Symmetry of the Spin Ladder

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We consider the symmetry breaking of pure states of quantum spin chain. We show that the strong mixing condition automatically leads to certain symmetry breaking. This explains the reason why the decay of two point correlation functions is slow for the spin ladder systems with an odd number of legs.

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1 Introduction

The study of low-dimensional materials is an active area of research. Recent experimental and theoretical outcome of the quasi one-dimensional material are stimulating. Among them, one of surprise is the discovery of spin ladder materials. See , for example, the review article [11] and the references therein . For some time, it has been known that the spin 1/2 antiferromagnetic quantum spin chain has either slow decay of correlation or breaking down of lattice translational symmetry. This observation was mainly supported by exact solution of integrable models. However , if higher spin systems are concerned the situation seems different. In fact , the Haldane's conjecture claims that the integer spin Heisenberg models has the unique ground state with the spectral gap and exponential decay of correlation . Even if the original assertition of this conjecture is still an open problem of mathematical physics, a few examples supporting the conjecture are found so far. (See , e.g. [3] and [17] .) Influenced by the Haldane's conjecture, H.J.Schultz considered the antiferromagnetic quantum spin chain on the ladder . ([25]) He argued that the certain spin ladder system with the even number of legs has exponential decay correlation while the case of the odd number of legs behaves in the same manner of the anitferromagnetic Heisenberg chain with the half odd integer spin. It is relatively recent that the spin ladder material such as $Sr_{n-1}Cu_nO_{2n-1}$ is produced and used in experiments. The experimental results support the qualitative difference of systems with even and odd number legs which was previously pointed out theoretically . From mathematical point of view, it is not clear that the relation of decay of correlation and the magnitude of spin is due to the special choice of Hamiltonians and that a universal structure exists enforcing slow decay of correlation for the half odd integer antiferromagnetic systems or the spin 1/2 ladder systems with an odd number of legs. So far theoretical results are based on exact diagonalization, solution of integrable models or (mathematically not yet verified) approximate methods such as Density Matrix Renormalization Group.

The aim of this paper is a trial to investigate the connection of decay of correlation , the(translation) symmetry breaking and the representation theoretic property of the gauge group $SU(2)$. We will show that strong mixing property (= rapid decay of two point correlation) of infinite volume pure states necessitates the break down of either gauge or translation symmetry if certain representation theoretic condition is satisfied . Our results explain partially the above mentioned difference of odd and even number legs etc.

We use the C^*-algebraic method for our analysis. The standard reference of this mathematical approach is [8] and [9]. The algebra of local observables is the infinite tensor product $\mathcal{A}_{loc}$ of the matrix algebras. For the usual quantum spin system with spin s, the one site algebra is the set of $2s+1$ by $2s+1$ matrices, $M_{2s+1}(\mathbf{C})$ and in this case , $\mathcal{A}_{loc}$ is denoted by $\mathcal{A}^s_{loc}$.

$$\mathcal{A}^s_{loc} = \bigotimes_{\mathbf{Z}} M_{2s+1}(\mathbf{C})$$

Each component of the tensor product above is specified with a lattice site $j \in \mathbf{Z}$.

For the ladder case, each quantum degree of freedom is attached to a site in the ladder L_n where L_n is a subset of $\mathbf{Z^2}$ defined by

$$L_n \quad = \quad \left\{ \ (j,a) \ \in \mathbf{Z^2} \ \mid \ j \in \mathbf{Z} \ a = 1,2,3....n \ \right\}$$

The positive integer n is called the number of legs. For simplicity we only consider the case that each site in L_n is equipped with the spin 1/2 degree of freedom. The algebra of local observables is denoted by $\mathcal{A}_{loc}^{L_n}$.

$$\mathcal{A}_{loc}^{L_n} = \bigotimes_{L_n} M_2(\mathbf{C})$$

Here each component of the above tensor product above is specified with a lattice site $(j,a) \in L_n$. In the both cases $\mathcal{A}_{loc}^{s}$ and $\mathcal{A}_{loc}^{L_n}$ the $SU(2)$ gauge action on the observable algebra is denoted by β_g ($g \in SU(2)$) .

A standard model Hamiltonian of the spin s antiferromagnetic chain is the Heisenberg Hamiltonian H^s

$$H = \sum_{j \in \mathbf{Z}} \sum_{\alpha = x,y,z} S_\alpha^{(j)} S_\alpha^{(j+1)}$$

where $S_\alpha^{(j)}$ is the spin operator at the site j. On the other hand the Hamiltonian for the ladder case is

$$H = J_1 \sum_{(j,a) \in L_n} \sum_{\alpha = x,y,z} S_\alpha^{(j,a)} S_\alpha^{(j+1,a)} \quad + \quad J_2 \sum_{j \in \mathbf{Z}} \sum_{a=1,2,3....n-1} \sum_{\alpha = x,y,z} S_\alpha^{(j,a)} S_\alpha^{(j,a+1)}$$

where J_1 and J_2 are real positive parameters. In the experimentally important case, the ratio $\frac{J_1}{J_2}$ of J_1 and J_2 is of order one and it may not be easy to work with rigorous expansion techniques in this region of parameters. So we will not consider any specific form of Hamiltonians but we assume the following uniform clustering property of the infinite volume states. By $\mathcal{A}_R$ the set of all local observables supported in the positive lattice sites $j = 0, 1, 2, 3, ...$ We also denote the set of all local observables supported in the negative lattice sites $j = -1, -2, -3, ...$ by $\mathcal{A}_L$. Let τ_j the lattice translation (j shift to the right). Note that if j is positive $\tau_j(\mathcal{A}_R) \subset \mathcal{A}_R$ and if j is negative $\tau_j(\mathcal{A}_L) \subset \mathcal{A}_L$

Definition 1.1 *Let φ be a translationally invariant state of $\mathcal{A}$. We say that φ is split if the following is valid.*

$$\lim_{k \to \infty} \sup_{\|Q\| \leq 1} \left| \sum_{i,j} \gamma_{ij} (\varphi(Q_j \tau_k(R_i)) - \varphi(Q_j)\varphi(R_i)) \right| = 0 \tag{1.1}$$

where Q is any local observable in $\mathcal{A}_{loc}$ written in the finite sum

$$Q = \sum_{i,j} \gamma_{ij} Q_j R_i \quad \gamma_{ij} \in \mathbf{C} \quad Q_j \in \mathcal{A}_L \quad R_i \in \mathcal{A}_R.$$

Note that the purity and translation invariance of φ implies that

$$\lim_{k \to \infty} \left| \sum_{i,j} \gamma_{ij} (\varphi(Q_j \tau_k(R_i)) - \varphi(Q_j)\varphi(R_i)) \right| = 0$$

however, the split condition does not always hold. The crucial part of the above assumption is uniformity of convergence. All the examples of pure states with exponential decay of

correlation we know satisfy the split property. By the exponential decay of correlation we mean

$$\lim_{|k|\to\infty} e^{M|k|}\,|\varphi(Q_1\tau_k(Q_2)) - \varphi(Q_1)\varphi(Q_2)| = 0$$

for any local observables Q_1 and Q_2 and a positive constant M.In the section 4 we will present examples of pure ground states which are split and non split . We are not aware of any counter example to the following conjecture.

Conjecture 1.2 *Let φ be a pure translationally invariant state such that for any local observable Q the following sum is finite.*

$$\sum_{k\in\mathbf{Z}} (|\varphi(Q\tau_k(Q)) - \varphi(Q)\varphi(Q)| < \infty \tag{1.2}$$

Then , the state φ is split.

The main result of this paper is as follows.

Theorem 1.3 *Suppose that the spin s is a half odd integer and consider a translationally invariant pure state φ of $\mathcal{A}^s_{loc}$.*
If φ is split, φ is not gauge invariant, namely , there exists an element g of $SU(2)$ such that $\varphi \circ \beta_g \neq \varphi$.

The same type of result is obtained for the ladder model.

Theorem 1.4 *Consider the ladder system $\mathcal{A}^{L_n}_{loc}$ with an odd number of legs. Suppose that φ is a translationally invariant pure state of $\mathcal{A}^{L_n}_{loc}$. If φ is split, φ is not gauge invariant.*

The above result means that one of the gauge symmetry , the split property , or the translation symmetry must break down but our proof does not tell which one does occur in the antiferromagnetic system . This certainly depends on the choice of the state φ or the Hamiltonian for which the state φ is a ground state. Intuitively the break down of the split property is plausible in antiferromagnetic system. This is because the quantum fluctuation of the antiferromagnetic system is so large that the systems behaves as if in the high temperature phase and all the symmetry of the Hamiltonian remains unbroken , which forces the lack of split property. This argument explains the reason why the decay of two point correlation is slow for the spin ladder material with an odd number of legs. We also claim that our result is complementary to a theorem of I.Affleck and E.Lieb in [2]. I.Affleck and E.Lieb discussed the connection of the spectral gap and the symmetry breaking for the standard antiferromagnetic Heisenberg chain with the half odd integer spin. Their argument does not tell any information on decay rate of correlation functions.

Our method also shows the following result.

