A SZEGÖ CONDITION FOR A MULTIDIMENSIONAL SCHRÖDINGER OPERATOR

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ABSTRACT. We consider spectral properties of a Schrödinger operator perturbed by a potential vanishing at infinity and prove that the corresponding spectral measure satisfies a Szegö type condition.

1. INTRODUCTION

In this short paper we consider a multidimesional Schrödinger operator in $L^2(\mathbb{R}^d)$ and introduce a version of a Szegö condition (see (2.4)). Szegö's condition is well known for spectral measures of Jacobi matrices (see, for example [8], [9], [10], [11], [12], [14]). Our condition seems to be comparable with the classical Szegö condition only for small energies. The corresponding condition for large energies is much weaker than the expected one and cannot be obtained by the methods of this paper.

One of the motivations of this article is the conjecture of B. Simon from [12]. Namely, let V be a real function on $\mathbb{R}^d$, $d \geq 2$, which obeys

$$\int |x|^{-d+1} |V(x)|^2 \, dx < \infty. \tag{1.1}$$

Then $-\Delta + V$ has the a.c. spectrum of infinite multiplicity essentially supported by $[0, \infty)$. For spherically symmetric potentials this result follows from the paper Deift-Killip [2], where the corresponding problem is solved for $d = 1$.

The conditions imposed on the potential in our main theorem 2.1 are much more restrictive than those in (1.1). When proving the main result we are able to use an analog of Buslaev-Faddeev-Zakharov trace formulae well known for one dimensional Schrödinger operators. The multidimensional case is reduced to a problem for a second order elliptic intergo-differential operator. One of the main difficulties of this appoach is the treatment of the "potential" type term which appears to be a dissipative integral operator depending on the spectral parameter. The corresponding Fredholm equation for the Jost functions might be not solvable for a discrete subset of the

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upper half of the complex plane. There might be a hope that the corresponding contribution to the trace formulae coming from this subset can be controled by some Lieb-Thirring ineqaulities. Fortunately the positivity of the imaginary parts of the points from this subset appears together with the "right" sign in the so-called "first" trace formula. The contribution of these points in the "second" trace formula is distructive and requires some upper estimates. This explains why in our main theorem we obtain condition involving the first power of the potential V rather than V^2.

2. THE MAIN RESULT

Let Ω_1 be the unit ball in $\mathbb{R}^d$ and let consider the operator $H+V = -\Delta + V$ on $L^2(\mathbb{R}^d \setminus \Omega_1)$ with the Dirichlet boundary conditions on $\partial\Omega_1 = \mathbb{S}^{d-1}$. We can assume without loss of generality from the very beginning that

$$V + \frac{\alpha_d}{r^2} = 0 \quad \text{for} \quad 1 < |x| < c_1,$$

where $\alpha_d = \frac{(d-1)^2}{4} - \frac{d-1}{2}$. Let $E_{H+V}(\delta)$, $\delta \subset \mathbb{R}$, be the spectral projection of the operator $H+V$. We construct a measure μ on the the real line such that for spherically symetric functions f

$$(E_{H+V}(\delta)f, f) = \int_\delta |F(\lambda)|^2 d\mu(\lambda), \quad \delta \subset \mathbb{R}, \tag{2.1}$$

where

$$F(\lambda) = \frac{1}{k}\int_0^{c_1} \sin(k(r-1))f(r)\, r^{(d-1)/2} dr, \quad \text{supp} f \subset \{x : 1 < |x| < c_1\}, \tag{2.2}$$

where $k^2 = \lambda$. Let $\mathbb{Q} = [0,1)^d$. Then cubes $\mathbb{Q}_n = \mathbb{Q} + n,\ n \in \mathbb{Z}^d$, form a partition of $\mathbb{R}^d$ with which we associate the classes of functions u such that the sequence of (quasi)norms $\{\|u\|_{L^q(\mathbb{Q}_n)}\}_{n=1}^\infty \in \ell^p$, $0 < p, q \leq \infty$. When proving the main result we need the boundedness of the operators in (7.2). For example this can be provided by the following local condition on V [1]

$$V \in \ell^\infty(\mathbb{Z}^d; L^r(\mathbb{Q})), \quad r > d/2. \tag{2.3}$$

This condition can be weakened by using the characterization of weak Hardy's weights in terms of capacities obtained by Maz'ya (see [6]).

