On singularly perturbed partial integro-differential equations with irregular singularity

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Abstract

We study exact asymptotic properties of solutions of singular singularly perturbed partial integro-differential equations of irregular type in the complex domain in its perturbation parameter. A result on $1-$summability is obtained for formal power series solutions of the given problem in the perturbation parameter.

Key words: Cauchy problem, partial integro-differential equation, Gevrey estimates, singular perturbation. 2000 MSC: 35C10, 35C20.

1 Introduction

In this paper, we study singular singularly perturbed partial integro-differential equations of the form

$$(1) \quad 4\epsilon t^2 \partial_t \partial_z^S X(t,z,\epsilon) + (2t\epsilon + 1)\partial_z^S X(t,z,\epsilon) = \sum_{(k_0,k_1)\in\mathcal{S}} b_{k_0,k_1}(z,\epsilon)\partial_t^{k_0}\partial_z^{k_1}X(t,z,\epsilon) + b(z,\epsilon)\partial_t \int_0^t X(t-\tau,z,\epsilon)X(\tau,z,\epsilon)d\tau,$$

where ϵ is a complex perturbation parameter on a neighborhood of the origin in $\mathbb{C}$, $S \geq 1$ is a positive integer, $\mathcal{S}$ is a finite subset of $\mathbb{N}^2$ and the coefficients $b_{k_0,k_1}(z,\epsilon), b(z,\epsilon)$ are holomorphic functions on a neighborhood of the origin in $\mathbb{C}^2$.

These equations (1) belong to a class of partial differential equations with irregular singularity at $t = 0$ for $\epsilon \neq 0$.

Our goal is to investigate exact asymptotics of actual solutions $X(t,z,\epsilon)$ of (1) in the ϵ variable whose coefficients belong to some Banach space of holomorphic functions on a domain which is a product of a sector and a disc, for given initial conditions $(\partial_z^j X)(t,0,\epsilon)$, $0 \leq j \leq S-1$.

This kind of asymptotic expansions are worked out to a large extent in the framework of singular perturbation of differential equations in the complex and real domain. For recent results in this fields, see [4],[5],[6],[7],[8],[16],[17]. But very few results are known in the context of singularly perturbed partial differential equations with complex parameter although there exists a huge literature on singularly perturbed PDEs with real parameter considered from several aspects both asymptotic and numeric, see [14].

Our approach is based on a resummation method of formal series

$$\hat{Y}(t,z,\epsilon) = \sum_{l\geq 0} Y_l(\epsilon,z)t^l$$

with respect to t, solution of an auxiliary regularly perturbed partial differential equation with irregular singularity at $t=0$, see (25). This resummation process is known as Borel summability, starting with the fundamental contributions of Écalle, see [10]. This method has been applied with success in several domains such as differential equations in the complex domain, see [2],[13],[15] and more recently in the framework of partial differential equations, see for instance [3],[9],[12].

The methodology of proof will follow the paper [7]. The first step will consist in the construction of $1-$sums of a formal series $\hat{Y}(t,z,\epsilon)$ in the t variable, solution of the auxiliary problem (25), for given formal series initial conditions $(\partial_z^j \hat{Y})(t,0,\epsilon)$, $0 \leq j \leq S-1$ (Theorem 2). These $1-$sums $Y_i(t,z,\epsilon)$, $i \in I$ where I is some finite subset of $\mathbb{N}$, are solutions of (25) and are holomorphic and bounded functions on products $T_i \times D(0,\rho) \times \mathcal{E}_i$, where $\mathcal{E}_i$ are sectors whose union form a good covering of a neighborhood of the origin in $\mathbb{C}$ and T_i are well chosen sectors. The main tool in the proof is a global Cauchy-Kowalevski theorem described in section 3 which is applied to the $1-$Borel transform of $\hat{Y}(t,z,\epsilon)$ with respect to t.

In a second step, we observe first that the functions $X_i(t,z,\epsilon) = Y_i(\epsilon t,z,\epsilon)$ are solutions of the singularly perturbed problem (1) on domains of the form $\mathcal{T} \times D(0,\rho) \times \mathcal{E}_i$, for all $i \in I$, where $\mathcal{T}$ is a sector. The key argument is that these functions X_i, $i \in I$, satisfy a cohomological criterion for $1-$summability in the perturbation parameter ϵ of formal series with coefficients in a Banach space (Theorem CC). This finally yields the existence and unicity of a formal series $\hat{X}(t,z,\epsilon)$ with respect to ϵ, with coefficients being holomorphic and bounded functions on $\mathcal{T} \times D(0,\rho)$ which is solution of (1). Moreover, the $1-$sums of $\hat{X}$ with respect to ϵ on $\mathcal{E}_i$ are the functions $\epsilon \mapsto X_i(\epsilon,t,z)$, for all $i \in I$, (Theorem 3).

2 A non-linear Cauchy-Kowalevski theorem

In this section, we describe a version of the Cauchy-Kowalevski theorem for non-linear partial differential equations for holomorphic functions.

Definition 1 *Let X,T be real numbers such that $X,T>0$. We define a vector space $G(T,X)$ of holomorphic functions on a neighborhood of the origin in $\mathbb{C}^2$. A formal series $U(t,x) \in \mathbb{C}[[t,x]]$,*

$$U(t,x) = \sum_{\beta,l\geq 0} u_{l,\beta} \frac{x^\beta}{\beta!}\frac{t^l}{l!}$$

belongs to $G(T,X)$, if the series

$$\sum_{\beta,l\geq 0} \frac{|u_{l,\beta}|}{(l+\beta)!} X^\beta T^l,$$

converges. We also define a norm on $G(T,X)$ as

$$||U(t,x)||_{(T,X)} = \sum_{\beta,l\geq 0} \frac{|u_{l,\beta}|}{(l+\beta)!} X^\beta T^l.$$

One can easily show that $(G((T,X)), ||.||_{(T,X)})$ *is a Banach space.*

Remark: Let $U(t,x)$ be in $G(T_0,X_0)$ for given $T_0,X_0>0$. Then, $U(t,x)$ also belong to the spaces $G(T,X)$ for all $X \leq X_0$ and $T \leq T_0$. Moreover, the maps $X \to ||U(t,x)||_{(T,X)}$ and $T \to ||U(t,x)||_{(T,X)}$ are increasing functions from $[0,X_0]$ (resp. $[0,T_0]$) into $\mathbb{R}_+$.

We state a theorem of Cauchy-Kowalevski with estimates for the solution in terms of the initial conditions.

Theorem (CK) *Consider a partial differential equation of the form*

$$\partial_x^S u(t,x) = \sum_{(k_0,k_1)\in\mathcal{S}} a_{(k_0,k_1)}(t,x)\partial_t^{k_0}\partial_x^{k_1}u(t,x) + a(t,x)(u(t,x)^2 + u(0,x)\partial_t u(t,x)) \tag{2}$$

where $\mathcal{S}$ *is a finite subset of* $\mathbb{N}\times\mathbb{N}$. *The coefficients* $a_{(k_0,k_1)}(t,x), a(t,x)$ *belong to* $G(T_0,X_0)$ *where* $T_0,X_0>0$. *We assume that the following assumption hold:*

(H1) *For all* $(k_0,k_1)\in\mathcal{S}$, *we have* $S \geq k_0+k_1$ *and* $S > k_1$.

We impose the initial conditions : For all $0 \leq k \leq S-1$,

$$(\partial_x^k u)(t,0) = \varphi_k(t), \tag{3}$$

where $\varphi_k(t)$ *belong to* $G(T_0,X_0)$, *for all* $0\leq k\leq S-1$.
Then, there exist $T,\delta>0$ *and* X *small enough such that, if*

$$||a_{(k_0,k_1)}(t,x)||_{(T_0,X_0)} \leq \delta \quad , \quad ||a(t,x)||_{(T_0,X_0)} \leq \delta,$$

for all $(k_0,k_1)\in\mathcal{S}$, *then the problem (2), (3) has a unique solution* $u(t,x)$ *in* $G(T,X)$.

We start with some preliminaries.

Lemma 1 *Let* $U_1(t,x)$, $U_2(t,x)$ *be in* $G(T,X)$. *Then, the product* $U_1(t,x)U_2(t,x)$ *belongs to* $G(T,X)$. *Moreover,*

$$||U_1(t,x)U_2(t,x)||_{(T,X)} \leq ||U_1(t,x)||_{(T,X)}||U_2(t,x)||_{(T,X)}. \tag{4}$$

In other words, the normed vector space $(G(T,X), ||.||_{(T,X)})$ *is in fact a Banach algebra.*

Proof For $k=1,2$, let

$$U_k(t,x) = \sum_{\beta,l\geq 0} u_{\beta,l}^k \frac{x^\beta}{\beta!}\frac{t^l}{l!}.$$

We have

$$U_1(t,x)U_2(t,x) = \sum_{\beta,l\geq 0} v_{\beta,l}\frac{x^\beta}{\beta!}\frac{t^l}{l!},$$

where

$$v_{\beta,l} = \sum_{\beta_1+\beta_2=\beta}\sum_{l_1+l_2=l}\left(\frac{u^1_{\beta_1,l_1}}{\beta_1!l_1!}\frac{u^2_{\beta_2,l_2}}{\beta_2!l_2!}\beta!l!\right).$$

By definition, we have

$$||U_1(t,x)||_{(T,X)}||U_2(t,x)||_{(T,X)} = \sum_{\beta,l\geq 0}\left(\sum_{\beta_1+\beta_2=\beta}\sum_{l_1+l_2=l}\frac{|u^1_{\beta_1,l_1}||u^2_{\beta_2,l_2}|}{(l_1+\beta_1)!(l_2+\beta_2)!}\right)X^\beta T^l,$$

and

$$||U_1(t,x)U_2(t,x)||_{(T,X)} = \sum_{\beta,l\geq 0} \left| \sum_{\beta_1+\beta_2=\beta} \sum_{l_1+l_2=l} \frac{u^1_{\beta_1,l_1} u^2_{\beta_2,l_2} l!\beta!}{l_1!\beta_1!l_2!\beta_2!} \right| \frac{X^\beta T^l}{(l+\beta)!}$$

On the other hand we have that

$$\frac{l!\beta!}{l_1!\beta_1!l_2!\beta_2} \leq \frac{(l+\beta)!}{(l_1+\beta_1)!(l_2+\beta_2)!} \tag{5}$$

for all $l_1, l_2 \geq 0$ such that $l_1 + l_2 = l$ and $\beta_1, \beta_2 \geq 0$ such that $\beta_1 + \beta_2 = \beta$.

