On Gevrey asymptotics for some nonlinear integro-differential equations

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Abstract

We construct formal power series solutions of nonlinear integro-differential equations for given initial conditions which are holomorphic functions on a strip in the complex plane $\mathbb{C}$. We give sufficient conditions on the shape of the equations and on the initial conditions under which there exist actual holomorphic solutions which are Gevrey asymptotic to the given formal series solutions with respect to the time variable on sectors in $\mathbb{C}$. We get $1-$Gevrey asymptotic expansions on sectors of opening less than π but not $1-$summability as it is the case for linear partial differential equations. Moreover, our approach yields global analytic solutions of these equations in both time and space variables.

Key words: Borel summability, Cauchy problem, Fourier transform, formal power series, Gevrey estimates. 2000 MSC: 35C10, 35C20.

1 Introduction

In the paper [4], W. Balser considered formal power series solutions $\hat{u}(t,z) = \sum_{k \geq 0} u_k(z) t^k$ of linear partial differential equations with constant coefficients

$$P(\partial_t, \partial_z)\hat{u}(t,z) = 0,$$

for given initial conditions $\hat{u}(0,z)$, assumed to be holomorphic functions near the origin in $\mathbb{C}$. He gave sufficient conditions for these formal series to be $\kappa-$summable in some directions $d \in \mathbb{R}$, with respect to t, when the given initial condition has an analytic continuation and exponential growth on some sector of infinite radius in $\mathbb{C}$.

Our goal is to extend this study to a family of nonlinear integro-differential equations. More precisely, we will consider partial differential equations of the form

$$(1) \qquad Q(\partial_t, \partial_z)u(t,z) = \int_0^t b(t-s,z)\partial_z^{s_0} u(s,z)ds + \int_0^t \partial_z^{s_1} u(t-s,z)\partial_z^{s_2} u(s,z)ds$$

where $Q(T,Z)$ is a polynomial in two variables with complex coefficients, s_0, s_1, s_2 are non negative integers, the function $b(t,z)$ is holomorphic with respect to t on the half-plane $\mathrm{Re}(t) > 0$, holomorphic with respect to z on a strip H_β of width $\beta > 0$ in $\mathbb{C}$ and has an asymptotic expansion $b(t,z) \sim \hat{b}(t,z) = \sum_{l \geq 0} b_l(z)t^l$, with holomorphic coefficients $b_l(z)$ on H_β when t tends to 0. We will study these equations for a given initial condition $u(0,z)$ which is holomorphic on H_β.

These equations belong to the class of so-called Volterra integro-differential initial value problems. There exists a huge literature concerning these equations, see [9], [20], and several authors have studied the regularity and asymptotic properties of their solutions in particular cases, see for instance [8], [13], [14].

In the class of equations we consider, we will focus on the situation when the degree of Q in Z is larger than it's degree in T. In that case the classical Cauchy-Kowalevski theorem does not hold and we cannot expect holomorphic solutions $u(t,z)$ of (1) at $t = 0$, even if the function $b(t,z)$ is holomorphic at $t = 0$. The strategy will be the same as in [4]. We first construct formal power series $\hat{u}(t,z) = \sum_{l \geq 0} u_l(z)t^l$ of the problem (1) where $b(t,z)$ is replaced with it's asymptotic expansion $\hat{b}(t,z)$. These formal series are in general divergent. Then, our goal will be to give sufficient conditions under which there exists an actual function $u(t,z)$, holomorphic with respect to t on the open right half-plane $\mathrm{Re}(t) > 0$, holomorphic with respect to z on H_β, having $\hat{u}(t,z)$ as asymptotic development as t tends to 0.

Now, we can state the main result of the paper (Theorem 1).

Theorem Let $P(T,X), R(X) \in \mathbb{C}[T,X]$ be polynomials with $\deg_T P(T,X) = n \geq 1$ and $s_0, s_1, s_2 \geq 0$ be non negative integers. We consider the following Cauchy problem

$$R(\partial_z)P(\partial_t,\partial_z)u(t,z) = \int_0^t b(t-s,z)\partial_z^{s_0}u(s,z)ds + \int_0^t \partial_z^{s_1}u(t-s,z)\partial_z^{s_2}u(s,z)ds \tag{2}$$

with initial conditions

$$u(0,z) = u_0(z) \quad , \quad (\partial_t^j u)(0,z) \equiv 0, \tag{3}$$

for all $1 \leq j \leq n-1$.