Theorem 2.1. *Let V be a real valued function on $\mathbb{R}^d \setminus \Omega_1$ which obeys* (2.3) *and such that*

$$\int_{\mathbb{R}^d\setminus\Omega_1} V_-^{(d+1)/2}(x)\, dx < \infty, \quad \int_{\mathbb{R}^d\setminus\Omega_1} V_+|x|^{-d+1}\, dx < \infty,$$

where $2V_\pm = |V| \pm V$. *Then*

$$\int_0^\infty \frac{\log(1/\mu'(t))\,dt}{(1+t^{3/2})\sqrt{t}} < \infty, \tag{2.4}$$

which is equivalent to the fact that the integral

$$\int_0^\infty \frac{\log\big(\frac{d}{d\lambda}(E_{H+V}(\lambda)f,f)\big)\,d\lambda}{(1+\lambda^{3/2})\sqrt{\lambda}} > -\infty, \tag{2.5}$$

is finite for any bounded spherically symmetric function $f \neq 0$ *with* $\mathrm{supp} f \subset \{x : 1 < |x| < c_1\}$.

Remark 1. Convergence of the integral in (2.4) guaranties that the a.c. spectrum of $-\Delta + V$ is essentially supported by $[0,\infty)$, since $\mu' > 0$ almost everywhere. The condition (2.4) gives a quantative information about the measure μ.

Remark 2. As it is easily seen from the proof, the multiplicity of the a.c. spectrum essentially supported by $[0,\infty)$ is infinite.

Remark 3. The equivalence of (2.4) and (2.5) follows from the fact that if F is defined as in (2.2) then the function $(1+\lambda^2)^{-1}\log(|F(\lambda)|)$ is in L^1 by a classical result from the book [4] (section IIIG2).

3. Reduction to a one-dimensional problem

In this section we assume that $V \in C_0^\infty$ and often use polar coordinates (r,θ), $x = r\theta \in \mathbb{R}^d$, $\theta \in \mathbb{S}^{d-1}$. Denote by $\{Y_j\}_{j=0}^\infty$ the orthonormal in $L^2(\mathbb{S}^{d-1})$ basis of (real) spherical funcions and let P_j be the orthogonal projection given by

$$P_j u(r,\theta) = Y_j(\theta)\int_{\mathbb{S}^{d-1}} Y_j(\theta')u(r,\theta')\,d\theta'.$$

Clearly $P_0 u$ depends only on r. Denote

$$V_1 = P_0 V P_0, \quad H_1 = P_0 H P_0,$$

$$V_{1,2} = P_0 V(I - P_0), \quad V_{2,1} = V_{1,2}^*,$$

$$V_2 = (I-P_0)V(I-P_0), \quad H_2 = (I-P_0)H(I-P_0).$$

Then the operator $H + V - z$ can be represented as a matrix:

$$H + V - z = \begin{pmatrix} H_1 + V_1 - z & V_{1,2} \\ V_{2,1} & H_2 + V_2 - z \end{pmatrix},$$

and the equation

$$(H+V-z)u = P_0 f, \quad \mathrm{Im}\, z \neq 0,$$

is equivalent to

(3.1) $(H_1+T_z-z)P_0u = P_0f, \qquad (H_2+V_2-z)^{-1}V_{2,1}P_0u = (P_0-I)u.$

Here the operator T_z is defined by

$$T_z = V_1 - V_{1,2}(H_2 + V_2 - z)^{-1}V_{2,1}$$

on $L^2((1,\infty), r^{d-1}\,dr)$.