Finally, from this inequality (5), we deduce that $||U_1(t,x)U_2(t,x)||_{(T,X)}$ converges and that the estimates (4) hold. □

Lemma 2 *Let $h_0, h_1 \in \mathbb{N}$ such that $h_0 \leq h_1$. The operator $\partial_t^{h_0}\partial_x^{-h_1}$ is a bounded linear operator from $(G(T,X), ||.||_{(T,X)})$ into itself. Moreover, there exists a constant $C > 0$ such that the estimates hold*

$$||\partial_t^{h_0}\partial_x^{-h_1} U(t,x)||_{(T,X)} \leq C X^{h_1} T^{-h_0} ||U(t,x)||_{(T,X)}, \tag{6}$$

for all $U(t,x) \in G(T,X)$.

Proof Let $U(t,x)$ be in $G(T,X)$. We have,

$$\partial_t^{h_0}\partial_x^{-h_1} U(t,x) = \sum_{\beta,l} u_{\beta-h_1,l+h_0} \frac{x^\beta}{\beta!}\frac{t^l}{l!}$$

By definition, we have

$$X^{h_1}T^{-h_0}||U(t,x)||_{(T,X)} = \sum_{\beta,l} \frac{|u_{\beta,l}|}{(l+\beta)!} X^{\beta+h_1}T^{l-h_0} = \sum_{\beta,l} \frac{|u_{\beta-h_1,l+h_0-\alpha}|}{((l+h_0)+\beta-h_1)!} X^\beta T^l$$

and

$$||\partial_t^{h_0}\partial_x^{-h_1} U(t,x)||_{(T,X)} = \sum_{\beta,l} \frac{|u_{\beta-h_1,l+h_0}|}{(l+\beta)!} X^\beta T^l.$$

Finally, from the inequality $(l + h_0 + \beta - h_1)! \leq (l + \beta)!$, we deduce that $||\partial_t^{h_0}\partial_z^{-h_1} U(t,x)||_{(T,X)}$ converges and that the estimates (6) hold. □

Lemma 3 *Let $h_1, h_2 \in \mathbb{N}$ and let $U(t,x)$ be in $G(T_0,X_0)$. Then, there exist $X, T > 0$ small enough (depending on h_1, h_2) such that the formal series $\partial_t^{h_1}\partial_x^{h_2} U(t,x)$ belongs to $G(T,X)$. Moreover, there exists a constant $K > 0$ such that*

$$||(\partial_t^{h_1}\partial_x^{h_2} U)(t,x)||_{(T,X)} \leq T^{-h_1}X^{-h_2} K ||U(t,x)||_{(T_0,X_0)}, \tag{7}$$

for all $U(t,x) \in G(T_0,X_0)$.

Proof Let $U(t,x)$ be in $G(T,X)$. By definition we have

$$||(\partial_t^{h_1}\partial_x^{h_2} U)(t,x)||_{(T,X)}$$
$$= T^{-h_1}X^{-h_2} \sum_{\beta,l} \left\{ \frac{((l+h_1)+\beta+h_2)!}{(l+\beta)!} \right\} \frac{|u_{\beta+h_2,l+h_1}|}{((l+h_1)+\beta+h_2)!} X^{\beta+h_2}T^{l+h_1}.$$

But we have
$$\frac{((l+h_1)+\beta+h_2)!}{(l+\beta)!} \sim (l+\beta)^{h_1+h_2},$$
as l,β tend to infinity. So that there exist $K, C_1, C_2 > 0$ such that
$$||(\partial_t^{h_1}\partial_x^{h_2}U)(t,x)||_{(T,X)} \leq T^{-h_1}X^{-h_2}K\sum_{\beta,l}\frac{|u_{\beta+h_2,l+h_1}|}{((l+h_1)+\beta+h_2)!}(C_2X)^{\beta+h_2}(C_1T)^{l+h_1}.$$

Due to the fact that $U(t,x) \in G(T_0,X_0)$, the series on the right hand side of the latter inequality is convergent if $C_2X \leq X_0$ and $C_1T \leq T_0$. The lemma follows. □

In the following, we put
$$w(t,x) = \sum_{j=0}^{S-1}\varphi_j(t)\frac{x^j}{j!},$$
where $\varphi_j(t)$ belong to $G(T_0,X_0)$.

Definition 2 *Consider the map $A : \mathbb{C}[[t,x]] \to \mathbb{C}[[t,x]]$ defined as follows*
$$A(v(t,x)) = \sum_{(k_0,k_1)\in\mathcal{S}} a_{(k_0,k_1)}(t,x)\partial_t^{k_0}\partial_x^{k_1-S}v(t,x)$$
$$+\sum_{(k_0,k_1)\in\mathcal{S}} a_{(k_0,k_1)}(t,x)\partial_t^{k_0}\partial_x^{k_1}w(t,x)+$$
$$a(t,x)\left((\partial_x^{-S}v(t,x)+w(t,x))^2+(\partial_x^{-S}v(0,x)+w(0,x))(\partial_t\partial_x^{-S}v(t,x)+\partial_t w(t,x))\right)$$
for all $v(t,x) \in \mathbb{C}[[t,x]]$.

Notice that if $v(t,x)$ is a solution of $A(v(t,x)) = v(t,x)$, then $u(t,x) = \partial_x^{-S}v(t,x) + w(t,x)$ is a solution of (2), (3) in Theorem (CK).

Proposition 1 *Assume that the assumption* (**H1**) *is satisfied. Then, there exist $R,\delta,X,T > 0$ such that, if*
$$||a(t,x)||_{(T_0,X_0)} \leq \delta, \quad ||a_{(k_0,k_1)}(t,x)||_{(T_0,X_0)} \leq \delta,$$
for all $(k_0,k_1) \in \mathcal{S}$, then the map $A : \mathbb{C}[[t,x]] \to \mathbb{C}[[t,x]]$ satisfy
i) *$A : B_R \to B_R$,*
ii) *For all $v_1, v_2 \in B_R$, we have $||A(v_2) - A(v_1)||_{(T,X)} \leq (1/2)||v_2 - v_1||_{(T,X)}$,*
where B_R is the closed ball with radius R, centered at 0 in $G(T,X)$.

Proof Proof of **i**). Due to Lemma 1,2, there exists a constant $C > 0$ such that

$$(8) \quad ||a_{(k_0,k_1)}(t,x)\partial_t^{k_0}\partial_x^{k_1-S}v(t,x)||_{(T,X)} \leq C||a_{(k_0,k_1)}(t,x)||_{(T,X)}X^{(S-k_1)}T^{-k_0}R \leq C\delta X^{S-k_1}T^{-k_0}R$$

for all $v(t,x) \in B_R$ and for all $(k_0,k_1) \in \mathcal{S}$. On the other hand, due to lemma 3, there exists a constant $C > 0$ such that

$$(9) \quad ||a_{(k_0,k_1)}(t,x)\partial_t^{k_0}\varphi_j(t)\frac{x^q}{q!}||_{(T,X)}$$
$$\leq C||a_{(k_0,k_1)}(t,x)\frac{x^q}{q!}||_{(T,X)}T^{-k_0}||\varphi_j(t)||_{(T_0,X_0)} \leq C\delta||\frac{x^q}{q!}||_{(T,X)}T^{-k_0}||\varphi_j(t)||_{(T_0,X_0)}$$

for all $0 \leq j, q \leq S-1$ and all $(k_0, k_1) \in \mathcal{S}$, for X, T small enough. Moreover there exists $C > 0$ such that

$$(10)\quad ||a(t,x)(\partial_x^{-S} v(t,x) + w(t,x))^2||_{(T,X)} \\ \leq C\delta((X^S R)^2 + 2X^S R ||w(t,x)||_{(T_0,X_0)} + (||w(t,x)||_{(T_0,X_0)})^2)$$

and

$$(11)\quad ||a(t,x)(\partial_x^{-S} v(0,x) + w(0,x))(\partial_t \partial_x^{-S} v(t,x) + \partial_t w(t,x))||_{(T,X)} \\ \leq C\delta\left(X^{2S} T^{-1} R^2 + 2X^S R T^{-1} ||w(t,x)||_{(T_0,X_0)} + T^{-1}(||w(t,x)||_{(T_0,X_0)})^2\right).$$

Conclusion: If $X, T, \delta, R > 0$ are small enough, the sum over $(k_0, k_1) \in \mathcal{S}$, $0 \leq j, q \leq S-1$ of the right hand side of the inequalities (8), (9), (10), (11) can be chosen lower than R, for all $v(t,x) \in G(T,X)$ with $||v||_{(T,X)} \leq R$. So that, if $v \in B_R$ then $A(v) \in B_R$.

