}\| + \log \|A_{3j+1}\| - \log \|A_{3j+1}A_{3j}\|| < \\ &\frac{12C}{\mu} + \frac{1}{2}\log \mu < \frac{1}{2}\log \mu_1 \end{aligned} \quad (3.8)$$

(it is for the last inequality in (3.8) that we need a largeness condition on $\mu$.)

Therefore, induction applies, and

$$\left| \log \left\| \prod_{j=N}^{1} A_j \right\| + \sum_{j=2}^{N_1-1} \log \|B_j\| - \sum_{j=1}^{N_1-1} \log \|B_{j+1}B_j\| \right| < C_1 \frac{3^{s-1}}{\mu_1} \quad (3.9)$$

Using (3.6) for each $B_j$ and $B_{j+1}B_j$ in (3.9) we obtain, after collecting terms,

$$\begin{aligned} &\left| \log \left\| \prod_{j=N}^{1} A_j \right\| + \sum_{j=2}^{N-1} \log \|A_j\| - \sum_{j=1}^{N-1} \log \|A_{j+1}A_j\| \right| \ < \\ &\qquad \frac{C_1 3^{s-1}}{\mu_1} + \frac{3C(3^{s-1}-2)}{\mu} + \frac{6C(3^{s-1}-1)}{\mu} \ < \ \frac{C_1 3^s}{\mu} \end{aligned} \quad (3.10)$$

if $C_1 = 5C$. ∎

In case of positive Lyapunov exponent, large deviation theorem provides us a posibility to apply avalanche principle to $M_N(x + jN\omega)$ for $x$ in a set of large measure and therefore pass on to a larger scale.

**LEMMA 6** *Let $\omega$ satisfy (2.1) and $L(E) > 100\kappa > 0$. Let $N > C\kappa^{-2}q$. Assume further $L_{2N}(E) > \frac{9}{10}L_N(E)$.*

*Then for $N_1$ s.t. $N|N_1$ and $N_1N^{-1} = m < e^{c\kappa q}$, we have*

$$|L_{N_1}(E) + \frac{m-2}{m}L_N(E) - 2\frac{m-1}{m}L_{2N}(E)| < C_1e^{-c\kappa q}. \tag{3.11}$$

**Remark.** Here $C$ is same as before, and $c$ is equal to $\frac{c}{2}$ from the large deviation theorem.

**Proof.** Apply the avalanche principle with

$$A_j = M_N(x + jN\omega, E)$$

and with $x$ restricted to the set $\Omega \subset \mathbb{T}$, s.t. for all $j \le m$

$$\left|\tfrac{1}{N}\log\|M_N(E; x + jN\omega)\| - L_N(E)\right| < \kappa \tag{3.12}$$
$$\left|\tfrac{1}{2N}\log\|M_{2N}(E; x + jN\omega)\| - L_{2N}(E)\right| < \kappa.$$

Thus from (2.2) and choice of $m$

$$\operatorname{mes}(\mathbb{T}\backslash\Omega) < 2me^{-c\kappa q} < Ce^{-\frac{c}{2}\kappa q}. \tag{3.13}$$

Since $\|A_j\| > e^{N(L_N(E)-\kappa)} > e^{\frac{99}{100}NL_N}$ and $|\log\|A_j\|+\log\|A_{j+1}\|-\log\|A_{j+1}A_j\|| < 4N\kappa+2N|L_N(E)-L_{2N}(E)| < \frac{6}{25}NL_N$, the avalanche principle applies. Thus, for sufficiently large $N$,

$$\left|\log\left\|\prod_{j=m}^{1} A_j\right\| + \sum_{j=2}^{m-1}\log\|A_j\| - \sum_{j=1}^{m-1}\log\|A_{j+1}A_j\|\right| < me^{-\frac{1}{2}NL_N}.$$

Integrating on $\Omega$, we get

$$\left|\int_\Omega \log\|M_{N_1}(E;x)\| + \sum_{j=2}^{m-1}\int_\Omega \log\|M_N(E;x+j\omega)\| \quad - \quad \sum_{j=1}^{m-1}\int_\Omega \log\|M_{2N}(E;x+j\omega)\|\right|$$
$$< \quad me^{-\frac{1}{2}NL_N}.$$

Therefore, recalling (3.13)

$$|L_{N_1}(E)+\frac{m-2}{m}L_N(E)-\frac{2(m-1)}{m}L_{2N}(E)| < \frac{m}{N_1}e^{-\frac{1}{2}NL_N}+Ce^{-\frac{c}{2}\kappa q} < C_1e^{-\frac{c}{2}\kappa q},$$

as claimed.