By using the unitary operator from $L^2((1,\infty), dr)$ to $L^2((1,\infty), r^{d-1}\,dr)$

$$Uu(r) = r^{-(d-1)/2}u$$

we reduce (3.1) to the problem for the following one-dimensional Schrödinger operator in $L^2(1,\infty)$

(3.2) $$A_zu(r) = -\frac{d^2u}{dr^2} + Q_zu, \quad u \in L^2(1,\infty),\ u(1) = 0,$$

where

$$Q_z = V_1 + \frac{\alpha_d}{r^2} - V_{1,2}(U^*H_2U + V_2 - z)^{-1}V_{2,1}, \quad \alpha_d = \frac{(d-1)^2}{4} - \frac{d-1}{2}.$$

By considering the potential

$$V - \frac{\alpha_d}{r^2} \quad \text{instead of} \quad V$$

without loss of generality we can assume that

(3.3) $$Q_z = V_1 - V_{1,2}(B + V_2 - z)^{-1}V_{2,1},$$

where

$$B = -\frac{d^2u}{dr^2} - \frac{\Delta_\theta u}{r^2}, \quad u(1,\theta) = 0.$$

Note that conditions on V will not change.

Then according to (3.1) we obtain

(3.4) $$P_0(H + V - z)^{-1}P_0 = U(A_z - z)^{-1}U^*.$$

We see also that if $\operatorname{supp} V \subset \{x \in \mathbb{R}^d : \ c_1 < |x| < c_2\}$, $c_1 > 1$, then for the operator (3.3) we have

$$Q_z = Q_z\chi = \chi Q_z,$$

where χ is an operator of multiplication by the characteristic function of (c_1, c_2), $c_1 > 0$. It is impotant for us that Q_z is an analytic operator valued function of z with the negative imaginary part in the upper half of the plane and positive one in the lower half plane.

4. Green's function.

Let us consider the equation

$$(A_z\psi)(r) := -\frac{d^2}{dr^2}\psi(r) + (Q_z\psi)(r) = z\psi(r), \quad r \geq 1,\ z \in \mathbb{C}, \tag{4.1}$$

with Q_z given by (3.3). For all z except perhaps a discret sequence of points, there exists a unique analytic in k solution $\psi_k(r)$, such that

$$\psi_k(r) = \exp(ikr), \quad k^2 = z,\ \text{Im}\, k > 0,\ \forall r > c_2.$$

Consider the resolvent operator $R(z) = (A_z - z)^{-1}$. If χ_{c_1} is the operator of multiplication by the characteristic function of $(1, c_1)$. Then $R(z)\chi_{c_1}$ is an integral operator whose kernel satisfies the relation:

$$G_z(r,s) = \begin{cases} \frac{1}{\psi_k(1)} \sin(k(r-1))\psi_k(s), & \text{for } r < s < c_1, \\ \frac{1}{\psi_k(1)}\psi_k(r)\sin(k(s-1)), & \text{for } s < \min\{c_1, r\}. \end{cases} \tag{4.2}$$

Indeed, assuming that supp$(f) \subset (1, c_1)$ we can easily check that the function

$$u(r) = \frac{1}{\psi_k(1)}\Big\{\int_r^\infty \sin(k(r-1))\psi_k(s)f(s)\,ds + \int_0^r \psi_k(r)\sin(k(s-1))f(s)\,ds\Big\}$$

satisfies the equation

$$-\frac{d^2}{dr^2}u(r) + (Q_z u)(r) - zu(r) = f(r), \quad r \geq 1,\ z \in \mathbb{C}, \tag{4.3}$$

and moreover $u(1) = 0$.

5. Wronskian and properties of the M-function.

Let now as above

$$Q_z = V_1 - V_{1,2}(B + V_2 - z)^{-1}V_{2,1}.$$

The function

$$M(k) = \frac{\psi_k'(1)}{\psi_k(1)}$$

is called the Weyl M-function of the operator (4.1). Let us consider the Wronskian

$$W[\overline{\psi_k}, \psi_k](r) = \overline{\psi_k'}(r)\psi_k(r) - \overline{\psi_k}(r)\psi_k'(r). \tag{5.1}$$

