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Continuous integral kernels for unbounded Schrödinger semigroups and their spectral projections

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By suitably extending a Feynman-Kac formula of Simon [Canadian Math. Soc. Conf. Proc. **28** (2000), 317–321], we study one-parameter semigroups generated by (the negative of) rather general Schrödinger operators, which may be unbounded from below and include a magnetic vector potential. In particular, a common domain of essential self-adjointness for such a semigroup is specified. Moreover, each member of the semigroup is proven to be a maximal Carleman operator with a continuous integral kernel given by a Brownian-bridge expectation. The results are used to show that the spectral projections of the generating Schrödinger operator also act as Carleman operators with continuous integral kernels. Applications to Schrödinger operators with rather general random scalar potentials include a rigorous justification of an integral-kernel representation of their integrated density of states – a relation frequently used in the physics literature on disordered solids.

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CONTENTS

0. INTRODUCTION

In non-relativistic quantum physics [19, 20, 47] a spinless (charged) particle with d-dimensional Euclidean configuration space $\mathbb{R}^d$, which is subjected to a scalar potential V, as well as to a magnetic field derived from a vector potential A, is characterized by a Schrödinger operator $H \equiv H(A, V)$. The latter is a linear, self-adjoint, second-order partial-differential operator acting on a dense domain in the Hilbert space $\mathrm{L}^2(\mathbb{R}^d)$ of Lebesgue square-integrable functions ψ on $\mathbb{R}^d$ [14, 7]. The spectrum of H corresponds physically to the possible values $E \in \mathbb{R}$ of the particle's energy. Useful information on a given Schrödinger operator H can be obtained by studying its semigroup $\{\mathrm{e}^{-tH}\}_{t\geq 0}$. As was convincingly demonstrated by Carmona [11] and Simon [40, 42], this, in turn, can be done very efficiently by using the Feynman-Kac(-Itô) formula [40, 13, 46, 10], which provides a probabilistic representation of $\mathrm{e}^{-tH}\psi$ in terms of a Brownian-motion expectation. Until present, the most systematic study along these lines is that of Simon [42]. It covers mostly situations without a magnetic field and where the scalar potential V is assumed to be Kato decomposable. The latter assumption assures in particular that the operator H is bounded from below and, hence, that $\{\mathrm{e}^{-tH}\}_{t\geq 0}$ is a family of bounded operators. Part of the regularity results in [42] were recently generalized to allow for rather general magnetic fields and an arbitrary open subset of $\mathbb{R}^d$ as the configuration space [10]. For additional regularity results see [23].

Some physically interesting situations, however, are modelled by scalar potentials which are not Kato decomposable and lead to Schrödinger operators that are unbounded from below. Here we only mention the Stark effect of atoms, electronic properties of disordered solids and the physically different, but mathematically closely related problem of classical diffusion in random media. For the first situation one uses a scalar potential with a term linear in the position [4, 14], and for the latter two situations the realizations of a suitable random scalar potential [29, 12, 36, 32, 46, 21, 22]. Gaussian random potentials are very popular examples thereof in the physics literature on disordered systems [39, 17, 33]. Since H is unbounded from below in these cases, the associated Schrödinger semigroup $\{\mathrm{e}^{-tH}\}_{t\geq 0}$ consists of unbounded operators. Among other things, the unboundedness of the operator exponentials e^{-tH} brings up new kinds of questions concerning domains, common cores for different t, etc. In fact, there are interesting analytic results on semigroups of unbounded linear operators even on abstract Hilbert and Banach spaces for more than two decades [35, 25, 18, 31] (see also Thm. 4.9 in [15]). However, it was only recently that Simon [43] singled out a maximal class of negative scalar potentials such that H is unbounded from below, but given an arbitrarily large (time) parameter $t > 0$ the operator exponential e^{-tH} still acts as an integral operator on functions ψ, which have sufficiently fast decay at infinity, and $\mathrm{e}^{-tH}\psi$ is given by a Feynman-Kac formula.

The present paper is in the spirit of Simon's note [43]. By suitably extending his Feynman-Kac formula we aim to achieve a better understanding of rather general unbounded Schrödinger semigroups $\{\mathrm{e}^{-tH}\}_{t\geq 0}$ on $\mathrm{L}^2(\mathbb{R}^d)$, which have remained widely unexplored up to now. To this end we consider a large class of scalar potentials which allows for the same fall-off towards minus infinity at infinity as was considered in [43]. In addition, the presence of rather general magnetic fields is admitted. Under these assumptions, we prove continuity of the Feynman-Kac-Itô integral kernel k_t of e^{-tH} and of the image function $\mathrm{e}^{-tH}\psi$, provided that $t > 0$ and ψ has sufficiently fast decay at infinity. Moreover, we extend the Feynman-Kac-Itô representation of $\mathrm{e}^{-tH}\psi$ to all ψ in the domain of the possibly unbounded operator e^{-tH}. This yields an alternative characterization of its domain and renders e^{-tH} the maximal Carleman operator induced by the integral kernel k_t. A theorem of Nussbaum [35] is applied to identify a common operator core for e^{-tH} for all $t \geq 0$. Lemma 1.7 and Theorem 1.10 summarize these results. Semigroup properties of the family $\{\mathrm{e}^{-tH}\}_{t\geq 0}$ are compiled in Theorem 1.12. Similar to Thm. B.7.8 in [42], we infer in Theorem 1.14 the existence and continuity of integral kernels for certain bounded functions of H, thereby allowing one to evaluate related traces in terms of integral kernels. In particular, all this is true for any spectral-projection operator $\chi_I(H)$ of H associated with a Borel set $I \subset \mathbb{R}$ which is bounded from above, see Corollary 1.16. Finally, the functional calculus is extended to integral kernels in Corollary 1.18. Applications to Schrödinger operators with rather general random scalar potentials yield a rigorous justification of some statements which are frequently used in the physics literature on disordered systems. Corollary 1.27 delivers an integral-kernel representation of the integrated density of states and Corollary 1.29, respectively its particularization to Gaussian random scalar potentials in Corollary 1.31, concerns properties of the integral kernel of the averaged semigroup.

The paper is organized as follows. Section 1 contains the basic notions, the precise formulations of the results mentioned in the previous paragraph and various comments. Sections 2 to 5 are devoted to the proofs.

1. Results and Comments

1.1. *Basic Notation and Definitions*

As usual, let $\mathbb{N} := \{1, 2, 3, \ldots\}$ denote the set of natural numbers. Let $\mathbb{R}$, respectively $\mathbb{C}$, denote the algebraic field of real, respectively complex numbers and let $\mathbb{Z}^d$ be the simple cubic unit-lattice in d dimensions, $d \in \mathbb{N}$. We fix a Cartesian co-ordinate system in d-dimensional Euclidean space $\mathbb{R}^d$ and define an open cube in $\mathbb{R}^d$ as a translate of the d-fold Cartesian product $I \times \ldots \times I$ of an open interval $I \subseteq \mathbb{R}$. In particular, $\Lambda_\ell(x)$ stands for the open cube in $\mathbb{R}^d$ with edge length $\ell > 0$ and centre $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. The Euclidean scalar product $x \cdot y := \sum_{j=1}^d x_j y_j$ of $x, y \in \mathbb{R}^d$ induces the Euclidean norm $|x| := (x \cdot x)^{1/2}$.

We denote the volume of a Borel subset $\Lambda \subseteq \mathbb{R}^d$ with respect to the d-dimensional Lebesgue measure as $|\Lambda| := \int_\Lambda \mathrm{d}x = \int_{\mathbb{R}^d} \mathrm{d}x\, \chi_\Lambda(x)$, where χ_Λ stands for the indicator function of Λ. In particular, if Λ is the strictly positive half-line, $\Theta := \chi_{]0,\infty[}$ denotes the left-continuous Heaviside unit-step function.

The Banach space $\mathrm{L}^p(\mathbb{R}^d)$, $p \in [1, \infty]$, consists of all Borel-measurable complex-valued functions $f : \mathbb{R}^d \to \mathbb{C}$ which are identified if their values differ only on a set of Lebesgue measure zero and which possess a finite norm $\|f\|_p := \left(\int_{\mathbb{R}^d} \mathrm{d}x\, |f(x)|^p\right)^{1/p} < \infty$, if $p < \infty$, and $\|f\|_\infty := \operatorname{ess\,sup}_{x \in \mathbb{R}^d} |f(x)| < \infty$, if $p = \infty$. We recall that $\mathrm{L}^2(\mathbb{R}^d)$ is a separable Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ given by $\langle f, g \rangle := \int_{\mathbb{R}^d} \mathrm{d}x\, f^*(x)\, g(x)$. Here the star denotes complex conjugation and the function f^* is defined pointwise by $f^*(x) := (f(x))^*$. We write $f \in \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^d)$, if $f \chi_\Lambda \in \mathrm{L}^p(\mathbb{R}^d)$ for any bounded Borel set $\Lambda \subset \mathbb{R}^d$. The uniform local Lebesgue spaces $\mathrm{L}^p_{\mathrm{unif,loc}}(\mathbb{R}^d)$ consist of all those $f \in \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^d)$ for which $\sup_{x \in \mathbb{Z}^d} \|f \chi_{\Lambda_1(x)}\|_p < \infty$. The Kato class [28, 2, 48, 23] over $\mathbb{R}^d$ may be defined as the vector space $\mathscr{K}(\mathbb{R}^d) := \left\{ f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d) : \lim_{t \downarrow 0} \varkappa_t(f) = 0 \right\}$, where $\varkappa_t(f) := \sup_{x \in \mathbb{R}^d} \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}\xi\, \mathrm{e}^{-|\xi|^2} |f(x + \xi\sqrt{s})|$. It obeys the inclusion $\mathscr{K}(\mathbb{R}^d) \subseteq \mathrm{L}^1_{\mathrm{unif,loc}}(\mathbb{R}^d)$ with equality if $d = 1$. We say that f belongs to $\mathscr{K}_{\mathrm{loc}}(\mathbb{R}^d)$, if $f \chi_\Lambda \in \mathscr{K}(\mathbb{R}^d)$ for any bounded Borel set $\Lambda \subset \mathbb{R}^d$. Moreover, f is called Kato decomposable, in symbols $f \in \mathscr{K}_\pm(\mathbb{R}^d)$, if $\sup\{0, f\} \in \mathscr{K}_{\mathrm{loc}}(\mathbb{R}^d)$ and $\sup\{0, -f\} \in \mathscr{K}(\mathbb{R}^d)$. Finally, $\mathscr{C}_0^\infty(\mathbb{R}^d)$ is the vector space of all functions $f : \mathbb{R}^d \to \mathbb{C}$ which are arbitrarily often differentiable and have compact supports $\operatorname{supp} f$.

The absolute value of a closed operator $F : \operatorname{dom}(F) \to \mathrm{L}^2(\mathbb{R}^d)$, with dense domain of definition $\operatorname{dom}(F) \subseteq \mathrm{L}^2(\mathbb{R}^d)$ and Hilbert adjoint F^*, is the positive operator $|F| := (F^*F)^{1/2}$. The (uniform) norm of a bounded operator $F : \mathrm{L}^2(\mathbb{R}^d) \to \mathrm{L}^2(\mathbb{R}^d)$ is defined as $\|F\| := \sup\big\{ \|Ff\|_2 \, : \, f \in \mathrm{L}^2(\mathbb{R}^d)\,,\, \|f\|_2 = 1 \big\}$.

DEFINITION 1.1. Let $d \in \mathbb{N}$. A *vector potential* A is a Borel-measurable, $\mathbb{R}^d$-valued function on $\mathbb{R}^d$ and a *scalar potential* V is a Borel-measurable, $\mathbb{R}$-valued function on $\mathbb{R}^d$. Furthermore,

(A) a vector potential A is said to satisfy property (A), if both its square $|A|^2$ and its divergence $\nabla \cdot A$ lie in the intersection $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d) \cap \mathscr{K}_{\mathrm{loc}}(\mathbb{R}^d)$. Here, $\nabla = (\partial_1, \ldots, \partial_d)$ stands for the gradient, which is supposed to act in the sense of distributions on $\mathscr{C}_0^\infty(\mathbb{R}^d)$.

(C) a vector potential A is said to satisfy property (C), if there exist real constants $B_{jk} = -B_{kj}$, where $j, k \in \{1, \ldots, d\}$, such that

$$A_k(x) = \frac{1}{2} \sum_{j=1}^{d} x_j \, B_{jk} \tag{1.1}$$

for all $x \in \mathbb{R}^d$ and all $k \in \{1, \ldots, d\}$. In other words, A generates a spatially constant magnetic field given by the skew-symmetric $d \times d$-matrix with entries $B_{jk} = \partial_j A_k - \partial_k A_j$.

(V) a scalar potential V is said to satisfy property (V), if it can be written as a sum

$$V = V_1 + V_2 \tag{1.2}$$

with V_1 being locally square-integrable and Kato decomposable,

$$V_1 \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d) \cap \mathscr{K}_{\pm}(\mathbb{R}^d)\,, \tag{1.3}$$

and V_2 obeying a sub-quadratic growth limitation in the following sense: for every $\varepsilon > 0$ there exists a finite constant $v_\varepsilon > 0$ such that

$$|V_2(x)| \leq \varepsilon |x|^2 + v_\varepsilon \tag{1.4}$$

for Lebesgue-almost all $x \in \mathbb{R}^d$.

Remarks 1.2. (i) For one space dimension, $d = 1$, there is no loss of generality in assuming $A = 0$ on account of gauge equivalence.

(ii) If $d \leq 3$, then $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d) \subseteq \mathscr{K}_{\mathrm{loc}}(\mathbb{R}^d)$.

(iii) Due to gauge equivalence we have contented ourselves in formulating the constant-magnetic-field condition (C) in the Poincaré gauge (1.1).

(iv) Property (C) implies property (A).

(v) Property (V) allows for a larger class of potentials than those considered in [43]. This is because (V) requires weaker local regularity properties. Yet, the crucial sub-quadratic growth limitation of $V(x)$ towards minus infinity as $|x| \to \infty$ is identical.

(vi) Even though a quadratic growth limitation instead of the stronger condition (1.4) would still yield a self-adjoint Schrödinger semigroup, we do not consider such situations, because the corresponding Feynman-Kac(-Itô) formula would not hold for an arbitrarily large time parameter t, cf. Sect. 5.13 in [27].

We base the definition of Schrödinger operators on the following proposition, whose proof is an application of Thm. 2.5 in [24].

PROPOSITION 1.3. *Let A be a vector potential with property* (A) *and let V be a scalar potential with property* (V). *Then the differential operator*

$$\mathscr{C}_0^\infty(\mathbb{R}^d) \ni \varphi \mapsto \frac{1}{2}\sum_{j=1}^{d}(\mathrm{i}\partial_j + \hat{A}_j)^2\varphi + \hat{V}\varphi \tag{1.5}$$

is essentially self-adjoint on $\mathrm{L}^2(\mathbb{R}^d)$. *Here* $\mathrm{i} = \sqrt{-1}$ *denotes the imaginary unit and a superposed hat on a function indicates the corresponding multiplication operator.*

DEFINITION 1.4. The self-adjoint closure of (1.5) on $\mathrm{L}^2(\mathbb{R}^d)$ is called the *(magnetic) Schrödinger operator* and denoted by $H(A, V)$.

As suggested in [43], we introduce vector spaces of $\mathrm{L}^p(\mathbb{R}^d)$-functions with a decay at infinity which is faster than that of some Gaussian function. These spaces are tailored for the, in general, unbounded Schrödinger semigroup $\{\mathrm{e}^{-tH(A,V)}\}_{t\geq 0}$ with V having property (V).

