# Lifshitz tails for the $1D$ Bernoulli-Anderson model

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**Abstract**

By using the adequate modified Prüfer variables, precise upper and lower bounds on the density of states in the (internal) Lifshitz tails are proven for a $1D$ Anderson model with bounded potential.

There have been numerous rigorous works about Lifshitz tails for the $1D$-Anderson model with bounded potentials (see [KW] for a collection of references). The aim of this note is to give a simple proof by passing to the normal form of the transfer matrix at a band edge and then using the adequate Prüfer variables in this regime. This allows to obtain quite precise estimates on the integrated density of states (IDS). The paper concludes with a brief outlook on how and why similar techniques lead to perturbative results about the IDS and the Lyapunov exponent in between the Lifshitz tails and the center of the band, a regime that will be studied in more detail elsewhere.

# 1 Result

Let $(H_\sigma)_{\sigma\in\Sigma}$ be a family of $L$-periodic real Jacobi matrices on $\ell^2(\mathbb{Z})$. Each $H_\sigma$ is specified by $2L$ real numbers $t_\sigma(l)\geq 0$ and $v_\sigma(l)$ where $l=1,\ldots,L$ such that for any state $\psi\in\ell^2(\mathbb{Z})$

$$(H_\sigma\psi)(n) \;=\; -t_\sigma(n+1)\psi(n+1)+v_\sigma(n)\psi(n)-t_\sigma(n)\psi(n-1)\;.$$

The eigenvalue equation $H_\sigma\psi=E\psi$ for $E\in\mathbb{R}$ can conveniently be rewritten using the transfer matrices $T^E_\sigma\in SL(\mathbb{R},2)$ over one period:

$$\begin{pmatrix} t_\sigma(L)\,\psi(L) \\ \psi(L-1)\end{pmatrix} \;=\; T^E_\sigma\;\begin{pmatrix} t_\sigma(1)\,\psi(1) \\ \psi(0)\end{pmatrix}\;.$$

The set $\Sigma$ is supposed to be finite here and on it be given a probability measure $\mathbf{p}=\sum_\sigma p_\sigma\delta_\sigma$. Of course, $p_\sigma\geq 0$ and $\sum_\sigma p_\sigma=1$. The Tychonov space $\Omega=\Sigma^{\times\mathbb{Z}}\times\{1,\ldots L\}$ is furnished with the probability measure $\mathbf{P}=\mathbf{p}^{\otimes\mathbb{Z}}\otimes\frac{1}{L}\sum_l\delta_l$. This measure is of the so-called Bernoulli type and it is invariant and ergodic w.r.t. the natural translation action of $\mathbb{Z}$ on $\Omega$. Associated to each configuration $\omega = ((\sigma_n)_{n\in\mathbb{Z}}, l) \in \Omega$ is a Hamiltonian $H_\omega$ obtained by juxtaposition of the periodic blocs according to the configuration $\omega$. More precisely, if $n = mL + l + k$ with $k = 1, \ldots, L$, then

$$(H_\omega\psi)(n) \;=\; -t_{\sigma_m}(k+1)\psi(n+1) + v_{\sigma_m}(k)\psi(n) - t_{\sigma_m}(k)\psi(n-1)\;,$$

with the convention that $t_{\sigma_m}(L+1) = t_{\sigma_{m+1}}(1)$. By construction, $(H_\omega)_{\omega\in\Omega}$ is a strongly continuous operator family for which the covariance relation w.r.t. the translations holds [PF, Bel]. Note that, even in the trivial case where $\mathbf{p}$ is supported by only one point $\sigma \in \Sigma$, one has a covariant operator family given by the $L$ shifts of the $L$-periodic operator $H_\sigma$. The spectrum of any such covariant family is $\mathbf{P}$-almost surely constant. Using approximate eigenfunctions, one easily verifies $\mathrm{spec}(H_\omega) \subset \bigcup_{\sigma,\, p_\sigma\neq 0} \mathrm{spec}(H_\sigma)$. Equality instead of inclusion holds if $H_\sigma$ is the sum of a periodic background operator and a random potential [PF]. In this situation, a band edge of the random operator is a band edge also for one of the periodic operators.