Lemma 6 may be iterated to get the following fact

**LEMMA 7** *Same assumptions as in Lemma 6.*
*Then*

$$|L_{N'} + L_N - 2L_{2N}| < e^{-c'\kappa q} + C\frac{N}{N'} \tag{3.14}$$

*holds for all* $N'$ *with* $N|N'$ *and* $\frac{N'}{N} < \exp\exp\frac{c}{2}\kappa q$.

**Proof.** (3.14) follows from (3.11) with $c' = c$ if $N' < e^{c\kappa q}N$. Thus we may assume $N' > e^{c\kappa q}N$. Take $N_1 \sim e^{c\kappa q}N$ in order to apply Lemma 6. Thus

$$|L_{N_1} + L_N - 2L_{2N}| < Ce^{-c\kappa q} \tag{3.15}$$

and

$$|L_{2N_1} + L_N - 2L_{2N}| < Ce^{-c\kappa q}$$

implying in particular

$$|L_{2N_1} - L_{N_1}| < 2Ce^{-c\kappa q}. \tag{3.16}$$

Replacing $N$ by $N_1$ and taking $N_2 \sim e^{c\kappa q}N_1$, we get similarly from Lemma 6

$$|L_{N_2} + L_{N_1} - 2L_{2N_1}| < Ce^{c\kappa q} \tag{3.17}$$

$$|L_{2N_2} - L_{N_2}| < 2Ce^{-c\kappa q}$$

and from (3.16), (3.17)

$$|L_{N_2} - L_{N_1}| < 5Ce^{-c\kappa q}. \tag{3.18}$$

Letting in general $N_s \sim e^{c\kappa q}N_{s-1}$, we obtain

$$|L_{N_s} + L_{N_{s-1}} + L_{2N_{s-1}}| < Ce^{-c\kappa q} \tag{3.19}$$

$$|L_{2N_s} - L_{N_s}| < 2Ce^{-c\kappa q} \tag{3.20}$$

$$|L_{N_s} - L_{N_{s-1}}| < 5Ce^{-c\kappa q}. \tag{3.21}$$

Consequently, from (3.18), (3.21)

$$|L_{N_s} - L_{N_1}| < 5Cse^{-c\kappa q}$$

and by (3.15)

$$|L_{N_s} + L_N - 2L_{2N}| < 6Cse^{-c\kappa q}. \tag{3.22}$$

To get (3.14) with $c' = \frac{c}{2}$, we may allow $s < e^{\frac{c}{2}\kappa q}$ in (3.22), hence the estimate holds for $N'$ as stated. $\blacksquare$

Lemma 7 will be sufficient for our induction step provided there exists an approximant $q$ with $e^{q_s} < q < \exp\exp c\kappa q_s$, where $q_s$ is the sequence of canonical rational approximants of $\omega$. For when this is not the case we need an additional statement.

**LEMMA 8** *In addition to the assumptions of Lemma 6, assume that $q|N$, $N < e^{c''\kappa q}$. Then*

$$|L_{N'} + L_N - 2L_{2N}| < C_2 e^{-\frac{c}{2}\kappa q} \tag{3.23}$$

*for all $5e^q N \leq N' \leq e^{-\frac{3}{2}q} q'$ with $N' = 3^s N$ or $\frac{N'}{2} = 3^s N$, where $q'$ is the next approximant after $q$.*

**Remark:** We assumed $N' = a3^s N,\ a = 1, 2$ for simplicity of formulation only. Lemma 8 holds as well for all $N'$ as in Lemma 5 with $\mu = e^{N(L_N - 2\kappa)}$.

**Proof:** The set $\Omega_N = \{x \in \mathbb{T} |\ \ |\frac{1}{N}\log\|M_N(E;x)\| - L_N(E)| > \kappa\}$ satisfies by (2.2) the measure estimate

$$\text{mes}\ \Omega_N < e^{-c\kappa q}. \tag{3.24}$$

Let us consider $v' = \sum_{|k|\leq N^2} \hat{v}(k)e^{2\pi i k x}$ a trigonometric polynomial of degree $N^2$, and let $M'_K$, $L'_K$, and $\Omega'_K$ be corresponding objects defined with $v$ replaced by $v'$.

Since

$$|\|M_N(x)\| - \|M'_N(x)\|| \leq \sup_x |v - v'|C^N,$$

we have that if $x \in \mathbb{T}\backslash(\Omega'_N \cup \Omega'_{2N})$, then

$$|\log\|M_K(x)\| - KL_K| < 2\kappa K,\ K = N, 2N, \tag{3.25}$$

However, $\Omega'_N$ admits a semi-algebraic description, therefore $\Omega'_N$ may be covered by at most $N^C$ intervals of size $< e^{-c\kappa q}$. The same holds for $\Omega'_N \cup \Omega'_{2N}$.