Theorem 1.5 *Consider the spin $\frac{1}{2}$ chain where the algebra of observables $\mathcal{A}_{loc}$ is*

$$\mathcal{A}_{loc} = \mathcal{A}^{\frac{1}{2}}_{loc} = \mathcal{A}^{L_1}_{loc}.$$

Let φ be a translationally invariant pure state. Suppose that it is invariant under a torus $U(1)$ (one dimensional abelian subgroup) in $SU(2)$, $\varphi \circ \beta_z = \varphi$ for any $z \in U(1)$. Then φ is either a product state or a non-split state.

Due to this result we will find that the ground state of the one-dimensional XY model is non-split. See Section 4.

Next we sketch the key point of our proof for Theorem 1.3 , 1.4. and mention related previous work. The proof is based on the three different notions, split property, the shift of $\mathcal{B}(\mathcal{H})$ and the finitely correlated states (=quantum markov states).

A standard argument of quasi-local algebra implies that a state φ is split if and only if φ is quasi-equivalent to the state $\varphi_L \otimes \varphi_R$ where φ_L and φ_R are the restriction of φ to $\mathcal{A}_L$ and $\mathcal{A}_R$. The latter condition is precisely the same condition of the local quantum field theory (See [16] ,[10] and [12] and the references therein.) Let $\pi(\ )$ be the GNS representation associated with a state φ. In our context, the split condition of the local QFT is as follows. Consider the von Neumann algebra M_1 generated by $\pi(\mathcal{A}_R)$ and the commutant $M_2 = \pi(\mathcal{A}_L)'$ of $\pi(\mathcal{A}_L)$. The inclusion of the von Neumann algebras $M_1 \subset M_2$ is split if it contains an intermediate type I factor N , $M_1 \subset N \subset M_2$. The inclusion is split if and only if the state φ is split.. The split was proved first for free (and , hence ,locally normal) relativistic quantum fields by D.Buchholtz in [10]. The standard split property was extensively studied by S.Doplicher and R.Longo in [12] and R.Longo obtained the solution of the factorial Stone - Weierstrass conjecture in [20] . There exists a vast number of results on local QFT based on the split property.(See [16] .) We should mention that our inclusion is not standard in sense of of S.Doplicher and R.Longo . This is due to lack of the Reeh-Schlieder property in our lattice models.

When φ is a translation invariant pure split state, the restriction φ_R to $\mathcal{A}_R$ is a type I factor state. Passing to the GNS representation and the restriction of τ_j ($j \geq 0$) to $\mathcal{A}_R$ we obtain the shift of $\mathcal{B}(\mathcal{H})$) in the sense of R.Powers ([24]). The close connection of the shift of the type I factor and the representation of the Cuntz algebra was clarified in a series of work of O.Bratteli,P.Jorgensen,A.Kishimoto , J.Price and R.Werner. See [5] [6] and [7] . They have shown that the set of conjugacy classes of the shift of $\mathcal{B}(\mathcal{H})$ can be identified with the equivalence classes of irreducible representations of the Cuntz algebra. The identification is based on the fact that any shift of $\mathcal{B}(\mathcal{H})$ with Powers index d is implemented by the generator of the Cuntz algebra O_d. Then following the construction of [23] , it is easy to show that a split state φ is a generalized finitely correlated state . The finitely correlated state which we will define later explicitly was introduced by M.Fannes, B.Nachtergaele, and R.Werner in [13], however, the construction of states is essentially same as of the quantum Markov states of L.Accradi . Historically speaking, in the middle of 1970's , L.Accardi investigated the quantum Markov states as a non-commutative generalization of classical Markov measures. His quantum Markov state is obtained by iteration of completely positive maps which play the role of the transfer matrix in classical statistical mechanics. Quite independently in [3] I.Affleck ,T.Kennedy, E.Lieb and H.Tasaki solved a spin antiferromagnetic quantum spin model as an affirmative example to the Haldane conjecture of the antiferromagnetic Heisenberg models with integer spin . They obtained the exact ground state of their model which they called 'Valence Bond Solid State'. Subsequently M.Fannes, B.Nachtergaele, and R.Werner noticed that valence bond states and the quantum Markov states can be treated in one framework. We show that the split state is constructed in the same way as the finitely correlated state. The classical counterpart to split states is now the Gibbs measure for the infinite range

interaction while the finitely correlated state of M.Fannes, B.Nachtergaele, and R.Werner corresponds to the finite range interaction. We can investigate the symmetry of split states using our realization as the generalized finitely correlated states.

The rest of this paper is devoted to the detail exposition of the above idea. The section 2 is the mathematical preliminary. In the section 3 we establish the realization of split states as as the generalized finitely correlated states. The symmetry of split states is also discussed. Our proof of Theorem 1.3 , 1.4 and 1.5 will be presented at the end of the section 3. In the section 4 we exhibit a few examples of (non-)split pure states. All the examples there are ground states of the quantum spin chain.

2 Mathematical Preliminary

We begin with introducing notations. The quantum spin model we consider is described by the UHF C^*-algebra denoted by $\mathcal{A}$. Here $\mathcal{A}$ is the C^*-completion of the infinite tensor product of the algebra $M_d(\mathbf{C})$ of d by d complex matrices.

$$\mathcal{A} = \overline{\bigotimes_{\mathbf{Z}} M_d(\mathbf{C})}^{C^*} \tag{2.1}$$

Each component of the tensor product is indexed by an integer j . Thus the element of $\mathcal{A}$ is a local physical observable . Let Q be a matrix in $M_d(\mathbf{C})$. By $Q^{(j)}$ we denote the following element of $\mathcal{A}$.

$$\cdots \otimes 1 \otimes 1 \otimes \underbrace{Q}_{\text{the j-th component}} \otimes 1 \otimes 1 \otimes \cdots \in \mathcal{A}$$

Given a subset Λ of $\mathbf{Z}$, $\mathcal{A}_\Lambda$ is defined as the C^*-subalgebra of $\mathcal{A}$ generated by all $Q^{(j)}$ with $Q \in M_n(\mathbf{C})$, $j \in \Lambda$. We also set

$$\mathcal{A}_{loc} = \cup_{|\Lambda|<\infty} \quad \mathcal{A}_\Lambda$$

where $|\Lambda|$ is the cardinality of Λ. Suppose that φ is a state of $\mathcal{A}$. The restriction of φ to $\mathcal{A}_\Lambda$ is denoted by φ_Λ.

$$\varphi|_{\mathcal{A}_\Lambda} \quad = \quad \varphi_\Lambda$$

We also set $\mathcal{A}_R = \mathcal{A}_{[0,\infty)}$, $\mathcal{A}_L = \mathcal{A}_{(-\infty,-1]}$, $\varphi_R = \varphi_{[0,\infty)}$ and $\varphi_L = \varphi_{(-\infty,-1]}$ The translation τ_j is an automorphism of $\mathcal{A}$ defined by

$$\tau_j(Q^{(k)}) = Q^{(j+k)} \ .$$

Thus by definition τ_1 leaves $\mathcal{A}_R$ globally invariant and τ_{-1} leaves $\mathcal{A}_L$ globally invariant . As we are working on a quasi-local algebra and each local subalgebra is a finite full matrix algebra , we can apply results of Section 2.6 and Section4.2 of [8]. We use Theorem 2.6.10 and Cororally 2.6.11 of [8]

Proposition 2.1 *(i) Suppose that a state φ of $\mathcal{A}$ is translationally invariant. φ is factor (the GNS representation is a factor representation) if and only if*

$$\lim_{|j|\to\infty} \varphi(Q_1\tau_j(Q_2)) = \varphi(Q_1)\varphi(Q_2) \tag{2.2}$$

for any Q_1 and Q_2 in $\mathcal{A}$.
(ii) Suppose that two states φ_1 and φ_2 are factor states of $\mathcal{A}$. Then the following conditions are equivalent.
(a) φ_1 and φ_2 are quasi-equivalent .
(b) For any $\epsilon > 0$ there exists k such that

$$|\varphi_1(Q) - \varphi_2(Q)| \le \epsilon \, \|Q\| \tag{2.3}$$

for any Q in $\mathcal{A}_{(-\infty,-k]\cup[k,\infty)}$.

The results of the same kind are valid for $\mathcal{A}_R$ and $\mathcal{A}_L$. If a state φ of $\mathcal{A}$ is factor , states φ_R of $\mathcal{A}_R$, φ_L of $\mathcal{A}_L$ and $\varphi_L \otimes \varphi_R$ of $\mathcal{A}$ are factor. The following is a straightforward consequence of the above proposition.

Proposition 2.2 *Let φ be a translationally invariant pure state of $\mathcal{A}$. The following conditions are equivalent.*
(i) The state φ is split in the sense specified in Definition 1.1.
(ii) φ is quasi-equivalent to the state $\varphi_L \otimes \varphi_R$.
(iii) The GNS representation of $\mathcal{A}_R$ associated with φ_R is type I factor.

Proof The equivalence of (i) and (ii) follows from Proposition 2.1.(ii). Suppose that φ is quasi-equivalent to the state $\varphi_L \otimes \varphi_R$. As we assumed that the state φ is pure both φ_L and φ_R are type I otherwise, if one of φ_L and φ_R is non-type I the tensor product cannot be type I. Thus we have only to show that the condition (iii) implies (ii).