DEFINITION 1.5. For each $p \in [1, \infty]$ we set

$$\mathrm{L}^p_{\mathrm{G}}(\mathbb{R}^d) := \Big\{ \psi \in \mathrm{L}^p(\mathbb{R}^d) : \text{ there exists } \rho \in]0, \infty[\text{ such that } \int_{\mathbb{R}^d} \mathrm{d}x\, \mathrm{e}^{\rho|x|^2}\, |\psi(x)|^p < \infty \Big\}. \tag{1.6}$$

Remarks 1.6. (i) Hölder's inequality yields the chain of inclusions

$$\mathrm{L}^\infty_{\mathrm{G}}(\mathbb{R}^d) \subseteq \mathrm{L}^q_{\mathrm{G}}(\mathbb{R}^d) \subseteq \mathrm{L}^p_{\mathrm{G}}(\mathbb{R}^d) \subseteq \mathrm{L}^1_{\mathrm{G}}(\mathbb{R}^d)\,, \tag{1.7}$$

if $1 \leq p \leq q \leq \infty$.

(ii) The space $\mathrm{L}^p_{\mathrm{G}}(\mathbb{R}^d)$ is dense in $\mathrm{L}^p(\mathbb{R}^d)$ for any $p \in [1, \infty]$ thanks to the inclusion

$$\mathscr{C}_0^\infty(\mathbb{R}^d) \subset \mathrm{L}^p_{\mathrm{G}}(\mathbb{R}^d)\,. \tag{1.8}$$

1.2. *Continuous integral kernels for unbounded Schrödinger semigroups and their spectral projections*

As a preparation for the Feynman-Kac-Itô formula (1.17) in Theorem 1.10 below we need to recall the *Brownian bridge* in $\mathbb{R}^d$ associated with the starting point $x \in \mathbb{R}^d$, the endpoint $y \in \mathbb{R}^d$ and the closed time interval $[0, t]$, where $t > 0$ is fixed but arbitrary. It may be defined as the $\mathbb{R}^d$-valued stochastic process whose d Cartesian components are independent and have continuous realizations $[0, t] \ni s \mapsto b_j(s) \in \mathbb{R}$, $j \in \{1, \dots, d\}$. Moreover, the j-th component b_j is distributed according to the Gaussian probability measure characterized by the mean function $[0, t] \ni s \mapsto x_j + (y_j - x_j)s/t$ and the covariance function $[0, t] \times [0, t] \ni (s, s') \mapsto \min\{s, s'\} - ss'/t$, see e.g. [40, 37, 46]. We denote the joint (product) probability measure of $b := (b_1, \dots, b_d)$ by $\mu^{0,t}_{x,y}$. Given $t > 0$, a vector potential A with property (A) and a scalar potential V with property (V), then the *Euclidean action functional*

$$S_t(A, V; b) := \mathrm{i} \int_0^t \mathrm{d}b(s) \cdot A(b(s)) + \frac{\mathrm{i}}{2} \int_0^t \mathrm{d}s\ (\nabla \cdot A)(b(s)) + \int_0^t \mathrm{d}s\ V(b(s)) \tag{1.9}$$

associated with these potentials is well defined for $\mu^{0,t}_{x,y}$-almost all paths b of the Brownian bridge. The first integral on the right-hand side of (1.9) is a stochastic line integral to be understood in the sense of Itô. The other two integrals with random integrands are meant in the sense of Lebesgue. The $\mu^{0,t}_{x,y}$-almost-sure existence of the integrals in (1.9) follows e.g. from Sects. 2 and 6 in [10] and the estimate

$$\int \mu^{0,t}_{x,y}(\mathrm{d}b)\ \left| \int_0^t \mathrm{d}s\ V_2(b(s)) \right| \le t v_\varepsilon + \varepsilon \int_0^t \mathrm{d}s \int \mu^{0,t}_{x,y}(\mathrm{d}b)\ |b(s)|^2 < \infty\,. \tag{1.10}$$

The latter is valid for all $\varepsilon > 0$ and relies on (1.4), Fubini's theorem and an explicit computation. As to the applicability of (1.4) in this estimate, we have used the basic fact that for $\mu^{0,t}_{x,y}$-almost every path b of the Brownian bridge the set $\{s \in [0, t] : b(s) \in \Lambda\}$ of time instances, for which b stays in a given Lebesgue-null set $\Lambda \subset \mathbb{R}^d$, is itself of Lebesgue measure zero in $[0, t]$, that is, $\int_0^t \mathrm{d}s\ \chi_\Lambda\big(b(s)\big) = 0$. We will make use of this fact in the following without further notice.

LEMMA 1.7. *Let A be a vector potential with property* (A) *and let V be a scalar potential with property* (V). *Finally, let $t > 0$. Then*

(i) *the function $k_t : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$, $(x, y) \mapsto k_t(x, y)$, where*

$$k_t(x, y) := \frac{\mathrm{e}^{-|x-y|^2/(2t)}}{(2\pi t)^{d/2}} \int \mu_{x,y}^{0,t}(\mathrm{d}b)\, \mathrm{e}^{-S_t(A,V;b)}, \tag{1.11}$$

is well defined in terms of a Brownian-bridge expectation, Hermitian in the sense that $k_t(x, y) = k_t^*(y, x)$ for all $x, y \in \mathbb{R}^d$, continuous and obeys the semigroup property

$$k_{t+t'}(x, z) = \int_{\mathbb{R}^d} \mathrm{d}y\, k_t(x, y)\, k_{t'}(y, z) \tag{1.12}$$

for all $x, z \in \mathbb{R}^d$ and all $t' > 0$.

(ii) *for every $\delta > 0$ there exists a finite constant $a_t^{(\delta)} > 0$, independent of $x, y \in \mathbb{R}^d$, such that the estimate*

$$|k_t(x, y)| \leq a_t^{(\delta)} \exp\left\{-\frac{|x-y|^2}{4t} + \delta|x|^2 + \delta|y|^2\right\} \tag{1.13}$$

holds for all $x, y \in \mathbb{R}^d$.

(iii) *the function k_t obeys*

$$k_t(x, \cdot) \in \mathrm{L}^\infty_{\mathrm{G}}(\mathbb{R}^d) \qquad \text{for all } x \in \mathbb{R}^d \tag{1.14}$$

and thus has the Carleman property (1.15) *below. Moreover, the mapping $\mathbb{R}^d \to \mathrm{L}^2(\mathbb{R}^d)$, $x \mapsto k_t(x, \cdot)$ is strongly continuous.*

Remarks 1.8. (i) The lemma is proven in Section 2.

(ii) Concerning the asserted continuity of k_t, the proof will even show that the function $]0, \infty[\times\mathbb{R}^d \times \mathbb{R}^d \ni (t, x, y) \mapsto k_t(x, y)$ is continuous.

(iii) The estimate (1.13) corresponds to Thm. 2.1 in [43].

(iv) Part (iii) of Lemma 1.7 continues to hold with $k_t(x, \cdot)$ replaced by $k_t(\cdot, x)$ thanks to the Hermiticity of k_t (for *all* $x, y \in \mathbb{R}^d$).

(v) While (1.14) follows (directly) from the estimate (1.13), the weaker *Carleman property* of k_t,

$$k_t(x, \cdot) \in \mathrm{L}^2(\mathbb{R}^d) \quad \text{for Lebesgue-almost all } x \in \mathbb{R}^d, \tag{1.15}$$

is already a consequence of the semigroup property, the Hermiticity and the continuity of k_t.

DEFINITION 1.9. Let $H(A,V)$ be the Schrödinger operator of Definition 1.4 and let $t \in \mathbb{R}$. Then the operator exponential $\mathrm{e}^{-tH(A,V)}$ is densely defined, self-adjoint and positive by the spectral theorem and the functional calculus for unbounded functions of unbounded self-adjoint operators (see e.g. Chap. 5 in [7]).

We are now in a position to give a probabilistic representation of $\mathrm{e}^{-tH(A,V)}$ by a Feynman-Kac-Itô formula.

THEOREM 1.10. *Let A be a vector potential with property* (A) *and let V be a scalar potential with property* (V). *Moreover, let $t > 0$ and let $\mathrm{e}^{-tH(A,V)}$ be given by Definition* 1.9. *Then*

(i) *the domain of $\mathrm{e}^{-tH(A,V)}$ is given by*

$$\mathrm{dom}\big(\mathrm{e}^{-tH(A,V)}\big) = \Big\{\psi \in \mathrm{L}^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} \mathrm{d}y\, k_t(\cdot, y)\,\psi(y) \in \mathrm{L}^2(\mathbb{R}^d)\Big\} \tag{1.16}$$

with k_t defined in (1.11). *Moreover, $\mathrm{L}^2_{\mathrm{G}}(\mathbb{R}^d) \subseteq \mathrm{dom}\big(\mathrm{e}^{-tH(A,V)}\big)$ is an operator core for $\mathrm{e}^{-tH(A,V)}$.*

(ii) *$\mathrm{e}^{-tH(A,V)}$ is the maximal Carleman operator induced by the continuous integral kernel* (1.11) *in the sense that*

$$\mathrm{e}^{-tH(A,V)}\psi = \int_{\mathbb{R}^d} \mathrm{d}y\, k_t(\cdot, y)\,\psi(y) \tag{1.17}$$

for all $\psi \in \mathrm{dom}\big(\mathrm{e}^{-tH(A,V)}\big)$ and that k_t has the Carleman property (1.15).

(iii) *the image $\mathrm{e}^{-tH(A,V)}\psi$ of any $\psi \in \mathrm{dom}\big(\mathrm{e}^{-tH(A,V)}\big)$ has a continuous representative in $\mathrm{L}^2(\mathbb{R}^d)$ given by the right-hand side of* (1.17). *If even $\psi \in \mathrm{L}^2_{\mathrm{G}}(\mathbb{R}^d)$, then, in addition, $\mathrm{e}^{-tH(A,V)}\psi \in \mathrm{L}^\infty_{\mathrm{G}}(\mathbb{R}^d)$.*

Remarks 1.11. (i) The proof of Theorem 1.10 is deferred to Section 3.

(ii) For the theory of Carleman operators we refer to [45, 3, 49]. We follow mostly the terminology and conventions of [49].

(iii) The right-hand side of (1.17) maps even any $\psi \in \mathrm{L}^1_{\mathrm{G}}(\mathbb{R}^d)$ (and hence any $\psi \in \mathrm{L}^p_{\mathrm{G}}(\mathbb{R}^d)$ for all $p \in [1, \infty]$) to an element of $\mathrm{L}^\infty_{\mathrm{G}}(\mathbb{R}^d)$. This fact is well known for the free case $A = 0$ and $V = 0$. It extends to the general situation of Theorem 1.10 simply by the basic estimate (1.13).

(iv) Theorem 1.10 extends the main result of [43], where the Feynman-Kac-Itô formula (1.17) was proven for $A = 0$ and $\psi \in \mathrm{L}^2_{\mathrm{G}}(\mathbb{R}^d)$ under somewhat more restrictive assumptions on the scalar potential V, see Remark 1.2(v).

(v) If $V_2 = 0$, then the scalar potential $V = V_1$ is Kato decomposable and $H(A, V_1)$ therefore bounded from below. Regularity properties of the associated *bounded* Schrödinger semigroup $\{e^{-tH(A,V_1)}\}_{t\geq 0}$ are well known and have been studied in great detail, see the seminal paper [42] and [23] for the non-magnetic case $A = 0$. Part of these results were extended to situations with rather general vector potentials in [10].

So far we have been concerned with the (possibly unbounded) operator exponential $e^{-tH(A,V)}$ for a fixed but arbitrary time parameter $t \in]0, \infty[$. Next we compile some semigroup properties of the family $\{e^{-tH(A,V)}\}_{t\geq 0}$.

THEOREM 1.12. *Assume the situation of Theorem* 1.10. *Then the family* $\{e^{-tH(A,V)}\}_{t\geq 0}$ *is a strongly continuous (one-parameter) semigroup of self-adjoint operators generated by the Schrödinger operator* $H(A, V)$ *in the following sense:*

(i) *the semigroup law*

$$e^{-(t+t')H(A,V)}\psi = e^{-tH(A,V)}\, e^{-t'H(A,V)}\psi \tag{1.18}$$

holds for all $t, t' \in [0, \infty[$ *and all* $\psi \in L^2_G(\mathbb{R}^d)$.

(ii) *the orbit mapping* $u_\psi : [0, \infty[\to L^2(\mathbb{R}^d)$, $t \mapsto u_\psi(t) := e^{-tH(A,V)}\psi$ *is strongly continuous (at* $t = 0$ *only from the right) for all* $\psi \in L^2_G(\mathbb{R}^d)$.

(iii) *for every* $\varphi \in \mathscr{C}_0^\infty(\mathbb{R}^d)$ *the orbit mapping* u_φ *is strongly differentiable (at* $t = 0$ *only from the right) and the unique solution of the linear initial-value problem*

$$\frac{d}{dt}\Phi(t) = -H(A, V)\Phi(t)\,, \qquad \Phi(0) = \varphi\,, \tag{1.19}$$

for a strongly differentiable (at $t = 0$ *only from the right) mapping* $\Phi : [0, \infty[\to \mathrm{dom}\big(H(A, V)\big)$, $t \mapsto \Phi(t)$.

Remarks 1.13. (i) The proof of Theorem 1.12 is given in Section 3.

(ii) Interesting analytic results on semigroups of unbounded operators on abstract Hilbert and Banach spaces were previously obtained in e.g. [35, 25, 18, 31].

In many situations it is useful to know that not only $e^{-tH(A,V)}$ has a continuous integral kernel but also certain bounded functions of $H(A, V)$.

THEOREM 1.14. *Assume the situation of Theorem* 1.10 *and let* $F \in L^\infty(\mathbb{R})$ *be a bounded function with an at least exponentially fast decay at plus infinity in the sense that the inequality*

$$|F(E)| \leq \gamma \min\{1, e^{-\tau E}\} \tag{1.20}$$

holds for Lebesgue-almost all $E \in \mathbb{R}$ *with some constants* $\gamma, \tau \in]0, \infty[$. *Furthermore, let* $F\big(H(A, V)\big)$ *be defined by the spectral theorem and the functional calculus. Then*

(i) $F\big(H(A, V)\big)$ *is a bounded Carleman operator induced by the continuous integral kernel* $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$, $(x, y) \mapsto f(x, y)$, *where*

$$f(x, y) := \big\langle k_t(\cdot, x), \mathrm{e}^{2tH(A,V)} F\big(H(A, V)\big) k_t(\cdot, y)\big\rangle \tag{1.21}$$

with arbitrary $t \in]0, \tau/2[$, *in the sense that*

$$F\big(H(A, V)\big)\psi = \int_{\mathbb{R}^d} \mathrm{d}y \; f(\cdot, y)\, \psi(y) \tag{1.22}$$

for all $\psi \in \mathrm{L}^2(\mathbb{R}^d)$ *and that* f *has the Carleman property* (1.15).

(ii) *the left-hand side of* (1.22) *has a continuous representative in* $\mathrm{L}^2(\mathbb{R}^d)$, *which is given by the right-hand side of* (1.22).

(iii) *for every* $w \in \mathrm{L}^\infty_{\mathrm{G}}(\mathbb{R}^d)$ *the product* $F\big(H(A, V)\big)\hat{w}$ *is a Hilbert-Schmidt operator with squared norm given by*

$$\mathrm{Trace}\big\{\hat{w}^* \big|F\big(H(A, V)\big)\big|^2 \hat{w}\big\} = \int_{\mathbb{R}^d} \mathrm{d}x \; |w(x)|^2 \int_{\mathbb{R}^d} \mathrm{d}y \; |f(x, y)|^2 . \tag{1.23}$$

Here $\hat{w}$ *denotes the bounded multiplication operator uniquely corresponding to* w, *and* $\hat{w}^*$ *denotes its Hilbert adjoint.*

Remarks 1.15. (i) The right-hand side of (1.21) is well defined and continuous in $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$ by Lemma 1.7(iii), Remark 1.8(iv), the boundedness of $\mathrm{e}^{2tH(A,V)} F\big(H(A, V)\big)$ and the continuity of the $\mathrm{L}^2(\mathbb{R}^d)$-scalar product $\langle \cdot, \cdot \rangle$. Moreover, (1.21) is independent of the chosen $t \in]0, \tau/2[$.