1) Assume that the conditions **(Hj)**, $0 \leq j \leq 3$, from section 3, hold for the polynomials P, R, which assert in particular that $R(ik)$ is invertible for all $k \in \mathbb{R}$ and that the polynomial $T \mapsto P(T^2, ik)$ has purely imaginary roots with at most linear growth as k tends to infinity.

2) Assume that $u_0(z)$ is the Fourier inverse transform of a function $w_0(k)$ that belongs to $E_{(\tilde{\beta},\tilde{\mu})}$ for a given $\tilde{\beta} > 0$ and for $\tilde{\mu} > 0$ large enough depending P,R,s_j, for $j = 0,1,2$, such that it's norm $||w_0||_{(\tilde{\beta},\tilde{\mu})}$ is small enough. In particular, $u_0(z)$ is holomorphic on the strip $H_{\tilde{\beta}}$.

3) Assume that $b(t,z)$ is the Fourier inverse transform (in the variable z) and a Laplace transform (in the variable t) of an even and smooth function $a(\tau,k)$ with respect to τ that belongs to $E_{(\nu,\tilde{\beta},\tilde{\mu})}$, for a given $\nu > 0$, such that it's norm $||a(\tau,k)||_{(\nu,\tilde{\beta},\tilde{\mu})}$ is small enough. Under these conditions, it follows that $b(t,z)$ has an asymptotic expansion $b(t,z) \sim \hat{b}(t,z) = \sum_{l \geq 0} b_l(z)t^l$, with holomorphic coefficients $b_l(z)$ on $H_{\tilde{\beta}}$ when t tends to 0.

Then, there exists a formal power series

$$\hat{u}(t,z) = \sum_{l \geq 0} u_l(z)\frac{t^l}{l!},$$

solution of the equation (2) where $b(t,z)$ is replaced by it's asymptotic expansion $\hat{b}(t,z)$, with initial condition (3), such that the coefficients $u_l(z)$ are holomorphic functions on a strip H_β, for some $0 < \beta < \tilde{\beta}$. Moreover, there exists an actual solution $u(t,z)$ of (2), (3), which is holomorphic with respect to t on the half-plane $\mathrm{Re}(t) > 0$ and with respect to z on H_β satisfying the following property : for all $0 < \theta < \pi/2$, all $\beta' < \beta$, there exist $C, K, R > 0$ such that

$$\sup_{z \in H_{\beta'}} |u(t,z) - \sum_{l=0}^{N-1} u_l(z) \frac{t^l}{l!}| \leq CK^N |t|^N N! \tag{4}$$

for all $t \in S_{0,\theta,R}$, all $N \geq 1$, where $S_{0,\theta,R}$ is a sector in $\mathbb{C}$ centered at 0, with bisecting direction $d = 0$, opening θ and radius R.

In our situation, we get $1-$Gevrey asymptotic for the solution $u(t,z)$ in the variable t on a sector with opening less than π in direction $d = 0$. This sector cannot be chosen with opening larger than π as it is the case for linear partial differential equations, see [4]. But this approach allows us to construct global analytic solutions to the problem in the variable z, for t in the right half-plane. In previous studies, local analytic solutions in the variables t, z near the origin were obtained, see [6], [16], [18], [19].

Moreover, we show that if the initial condition $u(0,z)$ and the coefficient $b(t,z)$ are holomorphic on a strip of width $\tilde{\beta}$, then the solution $u(t,z)$ of the problem will be holomorphic with respect to z on a strip of width $\beta < \tilde{\beta}$ with loss $\tilde{\beta} - \beta$ that does not depend on the initial data and the coefficient $b(t,z)$. In fact, one can show that if the initial condition $u(0,z)$ and the coefficient $b(t,z)$ are holomorphic on every strips $H_{\tilde{\beta}}$, for $\tilde{\beta} > 0$, the solution $u(t,z)$ will be an entire function in the variable z. Notice that there are very few results in the literature concerning entire solutions for non-linear Cauchy problems, see for instance [7].

The methodology we follow is similar to the one performed in the paper [11] by O. Costin and S. Tanveer for the Navier-Stokes equations. We use a Fourier transform in the space variable z and a modified Borel transform in the time variable t to transform the problem (2) into a differential equation with constant coefficients depending on a real parameter, see (14). This differential equation can be rewritten into an integral fixed point equation (24). The strategy will consist in constructing a suitable Banach space in which (24) has a unique solution and then apply a modified Laplace transform in time t and the Fourier inverse in space z to get a holomorphic solution $u(t,z)$ of the original problem (2) with the required asymptotic expansion at $t = 0$.