Let $\{\pi(\ ), \Omega, \mathcal{H}\}$ be the GNS triple for the state φ of $\mathcal{A}$ where $\pi(\ )$ is the representation , Ω is the GNS cyclic vector and $\mathcal{H}$ the representation space. $\pi(\mathcal{A}_R)''$is a factor. To see this, set $\mathcal{H}_R = \overline{\pi(\mathcal{A}_R)\Omega}$ and let $\pi_R(\mathcal{A}_R)$ be the restriction of $\pi(\mathcal{A}_R)$ to $\mathcal{H}_R$. Due to our assumption, the representation of $\pi(\ )$ of $\mathcal{A}$ is irreducible and the von Neumann algebra $\pi_R(\mathcal{A}_R)''$ generated by $\pi_R(\mathcal{A}_R)$ is a type I factor. The center of $\pi(\mathcal{A}_R)''$ is trivial because it commutes with both $\pi(\mathcal{A}_R)$ and $\pi(\mathcal{A}_L)$. Thus any subrepresentation of $\pi(\mathcal{A}_R)$ is quasi-equivalent to $\pi(\mathcal{A}_R)$ itself. In particular $\pi(\mathcal{A}_R)$ and $\pi_R(\mathcal{A}_R)$ are quasi-equivalent and $\pi(\mathcal{A}_R)''$ is a type I factor.

For any type I factor M on a separable Hilbert space $\mathcal{H}_0$ we can find separable Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$ and a unitary W from $\mathcal{H}_0$ to $\mathcal{H}_1 \otimes \mathcal{H}_2$ such that

$$WMW^{-1} = B(\mathcal{H}_1) \otimes 1_{\mathcal{H}_2} \quad , \quad WM'W^{-1} = 1_{\mathcal{H}_1} \otimes B(\mathcal{H}_2). \tag{2.4}$$

Due to these remarks, the state φ of $\mathcal{A}$ is equivalent to a product state $\psi_L \otimes \psi_R$ where ψ_L (resp. ψ_R) is a state of $\mathcal{A}_L$ (resp. ψ_R) which is quasi-equivalent to φ_L (resp. φ_R) . ∎

When φ_R is a type I factor state of $\mathcal{A}_R$, it yields a shift of $\mathcal{B}(\mathcal{H})$. So we next explain the relationship between the Cuntz algebra O_d and the shift of the set $\mathcal{B}(\mathcal{H})$ of all bounded linear operators on a separable Hilbert space $\mathcal{H}$. (See [5] , [6] and [7] for detail.) The Cuntz algebra O_d is a simple C^*-algebra generated by isometries satisfying

$$S_k^* S_l \ = \ \delta_{kl} 1 \quad , \quad \sum_{k=1}^{d} S_k S_k^* \ = \ 1 \tag{2.5}$$

The gauge action γ_U of the group $U(d)$ of d by d unitary matrices is defined via the following formula.

$$\gamma_U(S_k) \ = \ \sum_{l=1}^{d} U_{lk} S_l$$

where U_{kl} is the k l matrix element for U in $U(d)$. Consider the diagonal circle group $U(1) = \{ z \ \ |z| = 1 \ z \in \mathbf{C} \}$. The fixed point algebra ${O_d}^{U(1)}$ for this subgroup $U(1)$ of

$U(d)$ is the UHF algebra d_∞ which we will identify with $\mathcal{A}_R = \mathcal{A}_{[0,\infty)}$ later. We introduce the canonical endomorphism Θ of O_d determined by

$$\Theta(Q) = \sum_{k=1}^{d} S_k Q S_k^* \qquad Q \in O_d \tag{2.6}$$

Consider $\mathcal{B}(\mathcal{H})$ and an endomorphism $\tilde{\Theta}$ of $\mathcal{B}(\mathcal{H})$. Throughout this paper an endomorphism always means a unital normal *-endomorphism. The Powers index of the endomorphism $\tilde{\Theta}$ is d ($d \in \{1, 2, 3....\infty\}$) if the relative commutant $\tilde{\Theta}(\mathcal{B}(\mathcal{H}))' \cap \mathcal{B}(\mathcal{H})$ of $\tilde{\Theta}(\mathcal{B}(\mathcal{H}))$ in $\mathcal{B}(\mathcal{H})$ is isomorphic to the type I_d factor. (c.f. [24]) Any endomorphism $\tilde{\Theta}$ of $\mathcal{B}(\mathcal{H})$ with the Powers index d is implemented by generators of the Cuntz algebra O_d . Namely ,there is a non-degenerate representation of O_d on $\mathcal{H}$ and

$$\tilde{\Theta}(Q) = \sum_{k=1}^{d} S_k Q S_k^* \qquad Q \in \mathcal{B}(\mathcal{H})$$

See , for example , [5]. The implementation of the endomorphism is unique up to the gauge transformation of $U(d)$.

An endomorphism $\tilde{\Theta}$ of $\mathcal{B}(\mathcal{H})$ is called ergodic if the fixed point is trivial,

$$\left\{ Q \in \mathcal{B}(\mathcal{H}) \mid \tilde{\Theta}(Q) = Q \right\} = \mathbf{C}1_{\mathcal{H}}.$$

$\tilde{\Theta}$ is called a shift of $\mathcal{B}(\mathcal{H})$ if

$$\cap_{n=1}^{\infty} \tilde{\Theta}^n(\mathcal{B}(\mathcal{H})) \ = \ \mathbf{C}1_{\mathcal{H}}$$

If $\tilde{\Theta}$ is a shift it is ergodic. These notions are expressed in terms of representations of the Cuntz algebra implementing the endomorphism $\tilde{\Theta}$. $\tilde{\Theta}$ is ergodic if and only if O_d is weakly dense in $\mathcal{B}(\mathcal{H})$. $\tilde{\Theta}$ is shift iff the $U(1)$ fixed point algebra ${O_d}^{U(1)}$ is weakly dense in $\mathcal{B}(\mathcal{H})$.

We explain the relation between our quantum spin system and the shift of the type I factor.

Lemma 2.3 *Suppose that φ is a pure translationally invariant state of $\mathcal{A}$. We assume that φ is split. Let $\{\pi_R(\mathcal{A}_R), \Omega_R, \mathcal{H}_R\}$ be the GNS representation associated with φ_R of $\mathcal{A}_R$.*
(i) The translation τ_1 restricted to $\mathcal{A}_R$ is extendible to a normal unital endomorphism $\tilde{\Theta}$ of the type I factor $\pi_R(\mathcal{A}_R)''$. The normal extension $\tilde{\Theta}$ is a shift with the Powers index d.
(ii) There exists a representation of the Cuntz algebra O_d with the canonical generators S_j $(j = 1, 2, ...d)$ satisfying (2.5) $S_j \in \pi_R(\mathcal{A}_R)''$ $(j = 1, 2, ...d)$ and

$$\pi_R(\tau_1(Q)) = \sum_{k=1}^{d} S_k \pi_R(Q) S_k^* \qquad Q \in \mathcal{A}_R. \tag{2.7}$$

Proof
(i) We set $M = \pi_R(\mathcal{A}_R)''$. M is a type I factor due to Proposition 2.2. Note that $\pi_R(\mathcal{A}_R)$

may not act irreducibly on $\mathcal{H}_R$. Let $\{\pi(\mathcal{A}), \Omega, \mathcal{H}\}$ be the GNS triple associated with φ of $\mathcal{A}$. As φ is translationally invariant there exists the unitary U satisfying

$$U\pi(Q)U^* = \pi(\tau_1(Q)) \quad , \quad U\Omega = \Omega \ .$$

This equation gives rise to the normal extension of τ_1 to $\pi(\mathcal{A})''$. We denote this normal extension by $\tilde{\Theta}$. We also denote the restriction of $\tilde{\Theta}$ to $\pi(\mathcal{A}_R)''$ by the same symbol $\tilde{\Theta}$.

Obviously we may set $\Omega_R = \Omega$ and

$$\mathcal{H}_R = \overline{\pi(\mathcal{A}_R)\Omega} \quad , \quad \pi_R(Q) = \pi(Q)|_{\mathcal{H}_R} \quad Q \in \mathcal{A}_R.$$

The normal extension $\tilde{\Theta}$ of τ_1 on $\mathcal{A}_R$ to M is then introduced by the restriction of $\tilde{\Theta}$ to M acting on $\mathcal{H}_R$. It is easy to see that $\tilde{\Theta}$ is shift because φ_R is a factor state and $\tilde{\Theta}(M) = \pi_R(\mathcal{A}_{[1,\infty)})''$.

Next we show the Powers index of $\tilde{\Theta}$ is d .It suffices to show that the relative commutant $\tilde{\Theta}(M)' \cap M$ of $\tilde{\Theta}(M)$ in M is $\pi(\mathcal{A}_{\{0\}}) = M_d(\mathbf{C})$. There exists conditional expectations $P_{[1,k)}$ ($k = 2, 3, ...$) from $\mathcal{A}_R$ to $\mathcal{A}_{[1,k)^c}$. Let Q be an element of $\tilde{\Theta}(M)' \cap M$ and take Q_α from $\pi_R(\mathcal{A}_R)$ such that the weak limit of Q_α converges to Q , $w - \lim_\alpha Q_\alpha = Q$. By taking $P_{[1,k)}(Q_\alpha)$ in stead of Q_α we can assume that Q is in $\pi_R(\mathcal{A}_{[1,k)^c})''$. Thus

$$Q \in \pi_R(\mathcal{A}_{\{0\}}) \cup \cap_{k=2,3,...}\pi_R(\mathcal{A}_{[k+1,\infty}})'' = \pi_R(\mathcal{A}_{\{0\}})$$

In particular, the Powers index of $\tilde{\Theta}$ is d.