Proof of **ii**). Due to Lemma 1,2 there exists a constant $C > 0$ such that

$$(12)\quad ||a_{(k_0,k_1)}(t,x) \partial_t^{k_0} \partial_x^{k_1 - S}(v_2(t,x) - v_1(t,x))||_{(T,X)} \\ \leq C ||a_{(k_0,k_1)}(t,x)||_{(T,X)} X^{(S-k_1)} T^{-k_0} ||v_2(t,x) - v_1(t,x)||_{(T,X)},$$

for all $v_1(t,x), v_2(t,x) \in B_R$ and for all $(k_0, k_1) \in \mathcal{S}$. If we denote by

$$\mathcal{A}_{2,1}(t,x) = (\partial_x^{-S} v_2(0,x) + w(0,x))(\partial_t \partial_x^{-S} v_2(t,x) + w(t,x)) \\ - (\partial_x^{-S} v_1(0,x) + w(0,x))(\partial_t \partial_x^{-S} v_1(t,x) + w(t,x))$$

we also have

$$\mathcal{A}_{2,1}(t,x) = (\partial_x^{-S}(v_2(0,x) - v_1(0,x)))(\partial_t \partial_x^{-S} v_2(t,x) + w(t,x)) \\ + (\partial_t \partial_x^{-S}(v_2(t,x) - v_1(t,x)))(\partial_x^{-S} v_1(0,x) + w(0,x)).$$

We deduce that there exists a constant $C > 0$ such that

$$(13)\quad ||a(t,x)\left((\partial_x^{-S} v_2(t,x) + w(t,x))^2 - (\partial_x^{-S} v_1(t,x) + w(t,x))^2 + \mathcal{A}_{2,1}(t,x)\right)||_{(T,X)} \\ \leq ||a(t,x)||_{(T,X)} X^S ||v_2(t,x) - v_1(t,x)||_{(T,X)} ||\partial_x^{-S}(v_2(t,x) + v_1(t,x)) + 2w(t,x)||_{(T,X)} \\ + ||a(t,x)||_{(T,X)} X^S ||v_2(t,x) - v_1(t,x)||_{(T,X)} ||\partial_t \partial_x^{-S} v_2(t,x) + w(t,x)||_{(T,X)} \\ + ||a(t,x)||_{(T,X)} X^S T^{-1} ||v_2(t,x) - v_1(t,x)||_{(T,X)} ||\partial_x^{-S} v_1(0,x) + w(0,x)||_{(T,X)}.$$

Conclusion: If $X, T, \delta > 0$ are small enough, the sum over $(k_0, k_1) \in \mathcal{S}$ of the right hand side of the inequalities (12) and (13) can be chosen lower than $(1/2)||v_2(t,x) - v_1(t,x)||_{(T,X)}$. So that, if $v_1, v_2 \in B_R$ then $||A(v_2) - A(v_1)||_{(T,X)} \leq (1/2)||v_2(t,x) - v_1(t,x)||_{(T,X)}$. □

We are now in position to give the proof of Theorem (CK).

Proof Under the hypotheses of Theorem (CK), the assumptions of the proposition 1 are satisfied. From to the fact that $(G(T,X), ||.||_{(T,X)})$ is a Banach space, the closed ball (B_R, d) equipped with the metric $d(x,y) = ||y - x||_{(T,X)}$ is a complete metric space. From the proposition 1, the map A is a contraction from (B_R, d) into itself. From the classical fixed point theorem, we deduce that there exists a unique $v \in B_R$ such that $A(v) = v$. By construction, the formal series $u(t,x) = \partial_x^{-S} v(t,x) + w(t,x)$ is the solution of the problem (2), (3) and $u(t,x)$ belong to $G(T,X)$. □

3 A global Cauchy-Kowalevski theorem in a class of weighted spaces of holomorphic functions

First of all, we recall the definitions of the spaces of analytic functions in which we will choose our initial conditions and expect to find our solutions.

Let Ω be an open sector centered at the origin in $\mathbb{C}$ with bisecting direction $d \in \mathbb{R}$ and with infinite radius. We denote by $\bar{S}_{d,\delta} = \{t \in \mathbb{C} : |d - \arg(t)| \leq \delta\}$ a closed sector in Ω and by D_ρ an open disc of radius ρ centered at 0. Let $\bar{D}_{\rho'} \subset D_\rho$. We denote by $\mathcal{A}_{d,\delta}$ the vector space of holomorphic functions G on $\Omega \cup D_\rho$ such that

$$||G(t)||_{d,\delta} := M_0 \sup_{t \in \bar{S}_{d,\delta} \cup \bar{D}_{\rho'}} (1 + |t|^2) e^{-\nu |t|} |G(t)| < +\infty,$$

where $\nu > 0$ and $M_0 > 1$ is a constant such that

$$\int_0^{|t|} \frac{ds}{(1+s^2)(1+(|t|-s)^2)} \leq \frac{M_0}{(1+|t|^2)},$$

for all $t \in \bar{S}_{d,\delta} \cup \bar{D}_{\rho'}$. In the following, we will need a lemma from [9], Corollary 4.9.

Lemma 4 *The following inequality hold*

$$|| \int_0^t F(t-\tau) G(\tau) d\tau ||_{d,\delta} \leq ||F(t)||_{d,\delta} ||G(t)||_{d,\delta},$$

for all $F, G \in \mathcal{A}_{d,\delta}$.

Definition 3 *Let $X(t,z)$ be a holomorphic function on $(\Omega \cup D_\rho) \times D_r$. We say that $X(t,z)$ is weighted with respect to t on $(\Omega \cup D_\rho) \times D_r$, if the following inequalities hold.*

For all closed sector of infinite radius $\bar{S}_{d,\delta} = \{t \in \mathbb{C} : |d - \arg(t)| \leq \delta\}$ included in Ω and all closed disc $\bar{D}_{\rho'} \subset D_\rho$, there exist constants $C, T_0 > 0$ such that

$$M_0 \sup_{t \in \bar{S}_{d,\delta} \cup \bar{D}_{\rho'}, z \in \bar{D}_r} (1 + |t|^2) e^{-\nu |t|} |\partial_t^l X(t,z)| \leq C (1/T_0)^l l!$$

for all $l \geq 0$.

In the sequel we consider a partial integro-differential equation

$$(14) \quad \partial_z^S V(t,z) = \sum_{(k_0,k_1) \in \mathcal{S}} b_{k_0,k_1}(t,z) \partial_t^{k_0} \partial_z^{k_1} V(t,z) +$$
$$b_1(t,z) \partial_t \int_0^t V(t-\tau, z) V(\tau, z) d\tau + b_2(t,z) \partial_t \int_0^t V(t-\tau, z) C(\tau, z) d\tau$$

where $\mathcal{S}$ is a finite subset of $\mathbb{N}^2$, the coefficients

$$b_{k_0,k_1}(t,z) = \sum_{l \geq 0} b_{k_0,k_1,l}(t) z^l \quad , \quad b_1(t,z) = \sum_{l \geq 0} b_{1,l}(t) z^l \quad , \quad b_2(t,z) = \sum_{l \geq 0} b_{2,l}(t) z^l,$$

are holomorphic on a common domain $(G \cup D_\rho) \times D_r$ where G is an open sector with infinite radius centered at 0. The function

$$C(t,z) = \sum_{n \geq 0} C_n(t) z^n$$

is holomorphic on $(G \cup D_\rho) \times D_r$ and weighted with respect to t on $(G \cup D_\rho) \times D_r$.

We study formal series $V(t,z)$ in the variable z of the form

$$V(t,z) = \sum_{n \geq 0} \frac{V_n(t)}{n!} z^n, \tag{15}$$

with holomorphic coefficients $V_n(t)$, $n \geq 0$, on $G \cup D_\rho$, solutions of (14).

In the next lemma we describe formal series solutions of (14) in term of integro-differential relations.

Lemma 5 *The formal series $V(t,z)$ is a solution of (14) if and only if the following integro-differential relation is satisfied,*

$$\begin{aligned} \frac{V_{m+S}(t)}{m!} = \sum_{(k_0,k_1) \in \mathcal{S}} \sum_{l+n=m} b_{k_0,k_1,l}(t) \frac{\partial_t^{k_0} V_{n+k_1}(t)}{n!} + \\ \sum_{q_1+q_2+q_3=m} \frac{b_{1,q_3}(t)}{q_1! q_2!} \partial_t \int_0^t V_{q_1}(t-\tau) V_{q_2}(\tau) d\tau + \sum_{q_1+q_2+q_3=m} \frac{b_{2,q_3}(t)}{q_1!} \partial_t \int_0^t V_{q_1}(t-\tau) C_{q_2}(\tau) d\tau \end{aligned} \tag{16}$$

for all $m \geq 0$.

In the next proposition, we give sufficient conditions for the existence and uniqueness of formal series (15) that are solutions of (14).

Proposition 2 *We assume that the finite set $\mathcal{S}$ satisfies*

$$\mathcal{S} \subset \{(k_0, k_1, l) \in \mathbb{N}^3 : k_1 \leq S - 1\}.$$

Moreover, assume that holomorphic functions $V_\mu(t)$, for $0 \leq \mu \leq S-1$, are given on $G \cup D_\rho$. Then, there exists a unique sequence $(V_\mu(t))_{\mu \geq 0}$ of holomorphic functions which satisfies the integro-differential recurrence (16). As a result, for given holomorphic functions $V_\mu(t)$, for $0 \leq \mu \leq S-1$ on $G \cup D_\rho$, there exists a unique formal series (15), solution of (14).

The next result gives sufficient conditions under which there exist holomorphic solutions $V(t,z)$ to (14) which are weighted with respect to t.

Theorem 1 *Consider the partial integro-differential equation (14)*

$$\begin{aligned} \partial_z^S V(t,z) = \sum_{(k_0,k_1) \in \mathcal{S}} b_{k_0,k_1}(t,z) \partial_t^{k_0} \partial_z^{k_1} V(t,z) + \\ b_1(t,z) \partial_t \int_0^t V(t-\tau,z) V(\tau,z) d\tau + b_2(t,z) \partial_t \int_0^t V(t-\tau,z) C(\tau,z) d\tau \end{aligned}$$

where $\mathcal{S}$ is a finite subset of $\mathbb{N}^2$ which satisfies the hypothesis of Proposition 2. The coefficients $b_{k_0,k_1}(t,z), b_1(t,z), b_2(t,z)$ are holomorphic on a common domain $(G \cup D_\rho) \times D_r$ where G is an open sector with bisecting direction $d \in \mathbb{R}$ with infinite radius centered at 0 and D_ρ an open disc centered at 0 with radius ρ. The function $C(t,z)$ is holomorphic on $(G \cup D_\rho) \times D_r$ and weighted with respect to t on $(G \cup D_\rho) \times D_r$.