Note that $\overline{\psi_k}$ satisfies the equation (4.1) with $Q_{\overline{z}}$ and $\overline{z}$ instead of Q_z and z. Since ψ_k is a solution of the equation (4.1) we find

$$\frac{d}{dr}W[\overline{\psi_k}, \psi_k](r) = (z - \overline{z})\overline{\psi_k}(r)\psi_k(r) + (Q_{\overline{z}}\overline{\psi_k})(r)\psi_k(r) - \overline{\psi_k}(r)(Q_z\psi_k)(r).$$

So we obtain

$$\pm\text{Im}\,\{W[\overline{\psi_k},\psi_k](c_2)-W[\overline{\psi_k},\psi_k](c_1)\}\geq 0,\quad \text{for } \pm\,\text{Im}\,z\geq 0+, \tag{5.2}$$

which means that for all k we have the following inequality

$$\frac{k}{\text{Im}\,M(k)}\leq|\psi_k(1)|^2.$$

Moreover, if we represent the solution ψ_k for real k in the form

$$\psi_k(x)=a(k)e^{ikx}+b(k)e^{-ikx},\quad x<c_1,$$

then it follows from (5.2) that

$$|a|^2-|b|^2\geq 1.$$

Assume that Q_z is the above introduced operator with a smooth compactly supported V. Then for $k^2=z$

$$M(k)=\psi_k'(1)(\psi_k(1))^{-1}=ik(1-\rho(k))(1+\rho(k))^{-1},\quad \rho(k):=e^{-2ik}b(k)a(k)^{-1}.$$

The latter implies

$$\rho(k)=(ik-M(k))(ik+M(k))^{-1}.$$

Since $|a|^2-|b|^2\geq 1$ we obtain that for real k

$$|a(k)|^{-2}\leq 1-|\rho(k)|^2=$$

$$\frac{4k\text{Im}\,M}{|(ik+M(k)|^2}.$$

Note that since Im $M\geq 0$, then for any $k>0$ we have

$$|ik+M(k)|^2=k^2+|M|^2+2k\text{Im}\,M\geq k^2$$

and therefore

$$|a(k)|^{-2}\leq 4k^{-1}\big(\text{Im}\,M\big),\quad k>0. \tag{5.3}$$

Note also that

$$\text{Im}\,M(k)>0\quad \text{if Im}\,k^2>0. \tag{5.4}$$

Thus, there are constants $C_0\in\mathbb{R}$ and $C_1\geq 0$ and a positive measure μ, such that

$$\int\frac{d\mu(t)}{1+t^2}<\infty,$$

where

$$M(k)=C_0+C_1z+\int_{\mathbb{R}}\Big(\frac{1}{t-z}-\frac{t}{1+t^2}\Big)d\mu(t),\quad k^2=z. \tag{5.5}$$

Finally, note that $R(z) = P_0(U^*HU + V - z)^{-1}P_0$ and therefore we can write formally that

$$M(k) = \frac{\partial^2}{\partial r \partial s} G_z(r,s)|_{(1,1)} = (P_0(U^*HU + V - z)^{-1}P_0\delta_1', \delta_1'),$$

where δ_1' is the derivative of $\delta(r-1)$. Let χ_{c_1} be the characteristic function of $(1, c_1)$. The representation (4.2) for the resolvent operator gives us the representation for the operator $\chi_{c_1}P_0E_{U^*HU+V}(\delta)P_0\chi_{c_1}$, where $E_{U^*HU+V}(\delta)$ is the spectral measure of $U^*HU + V$:

$$(P_0E_{U^*HU+V}(\delta)P_0f, f) = \int_\delta |F(\lambda)|^2 d\mu(\lambda) \tag{5.6}$$

and where

$$F(\lambda) = \frac{1}{k}\int_0^{c_1} \sin(k(r-1))f(r)\,dr, \quad \operatorname{supp} f \subset (1, c_1), \ \ k^2 = \lambda.$$

Since F is a boundary value of an analytic function, we obtain that $F(\lambda) \neq 0$ for a.e. λ. This means that $E_{H+V}(\delta) \neq 0$ if $\mu' > 0$ a.e. on δ.