(ii) We first fix the matrix unit f_{ij} of $\pi_R(\mathcal{A}_{\{0\}})$ ($i, j = 1, 2, 3, ...d$) . As $\tilde{\Theta}$ is a shift with Powers index d, we have a representation of the Cuntz algebra O_d in M which implements $\tilde{\Theta}$ in the sense that

$$\tilde{\Theta}(Q) = \sum_{k=1}^{d} S_k Q {S_k}^* .$$

For the canonical S_j of this representation, we have the matrix units $e_{ij} = S_i S_j^*$ which belong to $\pi_R(\mathcal{A}_{\{0\}}) = \tilde{\Theta}(M)' \cap M$. Any matrix unit system in a finite dimensional full matrix algebra $M_d(\mathbf{C})$ is unitary conjugate to to another matrix unit system, we have a unitary w in $M_d(\mathbf{C})$ such that $f_{ij} = \gamma_w(e_{ij})$ for any $i, j = 1, 2, 3, ...d$. By replacing S_j with $\gamma_w(S_j)$ we obtain the Cuntz generators S_j satisfying (2.7) . Note that arbitrary elements of $\mathcal{A}_R$ and ${O_d}^{U(1)}$ are obtained from f_{ij} or e_{ij} and the action of the canonical endomorphism. ∎

3 Split States

In this section we show that the split pure state can be always constructed from composition of completely positive maps in the same way as quantum Markov states of L.Accardi in [1] or finitely correlated states of M.Fannes , B.Nachtergaele and R.Werner in [13].

Definition 3.1 *Let $\mathcal{K}$ be a separable Hilbert space and by $\mathcal{B}(\mathcal{K})$ we denote the set of all linear operators on $\mathcal{K}$. Suppose a normal completely positive map E from $M_d(\mathbf{C}) \otimes \mathcal{B}(\mathcal{K})$ to $\mathcal{B}(\mathcal{K})$ and a normal state ψ on $\mathcal{B}(\mathcal{K})$ are given. We assume that E is unital in the sense that*

$$E(1_{M_d} \otimes 1_{\mathcal{B}(\mathcal{K})}) = 1_{\mathcal{B}(\mathcal{K})}$$

and that the following invariance is valid for E and ψ

$$\psi(E(1_{M_d} \otimes Q)) = \psi(Q) \qquad \text{for } Q \in \mathcal{B}(\mathcal{K}) \ . \tag{3.1}$$

The following equation determines a translationally invariant state φ of $\mathcal{A}$.

$$\varphi(Q_0^{(j)} Q_1^{(j+1)} Q_2^{(j+2)} .... Q_l^{(j+l)}) = \psi(E(Q_0 \otimes E(Q_1 \otimes ....E(Q_l \otimes 1_{\mathcal{B}(\mathcal{K})}))...)) \tag{3.2}$$

The state φ determined by (3.2) is called the finitely correlated state associated with the triple $\mathcal{B}(\mathcal{K})$, E and ψ.

The positivity of the functional φ determined by (3.2) is guaranteed by the complete positivity of E. The operator E plays the role of the transfer matrix in classical spin systems so we will call E the transfer operator.

The quantum Markov state investigated in [1] corresponds to the case that the dimension $dim\ \mathcal{K}$ of $\mathcal{K}$ is d while the finitely correlated states of M.Fannes , B.Nachtergaele and R.Werner in [13] is the situation where the dimension of $\mathcal{K}$ is finite , $dim\ \mathcal{K} < \infty$. As was remarked in [13] the formula (3.2) gives rise to any translationally invariant state if we do not impose any condition on the algebra $\mathcal{B}(\mathcal{K})$ or some other data. In what follows we consider the following conditions. First we introduce operators $\tilde{E}$ and E_Q from $\mathcal{B}(\mathcal{K})$ to $\mathcal{B}(\mathcal{K})$ via the following equations.

$$\tilde{E}(R) = E(1_{M_d} \otimes R) \quad , \quad E_Q(R) = E(Q \otimes R)$$

where Q in $M_d(\mathbf{C})$ and any R in $\mathcal{B}(\mathcal{K})$.

The following notions are introuduced by M.Fannes, B.Nachtergaele and R.Werner in [13] and [14] .

Definition 3.2 *Let φ be a finitely correlated state associated with the triple $\mathcal{B}(\mathcal{K})$, E and ψ.*
(i) φ is purely generated if there exists a bounded linear operator V from $\mathcal{K}$ to $\mathbf{C}^d \otimes \mathcal{K}$ such that the transfer operator E is obtained via the following equation.

$$E(Q) = V^* Q V \qquad \text{for any } Q \text{ in } M_d(\mathbf{C}) \otimes \mathcal{B}(\mathcal{K}).$$

(ii) The triple of $\mathcal{B}(\mathcal{K})$, E and ψ is minimal if the non-trivial weakly closed subalgebra of $\mathcal{B}(\mathcal{K})$ containing $1_{\mathcal{B}(\mathcal{K})}$ and invariant under the operator E_Q with any $Q \in M_d(\mathbf{C})$ does not exist. In another word, let $\mathcal{B}_0$ be a subalgebra of $\mathcal{B}(\mathcal{K})$ such that $1_{\mathcal{B}(\mathcal{K})} \in \mathcal{B}_0$ and $E_Q(\mathcal{B}_0) \subset \mathcal{B}_0$ for any Q , then ${\mathcal{B}_0}'' = \mathcal{B}(\mathcal{K})$.

Proposition 3.3 *Let φ be a translationally invariant pure state of $\mathcal{A}$. The following conditions are equivalent.*
(i) φ is split .
(ii) φ is a purely generated finitely correlated state associated with the triple , $\mathcal{B}(\mathcal{K})$, E and ψ .

Proof of Proposition 3.3
Let $\{\pi(\mathcal{A}), \Omega, \mathcal{H}\}$ (resp.$\{\pi_R(\mathcal{A}_R), \Omega, \mathcal{H}_R\}$) be the GNS triple associated with φ (resp. φ_R). As we assumed that φ is pure translationally invariant , there exists a unitary U such that $U\pi(Q)U^* = \pi(\tau_1(Q))$ $Q \in \mathcal{A}$, $U\Omega = \Omega$ and

$$w - \lim_{|k|\to\infty} \pi(\tau_k(Q)) = \varphi(Q)1 \quad , \quad \lim_{|k|\to\infty} \psi(\tau_k(Q)) = \varphi(Q) \tag{3.3}$$

for any $Q \in \mathcal{A}$ and any state ψ which is quasi-equivalent to φ.

Now suppose that φ is split . Then φ_R is a type I factor state and the normal extension $\tilde{\Theta}$ of τ_1 on $\mathcal{B} = \pi_R(\mathcal{A}_R)''$ is a shift of $\mathcal{B}(\mathcal{H})$. (See Lemma 2.1.) We have a representation of the Cuntz algebra O_d in $\mathcal{B}$ such that the canonical generator S_j satisfies (2.7) .

Take a minimal projection q in the commutant of $\mathcal{B}$ such that $q\Omega \neq 0$. Consider the unit vector $\Omega_0 = cq\Omega$ where c is the normalization constant. The vector state associated with Ω_0 is a pure state of $\mathcal{A}_R$. Set $\mathcal{H}_0 = \overline{\pi_R(\mathcal{A}_R)\Omega_0}$ and the restriction of $\pi_R(\mathcal{A}_R)$ to $\mathcal{H}_0$ is an irreducible representation of $\mathcal{A}_R$ which we denote by π_0 . By definition π_0 is quasi-equivalent to π_R . Consider the subspace $\mathcal{K}$ of $\mathcal{H}_0$ generated by the vectors

$$S^*_{j_1} S^*_{j_2} S^*_{j_3} ....S^*_{j_l} \Omega_0$$

for $j_1, j_2, ...j_l \in \{1, 2, 3, ...d\}$. . Let P be the projection onto $\mathcal{K}$ and set $V_k = PS^*_k P = S^*_k P$. We obtain the completely positive map E from $M_d(\mathbf{C}) \otimes \mathcal{B}(\mathcal{K})$ to $\mathcal{B}(\mathcal{K})$ via the following equation.

$$E(e_{kl} \otimes Q) = V^*_k Q V_l \tag{3.4}$$

We show that the completely positive map E of (3.4) satisfies the minimality condition in Definition 3.2. Recall that $\pi_0(\mathcal{A}_R) = {O_d}^{U(1)}$ is weakly dense in $\mathcal{B}(\mathcal{H}_0) = {O_d}''$ so $P\pi_0(\mathcal{A}_R)P$ is weakly dense in $\mathcal{B}(\mathcal{K}) = P\mathcal{B}(\mathcal{H}_0)P$. Suppose that $\mathcal{B}_0$ is a subalgebra of $\mathcal{B}(\mathcal{K})$ containing the unit of $\mathcal{B}(\mathcal{K})$ and invariant under all E_Q . $\mathcal{B}_0$ contains $P\pi_0(\mathcal{A}_R)P$, $P\pi_0(\mathcal{A}_R)P \subset \mathcal{B}_0$, because $\mathcal{B}_0$ contains all the element of the form

$$PQ^{(0)}_0 Q^{(1)}_1 ....Q^{(m)}_m P = E_{Q_0} \circ E_{Q_1} \circ ....E_{Q_m}(1).$$

So we obtained the minimality.