1) Assume that the quantity

$$\sup_{t \in G \cup D_\rho} \frac{|\partial_t^{l_0} b_{k_0,k_1,l}(t)|}{l_0!} = |b|_{k_0,k_1,l,l_0} \quad , \quad \sup_{t \in G \cup D_\rho} \frac{|\partial_t^{l_0} b_{1,l}(t)|}{l_0!} = |b|_{1,l,l_0} \quad , \quad \sup_{t \in G \cup D_\rho} \frac{|\partial_t^{l_0} b_{2,l}(t)|}{l_0!} = |b|_{2,l,l_0},$$

exists and that

$$|b|_{k_0,k_1}(t,x) = \sum_{l_0,l \geq 0} |b|_{k_0,k_1,l,l_0} t^{l_0} x^l \quad , \quad |b|_1(t,x) = \sum_{l_0,l \geq 0} |b|_{1,l,l_0} t^{l_0} x^l \quad , \quad |b|_2(t,x) = \sum_{l_0,l \geq 0} |b|_{2,l,l_0} t^{l_0} x^l$$

are holomorphic functions on a neighborhood of the origin in $\mathbb{C}^2$ *for all* $(k_0,k_1) \in \mathcal{S}$*. We may assume that there exist* $X_0, T_0 > 0$ *such that* $|b|_{k_0,k_1}(t,x), |b|_1(t,x), |b|_2(t,x)$ *belong to* $G(T_0,X_0)$ *for all* $(k_0,k_1) \in \mathcal{S}$*. We also denote by*

$$||c||_{l,l_0} = \frac{||\partial_t^{l_0} C_l(t)||_{d,\delta}}{l_0!}$$

and

$$||C||(t,x) = \sum_{l_0,l \geq 0} ||c||_{l,l_0} t^{l_0} x^l.$$

2) Assume that $S \geq k_0 + k_1$ *and* $S > k_1$ *for all* $(k_0,k_1) \in \mathcal{S}$*. Moreover, consider the following auxiliary partial differential equation constructed from (14),*

$$\partial_z^S U(t,x) = \sum_{(k_0,k_1) \in \mathcal{S}} |b|_{k_0,k_1}(t,x) \partial_t^{k_0} \partial_x^{k_1} U(t,x) + |b|_1(t,x)((U(t,x))^2 + U(0,x)\partial_t U(t,x)) + |b|_2(t,x)(||C||(t,x)U(t,x) + ||C||(0,x)\partial_t U(t,x)). \tag{17}$$

Assume that

$$|||b|_{k_0,k_1}(t,x)||_{(T_0,X_0)} \leq \delta, \quad |||b|_1(t,x)||_{(T_0,X_0)} \leq \delta, \quad |||b|_2(t,x)||C||(t,x)||_{(T_0,X_0)} \leq \delta$$

for the real number $\delta > 0$ *constructed from Theorem (CK) applied to the equation (17).*

3) We make the following assumption on initial conditions. Let $V_\mu(t)$*, for* $0 \leq \mu \leq S-1$*, be given holomorphic functions which are weighted with respect to* t *on* $G \cup D_\rho$*.*

Then, the formal series

$$V(t,z) = \sum_{n \geq 0} \frac{V_n(t)}{n!} z^n$$

solution of (14) with initial conditions $(\partial_z^\mu V)(t,0) = V_\mu(t)$*,* $0 \leq \mu \leq S-1$*, constructed in the proposition 2 is convergent for all* $z \in D_r$ *for* r *small enough. Moreover, the function* $V(t,z)$ *is weighted with respect to* t *on* $(G \cup D_\rho) \times D_r$*.*

Proof In the first part of the proof, we show that the functions $V_n(t)$ constructed from the proposition 2 satisfy the following estimates: For all closed sector of infinite radius $\bar{S}_{d,\delta} \subset G$ centered at 0 with bisecting direction d, there exist $C, X_0, T_0 > 0$, independent of n, l, such that

$$||\partial_t^l V_n(t)||_{d,\delta} \leq C(1/X_0)^n (1/T_0)^l (l+n)!, \tag{18}$$

for all $l, n \geq 0$. From (18), we deduce that $V(t,z)$ converges for all $z \in D_r$ for r small enough and is weighted with respect to t on $(G \cup D_{\rho}) \times D_r$.

We consider now the sequence of quantities

$$w_{n_0,m_1} = ||\partial_t^{n_0} V_{m_1}(t)||_{d,\delta},$$

for $n_0, m_1 \geq 0$. We show that this sequence is in fact real valued and satisfies a multivariate recursion inequality.

From the Leibniz formula, we have that

$$\partial_t^{n_0}(b_{k_0,k_1,l}(t)\frac{\partial_t^{k_0} V_{n+k_1}(t)}{n!}) = \sum_{h_1+h_2=n_0} \frac{n_0!}{h_2!h_1!}\partial_t^{h_2} b_{k_0,k_1,l}(t)\frac{\partial_t^{h_1+k_0} V_{n+k_1}(t)}{n!}.$$

From the assumption 1), we deduce

$$||\partial_t^{n_0}(b_{k_0,k_1,l}(t)\frac{\partial_t^{k_0} V_{n+k_1}(t)}{n!})||_{d,\delta} \leq \sum_{h_1+h_2=n_0} n_0!|b|_{k_0,k_1,l,h_2}\frac{||(\partial_t^{h_1+k_0} V_{n+k_1})(t)||_{d,\delta}}{h_1!n!}.$$

On the other hand, we have

$$\partial_t^{n_0}(\frac{b_{1,q_3}(t)}{q_1!q_2!}\partial_t \int_0^t V_{q_1}(t-\tau)V_{q_2}(\tau)d\tau) = \sum_{h_1+h_2=n_0} \frac{n_0!}{h_2!h_1!}\partial_t^{h_2} b_{1,q_3}(t)\frac{\partial_t^{h_1+1}(\int_0^t V_{q_1}(t-\tau)V_{q_2}(\tau)d\tau)}{q_1!q_2!}$$

From the assumption 1), we deduce

$$||\partial_t^{n_0}(\frac{b_{1,q_3}(t)}{q_1!q_2!}\partial_t \int_0^t V_{q_1}(t-\tau)V_{q_2}(\tau)d\tau)||_{d,\delta} \leq \sum_{h_1+h_2=n_0} n_0!|b|_{1,q_3,h_2}\frac{||\partial_t^{h_1+1}(\int_0^t V_{q_1}(t-\tau)V_{q_2}(\tau)d\tau)||_{d,\delta}}{h_1!q_1!q_2!}.$$

We have

$$\partial_t^{h_1+1}(\int_0^t V_{q_1}(t-\tau)V_{q_2}(\tau)d\tau) = \sum_{l_1+l_2=h_1}(\partial_t^{l_1} V_{q_1})(0)\partial_t^{l_2} V_{q_2}(t) + \int_0^t (\partial_t^{h_1+1} V_{q_1})(t-\tau)V_{q_2}(\tau)d\tau.$$

From the inequalities

$$|(\partial_t^{l_1} V_{q_1})(0)| \leq ||\partial_t^{l_1} V_{q_1}(t)||_{d,\delta} \quad , \quad 1 \leq \frac{h_1!}{l_1!l_2!},$$

for $l_1, l_2 \geq 0$ with $l_1 + l_2 = h_1$, and the lemma 4, we get

$$||\partial_t^{h_1+1}(\int_0^t V_{q_1}(t-\tau)V_{q_2}(\tau)d\tau)||_{d,\delta} \leq (\sum_{l_1+l_2=h_1} \frac{h_1!}{l_1!l_2!}||\partial_t^{l_1} V_{q_1}(t)||_{d,\delta}||\partial_t^{l_2} V_{q_2}(t)||_{d,\delta}) + ||\partial_t^{h_1+1} V_{q_1}(t)||_{d,\delta}||V_{q_2}(t)||_{d,\delta}$$

In the same way as above, one can check that

$$||\partial_t^{n_0}(\frac{b_{2,q_3}(t)}{q_1!}\partial_t \int_0^t V_{q_1}(t-\tau)C_{q_2}(\tau)d\tau)||_{d,\delta} \leq \sum_{h_1+h_2=n_0} n_0!|b|_{2,q_3,h_2}\frac{||\partial_t^{h_1+1}(\int_0^t V_{q_1}(t-\tau)C_{q_2}(\tau)d\tau)||_{d,\delta}}{h_1!q_1!}.$$

and that

$$||\partial_t^{h_1+1}(\int_0^t V_{q_1}(t-\tau)C_{q_2}(\tau)d\tau)||_{d,\delta} \leq$$
$$(\sum_{l_1+l_2=h_1} \frac{h_1!}{l_1!l_2!}||\partial_t^{l_1}V_{q_1}(t)||_{d,\delta}||\partial_t^{l_2}C_{q_2}(t)||_{d,\delta}) + ||\partial_t^{h_1+1}V_{q_1}(t)||_{d,\delta}||C_{q_2}(t)||_{d,\delta}$$

Finally, from the latter estimates, we get the following inequalities for the numbers w_{n_0,n_1},

$$\frac{w_{n_0,m_1+S}}{n_0!m_1!} \leq A_{n_0,m_1} + B_{n_0,m_1} + C_{n_0,m_1}, \tag{19}$$

where

$$A_{n_0,m_1} = \sum_{(k_0,k_1)\in\mathcal{S}} \sum_{l+n=m_1} \sum_{h_1+h_2=n_0} |b|_{k_0,k_1,l,h_2} \frac{w_{h_1+k_0,n+k_1}}{h_1!n!},$$

and

$$B_{n_0,m_1} = \sum_{q_1+q_2+q_3=m_1} \sum_{h_1+h_2=n_0} |b|_{1,q_3,h_2}(\sum_{l_1+l_2=h_1} \frac{w_{l_1,q_1}w_{l_2,q_2}}{l_1!q_1!l_2!q_2!} + \frac{w_{h_1+1,q_1}w_{0,q_2}}{h_1!q_1!q_2!}),$$

and

$$C_{n_0,m_1} = \sum_{q_1+q_2+q_3=m_1} \sum_{h_1+h_2=n_0} |b|_{2,q_3,h_2}(\sum_{l_1+l_2=h_1} \frac{w_{l_1,q_1}||c||_{q_2,l_2}}{l_1!q_1!} + \frac{w_{h_1+1,q_1}||c||_{q_2,0}}{h_1!q_1!}),$$

for all $n_0, m_1 \geq 0$.

Consider now the auxiliary partial differential equation (17)

$$\partial_z^S U(t,x) = \sum_{(k_0,k_1)\in\mathcal{S}} |b|_{k_0,k_1}(t,x)\partial_t^{k_0}\partial_x^{k_1}U(t,x)+$$
$$|b|_1(t,x)((U(t,x))^2 + U(0,x)\partial_t U(t,x)) + |b|_2(t,x)(||C||(t,x)U(t,x) + ||C||(0,x)\partial_t U(t,x))$$

constructed from (14). Due to the assumption 2) in Theorem 1, we deduce that the hypotheses of Theorem (CK) in section 1 are satisfied for the latter equation (17). From Theorem (CK), we deduce the existence of a unique formal series $U(t,x) \in G(T,X)$, for $X, T > 0$ small enough, solution of (17), if $(\partial_z^k U)(t,0)$ are prescribed formal series which belong to $G(T_0, X_0)$, for all $0 \leq k \leq S-1$, for given $X_0, T_0 > 0$.