6. A trace inequality

In this section we assume that V is not a potential but the operator $\sum_{j=0}^n P_jV\sum_{j=0}^n P_j$, which approximate V for large n. It can be interpreted as an operator of multiplication by a matrix valued function of r. In this case the function V_1 remains the same as before. Since

$$\exp(-ikr)\psi_k(r) = 1 - \frac{1}{2ik}\int_r^\infty (1 - e^{2ik(s-r)})V_1(s)\,ds + o(1/k)$$

we obtain

$$a(k) = \lim_{r\to-\infty} \exp(-ikr)\psi_k(r) = 1 - \frac{1}{2ik}\int V_1\,dx + o(1/k),$$

as $k \to \infty$. Now let $i\beta_m$ and γ_j be zeros and poles of $a(k)$. Let $\mathfrak{B}$ be the corresponding Blaschke product

$$\mathfrak{B}(k) = \prod_m \frac{(k - i\beta_m)}{(k + i\beta_m)} \prod_j \frac{(k - \overline{\gamma_j})}{(k - \gamma_j)}.$$

Clearly $|\mathfrak{B}(k)| = 1,\ k \in \mathbb{R}$, and we obtain

$$\int_{-\infty}^{+\infty} \log(a(k)/\mathfrak{B}(k))\,dk = \frac{\pi}{2}\int V_1\,dx + 2\pi\Big(\sum \beta_n - \sum \operatorname{Im}\gamma_j\Big). \tag{6.1}$$

Let us show that $-\beta_n^2$ are the eigenvalues of a certain operator of a Schrödinger type. Namely, let $\hat{H}$ be the operator in $L^2(\mathbb{R}, L^2(\mathbb{S}^{d-1}))$

$$\hat{H} = -\frac{d^2u}{dr^2} - \chi_1\frac{\Delta_\theta}{r^2}, \quad (I - P_0)u(1,\cdot) = 0,$$

where χ_1 is the characteristic function of $(1,\infty)$. Obviously, if $s < c_1 < c_2 < r$, then the kernel of the operator $P_0(\hat{H}+V)^{-1}P_0$ equals

$$g(r,s,k) = -\frac{\exp ik(r-s)}{2ika(k)}.$$

On the other hand we can consider the expansion of g near the eigenvalue $-\beta_n^2$. Denote by $\phi_{n,j}(r,\theta)$, $j = 1,2\dots m$ the orthonormal system of eigenfunctions corresponding to $-\beta_n^2$. If $\phi_{n,j}^{(0)} = \int_{\mathbb{S}^{d-1}} \phi_{n,j}(r,\theta)\,d\theta$ then

$$g(r,s,k) = \frac{\sum_{j=1}^m \phi_{n,j}^{(0)}(r)\overline{\phi_{n,j}^{(0)}(s)}}{k^2+\beta_n^2} + g_0(r,s,k),$$

where $g_0(r,s,k) = O(1)$, as $k \to i\beta_n$. This proves that $a(k)$ is a meromorphic funcion and its zeros correspond to the eigenvalues $-\beta_n^2$ of the operator $\hat{H}+V$. Moreover the multiplicities of these zeros are equal to one.

By using Lieb-Thirring inequalities [5] we obtain

$$\sum \beta_n \le C \int_{\mathbb{R}^d} V_-^{(d+1)/2} dx.$$

Consequently, since Im $\gamma_j \ge 0$ the trace formula (6.1) leads to the inequality

$$\begin{aligned} \int_{-\infty}^{+\infty} \log|a(k)|\,dk \le \frac{\pi}{2}\int_{-\infty}^{+\infty} V_1\,dr \\ +C\int_{\mathbb{R}^d} V_-^{(d+1)/2}dx. \end{aligned} \tag{6.2}$$

Therefore for any pair of finite numbers $r_2 > r_1 \ge 0$

$$\begin{aligned} \int_{r_1}^{r_2} \frac{1}{2}\log\frac{k}{4\text{Im }M(k)}\,dk \le \int_{-\infty}^{+\infty} \log|a(k)|\,dk \le \frac{\pi}{2}\int_{-\infty}^{+\infty} V_1\,dr \\ +C\int_{\mathbb{R}^d} V_-^{(d+1)/2}dx. \end{aligned} \tag{6.3}$$

7. THE END OF THE PROOF OF THEOREM 2.1

Assume that our perturbation V is an arbitrary function satisfying conditions of Theorem 0.1. Then the Weyl function M can be defined for example as $M(k) = \frac{\partial^2}{\partial r\partial s} G_z(r,s)|_{(1,1)}$ where G_z is the integral kernel of the operator $P_0(U^*HU+V-z)^{-1}P_0$. The next proposition allows us to approximate V by compactly supported smooth functions V_n.