Next we consider the $\tilde{E}$ invariant state ψ of $\mathcal{B}(\mathcal{K})$. Let ω_0 is the vector state associated with Ω_0. We take the state ψ obtained by the following limit .

$$\psi(Q) = \lim_{L\to\infty} \frac{1}{L} \omega_0(\sum_{j=0}^{L-1} \tilde{E}^j(Q))$$

We claim that

$$\psi(PQP) = \varphi(Q) \qquad \text{for any } Q \text{ in } \mathcal{A}_R. \tag{3.5}$$

In fact it suffices to consider the following element Q of $\mathcal{A}_R$.

$$Q = e^{(0)}_{k_0 l_0} e^{(1)}_{k_1 l_1} e^{(2)}_{k_2 l_2} .... e^{(n)}_{k_n l_n} \tag{3.6}$$

where $e^{(j)}_{k_j l_j}$ is the matrix unit (of $\mathcal{A}_R$) located on the lattice site $j \geq 0$. As $\mathcal{A}_R$ is the $U(1)$ gauge invariant part of our Cuntz algebra O_d we have

$$e^{(0)}_{k_0 l_0} e^{(1)}_{k_1 l_1} e^{(2)}_{k_2 l_2} .... e^{(n)}_{k_n l_n} = S_{k_0} S_{k_1} .... S_{k_{n-1}} S_{k_n} S^*_{l_n} S^*_{l_{n-1}} ... S^*_{l_0}$$

Thus for Q of (3.6)

$$\begin{aligned} \omega_0(Q) &= \omega_0(PQP) \\ &= \omega_0(P S_{k_0} S_{k_1} .... S_{k_{n-1}} S_{k_n} S^*_{l_n} S^*_{l_{n-1}} ... S^*_{l_0} P) \\ &= \omega_0(V^*_{k_0} V^*_{k_1} ... V^*_{k_n} V_{l_n} ... V_{l_1} V_{l_0}) \\ &= \omega(E(e_{k_0 l_0} \otimes E(e_{k_1 l_1} \otimes ... E(e_{k_n l_n} \otimes 1_{\mathcal{B}(\mathcal{K})}) ....)) \end{aligned} \tag{3.7}$$

Next we take the spatial average for Q of (3.6) as

$$Q_L = \frac{\sum_{j=0}^{L-1} \tau_j(Q)}{L}$$

When we take the limit $L \to \infty$, the left hand side of converges to $\varphi(Q)$ due to (3.3) . If we set

$$\tilde{E}_L(R) = \frac{\sum_{j=0}^{L-1} \tilde{E}^L(R)}{L}$$

it is easy to see

$$\lim_{L\to\infty} \omega_0(\tilde{E}_L(Q)) = \lim_{L\to\infty} \omega_0(PQ_LP) = \varphi(Q)$$

Thus we obtain (3.5) . The equation (3.5) also implies that the state φ is determined by the equation (3.2) because again taking Q of (3.6) we have

$$\begin{aligned} \varphi(Q) &= \psi(PQP) \\ &= \psi(V^*_{k_0} V^*_{k_1} ... V^*_{k_n} V_{l_n} ... V_{l_1} V_{l_0}) \\ &= \psi(E(e_{k_0 l_0} \otimes E(e_{k_1 l_1} \otimes ... E(e_{k_n l_n} \otimes 1_{\mathcal{B}(\mathcal{K})}) ....)) \end{aligned} \tag{3.8}$$

The state ψ is a normal state of $\mathcal{B}(\mathcal{K})$ as φ is a normal state of $\mathcal{B}(\mathcal{H}_0)$ and $\mathcal{B}(\mathcal{K}) = P\mathcal{B}(\mathcal{H}_0)P$. This completes the first part of proof(from (i) to (ii)).

Our proof for the converse assertition of Proposition 3.3 is based on results of [13] and [7]. Consider a translationally invariant state φ of $\mathcal{A}$. If φ is a finitely correlated state purely generated by V and the normal state ψ of $\mathcal{B}(\mathcal{K})$, φ is generated by another $\tilde{V}$ and the faithful normal state $\tilde{\psi}$ of $\mathcal{B}(\tilde{\mathcal{K}})$. $\tilde{V}$ and $\tilde{\psi}$ are obtained via reduction by the support projection of the density matrix for ψ. Thus from the begining we may assume that the normal state ψ is faithful. See [13].

Another result we use here is the relationship of the state of the Cuntz algebra O_d and bounded operators V . Suppose we have a normal state ψ of $\mathcal{B}(\mathcal{K})$ and bounded operators V_k $(k = 1, 2, ...d)$ on $\mathcal{K}$ saysfying

$$\sum_{k=1}^{d} \quad {V_k}^* V_k = 1$$

Theorem 5.1 of [7] tells us that $\mathcal{K}$ can be embedded into a larger Hilbert space $\mathcal{H}$ carrying a representation $S_1, ...S_d$ of the Cuntz algebra O_d such that if $P : \mathcal{H} \to \mathcal{K}$ is the projection onto $\mathcal{K}$ we have

$$V_k = PS_k{}^*P = S_k{}^*P.$$

and the state ψ extends to the state $\tilde{\varphi}$ of O_d in the sense that

$$\varphi(S_{i_1}S_{i_2}...S_{i_n}S^*_{j_m}...S^*_{j_2}S^*_{j_1}) = \psi(V_{i_1}V_{i_2}...V_{i_n}V^*_{j_m}...V^*_{j_2}V^*_{j_1})$$

We now return to the proof of Proposition 3.3.

We assume that the normal state ψ is faithful. By the proof of Proposition 2.2 of [14] the fixed point of the CP map $\tilde{E}$ is trivial.

$$\{ Q \in \mathcal{B}(\mathcal{K}) \mid \tilde{E}(Q) = Q \} = \mathbf{C}$$

Due to Theorem 5.1 of [7] we have an irreducible representation $\pi(O_d)$ on $\mathcal{H}$ of the Cuntz algebra O_d which is the dilation of operators V_k. As we assumed that the state φ is pure , Theorem 7.1 of [7] implies that

$$PSp(\tilde{E}) \cap \mathbf{T} = \{1\}$$

where $PSp(\tilde{E})$ is the set of point spectrum of $\tilde{E}$. Theorem 6.3 of [7] tells us that the restriction of $\pi(O_d)$ to the UHF algebra $\mathcal{A}_R$ is irreducible. This means that the state φ_R of $\mathcal{A}_R$ is of type I. ∎

Next we consider the symmetry of split states. Let G be a compact group and $v(g)$ be a d-dimensional unitary representation of G. By β_g we denote the product action of G on the infinite tensor product $\mathcal{A}$ induced by $v(g)$.

$$\beta_g(Q) = (.... \otimes v(g) \otimes v(g) \otimes v(g) \otimes ...)Q(.... \otimes v(g)^{-1} \otimes v(g)^{-1} \otimes v(g)^{-1} \otimes ...)$$

for any Q in $\mathcal{A}$. On $\mathcal{A}_R$ the gauge action $\gamma_{v(g)}$ of the Cuntz algebra O_d and β_g coincide , $\gamma_{v(g)}(Q) = \beta_g(Q)$ for any $\mathcal{A}_R$.

Proposition 3.4 *Let φ be a translationally invariant pure state of $\mathcal{A}$. Suppose that φ is split and G invariant.*

$$\varphi(\beta_g(Q)) = \varphi(Q) \quad \textit{for any } g \textit{ in } G \textit{ and any } Q \textit{ in } \mathcal{A} \ .$$

There exists a Hilbert space $\mathcal{K}$, a projective unitary representation $u(g)$ of G on $\mathcal{K}$, one-dimensional unitary representation $\xi(g)$, a bounded linear operator V from $\mathcal{K}$ to $\mathbf{C}^d \otimes \mathcal{K}$ and a normal state $\tilde{\varphi}$ of $\mathcal{B}(\mathcal{K})$ such that the following statements are valid.
(i)If we set $E(R) = V^*RV$ for R in $M_d(\mathbf{C}) \otimes \mathcal{B}(\mathcal{K})$ φ is a finitely correlated state associated with the triple $\mathcal{B}(\mathcal{K})$, E and ψ . This triple is minimal.
(ii) The operator V intertwines the projective representation of G.

$$(\xi(g)v(g)) \otimes u(g)V = Vu(g) \tag{3.9}$$

Suppose , further, that G is one-dimensional torus $U(1)$ or a connected, simply connected semisimple Lie group. The projective representation $u(g)$ is a unitary representation.