If we expand $U(t,x)$ into a Taylor series in the variables t, x,

$$U(t,x) = \sum_{n_0,n_1\geq 0} U_{n_0,n_1} \frac{t^{n_0}x^{n_1}}{n_0!n_1!},$$

we get that

$$|b|_{k_0,k_1}(t,x)\partial_t^{k_0}\partial_x^{k_1}U(t,x) = \sum_{n_0,m_1} \mathcal{A}_{n_0,m_1} t^{n_0}x^{m_1},$$

with

$$\mathcal{A}_{n_0,m_1} = \sum_{h_1+h_2=n_0} \sum_{l+n_1=m_1} |b|_{k_0,k_1,l,h_2} \frac{U_{h_1+k_0,n_1+k_1}}{h_1!n_1!},$$

and

$$|b|_1(t,x)((U(t,x))^2 + U(0,x)\partial_t U(t,x)) = \sum_{n_0,m_1} \mathcal{B}_{n_0,m_1} t^{n_0}x^{m_1},$$

with
$$\mathcal{B}_{n_0,m_1} = \sum_{q_1+q_2+q_3=m_1} \sum_{h_1+h_2=n_0} |b|_{1,q_3,h_2} (\sum_{l_1+l_2=h_1} \frac{U_{l_1,q_1}U_{l_2,q_2}}{l_1!q_1!l_2!q_2!} + \frac{U_{h_1+1,q_1}U_{0,q_2}}{h_1!q_1!q_2!})$$
for all $n_0, m_1 \geq 0$. In the same way, we have
$$|b|_2(t,x)(||C||(t,x)U(t,x) + ||C||(0,x)\partial_t U(t,x)) = \sum_{n_0,m_1} \mathcal{C}_{n_0,m_1} t^{n_0} x^{m_1},$$
with
$$\mathcal{C}_{n_0,m_1} = \sum_{q_1+q_2+q_3=m_1} \sum_{h_1+h_2=n_0} |b|_{2,q_3,h_2} (\sum_{l_1+l_2=h_1} \frac{U_{l_1,q_1}||c||_{q_2,l_2}}{l_1!q_1!} + \frac{U_{h_1+1,q_1}||c||_{q_2,0}}{h_1!q_1!})$$
From this, we deduce the following multivariate nonlinear recurrence satisfied by the coefficients U_{n_0,n_1},

$$\frac{U_{n_0,m_1+S}}{n_0!m_1!} = \mathcal{A}_{n_0,m_1} + \mathcal{B}_{n_0,m_1} + \mathcal{C}_{n_0,m_1} \tag{20}$$

for all $n_0, m_1 \geq 0$. From the fact that $U(t,x) \in G(T,X)$, we deduce the following estimates for the coefficients U_{n_0,n_1}. There exists $C > 0$ such that

$$|U_{n_0,n_1}| \leq C(1/X)^{n_1}(1/T)^{n_0}(n_0+n_1)!, \tag{21}$$

for all $n_0, n_1 \geq 0$. From Theorem (CK), we see that the sequence $(U_{n_0,m_1})_{n_0,m_1}$ is uniquely determined by the sequences $(U_{n_0,m_1})_{n_0,m_1}$, $0 \leq m_1 \leq S-1$. We choose now the latter sequences as follows.
$$U_{n_0,m_1} := w_{n_0,m_1},$$
for all $0 \leq m_1 \leq S-1$, $n_0 \geq 0$.

From the assumption 3), we deduce that the series
$$U(t,0), \ldots, (\partial_z^{S-1}U)(t,0)$$
are convergent, which means in particular that they belong to $G(T_0,X_0)$ for $X_0, T_0 > 0$. Moreover, due to the relations (19) and (20), we deduce that

$$w_{n_0,m_1} \leq U_{n_0,m_1}, \tag{22}$$

for all $n_0, m_1 \geq 0$.

From the estimates (21), we finally deduce that there exist $C > 0$, such that the sequence w_{n_0,n_1} satisfies
$$w_{n_0,n_1} \leq C(1/X)^{n_1}(1/T)^{n_0}(n_0+n_1)!,$$
for all $n_0, n_1 \geq 0$, which yields the result. □

4 Analytic solutions in a small parameter of a singularly perturbed Cauchy problem

4.1 Laplace transform and asymptotic expansions

We recall the definition of Borel summability of formal series with coefficients in a Banach space, see [2].

Definition 4 *A formal series*

$$\hat{X}(t) = \sum_{j=0}^{\infty} \frac{a_j}{j!} t^j$$

with coefficients in a Banach space $(\mathbb{E}, ||.||_{\mathbb{E}})$ *is said to be* $1-$*summable with respect to* t *in the direction* $d \in [0, 2\pi)$ *if*

i) *There exists* $\rho \in \mathbb{R}_+$ *such that the following formal series, called formal Borel transform with respect to the variable* τ *of order* 1,

$$W(\tau) = \sum_{j=0}^{\infty} \frac{a_j \tau^j}{(j!)^2},$$

is absolutely convergent for $|\tau| < \rho$,

ii) *There exists* $\delta > 0$ *such that the series* $W(\tau)$ *can be analytically continued with respect to* τ *in a sector* $S_{d,\delta} = \{\tau \in \mathbb{C} : |d - \arg(\tau)| < \delta\}$. *Moreover, there exist* $\delta_1 < \delta$, $C > 0$, *and* $K > 0$ *such that*

$$||W(\tau)||_{\mathbb{E}} \leq Ce^{K|\tau|}$$

for all $\tau \in S_{d,\delta_1}$.

If this is so, the vector valued Laplace transform of order 1 of $W(\tau)$ in the direction d is defined by

$$\mathcal{L}^d(W)(t) = t^{-1} \int_0^{\infty(\gamma)} W(\tau) e^{-(\tau/t)} d\tau,$$

where the integration is made along the ray $\arg(\tau) = \gamma$ with $|d - \gamma| < \delta$, is called the $1-$sum of the formal series $\hat{X}(t)$. The function $\mathcal{L}^d(W)(t)$ is an holomorphic and bounded function on a sector

$$S_{d,\theta,R} = \{t \in \mathbb{C} : |t| < R \quad , \quad |d - \arg(t)| < \theta/2\},$$

where $\theta > \pi$ and $R > 0$. Moreover, for all $l \geq 0$, the function $\partial_t^l \mathcal{L}^d(W)(t)$ is asymptotic to $\partial_t^l \hat{X}(t)$ with Gevrey filtration of order 1, with respect to t on $S_{d,\theta,R}$. This means that for all $\theta_1 < \theta$, all $l \geq 0$, there exist $C, M > 0$, such that

$$||\partial_t^l \mathcal{L}^d(W)(t) - \sum_{p=0}^{n-1} \frac{a_{p+l}}{p!} t^p||_{\mathbb{E}} \leq CM^n n! |t|^n$$

for all $n \geq 1$ and $t \in S_{d,\theta_1,R}$.

In the next proposition, we give a criterion for a formal series to be $1-$summable. This is an application of a general moment summability result obtained in Theorem 38, p. 108 in [2].

Proposition 3 *Let* $\hat{X}(t) = \sum_{j=0}^{\infty} (a_j/j!) t^j$ *be a formal series with coefficients in a Banach space* $(\mathbb{E}, ||.||_{\mathbb{E}})$. *Let*

$$\mathcal{B}(\hat{X})(\tau) = \sum_{m \geq 0} \frac{a_m}{(2m)!} \tau^m$$

the modified Borel transform of order 1 which is a formal series with coefficients in $\mathbb{E}$. *Then,* $\hat{X}(t)$ *is* $1-$*summable with respect to* t *in the direction* $d \in [0, 2\pi)$ *if*

i) *There exists* $\rho \in \mathbb{R}_+$ *such that* $(\mathcal{B}\hat{X})(\tau)$ *is absolutely convergent for all* $\tau \in D_\rho$.

ii) *There exists $\delta > 0$ such that the series $(\mathcal{B}\hat{X})(\tau)$ can be analytically continued with respect to τ in a sector $S_{d,\delta} = \{\tau \in \mathbb{C} : |d - \arg(\tau)| < \delta\}$. Moreover, there exist $\delta_1 < \delta$, $C > 0$, and $K > 0$ such that*

$$||(\mathcal{B}\hat{X})(\tau)||_{\mathbb{E}} \leq Ce^{K|\tau|}$$

for all $\tau \in S_{d,\delta_1}$.

Moreover, if $X(t)$ is the $1-$sum of $\hat{X}$ defined on some sector $S_{d,\theta,R}$, with $\theta > \pi$, $R > 0$, then the following asymptotic expansion hold: For all $\theta_1 < \theta$, all positive integer $l \geq 0$, there exist $C, M > 0$, such that

$$||\partial_t^l X(t) - \sum_{p=0}^{n-1} \frac{a_{p+l}}{p!} t^p||_{\mathbb{E}} \leq CM^n n!|t|^n \tag{23}$$

for all $n \geq 1$, all $t \in S_{d,\theta_1,R}$.

Remark: The $1-$sum $X(t)$ can be expressed in term of the modified Borel transform using an integral operator

$$X(t) = \int_0^{\infty(d)} \frac{e(u/t)}{u} \mathcal{B}(\hat{X})(u) du$$

where $e(z)$ is some holomorphic function with exponential decay on the half-plane $\{\mathrm{Re}(z) > 0\}$, see Theorem 38 in [2].

4.2 Analytic solutions of a singular Cauchy problem with irregular singularity

Let $S \geq 1$ be an integer. Let $\mathcal{S}$ be a finite subset of $\mathbb{N}^2$ and let

$$b_{k_0,k_1}(z,\epsilon) = \sum_{n \geq 0} b_{k_0,k_1,n}(\epsilon) z^n \quad , \quad b(z,\epsilon) = \sum_{n \geq 0} b_n(\epsilon) z^n,$$

be holomorphic functions on a neighborhood of the origin in $\mathbb{C}^2$, for all $(k_0, k_1) \in \mathcal{S}$.