(ii) The proof of Theorem 1.14 is given in Section 4 and rests on a more general result, which is formulated as Lemma 4.1. This lemma is in the spirit of Thm. B.7.8 in [42], but, among others, we have relaxed a boundedness assumption in a suitable way. Theorem 1.14 itself may be viewed as a generalization of Thm. B.7.1(d) in [42] from Kato-decomposable scalar potentials to ones with property (V) and to vector potentials with property (A). But, whereas Thm. B.7.1(d) in [42] relies on resolvent techniques and requires the power-law decay $|F(E)| \leq \mathrm{const.}(1 + |E|)^{-\alpha}$ with $\alpha > d/2$ for energies E in the spectrum of H, we work with the semigroup and thus need the decay property (1.20).

COROLLARY 1.16. *Assume the situation of Theorem* 1.14 *and let* $I \subset \mathbb{R}$ *be a Borel set in the real line which is bounded from above,* $\sup I < \infty$. *Then Theorem* 1.14 *holds with* $F = \chi_I$, *that is, for the spectral projection* $\chi_I\big(H(A, V)\big)$

associated with the energy regime I of the Schrödinger operator $H(A, V)$. Denoting the corresponding continuous integral kernel (1.21) *by p_I, Eq.* (1.23) *takes the form*

$$\operatorname{Trace}\big[\hat{w}^* \chi_I\big(H(A,V)\big)\hat{w}\big] = \int_{\mathbb{R}^d} \mathrm{d}x\, |w(x)|^2\, p_I(x,x) \tag{1.24}$$

for all $w \in \mathrm{L}^\infty_{\mathrm{G}}(\mathbb{R}^d)$.

Remark 1.17. The proof of Corollary 1.16 is given in Section 4.

Finally, we note that the functional calculus extends to integral kernels.

COROLLARY 1.18. *Assume the situation of Theorem* 1.14. *Then*

$$f(x,y) = \int_{\mathbb{R}} \mathrm{d}\, p(E; x, y)\, F(E) \tag{1.25}$$

holds for all $x, y \in \mathbb{R}^d$ and all F obeying (1.20). *In addition,* (1.25) *holds for the function F given by $F(E) = \mathrm{e}^{-tE}$ with some arbitrary $t \in]0, \infty[$, in which case one has to set $f = k_t$. The right-hand side of* (1.25) *is to be understood as a Lebesgue-Stieltjes integral with respect to the complex "distribution" function $\mathbb{R} \ni E \mapsto p(E; x, y) := p_{]-\infty,E[}(x, y)$.*

Remark 1.19. The proof of Corollary 1.18 is given in Section 4.

1.3. *Applications to random Schrödinger operators*

The results of the previous subsection are nicely illustrated by random Schrödinger operators. In fact, certain random potentials of wide-spread use in the physics literature on disordered systems lead to Schrödinger operators which are almost surely unbounded from below and hence to Schrödinger semigroups which are almost surely unbounded from above.

DEFINITION 1.20. A *random scalar potential V* on $\mathbb{R}^d$ is a random field $V : \Omega \times \mathbb{R}^d \to \mathbb{R}$, $(\omega, x) \mapsto V^{(\omega)}(x)$, on a complete probability space $(\Omega, \mathscr{A}, \mathbb{P})$ which is measurable with respect to the product of the sigma-algebra $\mathscr{A}$ of event sets in Ω and the sigma-algebra of Borel sets in $\mathbb{R}^d$. Furthermore, a random scalar potential V is said to satisfy property

(S) if there exist two reals $p_1 > p(d)$ and $p_2 > p_1 d / \big[2(p_1 - p(d))\big]$ such that

$$\sup_{x \in \mathbb{Z}^d} \mathbb{E}\big[\|V\chi_{\Lambda_1(x)}\|_{p_1}^{p_2}\big] < \infty. \tag{1.26}$$

Here, $\mathbb{E}[X] := \int_\Omega \mathbb{P}(\mathrm{d}\omega)\, X^{(\omega)}$ denotes the expectation of a (complex-valued) random variable X on Ω, and the real $p(d)$ is defined as follows: $p(d) := 2$ if $d \leq 3$, $p(d) := d/2$ if $d \geq 5$ and $p(4) > 2$, otherwise arbitrary.

(E) if it is $\mathbb{R}^d$-ergodic with respect to the group of translations in $\mathbb{R}^d$, see [29].

(I) if

$$\sup_{x \in \mathbb{Z}^d} \mathbb{E}\big[\|V\chi_{\Lambda_1(x)}\|_{2\vartheta+1}^{2\vartheta+1}\big] < \infty, \tag{1.27}$$

where $\vartheta \in \mathbb{N}$ is the smallest integer with $\vartheta > d/4$.

(L) if the finiteness condition

$$\mathscr{L}_t := \operatorname*{ess\,sup}_{x \in \mathbb{R}^d} \mathbb{E}\big[\mathrm{e}^{-tV(x)}\big] < \infty \tag{1.28}$$

holds for all $t > 0$.

(G) if V is a Gaussian random field [1, 34] which is $\mathbb{R}^d$-homogeneous, has zero mean, $\mathbb{E}[\,V(0)] = 0$, and a covariance function $x \mapsto C(x) := \mathbb{E}[\,V(x)V(0)]$ that is continuous at the origin where it obeys $0 < C(0) < \infty$.

Remarks 1.21. (i) While property (S) will assure the applicability of the results in the previous subsection, property (I), respectively (L), is mainly a technical one needed for the existence of the integrated density of states in Proposition 1.25 below, respectively for the existence of the disorder-averaged semigroup in Corollary 1.29 below.

(ii) Given (E), property (I) simplifies to $\mathbb{E}\big[|V(0)|^{2\vartheta+1}\big] < \infty$ and property (L) to $\mathscr{L}_t = \mathbb{E}\big[\mathrm{e}^{-tV(0)}\big] < \infty$. Property (L) implies neither (S) nor (I) and vice versa. Moreover, if $d \neq 4$, property (I) in general does not imply property (S), even if property (E) is supposed. Given (E), a simple sufficient criterion for both (S) and (I) to hold is the finiteness

$$\mathbb{E}\big[|V(0)|^p\big] < \infty \tag{1.29}$$

of the p-th absolute moment for some real $p > \max\{3, d+1\}$. To prove this claim for property (S), we choose $p_1 = p_2 = p$ in (1.26). For (I) the claim follows from $2\vartheta \leq \max\{2, d\}$.

(iii) If V has property (G), then the standard Gaussian identity

$$\mathbb{E}\bigg[\exp\bigg\{\int_{\mathbb{R}^d}\zeta(\mathrm{d}x)\; V(x)\bigg\}\bigg] = \exp\bigg\{\frac{1}{2}\int_{\mathbb{R}^d}\zeta(\mathrm{d}x)\int_{\mathbb{R}^d}\zeta(\mathrm{d}y)\; C(x-y)\bigg\}. \tag{1.30}$$

holds for all (finite) complex Borel measures ζ on $\mathbb{R}^d$. Accordingly, property (G) implies properties (S), (I) and (L), see Remark 3.9(iii) in [26] for details. It also implies property (E), if the covariance function C decays at infinity.

In order to apply the results of the previous subsection we need the following

LEMMA 1.22. *Let V be a random scalar potential with property* (S). *Then for $\mathbb{P}$-almost every $\omega \in \Omega$ the realization $V^{(\omega)} : \mathbb{R}^d \to \mathbb{R}$, $x \mapsto V^{(\omega)}(x)$ is a scalar potential with property* (V).

Remark 1.23. The proof of the lemma is given in Section 5.

For a vector potential with property (A) and a random scalar potential with property (S) we thus infer from Proposition 1.3 and Definition 1.4 the existence of the *random (magnetic) Schrödinger operator* $H(A, V)$ given by the realizations $H(A, V^{(\omega)})$, which are essentially self-adjoint on $\mathscr{C}_0^\infty(\mathbb{R}^d)$ for $\mathbb{P}$-almost all $\omega \in \Omega$.

As an obvious consequence of Lemma 1.22 we note

COROLLARY 1.24. *Let A be a vector potential with property* (A) *and let V be a random scalar potential with property* (S). *Then the results of Lemma* 1.7*, Theorem* 1.10*, Theorem* 1.12*, Theorem* 1.14*, Corollary* 1.16 *and Corollary* 1.18 *apply for $\mathbb{P}$-almost every $\omega \in \Omega$ to the realization $H(A, V^{(\omega)})$ of the random Schrödinger operator as given by Definition* 1.4.

Corollary 1.24 is the basis for the rigorous derivations of two frequently used relations in the physics literature on disordered systems.

1.3.1. Integrated density of states. The first of these two relations is an integral-kernel representation of the integrated density of states of random Schrödinger operators. To formulate this representation, we first recall one possible definition of the integrated density of states in

PROPOSITION 1.25. *Let A be a vector potential with property* (C) *and let V be a random scalar potential with properties* (S), (E) *and* (I). *Let $\Gamma \subset \mathbb{R}^d$ be a bounded open cube and let $\hat{\chi}_\Gamma$ denote the bounded multiplication operator associated with the indicator function of Γ. Then the expectation value*

$$N(E) := \frac{1}{|\Gamma|}\, \mathbb{E}\Big\{ \mathrm{Trace}\Big[\hat{\chi}_\Gamma\, \chi_{]-\infty,E[}\big(H(A,V)\big)\hat{\chi}_\Gamma \Big]\Big\} \tag{1.31}$$

is well defined for every energy $E \in \mathbb{R}$ in terms of the spatially localized spectral projection associated with the half-line $]-\infty, E[$ of the random Schrödinger operator $H(A, V)$. Furthermore it is independent of Γ. The integrated density of states $E \mapsto N(E)$ *is the unbounded left-continuous distribution function of a positive Borel measure on the real line* $\mathbb{R}$.

Proof. We refer to Thm. 3.1 in [26] for the case $d \geq 2$ and to Thm. 5.20 in [36] for the case $d = 1$. ∎

Remark 1.26. Mostly, $N(E)$ is defined as the almost surely non-random quantity arising in the infinite-volume limit from the number of eigenvalues per volume (counting multiplicities) of a finite-volume restriction of $H(A, V^{(\omega)})$ below E. This definition coincides with the one in Proposition 1.25 above, as is shown in Cor. 3.3 of [26] under the present assumptions on A and V.

On account of Corollary 1.24 and (1.31) we conclude

COROLLARY 1.27. *Let A be a vector potential with property* (C) *and let V be a random scalar potential with properties* (S), (E) *and* (I). *Then the equality*

$$N(E) = \mathbb{E}\big[p(E; 0, 0)\big] \tag{1.32}$$

holds for all $E \in \mathbb{R}$, where $p^{(\omega)}(E; \cdot, \cdot) = p^{(\omega)}_{]-\infty, E[}$ denotes the continuous integral kernel of the spectral projection $\chi_{]-\infty, E[}\big(H(A, V^{(\omega)})\big)$. We recall that $p^{(\omega)}(E; \cdot, \cdot)$ exists for $\mathbb{P}$-almost all $\omega \in \Omega$ according to Corollary 1.24.

Remarks 1.28. (i) The corollary is proven in Section 5.

(ii) The representation (1.32) for the integrated density of states has been known previously from a rigorous point of view only under additional assumptions on the random scalar potential. For example, Remark VI.1.5 in [12] and Remark 3.4 in [26] require from the outset the $\mathbb{P}$-almost sure existence of continuous integral kernels for the spectral projections. A sufficient criterion for this requirement is that V is $\mathbb{P}$-almost surely Kato decomposable [42, 10]. Earlier derivations of the representation (1.32) by different authors require even stronger conditions on V, see Thms. 5.18 and 5.23 in [36]. The latter theorem, however, covers differential operators more general than Schrödinger operators.

(iii) To our knowledge, Corollary 1.27 provides the first rigorous derivation of the representation (1.32) for a wide class of random scalar potentials. As we have seen, this class includes also random potentials leading to Schrödinger operators which are $\mathbb{P}$-almost surely unbounded from below. For example, this is the case if V has properties (G) and (E) [29, 12, 36]. For such a choice of V the relation (1.32) is frequently taken for granted in the physics literature on disordered systems, see e.g. [39, 33, 17].

(iv) Corollary 1.27 strengthens Cor. 3.3 in [26] in the sense that Eq. (3.6) in [26] may be replaced by Eq. (3.7) in [26] without an additional assumption.

1.3.2. Disorder-averaged semigroup. The second application, for which Corollary 1.24 provides a rigorous justification, concerns, loosely speaking, the expectation value of the random operator exponential $\mathrm{e}^{-tH(A,V)}$.

COROLLARY 1.29. *Let A be a vector potential with property* (A) *and let V be a random scalar potential with properties* (S) *and* (L). *Moreover, let $t > 0$ and let $k_t^{(\omega)}$ denote the continuous integral kernel of* $\mathrm{e}^{-tH(A,V^{(\omega)})}$. *We recall that $k_t^{(\omega)}$ exists for $\mathbb{P}$-almost all $\omega \in \Omega$ according to Corollary* 1.24. *Then*

(i) *the disorder-averaged integral kernel $\overline{k_t} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$, $(x, y) \mapsto \overline{k_t}(x, y) := \mathbb{E}[k_t(x, y)]$ is well defined, Hermitian in the sense that $\overline{k_t}(x, y) = \overline{k_t}^*(y, x)$ for all $x, y \in \mathbb{R}^d$, continuous and dominated by the free heat kernel according to*

$$|\overline{k_t}(x, y)| \leq \mathscr{L}_t \, \frac{\mathrm{e}^{-|x-y|^2/(2t)}}{(2\pi t)^{d/2}} \tag{1.33}$$

for all $x, y \in \mathbb{R}^d$. In particular, $\overline{k_t}(x, \cdot) \in \mathrm{L}^\infty_{\mathrm{G}}(\mathbb{R}^d)$ for all $x \in \mathbb{R}^d$. The mapping $\mathbb{R}^d \to \mathrm{L}^2(\mathbb{R}^d)$, $x \mapsto \overline{k_t}(x, \cdot)$ is strongly continuous.

(ii) *the function $\overline{k_t}$ induces a bounded, self-adjoint and positive Carleman operator T_t on $\mathrm{L}^2(\mathbb{R}^d)$ in the sense that*

$$T_t \psi := \int_{\mathbb{R}^d} \mathrm{d}y \, \overline{k_t}(\cdot, y) \, \psi(y) \tag{1.34}$$

for all $\psi \in \mathrm{L}^2(\mathbb{R}^d)$ and that $\overline{k_t}$ has the Carleman property (1.15).

(iii) *the image $T_t \psi$ of any $\psi \in \mathrm{L}^2(\mathbb{R}^d)$ has a continuous representative in $\mathrm{L}^2(\mathbb{R}^d)$ given by the right-hand side of* (1.34). *If even $\psi \in \mathrm{L}^2_{\mathrm{G}}(\mathbb{R}^d)$, then one has in addition $T_t \psi \in \mathrm{L}^\infty_{\mathrm{G}}(\mathbb{R}^d)$ and the equality*

$$T_t \psi = \mathbb{E}\big[\mathrm{e}^{-tH(A,V)} \psi\big] \tag{1.35}$$

holds.

Remarks 1.30. (i) The corollary is proven in Section 5.

(ii) In view of the equality in (1.35), the operator T_t may be called the averaged semigroup (operator). One should note, however, that the one-parameter family $\{T_t\}_{t \geq 0}$ is not a semigroup in general.