Two fundemmental objects associated to a covariant family of $1D$ operators are the IDS $\mathcal{N}$ and the Lyapunov exponent [PF], the focus here being only on the former. One of the equivalent definitions of the Stieltjes function $\mathcal{N}$ is formula (5) in Section 2. Its support is $\mathrm{spec}(H_\omega)$. If $\mathbf{p} = \delta_\sigma$, then it is straight-forward to calculate the associated IDS, denoted $\mathcal{N}_\sigma$, by means of Bloch-Floquet theory.

Let now $E_\nu \in \mathbb{R}$ be a boundary point of $\mathrm{spec}(H_\nu)$ and $\mathrm{spec}(H_\omega)$. Hence $E_\nu$ can be either the bottom or the top of the spectrum or the boundary point of an *internal* spectral gap. Within any gap, the IDS $\mathcal{N}(E_\nu)$ is constant and therefore it is natural to study

$$\delta\mathcal{N}(\epsilon) \;=\; \pm\mathcal{N}(E_\nu \pm \epsilon) \mp \mathcal{N}(E_\nu)\;,$$

where the upper signs are chosen if $E_\nu$ is a lower band edge and the lower signs for an upper band edge. As above, one also introduces $\delta\mathcal{N}_\nu(\epsilon)$. Close to $E_\nu$, the IDS $\mathcal{N}$ is very small and its scaling is universal and called a Lifshitz tail, honoring the original contribution of L. Pastur's teacher.

**Theorem** *Let $E_\nu$ be a band edge of $H_\nu$ and $H_\omega$, but not of $H_\sigma$ if $\sigma \neq \nu$. Suppose moreover that the eigenvector of $T^{E_\nu}_\nu$ is not an eigenvector of $T^{E_\nu}_\sigma$ for any other $\sigma \neq \nu$. Then there exist constants $C > 0$ and $\epsilon_0 > 0$ such that for all $\epsilon \in [0, \epsilon_0]$*

$$\frac{1}{2}\,\delta\mathcal{N}_\nu(\epsilon)\, p_\nu^{\frac{1}{L\,\delta\mathcal{N}_\nu(\epsilon)}+1} \;\leq\; \delta\mathcal{N}(\epsilon) \;\leq\; C\,\delta\mathcal{N}_\nu(\epsilon)\, p_\nu^{\frac{1}{L\,\delta\mathcal{N}_\nu(\epsilon)}}\;. \qquad (1)$$

Because of the van Hove singularities, this result allows to read off the Lifshitz exponent

$$\lim_{\epsilon\to 0} \frac{\log(|\log(\delta\mathcal{N}(\epsilon))|)}{|\log(\epsilon)|} \;=\; \frac{1}{2}\;.$$

Moreover it gives precise bounds on the IDS and hence bounds on the so-called Lifshitz constants. For such asymptotics to hold, it is crucial that the transfer matrices $T^E_\sigma$ be uniformly bounded.

However, it should be possible to relax the condition that $(\Sigma, \mathbf{p})$ be discrete. By approximating a continuous measure $\mathbf{p}$ by a discrete one, a straight-forward adoption of the presented argument allows to obtain at least the lower bound. Finally, the (generic) conditions that $E_\nu$ is only a band edge of $H_\nu$ as well as the condition on the eigenvectors of $T_\sigma^{E_\nu}$ are not essential, but the statement of the result would be a bit more involved.

As will be discussed in Section 4, the estimates (1) are in a certain sense based on a deterministic argument, albeit taking place in a random model. The Lifshitz constants characterizing the true behavior of the IDS are model-dependent. One approach to calculate them is perturbation theory. As will be sketched in Section 4, this also allows to go beyond the Lifshitz tail regime.