Definition 5 *Let $V(\tau,\epsilon)$ be a holomorphic function on some polydisc $D(0,\tau_0) \times D(0,\epsilon_0)$ where $0 < \tau_0 < 1$ and $\epsilon_0 > 0$. Let $d \in [0, 2\pi)$ with $d \neq \pi$. Let U_d be a sector centered at 0, with bisecting direction d, with infinite radius and with small opening such that $-1 \notin U_d$. Let $\mathcal{E}$ be a sector centered at 0 such that $\mathcal{E} \subset D(0,\epsilon_0)$. Assume that the function $(\tau,\epsilon) \mapsto V(\tau,\epsilon)$ can be extended to an analytic and bounded function $(\tau,\epsilon) \mapsto V_{U_d,\mathcal{E}}(\tau,\epsilon)$ on $U_d \times \mathcal{E}$. Assume moreover, that there exist two constants $C, T_0 > 0$ such that*

$$\sup_{\epsilon \in \mathcal{E}} \sup_{t^2 \in D(0,\tau_0) \cup U_d} (1 + |t|^2) e^{-\nu|t|} |\partial_t^l (V_{U_d,\mathcal{E}}(t^2,\epsilon))| \leq C(1/T_0)^l l! \tag{24}$$

for all $l \geq 0$. We say that the set $\{V, V_{U_d,\mathcal{E}}, T_0, \tau_0, \epsilon_0\}$ is admissible.

We consider the following singular Cauchy problem

$$4t^2 \partial_t \partial_z^S Y_{U_d,\mathcal{E}}(t,z,\epsilon) + (2t+1)\partial_z^S Y_{U_d,\mathcal{E}}(t,z,\epsilon) = \sum_{(k_0,k_1) \in \mathcal{S}} b_{k_0,k_1}(z,\epsilon) \epsilon^{k_0} \partial_t^{k_0} \partial_z^{k_1} Y_{U_d,\mathcal{E}}(t,z,\epsilon) + b(z,\epsilon) \partial_t \int_0^t Y_{U_d,\mathcal{E}}(t-\tau,z,\epsilon) Y_{U_d,\mathcal{E}}(\tau,z,\epsilon) d\tau \tag{25}$$

for given initial conditions

$$(\partial_z^j Y_{U_d,\mathcal{E}})(t,0,\epsilon) = Y_{U_d,\mathcal{E},j}(t,\epsilon) \quad , \quad 0 \leq j \leq S-1 \tag{26}$$

The initial conditions $Y_{U_d,\mathcal{E},j}(t,\epsilon)$ are constructed as follows: for all $0 \leq j \leq S-1$, let $\{V_j, V_{U_d,\mathcal{E},j}, T_0, \tau_0, \epsilon_0\}$ be an admissible set. Let

$$V_j(\tau,\epsilon) = \sum_{m \geq 0} \frac{\chi_{m,j}(\epsilon)}{(2m)!}\tau^m$$

be it's Taylor expansion for $(\tau,\epsilon) \in D(0,\tau_0) \times D(0,\epsilon_0)$. We consider the formal series

$$\hat{Y}_j(t,\epsilon) := \sum_{m \geq 0} \frac{\chi_{m,j}(\epsilon)}{m!}t^m$$

for all $\epsilon \in D(0,\epsilon_0)$. We define $Y_{U_d,\mathcal{E},j}(t,\epsilon)$ as the $1-$sum of $\hat{Y}_j(t,\epsilon)$ in the direction d, which exists from Proposition 3 using the estimates (24).

We have the following result.

Theorem 2 *Let the following assumption hold: for all $(k_0,k_1) \in \mathcal{S}$ we have $S \geq 2k_0 + k_1$ and $S > k_1$. Moreover, we put*

$$|b|_{k_0,k_1}(r,\epsilon) = \sum_{n \geq 0} |b_{k_0,k_1,n}(\epsilon)| r^n \quad , \quad |b|(r,\epsilon) = \sum_{n \geq 0} |b_n(\epsilon)| r^n,$$

for $r > 0$ and we assume that there exists $r > 0$ such that

$$\lim_{\epsilon \to 0} |b|_{k_0,k_1}(r,\epsilon) = \lim_{\epsilon \to 0} |b|(r,\epsilon) = 0,$$

for all $(k_0,k_1) \in \mathcal{S}$. Let the initial data (26) constructed as above.

Then, the problem (25), (26) has a solution $Y_{U_d,\mathcal{E}}(t,z,\epsilon)$ which is a holomorphic and a bounded function on $U_{d,\theta,R} \times D(0,\rho) \times \mathcal{E}$, where

$$U_{d,\theta,R} = \{t \in \mathbb{C} : |t| < R \quad , \quad |d - \arg(t)| < \theta/2\},$$

for some $\theta > \pi$, $R, \rho > 0$. Moreover, there exists a sequence $(\chi_m(z,\epsilon))_{m \geq 0}$ of functions, which are independent of $U_d, \mathcal{E}$, which are bounded holomorphic functions on $D(0,\rho) \times D(0,\epsilon_0)$ and satisfies the following property: for all $l \geq 0$, there exist $C, A > 0$ such that

$$\sup_{z \in D(0,\rho)} |\partial_t^l Y_{U_d,\mathcal{E}}(t,z,\epsilon) - \sum_{m=0}^{n-1} \frac{\chi_{m+l}(z,\epsilon)}{m!}t^m| \leq CA^n n! |t|^n \tag{27}$$

for all $n \geq 1$, for all $t \in U_{d,\theta,R}$, all $\epsilon \in \mathcal{E}$.

Proof We consider the Cauchy problem

$$\partial_z^S V(t^2,z,\epsilon) = \sum_{(k_0,k_1) \in \mathcal{S}} \frac{b_{k_0,k_1}(z,\epsilon)}{t^2+1}\epsilon^{k_0}\partial_t^{2k_0}\partial_z^{k_1}V(t^2,z,\epsilon) \tag{28}$$

$$+ b(z,\epsilon)\partial_t \int_0^t V((t-\tau)^2,z,\epsilon)V(\tau^2,z,\epsilon)d\tau$$

for given initial conditions

$$(\partial_z^j V)(t^2,0,\epsilon) = V_{U_d,\mathcal{E},j}(t^2,\epsilon) \quad , \quad 0 \leq j \leq S-1. \tag{29}$$

From the fact that the initial conditions $(t,\epsilon) \mapsto V_j(t^2,z)$, $0 \leq j \leq S-1$ are holomorphic functions on $D(0,\tau_0^{1/2}) \times D(0,\epsilon_0)$ and satisfy the estimates (24), we deduce from Theorem 1, that if ϵ_0 is small enough, then the problem (28), (29) has a solution $(t,z,\epsilon) \mapsto V(t^2,z,\epsilon)$, which is holomorphic on $D(0,\tau_0^{1/2}) \times D(0,\rho) \times D(0,\epsilon_0)$ where $V(\tau,z,\epsilon)$ is a holomorphic function on $D(0,\tau_0) \times D(0,\rho) \times D(0,\epsilon_0)$. Let

$$V(t^2,z,\epsilon) = \sum_{m \geq 0} \frac{\chi_m(z,\epsilon)}{(2m)!} t^{2m}$$

it's Taylor expansion with respect to t on $D(0,\tau_0^{1/2})$. Now, we show that the formal series

$$\hat{Y}(t,z,\epsilon) = \sum_{m \geq 0} \frac{\chi_m(z,\epsilon)}{m!} t^m$$

satisfies the equation (25). Indeed, from the classical identity

$$\int_0^t (t-\tau)^{m_1} \tau^{m_2} d\tau = \frac{m_1! m_2!}{(1+m_1+m_2)!} t^{m_1+m_2+1},$$

we get

$$\partial_t \int_0^t \hat{Y}(t-\tau,z,\epsilon)\hat{Y}(\tau,z,\epsilon) d\tau = \sum_{j \geq 0} (\sum_{q_1+q_2=j} \chi_{q_1}(z,\epsilon)\chi_{q_2}(z,\epsilon)) \frac{t^j}{j!}$$

and also

$$\partial_t \int_0^t V((t-\tau)^2,z,\epsilon)V(\tau^2,z,\epsilon) d\tau = \sum_{j \geq 0} (\sum_{q_1+q_2=j} \chi_{q_1}(z,\epsilon)\chi_{q_2}(z,\epsilon)) \frac{t^{2j}}{(2j)!}.$$

On the other hand,

$$(4t^2\partial_t + 2t)\hat{Y}(t,z,\epsilon) = \sum_{j \geq 0} (2j)(2j-1)\chi_{j-1}(z,\epsilon) \frac{t^j}{j!}$$

and

$$t^2 V(t^2,z,\epsilon) = \sum_{j \geq 0} (2j)(2j-1)\chi_{j-1}(z,\epsilon) \frac{t^{2j}}{(2j)!},$$

which yields the result.

From the estimates (24) for the initial conditions $V_{U_d,\mathcal{E},j}(t^2,\epsilon)$, $0 \leq j \leq S-1$, we deduce from Theorem 1, that the holomorphic function $(t,z,\epsilon) \mapsto V(t^2,z,\epsilon)$ can be extended to a solution $(t,z,\epsilon) \mapsto V_{U_d,\mathcal{E}}(t^2,z,\epsilon)$ of (28), (29) where $V_{U_d,\mathcal{E}}(\tau,z,\epsilon)$ is a holomorphic function on $U_d \times D(0,\rho) \times \mathcal{E}$. Moreover, the function $V_{U_d,\mathcal{E}}$ satisfies in particular the following estimates: there exists a constant $C > 0$ such that

$$\sup_{z \in D(0,\rho)} \sup_{\epsilon \in \mathcal{E}} \sup_{t^2 \in D(0,\tau_0) \cup U_d} (1+|t|^2) e^{-\nu|t|} |(V_{U_d,\mathcal{E}}(t^2,z,\epsilon))| \leq C \tag{30}$$

From these latter estimates and the proposition 3, we deduce that the function $Y_{U_d,\mathcal{E}}(t,z,\epsilon)$ defined as the 1−sum of the formal series $\hat{Y}(t,z,\epsilon)$ in the direction d exists and is holomorphic and bounded function on $U_{d,\theta,R} \times D(0,\rho) \times \mathcal{E}$, for some $\theta > \pi$, $R, \rho > 0$. From the fact that $\hat{Y}(t,z,\epsilon)$ is a formal solution of the equation (25), we get from the general κ−summability theory, see [2], chapter 6.3, that the 1−sum $Y_{U_d,\mathcal{E}}(t,z,\epsilon)$ is also a solution of (25). Finally, the asymptotic expansion (27) follows from the estimates (23). □

4.3 Analytic solutions in a parameter of a singularly perturbed Cauchy problem

We define what will be called a good covering.