Proposition 7.1. *Let V satisfy the conditions of Theorem 2.1. Then there exists a sequence V_n of compactly supported smooth functions converging to V such that*

$$(V_n)_- \to V_- \ \text{ in } L^{(d+1)/2}(\mathbb{R}^d); \quad ((V_n)_+)_1 \to (V_+)_1 \text{ in } L^1((1,\infty)) \tag{7.1}$$

and such that the Weyl functions M_n, corresponding to V_n, converge uniformly to M when k^2 belongs to any compact subset of the upper half plane

$$M_n(k) \to M(k).$$

Therefore the sequence of measures μ_n converges weakly to the spectral measure μ.

Proof. Let us choose a sequence of functions $\{V_n\}_{n=1}^{\infty}$ satisfying (7.1) and let $\Gamma_0(z)$ and $\Gamma_n(z)$ be the resolvent operators of $B = U^*(-\Delta)U - \alpha_d/r^2$ and $B_n = B + V_n$ respectively. Recall that by δ_1' we denote the derivative of $\delta(r-1)$. Moreover let $W_n = \sqrt{|V_n|}$ and $S_n = \mathrm{sign} V_n$. We can also assume that $S_n = S$ is independent of n and W_n converges to $W = \sqrt{|V|}$ in L^2. The expression $\Gamma_0(z)\delta_1'$, Im $z \neq 0$, can be understood as the function

$$\Gamma_0(z)\delta_1' = \exp(ik(r-1)).$$

According to our assumptions

$$W_n\Gamma_0(z)\delta_1' \to W\Gamma_0(z)\delta_1',$$

in $L^2(\mathbb{R}^d)$ (with respect to any weight). Thus in order to prove that the Weyl functions

$$\begin{aligned} M_n(k) &= \frac{\partial^2}{\partial r \partial s} G_{n,z}(r,s)|_{(1,1)} = (\Gamma_n(z)\delta_1', \delta_1') \\ &= (\Gamma_0(z)\delta_1', \delta_1') - (W_n\Gamma_0(z)\delta_1', W_n S \Gamma_n(\overline{z})\delta_1') \end{aligned}$$

converge it is sufficient to show that $W_n S\Gamma_n(\overline{z})\delta_1'$ converges in $L^2(\mathbb{R}^d)$ to $WS(B+V-\overline{z})^{-1}\delta_1'$. The latter follows from the identity

$$W_n S\Gamma_n(\overline{z})\delta_1' = (S + W_n\Gamma_0(\overline{z})W_n)^{-1} W_n\Gamma_0(\overline{z})\delta_1'$$

and from the convergence

$$W_n\Gamma_0(\overline{z})W_n \to W\Gamma_0(\overline{z})W \tag{7.2}$$

under the assumption $V_n \to V$ in the class (2.3), as $n \to \infty$. □

Similarly we can prove the following

Proposition 7.2. *Let V be a compactly supported smooth function. Then the Weyl functions M_l corresponding to $\sum_{j=0}^{l} P_j V \sum_{j=0}^{l} P_j$*

converge uniformly to M when k^2 belongs to any compact subset of the upper half plane

$$M_l(k) \to M(k)$$

and therefore the sequence of measures μ_l converges weakly to the spectral measure μ constructed for V.