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## 2 Setup

First let us recall some of the analysis of the periodic operators $H_\sigma$. The eigenvalues of the transfer matrix $T_\sigma^E$ are $\lambda = \frac{1}{2}\left(\mathrm{Tr}(T_\sigma^E) + \sqrt{\mathrm{Tr}(T_\sigma^E)^2 - 4}\right)$ and $1/\lambda$. Hence, if $|\mathrm{Tr}(T_\sigma^E)| < 2$, there are complex conjugate eigenvalues $\lambda = e^{\imath\eta}$ and $e^{-\imath\eta}$ and the transfer matrix $T_\sigma^E$ is conjugate to a rotation by the phase $\eta$. This phase is also called the rotation number and one speaks of the elliptic case. Then $E \in \mathrm{spec}(H_\sigma)$ and the IDS $\mathcal{N}_\sigma$ at $E$ is equal to $\eta/(L\pi)$ up to a multiple of $1/L$ coming from the gap label [JM, Bel]. On the other hand, if $|\mathrm{Tr}(T_\sigma^E)| > 2$, the eigenvalues are both real. One of them has a modulus bigger than 1 and one smaller than 1. This is the hyperbolic case, $E \notin \mathrm{spec}(H_\sigma)$ and the transfer matrix is conjugate to the dilation $\begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}$. Again due to the gap labelling, one has $\mathcal{N}_\sigma(E) = m/L$ for some positive integer $m \leq L$.

Let now $E_\nu$ be a band edge of the operator $H_\nu$. One then has $|\mathrm{Tr}(T_\nu^{E_\nu})| = 2$ and $\lambda = \pm 1$ and is therefore in the so-called parabolic case. The transfer matrix $T_\nu^{E_\nu}$ has only one eigenvector denoted $\vec{v}$ as well as a principal vector $\vec{w}$ satisfying $(T_\nu^{E_\nu} - \lambda\mathbf{1})\vec{w} = \vec{v}$. Hence the basis change with $M = (\vec{v}\ \vec{w})$ conjugates $T_\nu^{E_\nu}$ to a Jordan normal form $M^{-1}T_\nu^{E_\nu}M = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$. As the energy varies around $E_\nu$, one is in either of the above elliptic or hyperbolic cases. However, the corresponding basis changes become singular at $E_\nu$ and it is better to work with a basis change into an object close to the parabolic normal form. It would be possible to simply work with the energy independent $M$, but for sake of more explicit formulas later on let us choose (which is easily seen to be possible for $|\epsilon| \leq \epsilon_0$ for some $\epsilon_0$) an energy dependent basis change $M_\epsilon$ such that

$$M_\epsilon^{-1} T_\nu^{E_\nu+\epsilon} M_\epsilon \;=\; \begin{pmatrix} \lambda(1-\kappa_\epsilon) & 1 \\ -\kappa_\epsilon & \lambda \end{pmatrix} , \qquad \kappa_\epsilon \;=\; 2 - \lambda\, \mathrm{Tr}(T_\nu^{E_\nu+\epsilon}) \,. \tag{2}$$

As $T_\nu^{E_\nu+\epsilon}$ is a polynomial in $\epsilon$, $M_\epsilon$ is analytic. If $\kappa_\epsilon < 0$, one is in the hyperbolic case and for $\kappa_\epsilon > 0$ in the elliptic one. In the latter the rotation number $\eta$ is then given by $e^{\imath\eta} = \frac{1}{2}\left(\lambda(2-\kappa_\epsilon) + \imath\sqrt{4\kappa_\epsilon - \kappa_\epsilon^2}\right)$. Band touching happens if $\kappa_\epsilon > 0$ for both positive and negative $\epsilon$.

Following [JSS], let us next define the Prüfer variables with some care. For $E \in \mathbb{R}$, let $u^E$ be the sequence of real numbers given via the recurrence relation $H_\omega u^E = E u^E$ and the initial condition $u^E(-1) = \sin(\theta^0)$ and $t_\omega(0) u^E(0) = \cos(\theta^0)$ where $t_\omega(n) = -\langle n|H_\omega|n-1\rangle$. The free Prüfer phases $\theta_\omega^{0,E}(n)$ and amplitudes $R_\omega^{0,E}(n) > 0$ are now defined by

$$R_\omega^{0,E}(n) \begin{pmatrix} \cos(\theta_\omega^{0,E}(n)) \\ \sin(\theta_\omega^{0,E}(n)) \end{pmatrix} = \begin{pmatrix} t_\omega(n) u^E(n) \\ u^E(n-1) \end{pmatrix} , \tag{3}$$

the above initial conditions as well as

$$-\frac{\pi}{2} < \theta_\omega^{0,E}(n+1) - \theta_\omega^{0,E}(n) < \frac{3\pi}{2} .$$