(iii) Assuming also properties (C) and (E), the diagonal of the kernel $\overline{k_t}$ is constant and given by the (two-sided) Laplace transform

$$\overline{k_t}(0,0) = \int_{\mathbb{R}} \mathrm{d}N(E)\, \mathrm{e}^{-tE} \tag{1.36}$$

of the integrated density of states. This follows from Lemma 5.1(ii) below, Corollary 1.18, integration by parts and Fubini's theorem. The latter two steps rely both on Lemma 4.2 below.

The content of Corollary 1.29 is often used in the physics literature on disordered solids and random media for the special case where V is a homogeneous Gaussian random potential, that is a random scalar potential with property (G). For this choice of V, the random Schrödinger operator $H(A, V)$ is $\mathbb{P}$-almost surely unbounded from below [29, 12, 36], but complies with the assumptions of Corollary 1.29 according to Remark 1.21(iii). The corresponding Carleman kernel $\overline{k_t}$ in Corollary 1.29 can then be made more explicit by applying Fubini's theorem and the standard Gaussian identity (1.30) with the finite measure ζ on $\mathbb{R}^d$ defined for $\mu_{x,y}^{0,t}$-almost every Brownian-bridge path b by its sojourn times $\zeta(\Lambda) := \int_0^t \mathrm{d}s\, \chi_\Lambda(b(s))$ in Borel sets $\Lambda \subseteq \mathbb{R}^d$. This leads to

COROLLARY 1.31. *Let A be a vector potential with property* (A) *and let V be a random scalar potential with property* (G). *Finally, let $t > 0$. Then the assertions of Corollary* 1.29 *hold with*

$$\overline{k_t}(x,y) = \frac{\mathrm{e}^{-|x-y|^2/(2t)}}{(2\pi t)^{d/2}} \int \mu_{x,y}^{0,t}(\mathrm{d}b)\, e^{-S_t(A,0;b)} \times \exp\left\{\frac{1}{2}\int_0^t \mathrm{d}s \int_0^t \mathrm{d}s'\, C\big(b(s) - b(s')\big)\right\} \tag{1.37}$$

for all $x, y \in \mathbb{R}^d$.

Remark 1.32. The integral kernel (1.37) obeys the inequality

$$|\overline{k_t}(x,y)| \le \mathrm{e}^{-|x-y|^2/(2t)}\ \overline{k_t}(0,0)\,\big|_{A=0}\,, \tag{1.38}$$

which is sharper, but less explicit than the estimate (1.33), when particularized to a Gaussian random potential. As to the validity of (1.38) we note that by the diamagnetic inequality it suffices to consider the situation with $A = 0$. The latter was treated in [33] by adapting an argument in the proof of Lemma 3.4 in [16].

2. Proof of Lemma 1.7

This section contains the probabilistic arguments which enter Lemma 1.7.

Proof (of Lemma 1.7). To begin with, we establish the bound (1.13). In so doing we also show that the Brownian-bridge functional $b \mapsto \exp\{-S_t(A, V; b)\}$ is $\mu_{x,y}^{0,t}$-integrable and hence (1.11) well defined. To this end, we successively apply the triangle and the Cauchy-Schwarz inequality to the (absolute square of the) Brownian-bridge expectation in (1.11)

$$\begin{aligned} \left| \int \mu_{x,y}^{0,t}(\mathrm{d}b)\, \mathrm{e}^{-S_t(A,V;b)} \right|^2 &\leq \left(\int \mu_{x,y}^{0,t}(\mathrm{d}b)\, |\mathrm{e}^{-S_t(A,V;b)}| \right)^2 \\ &= \left(\int \mu_{x,y}^{0,t}(\mathrm{d}b)\, \mathrm{e}^{-S_t(0,V;b)} \right)^2 \\ &\leq \int \mu_{x,y}^{0,t}(\mathrm{d}b)\, \mathrm{e}^{-S_t(0,2V_1;b)} \int \mu_{x,y}^{0,t}(\mathrm{d}b)\, \mathrm{e}^{-S_t(0,2V_2;b)} . \end{aligned} \tag{2.1}$$

It follows from Eq. (1.3.5) in [46] that

$$\int \mu_{x,y}^{0,t}(\mathrm{d}b)\, \mathrm{e}^{-S_t(0,2V_1;b)} \leq C_0(t) \exp\left\{|x-y|^2/(4t)\right\} \tag{2.2}$$

thanks to $V_1 \in \mathscr{K}_{\pm}(\mathbb{R}^d)$ by property (V). Here $C_0(t)$ is strictly positive and continuous in $t \in]0, \infty[$. Moreover, it is independent of $x, y \in \mathbb{R}^d$. As to the second expectation in the last line of (2.1), the inequality (1.4) and the proof of Thm. 2.1 in [43] give for all $\lambda > 0$ and all $\varepsilon \in]0, (\lambda t^2)^{-1}[$ the estimate

$$\int \mu_{x,y}^{0,t}(\mathrm{d}b)\, \mathrm{e}^{-S_t(0,\lambda V_2;b)} \leq \int \mu_{x,y}^{0,t}(\mathrm{d}b)\, \mathrm{e}^{S_t(0,\lambda|V_2|;b)} \leq \Upsilon(\lambda\varepsilon t^2)\, \mathrm{e}^{\lambda t v_\varepsilon}\, \mathrm{e}^{2\lambda\varepsilon t(|x|^2+|y|^2)} , \tag{2.3}$$

where $\Upsilon(\xi) := \int_0^1 \mathrm{d}\sigma\, [1 - 4\xi\sigma(1-\sigma)]^{-d/2}$ is increasing in ξ and finite for all $\xi \in [0, 1[$. Together with (2.2) and (2.1), the estimate (2.3) with $\lambda = 2$ establishes (1.13) for all $\delta \in]0, t^{-1}[$ by identifying δ with $2\varepsilon t$. For arbitrary $\delta \geq t^{-1}$ the estimate (1.13) then follows from the monotonicity of $\delta \mapsto \mathrm{e}^{\delta|x|^2+\delta|y|^2}$.

Next we prove the properties of k_t claimed in part (i) of the lemma. The Hermiticity and the semigroup property of k_t are a consequence of the time-reversal invariance and the Markov property of the Brownian bridge, respectively. This follows from the line of reasoning in the proof of Eqs. (1.3.6) and (1.3.7) in [46]. For the proof of the continuity of k_t we refer to Corollary 2.3 below.

Finally, we turn to the proof of part (iii). The claim (1.14) is immediate from the estimate (1.13). The semigroup property (1.12) and the Hermiticity give

$$\|k_t(x,\cdot) - k_t(z,\cdot)\|_2^2 = k_{2t}(x,x) - k_{2t}(z,x) - k_{2t}(x,z) + k_{2t}(z,z) \tag{2.4}$$

for all $x, z \in \mathbb{R}^d$. This equality together with the continuity of k_{2t} establishes the strong continuity of the mapping $\mathbb{R}^d \to \mathrm{L}^2(\mathbb{R}^d)$, $x \mapsto k_t(x, \cdot)$. ∎

Lemma 2.2 below is our basic technical result for deducing the already claimed continuity of k_t. It will also enter the proof of the Feynman-Kac-Itô formula in the next section. For both purposes Lemma 2.2 will provide an approximation argument. We use it to deduce the desired properties from corresponding ones of Schrödinger semigroups with regularized scalar potentials which are Kato decomposable.

DEFINITION 2.1. Given any real $R > 0$ and a scalar potential V with property (V), we define a *regularized scalar potential* $V_R \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d) \cap \mathscr{K}_\pm(\mathbb{R}^d)$ by setting

$$V_R := V_1 + V_{2,R}\,, \tag{2.5}$$

where its truncated part $x \mapsto V_{2,R}(x) := \Theta(R - |x|)\, V_2(x)$ lies in $\mathrm{L}^\infty(\mathbb{R}^d)$.

LEMMA 2.2. *Let A be a vector potential with property* (A) *and let V be a scalar potential with property* (V). *For $t > 0$, $R > 1$ and $x, y \in \mathbb{R}^d$ define the regularized kernel*

$$k_t^{(R)}(x, y) := \frac{\mathrm{e}^{-|x-y|^2/(2t)}}{(2\pi t)^{d/2}} \int \mu_{x,y}^{0,t}(\mathrm{d}b)\, \mathrm{e}^{-S_t(A, V_R; b)}\,. \tag{2.6}$$

Then for every triple $\tau_1, \tau_2, \tilde{\rho} \in]0, \infty[$ with $\tau_1 \leq \tau_2$ there exists $\rho \in]0, \infty[$ such that one has the uniform-type-of convergence

$$\lim_{R\to\infty} \sup_{x,y\in\mathbb{R}^d} \sup_{t\in[\tau_1,\tau_2]} \left[\mathrm{e}^{\rho|x|^2-\tilde{\rho}|y|^2} \left|k_t(x, y) - k_t^{(R)}(x, y)\right|\right] = 0\,. \tag{2.7}$$

Proof. Given a Hölder exponent $p \in]1, \infty[$, we denote by $p' := (1 - p^{-1})^{-1}$ its conjugate exponent. Moreover, we let $t \in [\tau_1, \tau_2]$ arbitrary. Then the triangle and the Hölder inequality yield

$$\begin{aligned}
&\left|\int \mu_{x,y}^{0,t}(\mathrm{d}b)\ \left[\mathrm{e}^{-S_t(A,V;b)} - \mathrm{e}^{-S_t(A,V_R;b)}\right]\right| \\
&\quad\leq \int \mu_{x,y}^{0,t}(\mathrm{d}b)\, \mathrm{e}^{-S_t(0,V_1;b)} \left|\mathrm{e}^{-S_t(0,V_2;b)} - \mathrm{e}^{-S_t(0,V_{2,R};b)}\right| \\
&\quad\leq \left[\int \mu_{x,y}^{0,t}(\mathrm{d}b)\, \mathrm{e}^{-S_t(0,pV_1;b)}\right]^{\frac{1}{p}} \left[\int \mu_{x,y}^{0,t}(\mathrm{d}b)\, \left|\mathrm{e}^{-S_t(0,V_2;b)} - \mathrm{e}^{-S_t(0,V_{2,R};b)}\right|^{p'}\right]^{\frac{1}{p'}}.
\end{aligned} \tag{2.8}$$

The first expectation in the last line of (2.8) is bounded according to

$$\left[\int \mu_{x,y}^{0,t}(\mathrm{d}b)\, \mathrm{e}^{-S_t(0,pV_1;b)}\right]^{1/p} \leq C_1 \exp\left\{|x-y|^2/(4\tau_1 p)\right\}, \tag{2.9}$$

confer (2.2). Here $C_1 \equiv C_1(p, \tau_1, \tau_2)$ is a finite constant. In order to bound the second expectation in the last line of (2.8) we employ the elementary inequality $|\mathrm{e}^r - \mathrm{e}^{r'}| \leq |r - r'|\, \mathrm{e}^{\max\{r,r'\}}$ for $r, r' \in \mathbb{R}$ together with $|V_{2,R}| \leq |V_2|$ and the Cauchy-Schwarz inequality. This gives

$$\begin{aligned}
&\int \mu_{x,y}^{0,t}(\mathrm{d}b)\, \left|\mathrm{e}^{-S_t(0,V_2;b)} - \mathrm{e}^{-S_t(0,V_{2,R};b)}\right|^{p'} \\
&\quad\leq \int \mu_{x,y}^{0,t}(\mathrm{d}b)\, \mathrm{e}^{S_t(0,p'|V_2|;b)} \left|S_t(0, V_2 - V_{2,R}; b)\right|^{p'} \\
&\quad\leq \left[\int \mu_{x,y}^{0,t}(\mathrm{d}b)\, \mathrm{e}^{S_t(0,2p'|V_2|;b)}\right]^{1/2} \left[\int \mu_{x,y}^{0,t}(\mathrm{d}b) \left|S_t(0, V_2 - V_{2,R}; b)\right|^{2p'}\right]^{1/2}.
\end{aligned} \tag{2.10}$$

The first expectation in the last line of (2.10) can be estimated as in (2.3),

$$\int \mu_{x,y}^{0,t}(\mathrm{d}b)\, \mathrm{e}^{S_t(0,2p'|V_2|;b)} \leq C_2^{2p'} \exp\left\{4p'\varepsilon\tau_2\left(|x|^2 + |y|^2\right)\right\}, \tag{2.11}$$

where $\varepsilon \in]0, (2p'\tau_2^2)^{-1}[$ is arbitrary and $C_2 \equiv C_2(p, \varepsilon, \tau_2)$ is another finite constant. Here we have used the monotonicity of the right-hand side of (2.3) in t. To bound the second expectation in the last line of (2.10) we observe that

$$|V_2(x) - V_{2,R}(x)| \leq (\varepsilon|x|^2 + v_\varepsilon)\, \Theta(|x| - R) \leq (\varepsilon + v_\varepsilon)\, \frac{|x|^4}{R^2} \tag{2.12}$$

for all $\varepsilon > 0$ and Lebesgue-almost all $x \in \mathbb{R}^d$. Here we have exploited $R > 1$ and the "Chebyshev" inequality $\Theta(\xi - 1) \leq \xi^2$, $\xi \in \mathbb{R}$. By the Jensen and the triangle inequality, Fubini's theorem and upon standardizing the Brownian bridge according to $b(s) =: t^{1/2}\,\tilde{b}(s/t) + x + (y-x)s/t$, the estimate (2.12) yields

$$\begin{aligned}
&\int \mu_{x,y}^{0,t}(\mathrm{d}b) \left|S_t(0, V_2 - V_{2,R}; b)\right|^{2p'} \\
&\quad\leq \left(\frac{(\varepsilon + v_\varepsilon)t}{R^2}\right)^{2p'} \int_0^t \frac{\mathrm{d}s}{t} \int \mu_{x,y}^{0,t}(\mathrm{d}b)\, |b(s)|^{8p'} \\
&\quad= \left(\frac{(\varepsilon + v_\varepsilon)t}{R^2}\right)^{2p'} \int_0^1 \mathrm{d}\sigma \int \mu_{0,0}^{0,1}(\mathrm{d}\tilde{b})\, |t^{1/2}\tilde{b}(\sigma) + x + (y-x)\sigma|^{8p'}.
\end{aligned} \tag{2.13}$$

This result and several applications of the elementary inequality

$$|r+r'|^\alpha \leq 2^\alpha\big(|r|^\alpha+|r'|^\alpha\big) \tag{2.14}$$

for $\alpha > 0$ and $r, r' \in \mathbb{R}^d$ show that there exist two further finite constants $C_3 \equiv C_3(p,\varepsilon)$ and $C_4 \equiv C_4(p,\varepsilon)$ such that

$$\left[\int \mu_{x,y}^{0,t}(\mathrm{d}b)\,\big|S_t(0,V_2-V_{2,R};b)\big|^{2p'}\right]^{1/(2p')} \leq \frac{\tau_2}{R^2}\,\big[C_3\tau_2^2+C_4\big(|x|^4+|y|^4\big)\big]\,. \tag{2.15}$$

Combining (2.8), (2.9), (2.10), (2.11) and (2.15), we obtain

$$\begin{aligned}&\left|\int \mu_{x,y}^{0,t}(\mathrm{d}b)\,\left[\mathrm{e}^{-S_t(A,V;b)}-\mathrm{e}^{-S_t(A,V_R;b)}\right]\right|\\ &\quad\leq \frac{C_1C_2\tau_2}{R^2}\,\big[C_3\tau_2^2+C_4\big(|x|^4+|y|^4\big)\big]\exp\left\{\frac{|x-y|^2}{4\tau_1 p}+2\varepsilon\tau_2\big(|x|^2+|y|^2\big)\right\}\end{aligned} \tag{2.16}$$

for all $t \in [\tau_1,\tau_2]$, all $\varepsilon \in]0,(2p'\tau_2^2)^{-1}[$ and all $x, y \in \mathbb{R}^d$. Another application of (2.14) and choosing $p = 2\tau_2/\tau_1 \geq 2$ then yields

$$\begin{aligned}&\sup_{t\in[\tau_1,\tau_2]}\left[\mathrm{e}^{\rho|x|^2-\tilde\rho|y|^2}\,|k_t(x,y)-k_t^{(R)}(x,y)|\right]\\ &\quad\leq \frac{C_1C_2\tau_2}{R^2(2\pi\tau_1)^{d/2}}\,\big[C_3\tau_2^2+C_4\big(|x|^4+|y|^4\big)\big]\\ &\qquad\times\exp\big\{-\big[1/(4\tau_2)-4\rho-8\varepsilon\tau_2\big]|x-y|^2-(\tilde\rho-4\rho-10\,\varepsilon\tau_2)|y|^2\big\}\end{aligned} \tag{2.17}$$

for all $\rho, \tilde\rho > 0$, all $\varepsilon \in]0,(2\tau_2-\tau_1)/(4\tau_2^3)[$ and all $x, y \in \mathbb{R}^d$. The assertion of the lemma now follows by choosing ρ and ε so small that $4\rho + 10\,\varepsilon\tau_2 < \min\{\tilde\rho, (4\tau_2)^{-1}\}$. ∎

Lemma 2.2 possesses an immediate corollary, which completes the proof of Lemma 1.7.