Proof of Proposition 3.4 We will use the notations of Proposition 3.3. but a different choice of the Hilbert space K and the transfer operator. Let $\{\pi_R(\mathcal{A}_R), \Omega, \mathcal{H}_R\}$ be the GNS triple associated with φ_R . As the state φ is split, we have a representation S_j of the Cuntz algebra in $\pi_R(\mathcal{A}_R)''$ which implements the shift τ_1 or $\tilde{\Theta}$. As φ_R is G invariant , there exists a unitary representation $u_0(g)$ on $\mathcal{H}_R$ such that

$$u_0(g)\pi_R(Q)u_0(g)^* = \pi_R(\beta_g(Q)) \quad , \quad u_0\Omega = \Omega$$

As τ_1 and β_g commute, $T_j = u_0(g)S_ju_0(g)^*$ also implements the same shift so there exists a unitary representation $w(g)$ of G in $\mathbf{C}^d$ such that

$$u_0(g)S_ju_0(g)^* = \gamma_{w(g)}(S_j) \tag{3.10}$$

where $\gamma_{w(g)}$ is the gauge automorphism of O_d. Consider another gauge action of G obtained by $\gamma_{v(g)}$. By definition for Q in ${O_d}^{U(1)} = \mathcal{A}_R$, we have

$$\gamma_{w(g)}(Q) = \gamma_{v(g)}(Q) \tag{3.11}$$

Thus there exists a scalar $\xi(g)$ such that

$$w(g)(g) = \xi(g)v(g) \quad , \quad \xi(g) \in \mathbf{C} \quad |\xi(g)| = 1 \tag{3.12}$$

$\xi(g)$ is a one-dimensional representation of G as both $w(g)$ and $v(g)$ are unitary representations of G.

As $u_0(g)$ gives rise to the continuous action on the type I factor $\pi(\mathcal{A}_R)'' = {O_d}''$ we have a projective unitary representation $u_1(g)$ of G in $\pi(\mathcal{A}_R)''$ which is the same G action on $\pi(\mathcal{A}_R)''$. As $u_0(g)u_1(g)^*S_j = S_ju_0(g)u_1(g)^*$ we have $u_0(g)u_1(g)^* \in \pi(\mathcal{A}_R)'$ and

$$u_1(g)S_ju_1(g)^* = \gamma_{w(g)}(S_j) \tag{3.13}$$

As is in the proof of Proposition3.3 we consider an irreducible representation $\{\pi_0, \mathcal{H}_0\}$ of $\mathcal{A}_R$ which is quasi-equivalent to the representation π associated with φ_R. Fix a unit vector Ω_0. Now let $\mathcal{K}$ be the closed subspace of $\mathcal{H}_0$ spanned by all the vectors

$$u_1(g_1)u_1(g_2)u_1(g_3).....u_1(g_k)S^*_{j_1}S^*_{j_2}S^*_{j_3}....S^*_{j_l}\Omega$$

for $g_1, g_2....g_k \in G$ and $l = 1, 2, 3, ...$. Let P be the projection onto $\mathcal{K}$. P commutes with $u_1(g)$ because

$$u_1(g)S^*_{j_1}S^*_{j_2}S^*_{j_3}....S^*_{j_l}\Omega = \gamma_{w(g)}(S^*_{j_1}S^*_{j_2}S^*_{j_3}....S^*_{j_l})u_1(g)\Omega_0 \tag{3.14}$$

We define $u(g)$ by the restriction of $u_1(g)$ to $\mathcal{K}$. The operator V from $\mathcal{K}$ to $\mathbf{C}^d \otimes \mathcal{K}$ is defined as before , namely, for a vector η in $\mathcal{K}$ we set

$$V\eta = ({S_1}^*P\eta, {S_2}^*P\eta, ..., {S_d}^*P\eta) \quad \in \mathbf{C}^d \otimes \mathcal{K} \tag{3.15}$$

Set $\mathcal{B} = \mathcal{B}(\mathcal{K}) = P\mathcal{B}(\mathcal{H}_0)P$ The normal invariant state for $\tilde{E}$ is constructed as in Proposition 3.3. The minimality follows from $\mathcal{B} = P\pi_0(\mathcal{A}_R)''P$. Then it is straightforward to check other claim of Proposition 3.4 (i) and (ii).

If G is one-dimensional torus $U(1)$ or a connected, simply connected semisimple Lie group, it is generally known that the continuous action of G on the type I factor is implemented by the inner unitary representation. (See Section 14 of [19]) ∎

Proof of Theorem 1.3 and Theorem 1.4. We show only Theorem 1.3 as our proof to Theorem 1.4. is essentially same. Suppose that the spin s is a half odd integer and φ is a translationally invariant , $SU(2)$ invariant , pure state. If it is split we can apply Proposition3.4. When G is $SU(2)$, the character $\xi(g)$ is trivial , $\xi(g) = 1$. Let $p(even)$ and $p(odd)$ (resp. $q(even)$ and $q(odd)$) be the projection to the integer and half odd integer spin space in $\mathcal{K}$ (resp. in $\mathbf{C}^d \otimes \mathcal{K}$). Note that

$$p(odd) = \sum_{s=\frac{1}{2},\frac{3}{2},\ldots} \int \overline{tr_s}(g)u(g)dg$$

$$p(even) = \sum_{s=0,1,2,3,\ldots} \int \overline{tr_s}(g)u(g)dg$$

where dg is the normalized Haar measure of $SU(2)$ and $tr_s(g)$ is the character (trace) of the spin s irreducible representation and $\overline{tr_s}(g)$ is its complex conjugate. It turns out that $p(even)$ and $p(odd)$ are in $P\pi(\mathcal{A})''P$ Obviously,

$$p(odd) + p(even) = 1_{\mathcal{K}} \quad , \quad q(odd) + q(even) = 1_{\mathbf{C}^d \otimes \mathcal{K}}$$

Note that $p(even)$ and $p(odd)$ are in $\mathcal{B}(\mathcal{K})$ as $u(g)$ is in $\mathcal{B}(\mathcal{K})$. The intertwining property (3.9) of $SU(2)$ representations implies

$$q(even)V = Vp(even) \quad , \quad q(odd)V = Vp(odd) \tag{3.16}$$

Obviously ,

$$q(even) = 1_{\mathbf{C}^d} \otimes p(odd) \quad , \quad q(odd) = 1_{\mathbf{C}^d} \otimes p(even)$$

Thus due to the invariance of ψ under $\tilde{E}$ we have

$$\psi(p(even)) = \psi(E(1_{\mathbf{C}^d} \otimes p(odd)) = \psi(p(odd)) \tag{3.17}$$

$$\psi(p(even)) = \psi(p(odd)) = \frac{1}{2}. \tag{3.18}$$

Consider the algebra $\mathcal{B}_0$ defined by

$$\mathcal{B}_0 = p(even)\mathcal{B}p(even) + p(odd)\mathcal{B}p(odd).$$

$\mathcal{B}_0$ is invariant by $\tilde{E}$ because

$$\tilde{E}(p(even)Q_1p(even)) = p(odd)\tilde{E}(Q_1)p(odd)$$

$$\tilde{E}(p(odd)Q_2p(odd)) = p(even)\tilde{E}(Q_2)p(even)$$

Due to the minimality we have $\mathcal{B}_0 = \mathcal{B}(\mathcal{K})$, which implies that either $p(odd) = 1$ or $p(even) = 1$. However the equation (3.18) leads to a contradiction. This completes the

proof of Theorem 1.3. ∎

Proof of Theorem 1.5 We consider the representation of $U(1)$ appearing in Proposition 3.4. Here we denote the character of $U(1)$ by $\rho(j)$ ($j = 0, \pm 1/2, \pm 1, ...$) and the restriction of the spin 1/2 $SU(2)$ to the torus is $\rho(1/2) \oplus \rho(-1/2)$. We denote the projection to the maximal subspace of $\mathcal{K}$ quasi-equivalent to $\rho(j)$ by $p(j)$. The projection to the subspace of $\mathbf{C}^2$ quasi-equivalent to $\rho(j)$ is denoted by $e(j)$.

According to Proposition 3.4. the representation $U(1)$ of in $\mathbf{C}^2 \otimes \mathcal{K}$ gives rise to the following identity,

$$e(k + \frac{1}{2}) + e(k - \frac{1}{2}) = 1 \ , \ \sum_j p(j) = 1$$

where k is originated from $\xi(g)$ of (3.9) . The intertwining property of V reads

$$\left(e(k + \frac{1}{2}) \otimes p(j - k - \frac{1}{2}) + e(k - \frac{1}{2}) \otimes p(j - k + \frac{1}{2}) \right) V = Vp(j)$$

When k is an integer we can use even-odd argument for $SU(2)$ in our proof of Theorem 1.3. which is a contradiction. So we assume that k is a half odd integer.

We first consider the case that $k = \frac{1}{2}$. So we have

$$e(1) + e(0) = 1 \quad (e(1) \otimes p(j-1) + e(0) \otimes p(j))V = Vp(j) \tag{3.19}$$

We claim that

$$\psi(V^* e(1) \otimes p(j-1)V) = \psi(V^* e(1) \otimes p(j)V) \tag{3.20}$$

In fact the equation (3.19) implies

$$\begin{aligned} & \psi(V^*(e(1) \otimes p(j-1) + e(0) \otimes p(j))V) \\ = \ & \psi(V^* V p(j)) = \psi(p(j)) \\ = \ & \psi(V^*(e(1) \otimes p(j) + e(0) \otimes p(j))V) \end{aligned}$$

where we used (3.1) in the last step. The equation (3.20) suggests the following equation.

$$\varphi(\frac{1}{2}(1 + \sigma_z^{(i)})) = \psi(V^* e(1) \otimes 1V) = 0 \tag{3.21}$$

for any i in $\mathbf{Z}$. This is a consequence of

$$0 \leq \sum_j \psi(V^* e(1) \otimes p(j)V) = \psi(V^* e(1) \otimes 1V) \leq 1$$

and of the equation (3.20) . The equation (3.21) tells us that the state φ is a ground state for the Hamiltonian H ,

$$H = \sum_i \frac{1}{2}(1 + \sigma_z^{(i)}).$$

So φ is a product state.