The interest will be on energies $E = E_\nu + \epsilon$ in the vicinity of the band edge $E_\nu$, namely $|\epsilon| \leq \epsilon_0$. Associated to the basis change (2), the $M_\epsilon$-modified Prüfer variables $(R_\omega^\epsilon(n), \theta_\omega^\epsilon(n)) \in \mathbb{R}_+ \times \mathbb{R}$ will be introduced next. Define a smooth function $m_\epsilon : \mathbb{R} \to \mathbb{R}$ with $m_\epsilon(\theta + \pi) = m_\epsilon(\theta) + \pi$ and $0 < C_1 \leq m'_\epsilon \leq C_2 < \infty$, by

$$r(\theta) \begin{pmatrix} \cos(m_\epsilon(\theta)) \\ \sin(m_\epsilon(\theta)) \end{pmatrix} = M_\epsilon \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} , \qquad r(\theta) > 0 , \qquad m_\epsilon(0) \in [-\pi, \pi) .$$

Then set $\theta_\omega^\epsilon(n) = m_\epsilon(\theta_\omega^{0,E_\nu+\epsilon}(n))$ and

$$R_\omega^\epsilon(n) \begin{pmatrix} \cos(\theta_\omega^\epsilon(n)) \\ \sin(\theta_\omega^\epsilon(n)) \end{pmatrix} = M_\epsilon \begin{pmatrix} t_\omega(n)\, u^{E_\nu+\epsilon}(n) \\ u^{E_\nu+\epsilon}(n-1) \end{pmatrix} , \tag{4}$$

where the dependence on the initial phase $\theta_\omega^\epsilon(0) = m_\epsilon(\theta^0)$ is suppressed. The oscillation theorem as proven in [JSS] implies that the IDS close to the band edge $E_\nu$ is given by

$$\mathcal{N}(E_\nu + \epsilon) = \frac{1}{\pi} \lim_{n\to\infty} \frac{1}{n} \mathbf{E}(\theta_\omega^\epsilon(n)) , \tag{5}$$

the expectation being taken w.r.t. $\mathbf{P}$. If $\mathbf{p} = \delta_\sigma$, this formula gives the IDS $\mathcal{N}_\sigma$ of the $L$-periodic operator $H_\sigma$. A similar formula allows to express the Lyapunov exponent in terms of the the Prüfer variables [JSS], but this will not be used here.

The $M_\epsilon$-modified phase shift dynamics $\mathcal{S}_{\epsilon,\sigma}(\theta)$ (with energy variation $\epsilon$ relative to the band edge $E_\nu$) is defined via the $M_\epsilon$-modified Prüfer phase with initial condition $\theta^0 = \theta$ by $\mathcal{S}_{\epsilon,\sigma}(\theta) = \theta_\omega^\epsilon(L) - L\,\pi\,\mathcal{N}_\sigma(E_\nu)$ where $\omega = ((\sigma_n)_{n\in\mathbb{Z}}, l = 0)$ and $\sigma_1 = \sigma$. Note that it verifies

$$\rho \begin{pmatrix} \cos(\mathcal{S}_{\epsilon,\sigma}(\theta)) \\ \sin(\mathcal{S}_{\epsilon,\sigma}(\theta)) \end{pmatrix} = M_\epsilon^{-1} T_\sigma^{E_\nu+\epsilon} M_\epsilon \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} , \qquad \rho > 0 . \tag{6}$$

One then obtains a discrete time random dynamical system $\mathcal{S}_{\epsilon,\omega}^m$ on $\mathbb{R}$ defined iteratively by:

$$\mathcal{S}_{\epsilon,\omega}^m(\theta) = \mathcal{S}_{\epsilon,\sigma_m}(\mathcal{S}_{\epsilon,\omega}^{m-1}(\theta)) , \qquad \mathcal{S}_{\epsilon,\omega}^0(\theta) = \theta .$$

Replacing the Prüfer phases in (5) by the phase shifts relative to the band edge $E_\nu$, the IDS in the vicinity of $E_\nu$ is given by