COROLLARY 2.3. *The function*

$$]0,\infty[\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{C},\qquad (t,x,y)\mapsto k_t(x,y) \tag{2.18}$$

is continuous under the assumptions of Lemma 1.7.

Proof. Since by assumption V_R lies in $\mathscr{K}_\pm(\mathbb{R}^d)$ and both $|A|^2$ and $\nabla \cdot A$ lie in $\mathscr{K}_{\mathrm{loc}}(\mathbb{R}^d)$, Thm. 6.1 in [10] for the case $d \geq 2$, respectively Prop. 1.3.5 in [46] for the case $d = 1$, guarantee the continuity of the function

$$]0, \infty[\times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}, \qquad (t, x, y) \mapsto k_t^{(R)}(x, y) \tag{2.19}$$

for all $R > 0$. But according to Lemma 2.2 the kernel $k_\bullet$ is the locally uniform limit of $k_\bullet^{(R)}$ as $R \to \infty$. Hence, $k_\bullet$ inherits the continuity properties of $k_\bullet^{(R)}$. ∎

3. Proofs of Theorem 1.10 and Theorem 1.12

Given the two probabilistic Lemmata 1.7 and 2.2, the additional arguments needed to prove Theorem 1.10 and Theorem 1.12 are purely analytic. First, we exploit the fact that the function k_t, as defined in Lemma 1.7, is a Carleman kernel [49].

LEMMA 3.1. *Let A be a vector potential with property* (A) *and let V be a scalar potential with property* (V). *For $t > 0$ we denote by K_t the integral operator induced by the kernel k_t with domain*

$$\mathrm{dom}(K_t) := \left\{ \psi \in \mathrm{L}^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} \mathrm{d}y \, k_t(\cdot, y) \, \psi(y) \in \mathrm{L}^2(\mathbb{R}^d) \right\} \tag{3.1}$$

and action

$$K_t \psi := \int_{\mathbb{R}^d} \mathrm{d}y \, k_t(\cdot, y) \, \psi(y) \tag{3.2}$$

for all $\psi \in \mathrm{dom}(K_t)$. Then K_t is a maximal Carleman operator, hence closed, and its domain is dense thanks to the inclusion

$$\mathrm{L}^2_{\mathrm{G}}(\mathbb{R}^d) \subseteq \mathrm{dom}(K_t) \, . \tag{3.3}$$

Moreover, the image $K_t \psi$ of any $\psi \in \mathrm{dom}(K_t)$ has a continuous representative in $\mathrm{L}^2(\mathbb{R}^d)$ given by the right-hand side of (3.2). *If even $\psi \in \mathrm{L}^2_{\mathrm{G}}(\mathbb{R}^d)$, then, in addition, $K_t \psi \in \mathrm{L}^\infty_{\mathrm{G}}(\mathbb{R}^d)$.*

Proof (of Lemma 3.1). By Lemma 1.7(i) and (iii) we know that k_t is a Hermitian Carleman kernel. Thus, Thm. 6.13(a) in [49] yields the closedness of the induced maximal Carleman operator K_t. The inclusion (3.3) is implied by Remark 1.6(i) and the inclusion $K_t \mathrm{L}^2_{\mathrm{G}}(\mathbb{R}^d) \subseteq \mathrm{L}^\infty_{\mathrm{G}}(\mathbb{R}^d)$, which we prove next. To do so, we note that (1.13) implies

$$\sup_{x \in \mathbb{R}^d} \left[\mathrm{e}^{\rho |x|^2} |k_t(x, y)| \right] \leq a_t^{(\delta)} \, \mathrm{e}^{(4\rho + 5\delta)|y|^2} \tag{3.4}$$

for all $\rho, \delta > 0$ with $\rho + \delta < 1/(16t)$ and all $y \in \mathbb{R}^d$. In deriving (3.4) we have also used the elementary inequality (2.14) with $r = x - y$, $r' = y$ and $\alpha = 2$.

Consequently, given any $\psi \in \mathrm{L}^2_{\mathrm{G}}(\mathbb{R}^d)$, we get

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}^d} \left| \mathrm{e}^{\rho|x|^2} (K_t \psi)(x) \right| \leq a_t^{(\delta)} \int_{\mathbb{R}^d} \mathrm{d}y \, \mathrm{e}^{(4\rho+5\delta)|y|^2} \, |\psi(y)| \, . \tag{3.5}$$

Now, choosing ρ and δ small enough, the right-hand side of (3.5) is finite since $\mathrm{L}^2_{\mathrm{G}}(\mathbb{R}^d) \subseteq \mathrm{L}^1_{\mathrm{G}}(\mathbb{R}^d)$ by Remark 1.6(i).

In order to complete the proof of the lemma we have to show the continuity of $K_t\psi$ for all $\psi \in \mathrm{dom}(K_t)$. To this end we observe

$$\left| (K_t\psi)(x) - (K_t\psi)(x') \right| \leq \|\psi\|_2 \, \|k_t(x,\cdot) - k_t(x',\cdot)\|_2 \tag{3.6}$$

by the triangle and the Cauchy-Schwarz inequality for all $x, x' \in \mathbb{R}^d$. The desired result now follows from the strong continuity of $x \mapsto k_t(x,\cdot)$ in Lemma 1.7(iii). ∎

We will eventually prove Theorem 1.10 by showing the operator equality $K_t = \mathrm{e}^{-tH(A,V)}$. As an initial step we recall Definition 2.1 and employ Lemma 2.2 in order to establish strong convergence of the regularized operator exponentials $\mathrm{e}^{-tH(A,V_R)}$ to K_t on $\mathrm{L}^2_{\mathrm{G}}(\mathbb{R}^d)$ as $R \to \infty$.

LEMMA 3.2. *Let $t > 0$, $\psi \in \mathrm{L}^2_{\mathrm{G}}(\mathbb{R}^d)$ and suppose the assumptions of Theorem* 1.10. *Then*

$$\lim_{R\to\infty} \|\mathrm{e}^{-tH(A,V_R)}\psi - K_t\psi\|_2 = 0 \tag{3.7}$$

holds.

Proof. We recall from Thm. 6.1 in [10] for the case $d \geq 2$, respectively from Eq. (6.6) in [40] or from Eqs. (1.3.3), (1.3.4) and Exercise 1.4.2 in [46] for the case $d = 1$, the Feynman-Kac-Itô formula for the bounded semigroup with the regularized potential

$$\mathrm{e}^{-tH(A,V_R)}\psi = \int_{\mathbb{R}^d} \mathrm{d}y \, k_t^{(R)}(\cdot, y)\, \psi(y) \, , \tag{3.8}$$

valid for all $\psi \in \mathrm{L}^2(\mathbb{R}^d)$. Now, given any $\psi \in \mathrm{L}^2_{\mathrm{G}}(\mathbb{R}^d)$ there exists $\tilde{\rho} > 0$ such that $\|\mathrm{e}^{\tilde{\rho}|\cdot|^2}\psi\|_1 < \infty$ by Remark 1.6(i). Lemma 2.2 then yields the existence of

$\rho > 0$ such that the right-hand side of the estimate

$$\begin{aligned}
\|\mathrm{e}^{-tH(A,V_R)}\psi - K_t\psi\|_2^2 &= \int_{\mathbb{R}^d} \mathrm{d}x \left| \int_{\mathbb{R}^d} \mathrm{d}y \, \left[k_t^{(R)}(x,y) - k_t(x,y)\right] \psi(y) \right|^2 \\
&\leq \int_{\mathbb{R}^d} \mathrm{d}x \, \mathrm{e}^{-2\rho|x|^2} \Bigg[\int_{\mathbb{R}^d} \mathrm{d}y \, \mathrm{e}^{\tilde{\rho}|y|^2} \, |\psi(y)| \\
&\qquad \times \mathrm{e}^{\rho|x|^2 - \tilde{\rho}|y|^2} \left| k_t^{(R)}(x,y) - k_t(x,y) \right| \Bigg]^2 \\
&\leq \left[\sup_{x,y \in \mathbb{R}^d} \left(\mathrm{e}^{\rho|x|^2 - \tilde{\rho}|y|^2} \left| k_t^{(R)}(x,y) - k_t(x,y) \right| \right) \right]^2 \\
&\qquad \times [\pi/(2\rho)]^{d/2} \, \|\mathrm{e}^{\tilde{\rho}|\cdot|^2} \, \psi\|_1^2
\end{aligned} \tag{3.9}$$

vanishes as $R \to \infty$. ∎

Remark 3.3. One can even show that the convergence in Lemma 3.2 holds with respect to the $\mathrm{L}^p(\mathbb{R}^d)$-norm for arbitrary $p \in [1, \infty]$, if one requires $\psi \in \mathrm{L}^p_{\mathrm{G}}(\mathbb{R}^d)$, see also Remark 1.11(iii).

The next lemma concerns a certain stability of strong-resolvent convergence. It will be the basis for an argument similar to the one provided by Thm. 3.1 in [43].

LEMMA 3.4. *For $n \in \mathbb{N}$ let A_n and A be self-adjoint operators acting on a complex Hilbert space and let $G : \mathbb{R} \to \mathbb{R}$ be a continuous function. Define $G(A_n)$ for $n \in \mathbb{N}$ and $G(A)$ via the spectral theorem and the functional calculus as self-adjoint operators. Then strong-resolvent convergence of A_n to A as $n \to \infty$ implies strong-resolvent convergence of $G(A_n)$ to $G(A)$.*

Proof. For $z \in \mathbb{C}$ with $\operatorname{Im} z \neq 0$ we define the bounded continuous function $R_z : \mathbb{R} \to \mathbb{C}$, $\lambda \mapsto R_z(\lambda) := (\lambda - z)^{-1}$. Hence, the composition $R_z \circ G$ is also a bounded and continuous function on $\mathbb{R}$. Therefore, $(R_z \circ G)(A_n) = R_z(G(A_n))$ converges strongly to $(R_z \circ G)(A) = R_z(G(A))$ as $n \to \infty$ by Thm. VIII.20(b) in [38] or Thm. 9.17 in [49]. ∎

Having these auxiliary results at our disposal, we can proceed to prove – as an intermediate step – Theorem 1.10(ii), which is analogous to the claim of Remark 1 after Thm. 1.2 in [43].

LEMMA 3.5. *Let $t > 0$. Under the assumptions of Theorem* 1.10 *one has $\mathrm{L}^2_{\mathrm{G}}(\mathbb{R}^d) \subseteq \operatorname{dom}\big(\mathrm{e}^{-tH(A,V)}\big)$ and the Feynman-Kac-Itô formula*

$$\mathrm{e}^{-tH(A,V)}\psi = K_t\psi \tag{3.10}$$

holds for all $\psi \in \mathrm{L}^2_{\mathrm{G}}(\mathbb{R}^d)$*. In particular,* $\mathrm{e}^{-tH(A,V)}$ *and thus* K_t *are both symmetric on* $\mathrm{L}^2_{\mathrm{G}}(\mathbb{R}^d)$.

Proof (of Lemma 3.5). The Schrödinger operators $H(A,V)$ and $H(A,V_R)$, $R>0$, are all essentially self-adjoint on $\mathscr{C}_0^\infty(\mathbb{R}^d)$ according to Proposition 1.3 and Definition 1.4. Moreover, $H(A,V_R)$ converges strongly to $H(A,V)$ on $\mathscr{C}_0^\infty(\mathbb{R}^d)$ as $R\to\infty$. This can be inferred from (1.4) and the estimate

$$\begin{aligned}\|H(A,V_R)\varphi - H(A,V)\varphi\|_2^2 &= \int_{\mathbb{R}^d} \mathrm{d}x\, \left|V_2^{(R)}(x) - V_2(x)\right|^2 |\varphi(x)|^2 \\ &\le \int_{\mathbb{R}^d} \mathrm{d}x\, \Theta(|x|-R)\left(\varepsilon|x|^2 + v_\varepsilon\right)^2 |\varphi(x)|^2 ,\end{aligned} \tag{3.11}$$

which is valid for all $\varepsilon>0$ and all $\varphi\in\mathscr{C}_0^\infty(\mathbb{R}^d)$. The right-hand side of (3.11) vanishes, if R is large enough. Therefore, Thm. VIII.25(a) in [38] implies that $H(A,V_R)$ converges to $H(A,V)$ in strong-resolvent sense as $R\to\infty$, and thus, thanks to Lemma 3.4, $\mathrm{e}^{-tH(A,V_R)}$ converges to $\mathrm{e}^{-tH(A,V)}$ as $R\to\infty$ in strong-resolvent sense for all $t>0$. Since the operators $\mathrm{e}^{-tH(A,V_R)}$ and $\mathrm{e}^{-tH(A,V)}$ are self-adjoint, strong-resolvent convergence is equivalent to $\mathrm{e}^{-tH(A,V)}$ being the strong-graph limit of $\mathrm{e}^{-tH(A,V_R)}$ as $R\to\infty$ by Thm. VIII.26 in [38]. Thus, by definition of this limit, the graph

$$\mathscr{G}_t := \left\{(\psi,\phi)\in \mathrm{L}^2(\mathbb{R}^d)\times \mathrm{L}^2(\mathbb{R}^d) : \psi\in\mathrm{dom}\big(\mathrm{e}^{-tH(A,V)}\big), \phi = \mathrm{e}^{-tH(A,V)}\psi\right\} \tag{3.12}$$

of $\mathrm{e}^{-tH(A,V)}$ consists of all pairs $(\psi,\phi)\in \mathrm{L}^2(\mathbb{R}^d)\times \mathrm{L}^2(\mathbb{R}^d)$ for which there exists a sequence $\{\psi_R\}_R$ with $\psi_R\in\mathrm{dom}\big(\mathrm{e}^{-tH(A,V_R)}\big) = \mathrm{L}^2(\mathbb{R}^d)$ such that

$$\lim_{R\to\infty}\big(\|\psi_R-\psi\|_2 + \|\mathrm{e}^{-tH(A,V_R)}\psi_R - \phi\|_2\big) = 0 . \tag{3.13}$$

According to Lemma 3.2 the convergence in (3.13) holds for every $\psi\in\mathrm{L}^2_{\mathrm{G}}(\mathbb{R}^d)$, if we set $\psi_R=\psi$ and $\phi = K_t\psi$, that is,

$$\mathscr{G}_t \supseteq \left\{(\psi,\phi)\in \mathrm{L}^2(\mathbb{R}^d)\times \mathrm{L}^2(\mathbb{R}^d) : \psi\in\mathrm{L}^2_{\mathrm{G}}(\mathbb{R}^d), \phi = K_t\psi\right\} . \tag{3.14}$$

This implies $\mathrm{L}^2_{\mathrm{G}}(\mathbb{R}^d)\subseteq \mathrm{dom}\big(\mathrm{e}^{-tH(A,V)}\big)$ and (3.10). Moreover, the restriction of the self-adjoint operator $\mathrm{e}^{-tH(A,V)}$ to $\mathrm{L}^2_{\mathrm{G}}(\mathbb{R}^d)$ yields a symmetric operator. ∎

Having settled Lemma 3.5, we are in a position to establish Theorem 1.12 on the semigroup properties of the family $\{\mathrm{e}^{-tH(A,V)}\}_{t\ge 0}$.