Definition 6 *For directions $0 \leq d_0 < \cdots < d_{\nu-1} < 2\pi$, $d_\nu = d_0$, we consider sectors*

$$\mathcal{E}_i = S_{(d_{i+1}+d_i)/2,(\pi+\delta_i)/2,\epsilon_0},$$

centered at 0, with bisecting direction $(d_{i+1}+d_i)/2$, radius ϵ_0 and opening $\pi+\delta_i$, where $\delta_i = d_{i+1}-d_i$, $0 \leq i \leq \nu-1$. Such a set of sectors $\{\mathcal{E}_i\}_{0\leq i\leq \nu-1}$ is called a good covering of the origin in $\mathbb{C}$.

Definition 7 *Let $\{\mathcal{E}_i\}_{0\leq i\leq\nu-1}$ be a good covering of the origin in $\mathbb{C}$. We consider a family of open sectors*

$$U_{f_i,\theta,R} := \{t \in \mathbb{C} : |t| < R \quad , \quad |f_i - \arg(t)| < \theta/2\},$$

where $f_i \in [0,2\pi)$, for $0 \leq i \leq \nu-1$, where $\theta > \pi$, $R > 0$, and let $\mathcal{T}$ be an open sector centered at 0 with radius R, which satisfy the following properties:

1) For all $0 \leq i \leq \nu-1$, $\arg(f_i) \neq \pi$.
2) For all $0 \leq i \leq \nu-1$, for all $t \in \mathcal{T}$, all $\epsilon \in \mathcal{E}_i$, we have $\epsilon t \in U_{f_i,\theta,R}$.

We say that the family $\{\{U_{f_i,\theta,R}\}_{0\leq i\leq\nu-1}, \mathcal{T}\}$ is associated to the good covering $\{\mathcal{E}_i\}_{0\leq i\leq\nu-1}$.

Let $S \geq 1$ be an integer. Let $\mathcal{S}$ be a finite subset of $\mathbb{N}^2$ and let

$$b_{k_0,k_1}(z,\epsilon) = \sum_{n\geq 0} b_{k_0,k_1,n}(\epsilon) z^n \quad , \quad b(z,\epsilon) = \sum_{n\geq 0} b_n(\epsilon) z^n,$$

be holomorphic functions on a neighborhood of the origin in $\mathbb{C}^2$, for all $(k_0,k_1) \in \mathcal{S}$. Let $\{\mathcal{E}_i\}_{0\leq i\leq\nu-1}$ be a good covering of the origin in $\mathbb{C}$. For all $0 \leq i \leq \nu-1$, we consider the following singularly perturbed Cauchy problem

$$(31) \quad 4\epsilon t^2 \partial_t \partial_z^S X_i(t,z,\epsilon) + (2t\epsilon+1)\partial_z^S X_i(t,z,\epsilon) = \sum_{(k_0,k_1)\in\mathcal{S}} b_{k_0,k_1}(z,\epsilon)\partial_t^{k_0}\partial_z^{k_1} X_i(t,z,\epsilon) + b(z,\epsilon)\partial_t \int_0^t X_i(t-\tau,z,\epsilon)X_i(\tau,z,\epsilon)d\tau,$$

for given initial conditions

$$(32) \qquad (\partial_z^j X_i)(t,0,\epsilon) = \varphi_{i,j}(t,\epsilon) \quad , \quad 0 \leq j \leq S-1$$

where the functions $\varphi_{i,j}(t,\epsilon)$ are constructed as follows. Let $\{\{U_{f_i,\theta,R}\}_{0\leq i\leq \nu-1},\mathcal{T}\}$ a family of open sets associated to $\{\mathcal{E}_i\}_{0\leq i\leq\nu-1}$. For all $0\leq j\leq S-1$, all $0\leq i\leq \nu-1$, let $\{V_j,V_{U_{f_i},\mathcal{E}_i,j},T_0,\tau_0,\epsilon_0\}$ be an admissible set. Let

$$V_j(\tau,\epsilon)=\sum_{m\geq 0}\frac{\chi_{m,j}(\epsilon)}{(2m)!}\tau^m$$

be it's Taylor expansion for $(\tau,\epsilon)\in D(0,\tau_0)\times D(0,\epsilon_0)$. We consider the formal series

$$\hat{Y}_j(t,\epsilon):=\sum_{m\geq 0}\frac{\chi_{m,j}(\epsilon)}{m!}t^m$$

for all $\epsilon\in D(0,\epsilon_0)$. We define $Y_{U_{f_i},\mathcal{E}_i,j}(t,\epsilon)$ as the $1-$sum of $\hat{Y}_j(t,\epsilon)$ in the direction f_i, which exists from Proposition 3 using the estimates (24). Then, we put

$$\varphi_{i,j}(t,\epsilon):=Y_{U_{f_i},\mathcal{E}_i,j}(\epsilon t,\epsilon),$$

for all $0\leq j\leq S-1$, all $0\leq i\leq\nu-1$. By construction, the function $\varphi_{i,j}(t,\epsilon)$ is a holomorphic and bounded function on $\mathcal{T}\times\mathcal{E}_i$, for all $0\leq j\leq S-1$, all $0\leq i\leq\nu-1$.

Proposition 4 *Let the following assumption hold: For all $(k_0,k_1)\in\mathcal{S}$ we have $S\geq 2k_0+k_1$ and $S>k_1$. Moreover, we put*

$$|b|_{k_0,k_1}(r,\epsilon)=\sum_{n\geq 0}|b_{k_0,k_1,n}(\epsilon)|r^n \quad , \quad |b|(r,\epsilon)=\sum_{n\geq 0}|b_n(\epsilon)|r^n,$$

for $r>0$ and we assume that there exists $r>0$ such that

$$\lim_{\epsilon\to 0}|b|_{k_0,k_1}(r,\epsilon)=\lim_{\epsilon\to 0}|b|(r,\epsilon)=0,$$

for all $(k_0,k_1)\in\mathcal{S}$. Let the initial data (32) constructed as above.

Then, for all $0\leq i\leq\nu-1$, the problem (31), (32) has a solution $X_i(t,z,\epsilon)$ such that $\partial_t^l X_i(t,z,\epsilon)$ is holomorphic and bounded on $\mathcal{T}\times D(0,\rho)\times\mathcal{E}_i$, for some $\rho>0$, for all $l\geq 0$. Moreover, for all $l\geq 0$, there exist two constants $C,A>0$ such that

$$\sup_{t\in\mathcal{T},z\in D(0,\rho)}|\partial_t^l X_{i+1}(t,z,\epsilon)-\partial_t^l X_i(t,z,\epsilon)|\leq CA^n n!|\epsilon|^n \tag{33}$$

for all $n\geq 0$, for all $\epsilon\in\mathcal{E}_i\cap\mathcal{E}_{i+1}$, for all $0\leq i\leq\nu-1$ (where by convention $X_\nu(t,z,\epsilon)=X_0(t,z,\epsilon)$ and $S_\nu=S_0$).

Proof For all $0\leq i\leq\nu-1$, we consider the Cauchy problem (25), (26) for the initial conditions $Y_{U_{f_i},\mathcal{E}_i,j}(t,\epsilon)$, for $0\leq j\leq S-1$. From Theorem 2, this problem has a solution $Y_{U_{f_i},\mathcal{E}_i}(t,z,\epsilon)$ which is holomorphic and bounded on $U_{f_i,\theta,R}\times D(0,\rho)\times\mathcal{E}_i$, for $\theta>0$, $R,\rho>0$. Now, we put

$$X_i(t,z,\epsilon):=Y_{U_{f_i},\mathcal{E}_i}(\epsilon t,z,\epsilon)$$

for all $(t,z,\epsilon)\in\mathcal{T}\times D(0,\rho)\times\mathcal{E}_i$. By construction, one can check that $X_i(t,z,\epsilon)$ is a solution of the Cauchy problem (31), (32) which is holomorphic and bounded on $\mathcal{T}\times D(0,\rho)\times\mathcal{E}_i$. Moreover, all the partial derivatives $\partial_t^l X_i(t,z,\epsilon)=\epsilon^l(\partial_t^l Y_{U_{f_i},\mathcal{E}_i})(\epsilon t,z,\epsilon)$ are also holomorphic and bounded functions on $\mathcal{T}\times D(0,\rho)\times\mathcal{E}_i$, for all $l\geq 0$.

In the rest of the proof, we show the estimates (33). From the estimates (27), there exists a sequence $(\chi_m(z,\epsilon))_{m\geq 0}$ of holomorphic and bounded functions on $D(0,\rho)\times D(0,\epsilon_0)$ such that the following estimates hold: for all $l \geq 0$, there exist constants $C, A > 0$ such that

$$\sup_{z\in D(0,\rho)} |\partial_t^l X_i(t,z,\epsilon) - \sum_{m=0}^{n-1} \frac{\chi_{m+l}(z,\epsilon)}{m!}(\epsilon t)^m \epsilon^l| \leq CA^n n! |\epsilon t|^n \tag{34}$$

for all $\epsilon \in \mathcal{E}_i$, all $t \in \mathcal{T}$, all $n \geq 1$, all $0 \leq i \leq \nu - 1$. From the estimates (34), for all $l \geq 0$, we get a constants $D, B > 0$ such that

$$\sup_{t\in\mathcal{T}, z\in D(0,\rho)} |\partial_t^l X_{i+1}(t,z,\epsilon) - \partial_t^l X_i(t,z,\epsilon)| \leq DB^n n! |\epsilon|^n \tag{35}$$

for all $\epsilon \in \mathcal{E}_i \cap \mathcal{E}_{i+1}$, all $n \geq 1$ and all $0 \leq i \leq \nu - 1$. □

5 Existence of formal series solutions and Gevrey asymptotic expansion in the parameter ϵ

In the following, for a given integer $K \geq 0$, a real $\rho > 0$ and $\mathcal{T}$ an open sector centered at 0, we will denote by $\mathbb{E}_K(\mathcal{T}\times D(0,\rho))$ the vector space of holomorphic and bounded functions $f(t,z)$ on $\mathcal{T}\times D(0,\rho)$ such that $\partial_t^l f(t,z)$ is holomorphic and bounded on $\mathcal{T}\times D(0,\rho)$ for all $0 \leq l \leq K$. This vector space $\mathbb{E}_K(\mathcal{T}\times D(0,\rho))$ will be equipped with the norm

$$||f(t,z)||_{\mathbb{E}_K(\mathcal{T}\times D(0,\rho))} = \sum_{h=0}^{K} \sup_{t\in\mathcal{T}, z\in D(0,\rho)} |\partial_t^h f(t,z)|.$$

The space $(\mathbb{E}_K(\mathcal{T}\times D(0,\rho)), ||.||_{\mathbb{E}_K(\mathcal{T}\times D(0,\rho))})$ is a Banach space. Moreover, for $K' < K$, the Banach space $\mathbb{E}_K(\mathcal{T}\times D(0,\rho))$ is continuously embedded in $\mathbb{E}_{K'}(\mathcal{T}\times D(0,\rho))$.