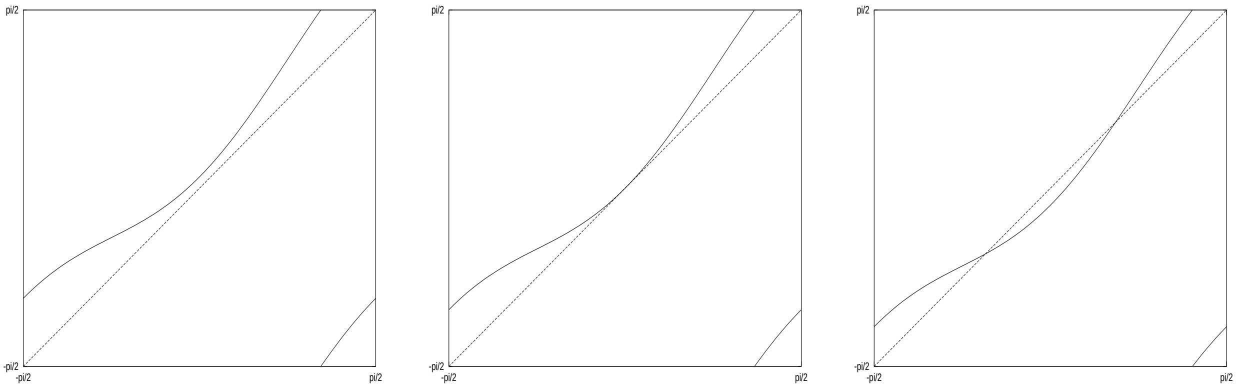


Figure 1: *Schematic plots of the phase shift dynamics $\mathcal{S}_{\epsilon,\nu}$ projected on $\mathbb{R}P(1) = [-\frac{\pi}{2}, \frac{\pi}{2})$ and for $\lambda = -1$ in the elliptic ($\kappa_\epsilon > 0$), parabolic ($\epsilon = 0$ and $\kappa_\epsilon = 0$) and hyperbolic ($\kappa_\epsilon < 0$) regime.*

$$\delta\mathcal{N}(\pm\epsilon) \;=\; \frac{1}{L}\,\frac{1}{\pi}\,\lim_{m\to\infty}\,\frac{1}{m}\,\mathbf{E}(\mathcal{S}^m_{\epsilon,\omega}(\theta))\;, \tag{7}$$

where the sign is chosen such that for positive $\epsilon$ one enters the spectrum of $H_\omega$.

Comparing (2) and (6), one notes that the phase shift dynamics $\mathcal{S}_{\epsilon,\nu}$ can be immediately calculated:

$$\tan(\mathcal{S}_{\epsilon,\nu}(\theta)) \;=\; \frac{-\kappa_\epsilon\cos(\theta) + \lambda\sin(\theta)}{\lambda(1-\kappa_\epsilon)\cos(\theta) + \sin(\theta)}\;.$$

Alternatively the cotangent may be used. For $\lambda = -1$ the curves are plotted in Fig. 1 for three different values of $\kappa_\epsilon$. Iterating $\mathcal{S}_{\epsilon,\nu}$ gives us a discrete time dynamical system on $\mathbb{R}$, which due to the periodicity relation may also be regarded as the lift of a dynamical system on $\mathbb{R}P(1) \cong [-\frac{\pi}{2}, \frac{\pi}{2})$. If $\kappa_\epsilon > 0$, there is no fixed point and therefore the dynamics $\mathcal{S}_{\epsilon,\nu}$ is conjugate to a rotation. The rotation number can be calculated explicitely, but it is roughly equal to $\pi$ over the number of iterations needed to go through one period. For $\kappa_\epsilon < 0$, there are two fixed points per period, one unstable and another one stable and (globally) attractive so that the rotation number is 0. In the parabolic case $\kappa_\epsilon = 0$ there is only one fixed point, instable to one side and stable to the other, and the rotation number is still 0. What was just described is locally simply a saddle node bifurcation.