Hence, because of our upper bound on $N$, there is a collection $\mathcal{J}$ of at most $N^C$ intervals $I \subset \mathbb{T}$ s.t.

$$\text{mes}\ (\mathbb{T}\backslash \bigcup_{I\in\mathcal{J}} I) < e^{-\frac{c}{2}\kappa q} \tag{3.26}$$

and if $x \in \bigcup_{I\in\mathcal{J}} I, |x-x'| < e^{-q}$, then $x' \in \mathbb{T}\backslash(\Omega'_N \cup \Omega'_{2N})$, therefore $x'$ satisfies (3.25). Observe next that since $|\omega - \frac{a}{q}| < \frac{1}{qq'}$ and $q|N$,

$$\|\ell N\omega\| < \frac{\ell e^{c''\kappa q}}{qq'} < e^{-q}$$

for

$$\ell < e^{-\frac{3}{2}q} q'. \tag{3.27}$$

Hence, fixing $x \in \bigcup_{I\in\mathcal{J}} I$, it follows from the preceding that $x' = x + \ell N\omega$ will satisfy (3.25) for all $\ell$ as in (3.27).

Denoting

$$A_\ell = M_N(x + \ell N\omega)$$

we have thus

$$\|A_\ell\| > e^{N(L_N - 2\kappa)} > e^{\frac{49}{50} N L_N} \tag{3.28}$$

$$|\log \|A_\ell\| - NL_N| < 2\kappa N \tag{3.29}$$

$$|\log \|A_{\ell+1}A_\ell\| - 2NL_N| < (6\kappa + 2|L_{2N} - L_N|)N < \tfrac{13}{50} N L_N \tag{3.30}$$

since $L_N > 100\kappa$.

Therefore, for $N'$ as in the Lemma, we may now aplly Lemma 5 to obtain

$$\left|\frac{1}{N'} \log \left\| \prod_{\ell=\frac{N'}{N}-1}^{0} A_\ell \right\| + \frac{1}{N'} \sum_{\ell=2}^{\frac{N'}{N}-1} \log \|A_\ell\| - \frac{1}{N'} \sum_{\ell=1}^{\frac{N'}{N}-1} \log \|A_{\ell+1}A_\ell\| \right| < C_1 e^{-98\kappa N} < \frac{1}{5} e^{-q}. \tag{3.31}$$

Integrating (3.31) in $x \in \bigcup_{I\in\mathcal{J}} I$, and recalling (3.26) and the lower bound on $N'$, we get

$$|L_{N'} + L_N - 2L_{2N}| < e^{-q} + Ce^{-\frac{c}{2}\kappa q} < C_2 e^{-\frac{c}{2}\kappa q} \tag{3.32}$$

∎

# 4 Proof of Theorem 1.

Assume $q_0$ is an approximant of $\omega$, thus

$$\left|\omega - \frac{a_0}{q_0}\right| < \frac{1}{q_0^2}, \qquad (a_0, q_0) = 1 \tag{4.1}$$

and $L(E) > 100\kappa > 0$. Here $\kappa$ is a small constant and we assume $q_0 > \kappa^{-2}$.

The construction below is described assuming $\omega \notin \mathbb{Q}$ but, as the reader will easily see, applies equally well for $\omega \in \mathbb{Q}$. In particular, the conclusion stated in Prop. 9, is valid in either case.

Since $100\kappa < L(E) \leq L_{2N} \leq L_N < C$ for any $N$, we may choose $N_0$, depending only on $\kappa, C, q_0$, satisfying

$$L_{2N_0} > \frac{9}{10} L_{N_0} \tag{4.2}$$

and

$$C\kappa^{-2} q_0 < N_0 < \kappa^{-C} q_0. \tag{4.3}$$

Set $q_{-1} = 0$. Starting from $q_0, N_0$, we construct a sequence of approximants $\{q_s\}$ of $\omega$ and integers $\{N_s\}$ such that

$$q_0 < N_0 < q_1 < \cdots < N_s < q_{s+1} < N_{s+1} < \cdots \tag{4.4}$$

$$q_{s+1} > e^{q_s} \tag{4.5}$$

$$C\kappa^{-2} q_s < N_s \sim q_s \text{ and, for } s \geq 1,\ N_{s-1} | N_s \tag{4.6}$$

$$|L_{N_{s+1}} + L_{N_s} - 2L_{2N_s}| < e^{-c_1 \kappa q_s} \tag{4.7}$$

$$|L_{2N_s} - L_{N_s}| < Ce^{-c_2 \kappa q_{s-1}} \tag{4.8}$$

$$|L_{N_{s+1}} - L_{N_s}| < e^{-c_3 \kappa q_{s-1}} \tag{4.9}$$

where $c' \gg c_1 > c_2 > c_3 > 0$. ($c' > 0$ the constant from Lemma 7).