Proof (of Theorem 1.12). (i) The validity of the semigroup law (1.18) on $\mathrm{L}^2_{\mathrm{G}}(\mathbb{R}^d)$ relies on the functional calculus for unbounded functions of unbounded self-adjoint operators, see e.g. Chap. 5 in [7], on Lemma 3.5 and on the inclusion $K_t \mathrm{L}^2_{\mathrm{G}}(\mathbb{R}^d) \subseteq \mathrm{L}^\infty_{\mathrm{G}}(\mathbb{R}^d)$, which was proven in Lemma 3.1. The latter two ensure that both sides of (1.18) are well defined on $\mathrm{L}^2_{\mathrm{G}}(\mathbb{R}^d)$.

(ii) Strong continuity of the orbit mapping u_ψ for $\psi \in \mathrm{L}^2_{\mathrm{G}}(\mathbb{R}^d)$ follows from the functional calculus, too, in that

$$\|u_\psi(t+h) - u_\psi(t)\|_2^2 = \int_{\mathbb{R}} \langle \psi, P(\mathrm{d}E)\psi\rangle \, (\mathrm{e}^{-(t+h)E} - \mathrm{e}^{-tE})^2 \tag{3.15}$$

for all $t \in [0, \infty[$ and all $h \in [-t, \infty[$. Here P denotes the projection-valued spectral measure of the Schrödinger operator $H := H(A, V)$, that is, $P(I) := \chi_I(H)$ for Borel sets $I \subseteq \mathbb{R}$. Indeed, the integral in (3.15) vanishes in the limit $h \to 0$ by the dominated-convergence theorem, because we may assume $h \in [-t, h_0]$ with some $h_0 \in]0, \infty[$ so that the function $\mathbb{R} \ni E \mapsto (1 + 2\,\mathrm{e}^{-(t+h_0)E})^2$ dominates the integrand of (3.15) and is $\langle \psi, P(\cdot)\psi\rangle$-integrable due to $\psi \in \mathrm{L}^2_{\mathrm{G}}(\mathbb{R}^d)$. In the special case $t = 0$, this procedure gives the only meaningful right-sided limit $h \downarrow 0$.

(iii) First we claim $\mathscr{C}_0^\infty(\mathbb{R}^d) \subset \mathrm{dom}(H\mathrm{e}^{-tH})$. Since $\mathscr{C}_0^\infty(\mathbb{R}^d) \subset \mathrm{dom}(\mathrm{e}^{-tH})$, this follows from Thm. 5.2.9(c) in [7], if

$$\int_{\mathbb{R}} \langle \varphi, P(\mathrm{d}E)\varphi\rangle \, (E\mathrm{e}^{-tE})^2 < \infty \tag{3.16}$$

for all $\varphi \in \mathscr{C}_0^\infty(\mathbb{R}^d)$. The latter holds true, because $(E\mathrm{e}^{-tE})^2 \leq E^2 + \mathrm{e}^{-2t_0 E}$ for all $E \in \mathbb{R}$ with some $t_0 > t$ and because $\mathscr{C}_0^\infty(\mathbb{R}^d) \subset \mathrm{dom}(H) \cap \mathrm{dom}(\mathrm{e}^{-t_0 H})$. Next we compute the strong derivative of u_φ for $\varphi \in \mathscr{C}_0^\infty(\mathbb{R}^d)$. To this end, we consider the squared norm

$$\begin{aligned} &\left\| h^{-1}\big(\mathrm{e}^{-(t+h)H}\varphi - \mathrm{e}^{-tH}\varphi\big) + H\mathrm{e}^{-tH}\varphi \right\|_2^2 \\ &\qquad = \int_{\mathbb{R}} \langle \varphi, P(\mathrm{d}E)\varphi\rangle \, \big[h^{-1}\big(\mathrm{e}^{-(t+h)E} - \mathrm{e}^{-tE}\big) + E\mathrm{e}^{-tE}\big]^2 \end{aligned} \tag{3.17}$$

for $h \in]-t, 1] \setminus \{0\}$ and claim that it vanishes in the limit $h \to 0$. (In the special case $t = 0$, the limit gives the only meaningful right-sided derivative.) This follows from the dominated-convergence theorem and the h-independent upper bound $2E^2\big(2 + \mathrm{e}^{-2tE} + 2\mathrm{e}^{-2(t+1)E}\big)$ for the integrand in (3.17). This bound is $\langle \varphi, P(\cdot)\varphi\rangle$-integrable as a function of E because of $\varphi \in \mathscr{C}_0^\infty(\mathbb{R}^d) \subset \mathrm{dom}(H)$ and (3.16).

It remains to show that u_φ is the *unique* solution of the initial-value problem (1.19). To this end, let Φ be an arbitrary solution of (1.19) and fix $t > 0$ arbitrary. By the above reasoning one has $\frac{\mathrm{d}}{\mathrm{d}s}\,\mathrm{e}^{-(t-s)H}g = H\mathrm{e}^{-(t-s)H}g$ in the strong sense for arbitrary $s \in]0, t[$ and arbitrary $g \in \mathscr{C}_0^\infty(\mathbb{R}^d)$. As a consequence, one finds

$$\frac{\mathrm{d}}{\mathrm{d}s} \langle \mathrm{e}^{-(t-s)H} g, \Phi(s)\rangle = \langle H\mathrm{e}^{-(t-s)H} g, \Phi(s)\rangle - \langle \mathrm{e}^{-(t-s)H} g, H\Phi(s)\rangle = 0 \tag{3.18}$$

by the assumptions on Φ and the self-adjointness of H. Hence, the fundamental theorem of calculus implies

$$\begin{aligned} 0 &= \int_0^t \mathrm{d}s \, \frac{\mathrm{d}}{\mathrm{d}s} \langle \mathrm{e}^{-(t-s)H} g, \Phi(s)\rangle = \langle g, \Phi(t)\rangle - \langle \mathrm{e}^{-tH} g, \Phi(0)\rangle \\ &= \langle g, \Phi(t)\rangle - \langle g, \mathrm{e}^{-tH}\varphi\rangle = \langle g, \Phi(t) - u_\varphi(t)\rangle . \end{aligned} \tag{3.19}$$

The denseness of $\mathscr{C}_0^\infty(\mathbb{R}^d)$ in $\mathrm{L}^2(\mathbb{R}^d)$ completes the proof of uniqueness. ∎

An immediate consequence of the just-proven Theorem 1.12 is

COROLLARY 3.6. *Assume the situation of Theorem* 1.10. *Then* $\mathrm{L}^2_{\mathrm{G}}(\mathbb{R}^d)$ *is an operator core for* $\mathrm{e}^{-tH(A,V)}$ *for all* $t > 0$.

Proof. By Theorem 1.12 and the symmetry of $\mathrm{e}^{-tH(A,V)}$ on $\mathrm{L}^2_{\mathrm{G}}(\mathbb{R}^d)$, see Lemma 3.5, all three assumptions of Thm. 1 in [35] are fulfilled by choosing there $\alpha = t \in]0, \infty[$, $S_t = \mathrm{e}^{-tH(A,V)}$ with $\mathrm{dom}(S_t) = \mathrm{L}^2_{\mathrm{G}}(\mathbb{R}^d)$ and $D = \mathrm{L}^2_{\mathrm{G}}(\mathbb{R}^d)$. In this context, we recall from Lemma 3.5 that $\mathrm{e}^{-tH(A,V)}$ is symmetric on $\mathrm{L}^2_{\mathrm{G}}(\mathbb{R}^d)$ and from Theorem 1.12 that the mapping $[0, \infty[\ni t \mapsto \langle \psi, u_\psi(t)\rangle$ is continuous – and hence Borel measurable – for every $\psi \in \mathrm{L}^2_{\mathrm{G}}(\mathbb{R}^d)$ due to the strong continuity of the orbit mapping u_ψ. Therefore the claim follows from Thm. 1 in [35]. ∎

The remaining part of the proof of Theorem 1.10 is provided by

LEMMA 3.7. *Assume the situation of Theorem* 1.10 *and let* K_t *be defined as in Lemma* 3.1. *Then one has the equality*

$$K_t = \mathrm{e}^{-tH(A,V)} . \tag{3.20}$$

Proof. We follow [3] or [45] and introduce the restriction $K_t^0 := K_t|_{\mathrm{dom}(K_t^0)}$ of the maximal Carleman operator K_t to the subspace

$$\mathrm{dom}(K_t^0) := \left\{\psi \in \mathrm{dom}(K_t) : \kappa_t\psi \in \mathrm{L}^1(\mathbb{R}^d)\right\} , \tag{3.21}$$

where the function $\mathbb{R}^d \ni x \mapsto \kappa_t(x) := \|k_t(x, \cdot)\|_2 = [k_{2t}(x, x)]^{1/2}$ is well defined and continuous because of Lemma 1.7(iii). The estimate (1.13) in Lemma 1.7 and Remark 1.6(i) imply $\mathrm{L}^2_{\mathrm{G}}(\mathbb{R}^d) \subseteq \mathrm{dom}(K_t^0)$. Thus, the Feynman-Kac-Itô formula from Lemma 3.5 leads to

$$\mathrm{e}^{-tH(A,V)}|_{\mathrm{L}^2_{\mathrm{G}}(\mathbb{R}^d)} = K_t|_{\mathrm{L}^2_{\mathrm{G}}(\mathbb{R}^d)} = K_t^0|_{\mathrm{L}^2_{\mathrm{G}}(\mathbb{R}^d)} \subseteq K_t^0 \,. \tag{3.22}$$

Here, as usual, the notation $A \subseteq B$ means that the operator B is an extension of the operator A. By Thm. 10.1 in [45] the operator K_t^0 is symmetric, hence closable. Taking the closure of (3.22) with respect to the graph norm and exploiting Corollary 3.6, we get $\mathrm{e}^{-tH(A,V)} \subseteq \overline{K_t^0}$. Since K_t^0 is symmetric, so is its closure $\overline{K_t^0}$. Therefore we conclude

$$\mathrm{e}^{-tH(A,V)} = \overline{K_t^0}\,, \tag{3.23}$$

because self-adjoint operators are maximally symmetric. Furthermore, we observe the equalities $\overline{K_t^0} = \big(\overline{K_t^0}\big)^* = (K_t^0)^* = K_t$, which hold according to (3.23), Thm. VIII.1(c) in [38] and Thm. 10.1 in [45]. This completes the proof. ∎

Finally, we gather our previous results to complete the

Proof (of Theorem 1.10). Corollary 3.6 has established that $\mathrm{L}^2_{\mathrm{G}}(\mathbb{R}^d)$ is an operator core for $\mathrm{e}^{-tH(A,V)}$. The remaining assertions of Theorem 1.10 follow from Lemma 3.7, Lemma 3.1 and Lemma 1.7(iii). ∎

4. Proofs of Theorem 1.14, Corollary 1.16 and Corollary 1.18

The following lemma is in the spirit of Thm. B.7.8 in [42], but, among others, we do not assume that the operator M is bounded.

LEMMA 4.1. *Let M be the maximal self-adjoint Carleman operator induced by the Borel-measurable and Hermitian integral kernel $m : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ in the sense that*

$$\begin{aligned} \mathscr{C}_0^\infty(\mathbb{R}^d) \ &\subset \ \mathrm{dom}(M) := \left\{ \psi \in \mathrm{L}^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} \mathrm{d}y\, m(\cdot, y)\, \psi(y) \in \mathrm{L}^2(\mathbb{R}^d) \right\}, \\ M\psi \ &= \ \int_{\mathbb{R}^d} \mathrm{d}y\, m(\cdot, y)\, \psi(y) \end{aligned} \tag{4.1}$$

for all $\psi \in \mathrm{dom}(M)$, $m(x, y) = m^*(y, x)$ for Lebesgue-almost all pairs $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$ and m has the Carleman property (1.15). *Assume further that $x \mapsto m(\cdot, x)$ defines a strongly continuous mapping from $\mathbb{R}^d$ to $\mathrm{L}^2(\mathbb{R}^d)$. Finally, let B be a bounded operator on $\mathrm{L}^2(\mathbb{R}^d)$ such that MB and MB^* are also bounded and that MBM admits a bounded closed extension $\overline{MBM}$ to all of $\mathrm{L}^2(\mathbb{R}^d)$. Then*

(i) *$\overline{MBM}$ is a bounded Carleman operator induced by the continuous integral kernel $\beta : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$, $(x, y) \mapsto \beta(x, y) := \langle m(\cdot, x), Bm(\cdot, y)\rangle$ in the* *sense that*

$$\overline{MBM}\psi = \int_{\mathbb{R}^d} \mathrm{d}y\, \beta(\cdot, y)\, \psi(y) \tag{4.2}$$

for all $\psi \in \mathrm{L}^2(\mathbb{R}^d)$ *and* β *has the Carleman property* (1.15).

(ii) *the left-hand side of* (4.2) *has a continuous representative in* $\mathrm{L}^2(\mathbb{R}^d)$*, which is given by the right-hand side of* (4.2).