# 3 Proof

Fig. 2 shows the elliptic dynamics $\mathcal{S}_{\epsilon,\nu}$ as well as a second dynamics which is hyperbolic. The latter should be thought of as representing those of the hyperbolic $\mathcal{S}_{\epsilon,\sigma}$, $\sigma \neq \nu$, which is closest to the elliptic case. Let us first argue that the hypothesis imply that Fig. 2 is qualitatively correct. Indeed, $E_\nu$ is supposed to be a band edge only of $H_\nu$ so that $\mathcal{S}_{\epsilon,\sigma}$ is hyperbolic for all $\sigma \neq \nu$ as long as $|\epsilon| \leq \epsilon_0$ for some adequately chosen $\epsilon_0$. Furthermore $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector

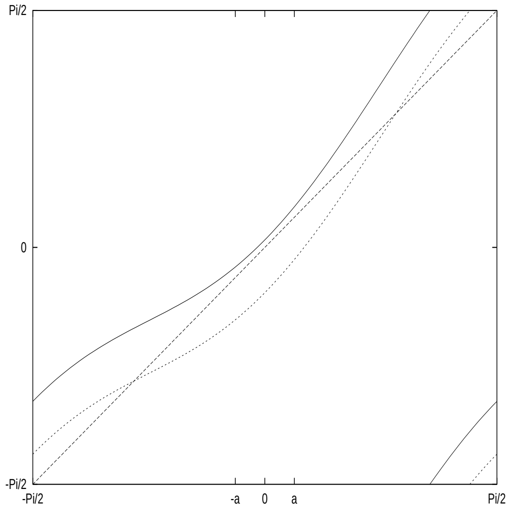


Figure 2: *Plot of the dynamics $\mathcal{S}_{\epsilon,\nu}$ just in the elliptic regime, and the next closest hyperbolic dynamics (dotted curve). This is the relevant sitution for the study of the Lifshitz tails.*

of $MT_\nu^{E_\nu}M^{-1}$ which implies that the fixed point of the parabolic map $\mathcal{S}_{0,\nu}$ is $\theta = 0$. For $\epsilon > 0$, the map $\mathcal{S}_{\epsilon,\nu}$ is given by shifting the graph of $\mathcal{S}_{0,\nu}$ into the elliptic regime. By hypothesis, $M^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is not an eigenvector of $T_\sigma^{E_\nu}$ for any $\sigma \neq \nu$, hence the fixed points of $\mathcal{S}_{\epsilon,\sigma}$, for $\sigma \neq \nu$ and $|\epsilon| \leq \epsilon_0$, are bounded away from $\theta = 0$. These facts are resumed in Fig. 2.

Next let us briefly present the main argument qualitatively. According to (7), the IDS is given by the mean rotation number, the average being taken w.r.t. the probability measure choosing the upper and lower graph in Fig. 2 randomly. Very close to the band edge, the dynamics $\mathcal{S}_{\epsilon,\nu}$ is only slightly in the elliptic regime and many iterations are necessary in order to complete one rotation. During most of these iterations, the angle is in a small interval $I = [-a, a]$ close to the origin. If at any of these iterations any of the other dynamics is chosen, the angle is immediately again outside and to the left of $I$ (this will be the definition of $I$). Hence the only way to go through $I$ and hence complete a rotation is to always choose the dynamics $\mathcal{S}_{\epsilon,\nu}$ until the angle is to the right of $I$. This happens with a very small probability which leads to the precise form of the Lifshitz tails.

The proof of the lower bound now goes as follows. The rotation number of $\mathcal{S}_{\epsilon,\nu}$ is equal to $\pi L\,\delta\mathcal{N}_\nu(\epsilon)$. The number of iterations needed to complete one rotation is $K = \left[\frac{1}{L\,\delta\mathcal{N}_\nu(\epsilon)}\right] + 1$ (here $[b]$ denotes the integer part of $b \in \mathbb{R}$). Then $\mathcal{S}^K_{\epsilon,\nu}(\theta) \geq \theta + \pi$, but $\mathcal{S}^{K-2}_{\epsilon,\nu}(\theta) \leq \theta + \pi$ for all $\theta \in \mathbb{R}$. For sake of concreteness, suppose that $E_\nu$ is a lower band edge so that the sign in (7) is $+$. Next let us set $m = KN$ in (7) and decompose $\mathcal{S}^{KN}_{\epsilon,\omega}(\theta)$ into a telescopic sum:

$$\delta\mathcal{N}(\epsilon) \;=\; \frac{1}{\pi}\,\frac{1}{L}\,\frac{1}{K}\,\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mathbf{E}\left(\mathcal{S}^{Kn}_{\epsilon,\omega}(\theta) - \mathcal{S}^{K(n-1)}_{\epsilon,\omega}(\theta)\right)\;. \tag{8}$$