In this section, we establish the existence of formal power series solution $\hat{X}(t,z,\epsilon) \in \mathbb{E}_0(\mathcal{T}\times D(0,\rho))[[\epsilon]]$ of the equation (31) which satisfies the property that the solutions $X_i(t,z,\epsilon)$ of the problem (31), (32) has $\hat{X}$ as Gevrey asymptotic expansion of order 1 on $\mathcal{E}_i$.

The proof is based on a cohomological criterion for summability of formal series with coefficients in a Banach space, see [2], page 121, which is known as the Sibuya-Malgrange theorem in the literature.

Theorem (CC) *Let $(\mathbb{E}, ||.||_{\mathbb{E}})$ be a Banach space over $\mathbb{C}$. Assume that a collection $(d_i, \mathcal{E}_i, f_i)$ is given with*

- Directions $0 \leq d_0 < \cdots < d_{\nu-1} < 2\pi$, $d_\nu = d_0$.
- For all $0 \leq i \leq \nu - 1$, sectors $\mathcal{E}_i = S_{(d_{i+1}+d_i)/2,(\pi+\delta_i)/2,\epsilon_0}$ where $\delta_i = d_{i+1} - d_i$.
- For all $0 \leq i \leq \nu - 1$, holomorphic functions f_i from $\mathcal{E}_i$ into the Banach space $(\mathbb{E}, ||.||_{\mathbb{E}})$.

For all $0 \leq i \leq \nu - 1$, consider the sectors $Z_i = \mathcal{E}_{i+1} \cap \mathcal{E}_i$ and the holomorphic functions $h_i = f_{i+1} - f_i$ on Z_i (which is a sector of opening π) (By convention, $S_\nu = S_0$ and $f_\nu = f_0$).
1) Assume that the functions $f_i(\epsilon)$ are bounded as $\epsilon \in \mathcal{E}_i$ tends to the origin in $\mathbb{C}$, for all $0 \leq i \leq \nu - 1$.

2) Assume that the functions $h_i(\epsilon)$ have the formal series $\hat{0}$ as Gevrey asymptotic expansion of order 1 on Z_i, for all $0 \leq i \leq \nu - 1$. This means that there exist constants $C, A > 0$ such that

$$||h_i(\epsilon)||_{\mathbb{E}} \leq CA^n n! |\epsilon|^n$$

for all $\epsilon \in Z_i$, all $n \geq 1$, all $0 \leq i \leq \nu - 1$. Then, for all $0 \leq i \leq \nu - 1$, the functions $f_i(\epsilon)$ are the $1-$sums on $\mathcal{E}_i$ of a $1-$summable formal series $\hat{f}(\epsilon)$ in ϵ with coefficients in the Banach space $\mathbb{E}$.

We are now in position to state the main result of this paper.

Theorem 3 *Under the same hypotheses as in the proposition 4, there exists a formal series*

$$\hat{X}(t,z,\epsilon) = \sum_{k \geq 0} X_k(t,z)\epsilon^k \in \mathbb{E}_0(\mathcal{T} \times D(0,\rho))[[\epsilon]]$$

which satisfies the singularly perturbed equation

$$(36) \quad 4\epsilon t^2 \partial_t \partial_z^S \hat{X}(t,z,\epsilon) + (2t\epsilon + 1)\partial_z^S \hat{X}(t,z,\epsilon) \\ = \sum_{(k_0,k_1) \in \mathcal{S}} b_{k_0,k_1}(z,\epsilon)\partial_t^{k_0}\partial_z^{k_1}\hat{X}(t,z,\epsilon) + b(z,\epsilon)\partial_t \int_0^t \hat{X}(t-\tau,z,\epsilon)\hat{X}(\tau,z,\epsilon)d\tau$$

and which has the $\mathbb{E}_0(\mathcal{T} \times D(0,\rho))-$valued functions $\epsilon \mapsto X_i(t,z,\epsilon)$ as $1-$sums on $\mathcal{E}_i$, for all $0 \leq i \leq \nu - 1$.

Proof We consider the sequence $(X_i(t,z,\epsilon))_{0 \leq i \leq \nu-1}$ of functions constructed in the proposition 4. Let us define $f_i(\epsilon) := ((t,z) \mapsto X_i(t,z,\epsilon))$ which is, by construction, a holomorphic and bounded function from $\mathcal{E}_i$ into the Banach space $\mathbb{E}_K(\mathcal{T} \times D(0,\rho))$, for all $0 \leq i \leq \nu - 1$, for all $K \geq 0$. Moreover, from the estimates (33), we deduce that the functions $h_i(\epsilon) = f_{i+1}(\epsilon) - f_i(\epsilon)$ have $\hat{0}$ as Gevrey asymptotic expansion of order 1 on Z_i, for $0 \leq i \leq \nu - 1$, as functions with values in $\mathbb{E}_K(\mathcal{T} \times D(0,\rho))$, for all $K \geq 0$.

From the upper mentioned theorem (CC), we deduce that there exists a formal series $\hat{f}(\epsilon) \in \mathbb{E}_K(\mathcal{T} \times D(0,\rho))[[\epsilon]]$, for all $K \geq 0$, such that the functions $f_i(\epsilon)$ are the $1-$sums on $\mathcal{E}_i$ of $\hat{f}(\epsilon)$ as $\mathbb{E}_K(\mathcal{T} \times D(0,\rho))-$valued functions. We put $\hat{X}(t,z,\epsilon) := \hat{f}(\epsilon)$.
In the sequel, it remains to show that the formal series $\hat{X}(t,z,\epsilon)$ satisfies the equation (36).
The first step will consist in showing that $\partial_t^{h_1}\partial_z^{h_2}X_i(t,z,\epsilon)$ is the $1-$sum of $\partial_t^{h_1}\partial_z^{h_2}\hat{X}(t,z,\epsilon)$ for all $(h_1,h_2) \in \mathcal{S}$ as $\mathbb{E}_0(\mathcal{T} \times D(0,\rho))-$valued functions. Indeed, let us denote by $\hat{X}^N(t,z,\epsilon)$ the partial sum of $\hat{X}$ with respect to ϵ of degree $N \geq 0$. By construction, the function $X_i(t,z,\epsilon)$ is the $1-$sum of $\hat{X}(t,z,\epsilon)$ in the variable ϵ with coefficients in $\mathbb{E}_K(\mathcal{T} \times D(0,\rho))$, for all $K \geq 0$.

We deduce that estimates of the following form hold,

$$\sup_{t \in \mathcal{T}, z \in D(0,\rho)} |\partial_t^{h_1}X_i(t,z,\epsilon) - \partial_t^{h_1}\hat{X}^{N-1}(t,z,\epsilon)| \leq CA^N N! |\epsilon|^N,$$

for all $\epsilon \in \mathcal{E}_i$, all $N \geq 1$ and $0 \leq h_1 \leq K$. Moreover, from the Cauchy formula, we deduce that there exist $C_1 > 0$, $0 < \rho' < \rho$ such that

$$\sup_{t \in \mathcal{T}, z \in D(0,\rho')} \left| \partial_t^{h_1}\partial_z^{h_2}X_i(t,z,\epsilon) - \partial_t^{h_1}\partial_z^{h_2}\hat{X}^{N-1}(t,z,\epsilon) \right| \\ \leq C_1 \sup_{t \in \mathcal{T}, z \in D(0,\rho)} \left| \partial_t^{h_1}X_i(t,z,\epsilon) - \partial_t^{h_1}\hat{X}^{N-1}(t,z,\epsilon) \right| \leq CA^N N! |\epsilon|^N$$

for all $\epsilon \in \mathcal{E}_i$, all $N \geq 1$ and $(h_1,h_2) \in \mathcal{S}$. This means, that for all $(h_1,h_2) \in \mathcal{S}$, $\partial_t^{h_1}\partial_z^{h_2}X_i(t,z,\epsilon)$ is the $1-$sum of $\partial_t^{h_1}\partial_z^{h_2}\hat{X}(t,z,\epsilon)$ where the sum and the formal series are considered with coefficients in the Banach space $\mathbb{E}_0(\mathcal{T} \times D(0,\rho'))$.

On the other hand, we have

$$\partial_t \int_0^t X_i(t-\tau,z,\epsilon)X_i(\tau,z,\epsilon)d\tau = X_i(0,z,\epsilon)X_i(t,z,\epsilon) + \int_0^t (\partial_t X_i)(t-\tau,z,\epsilon)X_i(\tau,z,\epsilon)d\tau.$$

From the general results of $\kappa-$summability theory, see [2], section 4.5, we also deduce that $X_i(0,z,\epsilon)X_i(t,z,\epsilon)$ is the $1-$sum of $\hat{X}(0,z,\epsilon)\hat{X}(t,z,\epsilon)$ and $\int_0^t (\partial_t X_i)(t-\tau,z,\epsilon)X_i(\tau,z,\epsilon)d\tau$ is the $1-$sum of $\int_0^t (\partial_t \hat{X})(t-\tau,z,\epsilon)\hat{X}(\tau,z,\epsilon)d\tau$ as $\mathbb{E}_0(\mathcal{T} \times D(0,\rho'))-$valued function. Finally, from the general properties of $\kappa-$summability, we deduce that $\hat{X}(t,z,\epsilon)$ satisfies the same equation (36) as it's $1-$sum $X_i(t,z,\epsilon)$ on $\mathcal{E}_i$. □

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