We next consider the case that $k > \frac{1}{2}$ We have a positive integer $m = k + \frac{1}{2}$ such that

$$e(m+1) + e(m) = 1 \quad (e(m+1) \otimes p(j-1) + e(m) \otimes p(j))V = Vp(j+m)$$

Then, as before ,we have

$$\sum_{j=n}^{\infty} \psi(V^*(e(m+1) \otimes p(j-1) + e(m) \otimes p(j))V) = \sum_{j=n}^{\infty} \psi(p(j+m)).$$

On the other hand,

$$\psi(p(j+m)) = \psi(V^*(e(m+1) \otimes p(j+m) + e(m) \otimes p(j+m))V)$$

It turns out that

$$\sum_{j=n}^{n+m} \psi(V^*(e(m+1) \otimes p(j-1))V) + \sum_{j=n}^{n+m-1} \psi(V^*(e(m) \otimes p(j)V)) = 0 \tag{3.22}$$

As each summand in (3.22) is positive we have

$$\psi(V^*(e(m+1) \otimes p(j))V) = \psi(V^*(e(m) \otimes p(j))V) = 0$$

However, this is a contradiction to the following equation,

$$1 = \sum_j (\psi(V^*(e(m+1) \otimes p(j))V) + \psi(V^*(e(m) \otimes p(j))V))$$

The case of the negative k can be handled in the same manner. Thus we have proved Theorem 1.5. ■

4 Examples

In this section we present a few examples of split and non-split pure states. As these are constructed as infinite volume ground states of spin models, we begin with explaining the mathematical physical definition of ground states. (For more detail , see [9].) We consider the translationally invariant Hamiltonian. For simplicity we assume that the interaction is of nearest neighbor. So suppose a selfadjoint element $h_0 = h_0^*$ in $\mathcal{A}_{\{0,1\}}$ is given and consider the finite volume Hamiltonian $H_{[n,m]}$ on the interval $[n, m]$ is determined by

$$H_{[n,m]} = \sum_{j=n}^{m-1} h_j \ .$$

where we set $\tau_j(h_0) = h_j$. The formal infinite volume Hamiltonian will be denoted by

$$H = \sum_{j \in \mathbf{Z}} h_j \tag{4.1}$$

The time evolution $\alpha_t(Q)$ of $Q \in \mathcal{A}$ is obtained via the thermodynamic limit ,

$$\alpha_t(Q) = \lim_{\Lambda \to \mathbf{Z}} e^{itH_\Lambda} Q e^{-itH_\Lambda}.$$

The generator δ of α_t is $\delta(Q) = [H, Q]$ which is densely defined. α_t will be referred to as the time evolution . Formally the one-parameter group α_t of automorphisms of $\mathcal{A}$ corresponds to

$$e^{itH} Q e^{-itH} = \alpha_t(Q).$$

See [9] .

A state φ is invariant (under α_t) if $\varphi(\alpha_t(Q)) = \varphi(Q)$ for any t and any Q in $\mathcal{A}$. The following condition is the standard definition of the ground state for infinite spin systems.

Definition 4.1 *Let φ be a state of $\mathcal{A}$. φ is a ground state for the Hamiltonian (4.1) if and only if*

$$\varphi(Q^*[H, Q]) \geq 0 \qquad \text{for any } Q \text{ in } \mathcal{A}_{loc}. \tag{4.2}$$

The ground state is characterized by minimization of the energy in the following sense.

Proposition 4.2 *The following conditions for a state φ of $\mathcal{A}$ are equivalent.*
(i) The state φ is a ground state in the sense specified in Definition 4.1.
(ii) For any interval $[n, m]$,

$$\varphi(H_{[n,m]}) = \inf \quad \psi(H_{[n,m]}) \tag{4.3}$$

where the inf *is taken among all the states ψ satisfying*

$$\varphi(Q) = \psi(Q) \qquad \text{for any } Q \text{ in } \mathcal{A}_{(-\infty, n+1] \cup [m-1, \infty)}$$

Furthermore if the state φ is translationally invariant, the following condition is equivalent to (i) and (ii).
(iii) The state φ minimizes the mean energy .

$$\varphi(h_0) = \inf \quad \psi(h_0) \tag{4.4}$$

where the infimum is taken among all the translationally invariant states ψ of $\mathcal{A}$.

The set of all ground states is a weakly compact convex set in the state space of $\mathcal{A}$. The extremal ground state is a pure state.

We present the example of non split pure states. We consider the exactly solvable XY model. The Hamiltonian H_{XY} of the XY model is determined by the following equation.

$$H_{XY} = -\sum_{j\in\mathbf{Z}} \left\{\sigma_x^{(j)}\sigma_x^{(j+1)} + \sigma_y^{(j)}\sigma_y^{(j+1)}\right\} - 2\lambda \sum_{j\in\mathbf{Z}} \sigma_z^{(j)} \tag{4.5}$$

where λ is a real parameter (an external magnetic field) , $\sigma_x^{(j)}$, $\sigma_y^{(j)}$ and $\sigma_z^{(j)}$ are Pauli spin matrices at the site j. For the finite chain the XY model is equivalent to the free Fermion via the Jordan Wigner transformation. For the infinite chain the equivalence is not literally correct due to the infinite product of $\sigma_z^{(j)}$ in the Jordan Wigner transformation. Nevertheless we have shown that the ground state is unique in [4]. When $|\lambda| \geq 1$ it is easy to see that the ground state is a product state. If $|\lambda| < 1$ the ground state is not a product state . On the other hand the Hamiltonian H_{XY} of the XY model is invariant under the rotation around the z axis. The infinitesimal generator of the rotation commutes with the Hamiltonian H_{XY}

$$[H_{XY}, N] = 0$$

where

$$N = \sum_{j\in\mathbf{Z}} \sigma_z^{(j)}$$

By the uniqueness mentioned above, the ground state is invariant under the rotation around the z axis. Thus due to Theorem 1.5 the ground state is non-split.

When $\lambda = 0$ the one-side restriction of the ground state gives rise to the type III_1 state. To see this, we briefly explain the connection of the ground state of the XY model and that of the massless free Fermion. Consider the automorphism Θ of $\mathcal{A}$ determined by

$$\Theta(\sigma_x^{(j)}) = -\sigma_x^{(j)} \quad \Theta(\sigma_y^{(j)}) = -\sigma_y^{(j)} \quad \Theta(\sigma_z^{(j)}) = \sigma_z^{(j)} \tag{4.6}$$

for any j. As the Hamiltonian H_{XY} of the XY model is invariant under Θ the unique ground state φ is invariant under Θ. Let $\mathcal{A}^{(+)}$ be the fixed point subalgebra of $\mathcal{A}$ under Θ.

$$\mathcal{A}^{(+)} = \{Q \in \mathcal{A} \ \mid \ \Theta(Q) = Q\}$$

Next conisder the CAR (canonical anti-commutation relation) algebra $\mathcal{A}_{CAR}$. It is the algebra generated by Fermion creation -anihilation operators on the lattice $\mathbf{Z}$. Here creation -anihilation operators are denoted by c_j^* and c_j .

$$c_jc_k + c_kc_j = 0 \ , \ c_j^*c_k^* + c_k^*c_j^* = 0 \ , \ c_j^*c_k + c_kc_j^* = \delta_{jk}$$

By abuse of notation we denote the parity automorphism of the CAR algebra with the same symbol Θ.

$$\Theta(c_j) = -c_j \quad , \quad \Theta(c_j^*) = -c_j^*$$

and the even part is denoted by $\mathcal{A}_{CAR}^{(+)}$. We can identify $\mathcal{A}^{(+)}$ and $\mathcal{A}_{CAR}^{(+)}$ via the equation.

$$\sigma_x^{(j)}\sigma_x^{(j+1)} = (c_j - c_j^*)(c_{j+1} + c_{j+1}^*) \quad \sigma_z^{(j)} = 2c_j^*c_j - 1$$

Consider the free Fermion Hamiltonian

$$H_{CAR} = -\sum_{j\in\mathbf{Z}} \left\{ c_j^* c_{j+1} + c_{j+1}^* c_j \right\}$$

The unique ground state of H_{CAR} is the Fock state which we denote ω. The restriction of ω and φ to $\mathcal{A}^{(+)}$ are same.

$$\omega(Q) = \varphi(Q) \qquad \text{for } Q \text{ in } \mathcal{A}^{(+)} .$$

On the other hand the computation of A.Wassermann in [26] implies that the restriction of ω to $\mathcal{A}_R^{(+)}$ gives rise to a type III factor representation. In fact recall that the type of the representation the quasi-free state is determined by the spectrum of the operator associated with two point function. the positive part of the spectrum of QPQ and PQP are same. In the present case, Q is the projection to positive interger subspace in $l^2(\mathbf{Z})$ and P is the projection to the positive energy in the one particle space.