With probability $p_\nu^K$ one has $\sigma_j = \nu$ for $j = K(n-1)+1, \ldots, Kn$. In this case, one rotation is completed and hence each summand can be bounded from below by $\pi p_\nu^K$. Elementary inequalities now imply the lower bound in (1).

In order to prove the upper bound, set $a = \max\{\theta \leq 1 \,|\, \mathcal{S}_{\sigma,\epsilon}(\theta) < -\theta \;\; \forall\, |\epsilon| \leq \epsilon_0,\; \sigma \neq \nu\}$ and $M = \inf\{m \geq 1 \,|\, \mathcal{S}^m_{\epsilon,\nu}(a) \geq \pi - a \;\; \forall\, 0 \leq \epsilon \leq \epsilon_0\}$. By construction, the only way to cross $I = [-a, a]$ is to chose $\sigma = \nu$ at least $K = \left[\frac{1}{L\,\delta\mathcal{N}_\nu(\epsilon)}\right] - M$ times. This happens with probability $p_\nu^K$. If this event occurs, the accumulated phase shift is of order $2a$ which can simply be bounded above by $\pi$. Hence using the same decomposition as in (8), but with the different $K$, one gets the bound $\delta\mathcal{N}(\epsilon) \leq \frac{1}{LK} p_\nu^K$. As $M$ is finite, this implies the upper bound in (1).

# 4 Outlook

The upper bound in the above argument exploits the following fact: the only way to cross the critical region $I = [-a, a]$ is by sucessively choosing the favorable branch $\mathcal{S}_{\epsilon,\nu}$. As this is the *only* way, it seems adequate to speak of a deterministic estimate. A more complete analysis of the mean rotation number would also have to take into account what happens in the remainder $S^1 \backslash I$. Obviously that heavily depends on the precise form of the maps $\mathcal{S}_{\epsilon,\sigma}$ as well as the probability measure $\mathbf{p}$.

A situation in which an analysis becomes feasable is perturbation theory. Suppose that dynamics $\mathcal{S}_{\lambda,\epsilon,\sigma}$ depend on a supplementary small parameter $\lambda$ giving the order of the $L^\infty$-distance between all the maps $\mathcal{S}_{\lambda,\epsilon,\sigma}$. Then the random dynamics in $S^1 \backslash I$ can be analysed perturbatively in $\lambda$ allowing to calucalate the Lipshitz constants perturbatively. This situation arises for example in the Anderson model in the weak coupling limit, namely $H_\omega = H_0 + \lambda V_\omega$ with some $L$-periodic backround operator $H_0$ and a random potential $V_\omega$. Then the gaps of $H_0$ remain open for sufficiently small $\lambda$ and the IDS in the gap is still given by the gap label of $H_0$. Within this framework it is also possible to study the IDS away from the Lifshitz tails. If all the dynamics at a given energy $E$ are elliptic, then the IDS is simply given by $\mathcal{N}(E) = \int d\mathbf{p}(\sigma)\, \mathcal{N}_\sigma(E) + \mathcal{O}(\lambda^2)$ (this can be proven along the lines of Sec. 4.5 of [JSS]). The situation becomes more interesting at energies where there are both elliptic and hypobolic phase shift dynamics. Then the mean rotation number can be calculated perturbatively via a classical ruin problem associated to the passage through an (appropriately chosen) interval $I$. The Lifshitz tails are recovered on one extreme (where an error leads to immediate ruin), but this allows moreover to control a cascade of large deviation regimes between the Lifshitz tails and the band center. By the same techniques the Lyapunov exponent can be computed perturbatively, completing thus the results of [PF, Thm. 14.6] and [JSS, Sec. 4.6]. The work giving a detailed analysis corresponding to these ideas is under preparation.

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