Denoting $q_{s+1} > e^{q_s}$ the smallest approximant of $\omega$ satisfying (4.5), we distinguish 2 cases.

**Case I**: $q_{s+1} < e^{10q_s}$

Take $N_{s+1}$ satisfying (4.6), hence $e^{q_s} < N_{s+1} < e^{11q_s} N_s$. Since $|\omega - \frac{a_s}{q_s}| < \frac{1}{q_s^2}$ and $N_s$ satisfies (4.6), (4.8), Lemma 7 applies with $q = q_s$, $N = N_s, N' = N_{s+1}$. Thus from (3.14)

$$|L_{N_{s+1}} + L_{N_s} - 2L_{2N_s}| < e^{-c'\kappa q_s} + e^{-\frac{1}{2}q_s} < 2e^{-c'\kappa q_s} < e^{-c_1\kappa q_s} \tag{4.10}$$

and similarly with $N_{s+1}$ replaced by $2N_{s+1}$.

From (4.8), (4.10)

$$|L_{N_{s+1}} - L_{N_s}| < 2e^{-c'\kappa q_s} + 2Ce^{-c_2\kappa q_{s-1}} < e^{-c_3\kappa q_{s-1}}$$

and also

$$|L_{2N_{s+1}} - L_{N_{s+1}}| < 4e^{-c'\kappa q_s} < e^{-c_2\kappa q_s}.$$

**Case II**: $q_{s+1} \geq e^{10q_s}$.

Take again $N_{s+1}$ satisfying (4.6).

In this situation, we may not be able to apply Lemma 7 immediately and we perform some intermediate steps. Denote $q_s \leq q \leq e^{q_s}$ the approximant preceding $q_{s+1}$ and consider a first intermediate scale

$$N \sim \max(\kappa^{-2}q, e^{5c_1\kappa q_s}), q|N. \tag{4.11}$$

Thus, as in case (I)

$$|L_N + L_{N_s} - 2L_{2N_s}| < e^{-c'\kappa q_s} + e^{-4c_1\kappa q_s} < 2e^{-4c_1\kappa q_s} \tag{4.12}$$

and

$$|L_{2N} + L_{N_s} - 2L_{2N_s}| < 2e^{-4c_1\kappa q_s}. \tag{4.13}$$

A second scale $N'' \geq N$ is introduced as follows

If $q_{s+1} \leq e^{4q}$, let $N'' = N$.

If $q_{s+1} > e^{4q}$, let $N'' \sim e^{-2q}q_{s+1}$, with $N'' = 3^b N$. In the second case, we have conditions of Lemma 8 satisfied, and therefore (3.23) holds for both $N''$ and $2N''$. Therefore, we also have:

$$|L_{N''} - L_{2N''}| < Ce^{-\frac{c}{2}\kappa q}. \tag{4.14}$$

Next, apply Lemma 7 with $N = N''$ and $N' = N_{s+1} < C\kappa^{-2}e^{2q}N''$. Thus

$$|L_{N_{s+1}} + L_{N''} - 2L_{2N''}| < e^{-c'\kappa q} + C\frac{N''}{N_{s+1}} < 2e^{-c'\kappa q} \tag{4.15}$$

and similarly with $N_{s+1}$ replaced by $2N_{s+1}$.

Collecting the estimates (4.12), (4.13), (3.23) with $N' = N'', 2N''$, and (4.15), we obtain that

$$|L_{N_{s+1}} + L_{N_s} - 2L_{2N_s}| < 6e^{-4c_1\kappa q_s} + Ce^{-\frac{c}{2}\kappa q} + 2e^{-c\kappa q} < e^{-c_1\kappa q_s}$$

and similarly with $N_{s+1}$ replaced by $2N_{s+1}$. Therefore, in both cases I, II, (4.7) holds. (4.8) and (4.9) are then obtained as in case (I). This completes the construction.