Thus the state φ_R cannot be of type I as its restriction to $\mathcal{A}_R^{(+)}$ is non-type I. This is a proof of the non split property of φ_R

Theorem 4.3 *Let φ be the unique ground state of the XY model H_{XY}. φ_R gives rise to a type III_1 representation.*

Next we consider a generalization of the AKLT model of [3] which was considered in [23] . Here we only give an abstract condition for Hamiltonians. We return to the general nearest neighbor interaction (4.1) .

Assumption 4.4 *(i) We assume that h_j is positive , thus $H_{[n,m]} \geq 0$ for any $[n,m]$.*
(ii) The dimension of the kernel of $H_{[n,m]}$ (the multiplicity of zero eigenvalue of $H_{[n,m]}$ as a finite matrix) is greater than one and uniformly bounded in n and m.

$$1 \leq \sup_{n,m} dim\, ker\, H_{[n,m]} \ < \ \infty \tag{4.7}$$

Note that the condition (i) is not a constraint. It fixes a constant which is physically irrelevant. In the condition (ii) we regard $H_{[n,m]}$ as a d^{m-n+1} by d^{m-n+1} matrix as we are considering the nearest neighbor interaction.

Definition 4.5 *Suppose that $h_j \geq 0$. A state φ of $\mathcal{A}$ is a zero energy state if and only if for any j in* $\mathbf{Z}$

$$\varphi(h_j) = 0 \tag{4.8}$$

Zero energy states may not exist for arbitrary Hamiltonians. The geometric structure of the set of all zero energy states was investigated in [15] . If φ is a zero energy state , it is a ground state. If a zero energy state exists , any translationally invariant ground state is a zero energy state, however, non-translationally zero energy state may exist. For example , the massive ferromagnetic XXZ model has an infinite number of non-translationally invariant zero energy states. (See [15].) The first inequality in (4.7) of Assumption

4.4.(ii) is the existence of the zero energy state. Examples of Hamiltonians with zero energy states are given in [3] , [13]. The ladder case is discussed by A.K.Kolezhuk and H.J. Mikeska in [18]. The main result of the paper [23] is expressed in the present context as follows.

Theorem 4.6 *Suppose that Assumption 4.4 is valid. Let φ be a translationally invariant ground state . Assume that φ is mixing in the sense that*

$$\lim_{|j|\to\infty} \quad \varphi(Q_1\tau_j(Q_2)) \quad = \quad \varphi(Q_1)\varphi(Q_2) \tag{4.9}$$

for any local Q_1 , Q_2 in $\mathcal{A}$.

The state φ is split and pure. In fact , the auxiliary Hilbert space $\mathcal{K}$ in Definition 4.1 is finite dimensional. The two point function decays exponentially fast.

$$\sup_{j\in\mathbf{Z}} |\varphi(Q_1\tau_j(Q_2)) - \varphi(Q_1)\varphi(Q_2)|\, e^{m|j|} \;<\; \infty \tag{4.10}$$

for any local Q_1 and Q_2 .

The above result is a converse to a result of M.Fannes, B.Nachetergaele and R.Werner in [13]. They have shown that if φ is a pure translationally invariant finitely correlated state with the finite dimensional auxiliary Hilbert space $\mathcal{K}$, there exists a projection P in $\mathcal{A}_{loc}$ such that

$$\varphi(P_j) = 0$$

where $P_j = \tau_j(P)$. Thus φ is a ground state for the Hamiltonian $H = \sum_j P_j$.

Finally we present an example for infinite dimensional auxilially space $\mathcal{K}$. Such states can be constructed via the pure state extension of classical Gibbs measures. (See [21] and [22] .) For simplicity , we only consider the spin 1/2 case. Let $\mathcal{B}$ be the abelian C^*-subalgebra generated by all $\sigma_z^{(j)}$. We set $\mathcal{B}_\Lambda = \mathcal{A}_\Lambda \cap \mathcal{B}$. By the pure state extension of a classical Gibbs measure μ we mean a pure state φ of $\mathcal{A}$ such that the restriction of φ to $\mathcal{B}$ is the one-dimensional classical Gibbs measure μ.

To be more specific , we introduce a class of interactions of the spin 1/2 classical spin. Let X be the configuration space of the classical spin on $\mathbf{Z}$.

$$X = \prod_{\mathbf{Z}} \{1, -1\}$$

A spin configuration will be denoted by σ , $\sigma \in X$. The projection of the spin configuration σ to the j th component is denoted by $\sigma^{(j)}$ The space $C(X)$ of continuous functions on X can be naturally identified with $\mathcal{B}$ and we identify $\sigma^{(j)} = \sigma_z^{(j)}$. Any state of $C(X)$ is a Radon measure on X. Let A be a finite subset of $\mathbf{Z}$. We set

$$\sigma_\alpha(A) = \prod_{k\in A} \sigma_\alpha^{(k)} \quad \alpha = x, y, z$$

By interaction we mean a map J_A from finite subsets A of $\mathbf{Z}$ to $\mathbf{R}$.

Assumption 4.7 *We assume that the interaction J_A is translationally invariant $J_A = J_{A+k}$ for any k in $\mathbf{Z}$ such that*

$$\sum_{A\cap(-\infty,-1]\neq\emptyset \ A\cap[0,\infty)\neq\emptyset} |J_A| \quad < \quad \infty \tag{4.11}$$

Given a translationally invariant interaction J_A satisfying (4.11) the Gibbs measure associated to J_A is a measure μ formally determined by

$$\int d\mu(\sigma) f(\sigma) = \frac{1}{Z} \int \prod_{j\in\mathbf{Z}} d\sigma^{(j)} e^{-H(\sigma)} f(\sigma) \tag{4.12}$$

where $f(\sigma)$ is a function in $C(X)$ and Z is the (infinite) normalization constant and $-H(\sigma)$ is the total energy for the configuration σ

$$H(\sigma) = \sum_A J_A \sigma(A)$$

For the infinite system the above expression is divergent and one way to characterize the Gibbs measure μ is the following equation

$$\frac{d\mu(\sigma_j)}{d\mu(\sigma)} = exp(2 \sum_{j:j\in A} J_A \sigma(A)) \tag{4.13}$$

where σ_j is the configuration obtained by the spin flip at site j and the lefthand side $\frac{d\mu(\sigma_j)}{d\mu(\sigma)}$ is the Radon Nikodym derivative.

In [21] we considered the pure state φ_μ determined by

$$\varphi_\mu(\sigma_z(A)\sigma_x(B)) = \int d\mu(\sigma)\sigma(A) exp(\sum_{|B\cap C|:odd} J_B \sigma(B)) \tag{4.14}$$

Theorem 4.8 *Let J_A be a translationally invariant interaction J_A satisfying (4.11) and μ be the (unique) Gibbs measure associated with J_A. The equation (4.14) gives rise to a translationally invariant pure state φ_μ. The state φ_μ is split.*

Sketch of Proof
The assumption (4.11) is the bounded surface energy condition which leads to uniqueness of the Gibbs measure. When the Gibbs measure is unique , the state φ_μ is pure . (See [21].)

The split property of φ_μ follows from that of the measure μ. In fact set

$$V = exp(\sum_{A\cap(-\infty,-1]\neq\emptyset \ A\cap[0,\infty)\neq\emptyset} J_A \sigma(A))$$

There exists states ψ_L of $\mathcal{A}_L$ and ψ_R of $\mathcal{A}_R$ such that

$$\frac{1}{\varphi_\mu(V)} \varphi_\mu(V^{\frac{1}{2}} Q V^{\frac{1}{2}}) = \psi_L \otimes \psi_R(Q) \tag{4.15}$$

for any Q in $\mathcal{A}$. This is more or less obvious from the formal expression (4.12) of the Gibbs measure μ. ∎

Remark 4.9 *(i) We have shown that the pure split state in Theorem 4.8. is a ground state of a suitable (possibly infinite range) Hamiltonian in [21]. For example, consider the XYZ model with Ising like anisotropy.*

$$H_{XYZ} = -\sum_j \{\sigma_x^{(j)}\sigma_x^{(j+1)} + \delta_1\sigma_y^{(j)}\sigma_y^{(j+1)} + \delta_2\sigma_z^{(j)}\sigma_z^{(j+1)}\}$$

with small $|\delta_1|$ and $|\delta_2|$. There exists two pure translationally invariant ground states for H_{XYZ} and one of them is realized as φ_μ for a suitable interaction and the other is $\varphi_\mu \circ \Theta$ where Θ is an automorphism of $\mathcal{A}$ determined by (4.6) . The fact that an infinite range classical interaction enters in the ground state of a quantum finite range Hamiltonians is not surprise. In general one-dimensional quantum models are equivalent to two dimensional classical spin models but at the level of correlation functions, they are obtained via projection of two dimensional Gibbs measures (of classical finite range interactions) to one dimensional line. The situation of previous Theorem 4. 8. is that the projected measure is still a Gibbs measure of another infinite range classical interaction on the one dimensional lattice.
(ii) A folklore is that the decay rate of two point correlation functions of the Gibbs measure for some long range interaction is not exponential while the state of Theorem 4.6 exhibits the exact exponential decay of correlation so we obtain the example of finitely correlated states with the infinite dimensional auxiliary space $\mathcal{K}$.

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