Proof. Let us denote $V_l = \sum_{j=0}^{l} P_j V \sum_{j=0}^{l} P_j$ let $\Gamma_0(z)$ and let $\Gamma_l(z)$ be the resolvent operators of $B = U^*(-\Delta)U - \alpha_d/r^2$ and $B_l = B + V_l$ respectively. As in Proposition 7.1 the expression $\Gamma_0(z)\delta_1'$, Im $z \neq 0$, is understood as the function $\Gamma_0(z)\delta_1' = \exp(ik(r-1))$. According to our assumptions

$$V_l\Gamma_0(z)\delta_1' \to V\Gamma_0(z)\delta_1',$$

in $L^2(\mathbb{R}^d)$. Thus in order to prove that the Weyl functions

$$M_l(k) = \frac{\partial^2}{\partial r\partial s}G_{n,z}(r,s)|_{(1,1)} = (\Gamma_l(z)\delta_1', \delta_1')$$
$$= (\Gamma_0(z)\delta_1', \delta_1') - (V_l\Gamma_0(z)\delta_1', \Gamma_l(\overline{z})\delta_1')$$

converge, it is sufficient to show that $\Gamma_l(\overline{z})\delta_1'$ converges to $(B+V-\overline{z})^{-1}\delta_1'$ in $L^2(\mathbb{R}^d)$. The latter follows from the identity

$$\Gamma_l(\overline{z})\delta_1' = (B+V-\overline{z})^{-1}\delta_1' - \Gamma_l(\overline{z})(V_l - V)\Gamma_0(\overline{z})\delta_1'$$

and from the bound

$$||\Gamma_l(\overline{z})|| \leq C.$$

□

Finally according to Proposition 7.1 and 7.2 we observe that there exists a sequence of measures μ_l weakly convergent to μ, such that for any fixed $c > 0$

$$\int_0^c \frac{\log(1/\mu_l'(t))\,dt}{(1+t^{3/2})\sqrt{t}} < C(V), \quad \forall n,$$

where $C(V)$ is independent of c. Therefore

$$\int_0^\infty \frac{\log(1/\mu'(t))\,dt}{(1+t^{3/2})\sqrt{t}} < \infty.$$

REFERENCES

[1] M.Sh. Birman, *Discrete Spectrum in the gaps of a continuous one for perturbations with large coupling constants*, Advances in Soviet Mathematics, **7** (1991), 57-73.

[2] P. Deift and R. Killip, *On the absolutely continuous spectrum of one -dimensional Schrödinger operators with square summable potentials*, Commun. Math. Phys. **203** (1999), 341-347.

[3] R. Killip and B. Simon, *Sum rules for Jacobi matrices and their applications to spectral theory*, Annals of Math., to appear.

[4] P. Koosis, *The logarithmic integral I*, Cambidge university press (1988).

[5] E.H. Lieb and W. Thirring, *Inequalities for the moments of the eigenvalues of the Schrödinger Hamiltonian and their relation to Sobolev inequalities*, Studies in Math. Phys., Essays in Honor of Valentine Bargmann, Princeton, (1976) 269-303.
[6] V. Maz'ya, *Sobolev Spaces*, Springer-Verlag, Berlin Heidelberg New York Tokio (1985).
[7] S. Molchanov, M. Novitskii and B. Vainberg, *First KdV Integrals and Absolutely Continuous Spectrum for 1-D Schrdinger Operator*, Comm. Math. Phys. **216** (2001), no. 1, 195-213.
[8] G. Szegő, *Beiträge zue Theorie der Toeplitzschen Formen, II*, Math. Z. **9** (1921), 167-190.
[9] G. Szegő, *Orthogonal Polynomials*, 4th edition. American Mathematical Society, Colloquium Publications, Vol. XXIII. American Mathematical Society, Providence, R.I., 1975.
[10] D. Damanik and D. Hundertmark, *Bound states and the Szegö condition for Jacobi matrices and Schrödinger operators*, preprint.
[11] B. Simon and R. Killip, *Sum rules for Jacobi matrices and their applications to spectral theory*, Annals of Math., to appear.
[12] B. Simon, *Schrdinger operators in the twentieth century*, J. Math. Phys. **41** (2000), no. 6, 3523-3555.
[13] B. Simon, *Some Schrödinger operators with dense point spectrum*, Proc. Am. Math. Soc. **125** (1997), no.1, 203-208.
[14] B. Simon and A. Zlatos, *Sum rules and the Szegö condition for orthogonal polynomials on the real line*, preprint.

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