(iii) *for any* $w \in \mathrm{L}^\infty(\mathbb{R}^d)$ *with* $\int_{\mathbb{R}^d\times\mathbb{R}^d} \mathrm{d}x\mathrm{d}y\, |w(x)|^2\, |m(x,y)|^2 < \infty$ *the product* $\overline{MBM}\hat{w}$ *is a Hilbert-Schmidt operator with squared norm given by*

$$\mathrm{Trace}\big[\hat{w}^*|\overline{MBM}|^2\hat{w}\big] = \int_{\mathbb{R}^d} \mathrm{d}x\, |w(x)|^2 \int_{\mathbb{R}^d} \mathrm{d}y\, |\beta(x,y)|^2 . \tag{4.3}$$

Here $\hat{w}$ *is the bounded multiplication operator uniquely corresponding to* w*, and* $\hat{w}^*$ *denotes its Hilbert adjoint.*

Proof. The strong continuity of the mapping $\mathbb{R}^d \to \mathrm{L}^2(\mathbb{R}^d)$, $x \mapsto m(\cdot, x)$, the triangle and the Cauchy-Schwarz inequality imply the continuity of the function $\mathscr{M} : \mathbb{R}^d \to \mathbb{R}$, $x \mapsto \mathscr{M}(x) := \|m(\cdot, x)\|_2$ because $|\mathscr{M}(x) - \mathscr{M}(x')| \leq \|m(\cdot, x) - m(\cdot, x')\|_2$. Now, for every $\varphi \in \mathscr{C}_0^\infty(\mathbb{R}^d)$ and every $\psi \in \mathrm{L}^2(\mathbb{R}^d)$ the Cauchy-Schwarz inequality provides the estimate

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} \mathrm{d}x\mathrm{d}y\, |\psi(y)|\, |m(y,x)|\, |\varphi(x)| \leq \|\psi\|_2\, \|\varphi\|_2\, \|\mathscr{M}\chi_{\mathrm{supp}\,\varphi}\|_2 < \infty \tag{4.4}$$

due to the continuity of $\mathscr{M}$. Therefore, (4.1) and Fubini's theorem yield

$$\langle M\varphi, \psi\rangle = \int_{\mathbb{R}^d} \mathrm{d}x\, \varphi^*(x)\, \langle m(\cdot, x), \psi\rangle\, , \tag{4.5}$$

where the scalar product in the integrand is well defined, because, by hypothesis, $m(\cdot, x) \in \mathrm{L}^2(\mathbb{R}^d)$ for all $x \in \mathbb{R}^d$. Next, we consider a sequence $(\psi_n)_{n\in\mathbb{N}} \subset \mathscr{C}_0^\infty(\mathbb{R}^d)$ with $\lim_{n\to\infty} \|\psi_n - \psi\|_2 = 0$ and $\sup_{n\in\mathbb{N}}\{\|\psi_n\|_2\} \leq 2\|\psi\|_2$. From the boundedness of $\overline{MBM}$, the continuity of the scalar product $\langle\cdot,\cdot\rangle$ and (4.5) we conclude

$$\begin{aligned}
\langle \varphi, \overline{MBM}\psi\rangle &= \lim_{n\to\infty} \langle \varphi, MBM\psi_n\rangle \\
&= \lim_{n\to\infty} \langle M\varphi, BM\psi_n\rangle \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^d} \mathrm{d}x\, \varphi^*(x)\, \langle m(\cdot, x), BM\psi_n\rangle \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^d} \mathrm{d}x\, \varphi^*(x)\, \langle MB^* m(\cdot, x), \psi_n\rangle\, .
\end{aligned} \tag{4.6}$$

Since

$$\sup_{n\in\mathbb{N}}\left|\langle MB^* m(\cdot,x),\psi_n\rangle\right| \le 2\,\|MB^*\|\,\|\psi\|_2\,\mathscr{M}(x) \tag{4.7}$$

for all $x \in \mathbb{R}^d$, MB^* is bounded and $\mathscr{M}$ is continuous, the dominated-convergence theorem and the continuity of the scalar product yield

$$\langle \varphi, \overline{MBM}\psi\rangle = \int_{\mathbb{R}^d} \mathrm{d}x\; \varphi^*(x)\,\langle MB^* m(\cdot,x),\psi\rangle \tag{4.8}$$

for all $\varphi \in \mathscr{C}_0^\infty(\mathbb{R}^d)$ and all $\psi \in \mathrm{L}^2(\mathbb{R}^d)$. Moreover, the function $\mathbb{R}^d \ni x \mapsto \langle MB^* m(\cdot,x),\psi\rangle$ belongs to $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^d)$, confer (4.7), so that the lemma of Du Bois-Reymond – also known as the fundamental lemma of the calculus of variations, see e.g. Lemma 3.26 in [5] – implies

$$\begin{aligned}\big(\overline{MBM}\psi\big)(x) &= \langle MB^* m(\cdot,x),\psi\rangle \\ &= \int_{\mathbb{R}^d}\mathrm{d}y \left[\int_{\mathbb{R}^d}\mathrm{d}z\; m(y,z)\,\big(B^* m(\cdot,x)\big)(z)\right]^* \psi(y) \\ &= \int_{\mathbb{R}^d}\mathrm{d}y\; \langle m(\cdot,x), Bm(\cdot,y)\rangle\,\psi(y)\end{aligned} \tag{4.9}$$

for Lebesgue-almost all $x \in \mathbb{R}^d$ and all $\psi \in \mathrm{L}^2(\mathbb{R}^d)$. To get the last equality, we have also used the Hermiticity, $m(x,y) = m^*(y,x)$ for Lebesgue-almost all pairs $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$. This proves (4.2).

The Carleman property (1.15) for β follows from part (iii) of the lemma (to be proven below). Indeed, since m is Hermitian and since $\mathscr{M}$ is continuous, one may choose $w = \chi_\Lambda$ in (4.3) for an arbitrary bounded Borel subset $\Lambda \subset \mathbb{R}^d$. This completes the proof of part (i).

The proof of assertion (ii) follows from the first equality in (4.9), the fact that the mapping $\mathbb{R}^d \to \mathrm{L}^2(\mathbb{R}^d)$, $x \mapsto m(\cdot,x)$, is strongly continuous, MB^* is bounded and $\langle\cdot,\cdot\rangle$ is continuous.

For the proof of assertion (iii) we exploit our assumption on w, the maximality of the Carleman operator M, (4.1) and Thm. VI.23 in [38] to conclude that $M\hat{w}$ is Hilbert-Schmidt. Therefore, $MBM\hat{w} = \overline{MBM}\hat{w}$ is Hilbert-Schmidt, too, by the boundedness of MB and the Hölder inequality for Schatten norms, see e.g. Thm. 2.8 in [41]. Thanks to $w \in \mathrm{L}^\infty(\mathbb{R}^d)$ and Eq. (4.2) we have $\overline{MBM}\hat{w}\psi = \int_{\mathbb{R}^d}\mathrm{d}y\,\beta(\cdot,y)\,w(y)\,\psi(y)$ for all $\psi \in \mathrm{L}^2(\mathbb{R}^d)$. Hence (4.3) follows from an anew application of Thm. VI.23 in [38]. ∎

After these preparations it is easy to deduce Theorem 1.14 as a special case.

Proof (of Theorem 1.14). We apply Lemma 4.1 with the choices $M = \mathrm{e}^{-tH(A,V)}$ and $B = \mathrm{e}^{2tH(A,V)}F\big(H(A,V)\big)$, where $t \in]0,\tau/2[$.

This is allowed, because Theorem 1.10 ensures that $\mathrm{e}^{-tH(A,V)}$ is a maximal Carleman operator with the required properties, recall Remark 1.6(ii), Lemma 1.7 and Remark 1.8(iv).

Furthermore, we observe from (1.20) and the functional calculus for unbounded functions of unbounded self-adjoint operators, see e.g. Chap. 5 in [7], that the operator product $B = \mathrm{e}^{2tH(A,V)} F\big(H(A,V)\big)$ is bounded. The functional calculus also guarantees that the two operator products MB and MB^* are bounded and that the equality $MBM = F\big(H(A,V)\big)$ holds on $\mathrm{dom}(M)$. The latter implies the boundedness of $\overline{MBM} = F\big(H(A,V)\big)$, because $F \in \mathrm{L}^\infty(\mathbb{R})$.

Finally, the finiteness of the integral $\int_{\mathbb{R}^d\times\mathbb{R}^d} \mathrm{d}x\mathrm{d}y\, |w(x)|^2 |k_t(x,y)|^2$ for all $w \in \mathrm{L}^\infty_{\mathrm{G}}(\mathbb{R}^d)$ follows from the estimate (1.13) with sufficiently small $\delta > 0$, inequality (2.14) and Remark 1.6(i). Thus, all assumptions of Lemma 4.1 are fulfilled and Theorem 1.14 holds with $f = \beta$ and for all $w \in \mathrm{L}^\infty_{\mathrm{G}}(\mathbb{R}^d)$. ∎

Next we show how to deduce Corollary 1.16 from Theorem 1.14.

Proof (of Corollary 1.16). Clearly, choosing $F = \chi_I$ in Theorem 1.14 is in accordance with (1.20) because of $\sup I < \infty$. Therefore, part (i) of this theorem yields the existence and continuity of the integral kernel p_I of $\chi_I\big(H(A,V)\big)$. To derive (1.24) we note that the operator $\hat{w}^*\chi_I\big(H(A,V)\big)\hat{w}$ is trace class by Theorem 1.14(iii) and $\chi_I^2 = \chi_I$. Moreover, thanks to $w \in \mathrm{L}^\infty_{\mathrm{G}}(\mathbb{R}^d)$ the $\mathrm{L}^2(\mathbb{R}^d \times \mathbb{R}^d)$-function $(x,y) \mapsto w^*(x) p_I(x,y) w(y)$ is an integral kernel for $\hat{w}^*\chi_I\big(H(A,V)\big)\hat{w}$. Recalling that $\Lambda_\ell(x)$ is the open cube in $\mathbb{R}^d$ with edge length $\ell > 0$ and centre $x \in \mathbb{R}^d$, an application of Thm. 3.1 in [8], see also [9] or [6], gives the equality

$$\begin{aligned}&\mathrm{Trace}\big[\hat{w}^*\chi_I\big(H(A,V)\big)\hat{w}\big]\\ &\quad= \int_{\mathbb{R}^d} \mathrm{d}x \, \lim_{\ell\downarrow 0} \ell^{-2d} \int_{\Lambda_\ell(x)\times\Lambda_\ell(x)} \mathrm{d}x'\mathrm{d}y'\, w^*(x')\, p_I(x',y')\, w(y') \,. \end{aligned} \tag{4.10}$$

The continuity of p_I and the Lebesgue differentiation theorem, see e.g. Sects. I.1.3 and I.1.8 in [44], now complete the proof because

$$\begin{aligned}&\lim_{\ell\downarrow 0} \ell^{-2d} \int_{\Lambda_\ell(x)\times\Lambda_\ell(x)} \mathrm{d}x'\mathrm{d}y'\, w^*(x')\, p_I(x',y')\, w(y')\\ &\qquad\qquad = p_I(x,x)\, \lim_{\ell\downarrow 0} \left| \ell^{-d} \int_{\Lambda_\ell(x)} \mathrm{d}x'\, w(x') \right|^2\\ &\qquad\qquad = p_I(x,x)\, |w(x)|^2 \end{aligned} \tag{4.11}$$

for Lebesgue-almost all $x \in \mathbb{R}^d$. ∎

Now we are concerned with the second corollary to Theorem 1.14.

Proof (of Corollary 1.18). We fix $x, y \in \mathbb{R}^d$. In the first case we apply the functional calculus to the right-hand side of (1.21). This gives

$$f(x, y) = \int_{\mathbb{R}} \mathrm{d}\vartheta_t(E; x, y)\, \mathrm{e}^{2tE}\, F(E) \tag{4.12}$$

for any $t \in]0, \tau/2[$ with the complex spectral "distribution" function $\vartheta_t(E; x, y) := \big\langle k_t(\cdot, x), \chi_{]-\infty,E[}\big(H(A, V)\big)k_t(\cdot, y)\big\rangle$. Here, $\tau > 0$ is the constant required to exist for F in (1.20). In particular, for $F = \chi_{]-\infty,E_0[}$ with $E_0 \in \mathbb{R}$, Eq. (4.12) takes the form

$$p(E_0; x, y) = \int_{-\infty}^{E_0} \mathrm{d}\vartheta_t(E; x, y)\, \mathrm{e}^{2tE}\,. \tag{4.13}$$

This equation holds for arbitrary $t > 0$, because τ can be chosen arbitrarily large in this particular case. Taken together, (4.12) and (4.13) yield the claim (1.25).

In the second case we may write

$$k_t(x, y) = \big\langle k_{t/2}(\cdot, x), k_{t/2}(\cdot, y)\big\rangle = \int_{\mathbb{R}} \mathrm{d}\vartheta_{t/2}(E; x, y) = \int_{\mathbb{R}} \mathrm{d}\,p(E; x, y)\, \mathrm{e}^{-tE} \tag{4.14}$$

for all $t > 0$. Here, the first equality is due to the Hermiticity and the semigroup property of the kernel k_t, the second equality is just the definition of $\vartheta_{t/2}$ and the last equality follows from (4.13). ∎

For convenience, we formulate and prove simple estimates on the integral kernel of a spectral projection in the remainder of this section. We will only need these estimates for the applications to random Schrödinger operators.

LEMMA 4.2. *Assume the situation of Corollary* 1.16. *Then the diagonal of the continuous integral kernel p_I of the spectral projection $\chi_I\big(H(A, V)\big)$ obeys the estimates*

$$0 \le p_I(x, x) \le \mathrm{e}^{t \sup I}\, k_t(x, x) \tag{4.15}$$

for all $x \in \mathbb{R}^d$ with any $t \in]0, \infty[$.

Proof. Fix $x \in \mathbb{R}^d$ arbitrary, pick $\varphi \in \mathscr{C}_0^\infty(\mathbb{R}^d)$ and define $\varphi_x^{(\varepsilon)}$ by $\varphi_x^{(\varepsilon)}(y) := \varepsilon^{-d}\varphi\big((y - x)/\varepsilon\big)$ for every $y \in \mathbb{R}^d$ and every $\varepsilon \in]0, 1]$. Then $\{\varphi_x^{(\varepsilon)}\}_{\varepsilon\in]0,1]} \subset \mathrm{L}^2(\mathbb{R}^d)$ is a family of approximating delta functions at $x \in \mathbb{R}^d$. By the continuity of p_I and the dominated-convergence theorem one gets the representation

$$p_I(x, x) = \lim_{\varepsilon\downarrow 0}\langle\varphi_x^{(\varepsilon)}, \chi_I\big(H(A, V)\big)\varphi_x^{(\varepsilon)}\rangle\,. \tag{4.16}$$

The same arguments yield

$$k_t(x,x) = \lim_{\varepsilon\downarrow 0}\langle \varphi_x^{(\varepsilon)}, \mathrm{e}^{-tH(A,V)}\varphi_x^{(\varepsilon)}\rangle \tag{4.17}$$

for any $t \in]0,\infty[$. The claim (4.15) now follows from the functional calculus and the elementary inequalities

$$0 \le \chi_I(E) \le \mathrm{e}^{t(\sup I - E)} \tag{4.18}$$

for all $E \in \mathbb{R}$. ∎

5. PROOFS OF LEMMA 1.22, COROLLARY 1.27 AND COROLLARY 1.29

Proof (of Lemma 1.22). We mimic the proof of [30], see also Prop. V.3.2 in [12]. By the definition of $p(d)$ in property (S) and since $(d/2)p_1/[p_1 - p(d)] < p_2$, we can find $\nu \in]0,2[$ and $r \in]p(d), p_1[$ such that

$$\frac{d}{\nu}\,\frac{p_1}{p_1 - r} < p_2\,. \tag{5.1}$$

Next, we pick a constant $c \in]0,\infty[$ and define

$$V_2^{(\omega)}(x) := V^{(\omega)}(x)\,\Theta\big(c(1+|x|^\nu) - |V^{(\omega)}(x)|\big)\,, \tag{5.2a}$$

$$V_1^{(\omega)}(x) := V^{(\omega)}(x) - V_2^{(\omega)}(x) \tag{5.2b}$$

for all $\omega \in \Omega$ and all $x \in \mathbb{R}^d$. Clearly, for every $\omega \in \Omega$ the realization $V_2^{(\omega)}$ satisfies (1.4) for all $\varepsilon > 0$. We will show below that $V_1^{(\omega)} \in \mathrm{L}^r_{\mathrm{unif,loc}}(\mathbb{R}^d)$ for $\mathbb{P}$-almost all $\omega \in \Omega$. This proves the lemma, because $\mathrm{L}^r_{\mathrm{unif,loc}}(\mathbb{R}^d) \subseteq \mathscr{K}(\mathbb{R}^d)$, see e.g. Eq. (A 21) in [42] for $d \ge 2$ and note $\mathscr{K}(\mathbb{R}) = \mathrm{L}^1_{\mathrm{unif,loc}}(\mathbb{R})$.