As a consequence of (4.7) with $s = 0$ and (4.9)

$$|L(E) + L_{N_0}(E) - 2L_{2N_0}(E)| <$$
$$|L_{N_1} + L_{N_0} - 2L_{2N_0}| + \sum_{s\geq 1} |L_{N_{s+1}} - L_{N_s}| < e^{-c_1\kappa q_0} + \sum_{s\geq 0} e^{-c_3\kappa q_s} < 2e^{-c_3\kappa q_0}.$$

Observe also that the assumption $L(E) > 100\kappa > 0$ in the beginning of this section could have been replaced by an assumption

$$L_N(E) > 100\kappa$$

for some $N$ chosen at least $\kappa^{-C}q_0$, $C$ some constant, as it is sufficient for the existence of $N_0$ satisfying (4.2),(4.3).

The conclusion is the following

**PROPOSITION 9** *Assume* $|\omega - \frac{a}{q}| < \frac{1}{q^2}$, $0 < \kappa < \frac{1}{100}, q > C\kappa^{-2}$ *and* $L_N(\omega, E) > \kappa$ *for some* $N > \kappa^{-C}q$. *Then there is* $N_0 < \kappa^{-C}q$, *and depending only on* $\kappa, C, q$, *s.t.*

$$|L(E) + L_{N_0}(E) - 2L_{2N_0}(E)| < e^{-c\kappa q}. \tag{4.16}$$

We can now finish the proof of the first statement of Theorem 1.

We may assume $\omega \notin \mathbb{Q}$. If $E_\alpha \to E$, then always, by subharmonicity, $\limsup L(E_\alpha) \leq L(E)$. We may therefore assume $L(E) > \kappa > 0$. Let $q > C\kappa^{-2}$ be an approximant of $\omega$. Taking $N > \kappa^{-C}q$, we have $L_N(E) > \kappa$ and hence also $L_N(E_\alpha) > \kappa$ for $\alpha > \alpha_0$. One may then choose $N_0$ s.t. (4.16) holds for both $E$ and $E_\alpha$. Thus

$$\begin{aligned}|L(E) - L(E_\alpha)| \ &\leq |L_{N_0}(E) - L_{N_0}(E_\alpha)| + 2|L_{2N_0}(E) - L_{2N_0}(E_\alpha)| + 2e^{-c\kappa q} \\ &\leq C(\kappa)^q |E - E_\alpha| + 2e^{-c\kappa q}\end{aligned}$$

Thus $\limsup_\alpha |L(E) - L(E_\alpha)| \leq 2e^{-c\kappa q}$ and, letting $q \to \infty$, the result follows. To prove the second statement of Theorem 1, we assume $(\omega_\alpha, E_\alpha) \to (\omega_0, E_0)$. Note that since for each $N$, $L_N(\omega, E)$ is a subharmonic function in both variables, therefore, $L(\omega, E) = \inf_N L_N(\omega, E)$ is upper semicontinuous, so $\limsup_\alpha L(\omega_\alpha, E_\alpha) \leq L(\omega_0, E_0)$. Therefore we may assume $L(\omega_0, E_0) > \kappa > 0$. Let $q > C\kappa^{-2}$ be an approximant of $\omega$, hence

$$\left|\omega_0 - \frac{a}{q}\right| < \frac{1}{q^2}.$$

Taking again $N > \kappa^{-C}q$, we have $L_N(\omega_0, E_0) > \kappa$, hence $L_N(\omega_\alpha, E_\alpha) > \kappa$ and $|\omega_\alpha - \frac{a}{q}| < \frac{1}{q^2}$ for $\alpha > \alpha_0$.

Fixing any $\alpha > \alpha_0$, we may find $N_0 < \kappa^{-C}q$ s.t.

$$|L(\omega_0, E_0) + L_{N_0}(\omega_0, E_0) - 2L_{2N_0}(\omega_0, E_0)| < e^{-c\kappa q}$$

and

$$|L(\omega_\alpha, E_\alpha) + L_{N_0}(\omega_\alpha, E_\alpha) - 2L_{2N_0}(\omega_\alpha, E_\alpha)| < e^{-c\kappa q}.$$

Hence

$$|L(\omega_0, E_0) - L(\omega_\alpha, E_\alpha)| < C(\kappa)^q(|\omega_0 - \omega_\alpha| + |E_0 - E_\alpha|) + 2e^{-c\kappa q}$$
$$\limsup_\alpha |L(\omega_0, E_0) - L(\omega_\alpha, E_\alpha)| < 2e^{-c\kappa q}.$$

Letting $q \to \infty$, it follows that $L(\omega_0, E_0) = \lim_\alpha L(\omega_\alpha, E_\alpha)$. ∎

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