In this proof we use the abbreviation $\Lambda(y) := \Lambda_1(y)$ for the open unit cube in $\mathbb{R}^d$ with centre $y \in \mathbb{R}^d$. To prove $\mathbb{P}\big[V_1 \in \mathrm{L}^r_{\mathrm{unif,loc}}(\mathbb{R}^d)\big] = 1$ we apply the "Chebyshev-Markov" inequality $\Theta(\xi - 1) \le |\xi|^\kappa$ with $\kappa = p_1 - r > 0$ to obtain for all $\omega \in \Omega$ the estimate

$$\|V_1^{(\omega)}\chi_{\Lambda(y)}\|_r^r = \int_{\Lambda(y)} \mathrm{d}x\,|V^{(\omega)}(x)|^r\,\Theta\left(\frac{|V^{(\omega)}(x)|}{c(1+|x|^\nu)} - 1\right) \le \frac{\tilde{c}^r\,\|V^{(\omega)}\chi_{\Lambda(y)}\|_{p_1}^{p_1}}{(1+|y|^\nu)^{p_1-r}} \tag{5.3}$$

for all $y \in \mathbb{Z}^d$ with some constant $\tilde{c} \in]0, \infty[$, which is independent of $y \in \mathbb{Z}^d$. This implies

$$\begin{aligned}
\sum_{y\in\mathbb{Z}^d} \mathbb{P}\big[\|V_1\chi_{\Lambda(y)}\|_r > 1\big] \;&\leq\; \sum_{y\in\mathbb{Z}^d} \mathbb{E}\left[\Theta\left(\frac{\tilde{c}\,\|V\chi_{\Lambda(y)}\|_{p_1}^{p_1/r}}{(1+|y|^{\nu})^{(p_1-r)/r}} - 1\right)\right] \\
&\leq\; \tilde{c}^q \sum_{y\in\mathbb{Z}^d} \frac{\mathbb{E}\big[\|V\chi_{\Lambda(y)}\|_{p_1}^{p_1 q/r}\big]}{(1+|y|^{\nu})^{(p_1-r)q/r}}\,. \qquad (5.4)
\end{aligned}$$

In order to get the second inequality in (5.4), we used the "Chebyshev-Markov" inequality with $\kappa = q$, where q is chosen such that

$$\frac{d}{\nu}\,\frac{p_1}{p_1 - r} < \frac{p_1 q}{r} < p_2\,. \qquad (5.5)$$

The numerator in the second line of (5.4) is uniformly bounded in $y \in \mathbb{Z}^d$ due to the right inequality in (5.5), Jensen's inequality and property (S). The left inequality in (5.5) then assures that the series in the second line of (5.4) is summable, which implies by the first Borel-Cantelli lemma

$$\mathbb{P}\big[\|V_1\chi_{\Lambda(y)}\|_r > 1 \;\text{ for infinitely many }\; y \in \mathbb{Z}^d\big] = 0\,. \qquad (5.6)$$

This delivers

$$\begin{aligned}
\mathbb{P}\left[\sup_{y\in\mathbb{Z}^d} \|V_1\chi_{\Lambda(y)}\|_r = \infty\right] \;&=\; \mathbb{P}\big[\|V_1\chi_{\Lambda(y_0)}\|_r = \infty \;\text{ for some }\; y_0 \in \mathbb{Z}^d\big] \\
&\leq\; \sum_{y\in\mathbb{Z}^d} \mathbb{P}\big[\|V_1\chi_{\Lambda(y)}\|_r = \infty\big] \\
&\leq\; \sum_{y\in\mathbb{Z}^d} \mathbb{P}\big[\|V\chi_{\Lambda(y)}\|_{p_1} = \infty\big] \\
&=\; 0\,, \qquad (5.7)
\end{aligned}$$

where we have used the countable subadditivity of $\mathbb{P}$ for the first inequality and $|V_1| \leq |V|$ as well as $r < p_1$ for the second inequality. The last equality in (5.7) follows from property (S). Thus, we have shown

$$\mathbb{P}\big[V_1 \in \mathrm{L}^r_{\mathrm{unif,loc}}(\mathbb{R}^d)\big] = 1\,. \qquad \blacksquare \qquad (5.8)$$

For the proof of Corollary 1.27 and Corollary 1.29 we need suitable measurability properties of the involved integral kernels, which we establish in

LEMMA 5.1. *Let A be a vector potential with property* (A) *and let V be a random scalar potential with property* (S). *Then there exists $\Omega_0 \in \mathscr{A}$ with $\mathbb{P}(\Omega_0) = 1$ such that for every $\omega \in \Omega_0$*

(i) *the operator exponential $\mathrm{e}^{-tH(A,V^{(\omega)})}$ has a continuous integral kernel $k_t^{(\omega)}$ for any $t > 0$ and the mapping*

$$\begin{array}{ccc} \Omega_0 \times]0, \infty[\times \mathbb{R}^d \times \mathbb{R}^d & \to & \mathbb{C} \\ (\omega, t, x, y) & \mapsto & k_t^{(\omega)}(x, y) \end{array} \tag{5.9}$$

is $\mathscr{A}_0 \otimes \mathscr{B}(]0, \infty[) \otimes \mathscr{B}(\mathbb{R}^d) \otimes \mathscr{B}(\mathbb{R}^d)$-measurable.

(ii) *the spectral projection $\chi_{]-\infty,E[}\big(H(A, V^{(\omega)})\big)$ has a continuous integral kernel $p^{(\omega)}(E; \cdot, \cdot)$ for any $E \in \mathbb{R}$ and the mapping*

$$\begin{array}{ccc} \Omega_0 \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d & \to & \mathbb{C} \\ (\omega, E, x, y) & \mapsto & p^{(\omega)}(E; x, y) \end{array} \tag{5.10}$$

is $\mathscr{A}_0 \otimes \mathscr{B}(\mathbb{R}) \otimes \mathscr{B}(\mathbb{R}^d) \otimes \mathscr{B}(\mathbb{R}^d)$-measurable.
Here, $\mathscr{A}_0$ is the restriction of the sigma-algebra $\mathscr{A}$ of Ω to Ω_0, and given any Borel set $B \subseteq \mathbb{R}^d$ we denote by $\mathscr{B}(B)$ the sub-sigma-algebra of Borel sets in $\mathbb{R}^d$ which are contained in B.

Proof. The existence and continuity of the integral kernels is guaranteed by Corollary 1.24, Lemma 1.7, Theorem 1.10 and Corollary 1.16 (see also Corollary 1.18). The measurability claimed in (i) follows from the Brownian-bridge representation (1.11) for $k_t^{(\omega)}$. The claim of (ii) follows from (i), Corollary 1.18 and the invertibility of the Laplace transformation. ∎

Proof (of Corollary 1.27). We fix $E \in \mathbb{R}$ arbitrary. Lemma 5.1(ii) guarantees the existence, continuity and suitable measurability properties of the integral kernel $p^{(\omega)}(E; \cdot, \cdot)$ of the spectral projection $\chi_{]-\infty,E[}\big(H(A, V^{(\omega)})\big)$ for all $\omega \in \Omega_0 \in \mathscr{A}$ with $\mathbb{P}(\Omega_0) = 1$. Eq. (1.24) and Proposition 1.25 imply that

$$N(E) = \mathbb{E}\left[\int_\Gamma \frac{\mathrm{d}x}{|\Gamma|}\, p(E; x, x)\right] \tag{5.11}$$

is finite. Now the claim follows from Fubini's theorem, because $p^{(\omega)}(E; x, x) \geq 0$ for all $\omega \in \Omega_0$ and all $x \in \mathbb{R}^d$, see Lemma 4.2, and because $\mathbb{E}[p(E; x, x)]$ is independent of $x \in \mathbb{R}^d$ due to the $\mathbb{R}^d$-ergodicity of V. ∎

Proof (of Corollary 1.29). We fix $t > 0$ arbitrary. Lemma 5.1(i) guarantees the existence, continuity and suitable measurability properties of the integral kernel $k_t^{(\omega)}$ of the operator exponential $\mathrm{e}^{-tH(A,V^{(\omega)})}$ for all $\omega \in \Omega_0 \in \mathscr{A}$ with

$\mathbb{P}(\Omega_0) = 1$. Jensen's inequality, Fubini's theorem and property (L) imply for $\mu_{x,y}^{0,t}$-almost every path b of the Brownian bridge the estimate

$$\mathbb{E}\left[\exp\left\{-\int_0^t \mathrm{d}s\; V(b(s))\right\}\right] \le \int_0^t \frac{\mathrm{d}s}{t}\; \mathbb{E}\big[\exp\{-tV(b(s))\}\big] \le \mathscr{L}_t < \infty\,, \tag{5.12}$$

which shows that the integral kernel $\overline{k_t}$ is well defined and obeys the inequality

$$|\overline{k_t}(x,y)| \le \mathbb{E}\big[|k_t(x,y)|\big] \le \mathscr{L}_t\, \frac{\mathrm{e}^{-|x-y|^2/(2t)}}{(2\pi t)^{d/2}} \tag{5.13}$$

for all $x, y \in \mathbb{R}^d$, thereby proving (1.33). The Hermiticity of $\overline{k_t}$ is inherited from that of k_t, see Lemma 1.7(i). The estimate (5.13) also yields $\overline{k_t}(x,\cdot) \in \mathrm{L}^\infty_{\mathrm{G}}(\mathbb{R}^d)$ for all $x \in \mathbb{R}^d$, and hence the Carleman property (1.15) for $\overline{k_t}$. We defer the proof of the continuity of $\overline{k_t}$ to the end, but exploit its consequences right now. Jensen's inequality, Fubini's theorem and the almost-surely applicable Markov property (1.12) yield the estimate

$$\begin{aligned} \|\overline{k_t}(x,\cdot) - \overline{k_t}(z,\cdot)\|_2^2 \;&\le\; \int_{\mathbb{R}^d} \mathrm{d}y\; \mathbb{E}\big[|k_t(x,y) - k_t(z,y)|^2\big] \\ &=\; \overline{k_{2t}}(x,x) - \overline{k_{2t}}(z,x) - \overline{k_{2t}}(x,z) + \overline{k_{2t}}(z,z)\,, \end{aligned} \tag{5.14}$$

showing that the continuity of $\overline{k_{2t}}$ implies the strong continuity of the mapping $\mathbb{R}^d \to \mathrm{L}^2(\mathbb{R}^d)$, $x \mapsto \overline{k_t}(x,\cdot)$.

The estimate (5.13) delivers

$$|T_t\psi| \le \mathscr{L}_t\, \mathrm{e}^{-tH(0,0)}|\psi| \tag{5.15}$$

for all $\psi \in \mathrm{L}^2(\mathbb{R}^d)$, where T_t is defined as in (1.34). Consequently, T_t is a bounded Carleman operator on $\mathrm{L}^2(\mathbb{R}^d)$. Moreover, T_t is self-adjoint because of the Hermiticity of $\overline{k_t}$ and an interchange of integrations thanks to (5.13) and Fubini's theorem. The continuity of any image $T_t\psi$ follows from the strong continuity of $\overline{k_t}(x,\cdot)$ by proceeding along the lines of Eq. (3.6) in the proof of Lemma 3.1.

Now let $\psi \in \mathrm{L}^2_{\mathrm{G}}(\mathbb{R}^d)$ so that the equality $T_t\psi = \mathbb{E}\big[\mathrm{e}^{-tH(A,V)}\psi\big]$ follows from (1.17) and an interchange of integrations. This interchange is again allowed by Fubini's theorem and (5.13). The inequalities (5.13) and (2.14) imply that $T_t\psi \in \mathrm{L}^\infty_{\mathrm{G}}(\mathbb{R}^d)$ for all $\psi \in \mathrm{L}^2_{\mathrm{G}}(\mathbb{R}^d)$. Remark 1.11(iii) applies accordingly.

Next we establish the positivity of T_t. Given any $\psi \in \mathrm{L}^2_{\mathrm{G}}(\mathbb{R}^d)$, one deduces from the just-proven equality (1.35), the estimate (5.13) and Fubini's theorem that $\langle\psi, T_t\psi\rangle = \mathbb{E}\big[\langle\psi, \mathrm{e}^{-tH(A,V)}\psi\rangle\big] \ge 0$, where the lower bound follows from the positivity of $\mathrm{e}^{-tH(A,V^{(\omega)})}$ for $\mathbb{P}$-almost all $\omega \in \Omega$. Now, the denseness of $\mathrm{L}^2_{\mathrm{G}}(\mathbb{R}^d)$ in $\mathrm{L}^2(\mathbb{R}^d)$, the boundedness of T_t and the continuity of the scalar product yield $\langle\psi, T_t\psi\rangle \ge 0$ for all $\psi \in \mathrm{L}^2(\mathbb{R}^d)$.

Finally, we turn to the postponed proof of the continuity of the mapping $\mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$, $(x, y) \mapsto \overline{k_t}(x, y)$. This continuity will follow from Lemma 5.1(i) and the dominated-convergence theorem, provided we show

$$\mathbb{E}\Big[\sup_{x,y\in\mathscr{K}} |k_t(x, y)|\Big] < \infty \tag{5.16}$$

for any bounded set $\mathscr{K} \subset \mathbb{R}^d \times \mathbb{R}^d$. In order to do so, let us fix $\omega \in \Omega_0$ and $x, y \in \mathscr{K}$ arbitrary. By using (1.11), the triangle inequality, Jensen's inequality and Fubini's theorem, we get

$$\begin{aligned} |k_t^{(\omega)}(x, y)| &\le (2\pi t)^{-d/2} \int_0^t \frac{\mathrm{d}s}{t} \int \mu_{x,y}^{0,t}(\mathrm{d}b)\, \mathrm{e}^{-tV^{(\omega)}(b(s))} \\ &= (2\pi t)^{-d/2} \int_0^1 \mathrm{d}\sigma \int_{\mathbb{R}^d} \mathrm{d}z\, g_\sigma\big(z - m_{x,y}(\sigma)\big)\, \mathrm{e}^{-tV^{(\omega)}(z)}, \end{aligned} \tag{5.17}$$

where the equality follows from an explicit computation with $m_{x,y}(\sigma) := x + (y - x)\sigma$ and

$$g_\sigma(z) := \frac{\exp\{-|z|^2/[2(1-\sigma)\sigma t]\}}{[2\pi(1-\sigma)\sigma t]^{d/2}}. \tag{5.18}$$

Next we apply Hölder's inequality with the conjugated exponents $p \in]1, \infty[$ and $p' := (1 - p^{-1})^{-1}$ to the integral with respect to z in (5.17), which yields the upper bound

$$\left(\int_{\mathbb{R}^d} \mathrm{d}z\, \mathrm{e}^{-ptV^{(\omega)}(z)}\, \mathrm{e}^{-p|z|}\right)^{1/p} \left(\int_{\mathbb{R}^d} \mathrm{d}z\, \mathrm{e}^{p'|z|} \big|g_\sigma\big(z - m_{x,y}(\sigma)\big)\big|^{p'}\right)^{1/p'}. \tag{5.19}$$

The second integral in (5.19) is bounded from above by

$$\mathrm{e}^{p'\max\{|x|,|y|\}} \int_{\mathbb{R}^d} \mathrm{d}z\, \mathrm{e}^{p'|z|}\, |g_\sigma(z)|^{p'} \le \mathrm{e}^{p'\max\{|x|,|y|\}}\, [(1-\sigma)\sigma t]^{(1-p')d/2}\, I_{p'}, \tag{5.20}$$

where $I_{p'} := (2\pi)^{-d/2} \int_{\mathbb{R}^d} \mathrm{d}\zeta\, \mathrm{e}^{-p'(|\zeta|^2 - |\zeta|\sqrt{t})/2} < \infty$ for any $p' > 1$. This gives the estimate

$$\begin{aligned} \mathbb{E}\Big[\sup_{x,y\in\mathscr{K}} |k_t(x, y)|\Big] &\le (2\pi t)^{-d/2} I_{p'}^{1/p'} \Big(\sup_{z\in\mathscr{K}} \mathrm{e}^{|z|}\Big) \int_0^1 \mathrm{d}\sigma [(1-\sigma)\sigma t]^{-d/(2p)} \\ &\quad \times \mathbb{E}\left[\left(\int_{\mathbb{R}^d} \mathrm{d}z\, \mathrm{e}^{-ptV(z)}\, \mathrm{e}^{-p|z|}\right)^{1/p}\right]. \end{aligned} \tag{5.21}$$

The expectation value on the right-hand side of (5.21) is finite for any $p > 1$ by Jensen's inequality, property (L) and Fubini's theorem. Therefore (5.16) follows from the boundedness of $\mathscr{K}$ and by choosing $p > \max\{1, d/2\}$. ∎

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