In the first part of the paper, we introduce Banach spaces $E_{(\nu,\beta,\mu)}$ of functions $h(\tau,k)$ which are exponentially decreasing in the variable $k \in \mathbb{R}$ and exponentially increasing functions in the variable $\tau \in \mathbb{R}$. Similar Banach spaces have been introduced in several recent works, see [2], [10], [11], [12].

In a second part, we construct solutions $h(\tau,k)$ of the fixed point equation (24) in the Banach spaces introduced in the first part. To ensure the holomorphy of the solutions near the origin in $\mathbb{C}$ with respect to τ, we use the crucial property of functions in $E_{(\nu,\beta,\mu)}$ to have a decreasing rate in the variable k depending on the position of τ. Notice that a similar phenomenon of space regularity that depends on the position of the time variable has been observed for the Navier Stokes equations, see [2], [15]. We show that the constructed solutions then solve the parameter depending differential equation (14).

In a third part, we use a modified Laplace transform $X_h(t,k)$ on the functions $\tau \mapsto h(\tau,k) \in E_{(\nu,\beta,\mu)}$ which is a particular acceleration operator of J. Écalle involving the heat kernel. Following the classical results of [21], we analyse it's asymptotic expansion at $t = 0$ using the analyticity of h near $\tau = 0$ as in the classical Watson lemma. Finally, we express the holomorphic solution $u(t,z)$ of the problem (2) as the Fourier inverse transform of $X_h(t,k)$ with respect to k, where h solves (24).

2 Weighted Banach spaces

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a decreasing function on $\mathbb{R}_+$ which is derivable on an interval $(0,\rho)$, with $\rho > 0$, continuous on $\mathbb{R}_+$ and satisfies the following properties : there exists $M_0 > 0$ such that $\varphi'(x) \leq -M_0$, for all $x \in (0,\rho)$, and $\varphi(x) = 0$, for all $x > \rho$. An example of such a function is given by

$$\varphi(x) = \begin{cases} M_0(\rho - x) & \text{if } 0 \leq x \leq \rho \\ 0 & \text{if } x > 0 \end{cases}$$

Definition 1 *Let $\nu, \beta, \mu > 0$ be real numbers. Let $D(0,\rho)$ be the disc centered at 0 with radius $\rho > 0$ in $\mathbb{C}$. We denote by $E_{(\nu,\beta,\mu)}$ the vector space of functions $h(t,k)$ from $(\bar{D}(0,\rho) \cup \mathbb{R}) \times \mathbb{R}$ into $\mathbb{C}$, which are analytic with respect to t on $D(0,\rho)$, continuous with respect to t on $\bar{D}(0,\rho) \cup \mathbb{R}$ and continuous with respect to k on $\mathbb{R}$, such that*

$$||h(t,k)||_{(\nu,\beta,\mu)} = \sup_{t \in \bar{D}(0,\rho) \cup \mathbb{R}} \sup_{k \in \mathbb{R}} (1+|k|)^{\mu}(1+|t|^2)\exp(\beta|k| - \nu|t| + |k|\varphi(|t|))|h(t,k)|$$

is finite. The space $E_{(\nu,\beta,\mu)}$ equipped with the norm $||.||_{(\nu,\beta,\mu)}$ is a Banach space. We denote also by $E^{\mathbb{R}}_{(\nu,\beta,\mu)}$ the vector space of continuous functions $h(t,k)$ from $\mathbb{R}^2$ into $\mathbb{C}$ such that

$$||h(t,k)||^{\mathbb{R}}_{(\nu,\beta,\mu)} = \sup_{t \in \mathbb{R}} \sup_{k \in \mathbb{R}} (1+|k|)^{\mu}(1+|t|^2)\exp(\beta|k| - \nu|t| + |k|\varphi(|t|))|h(t,k)|$$

is finite.

Remark : Similar Banach spaces have been introduced in several works like [2], [11], [12]. As in [11], we introduce the following vector space.

Definition 2 *Let $\beta, \mu \in \mathbb{R}$. We denote by $E_{(\beta,\mu)}$ the vector space of continuous functions $h : \mathbb{R} \to \mathbb{C}$ such that*

$$||h(k)||_{(\beta,\mu)} = \sup_{k \in \mathbb{R}} (1+|k|)^{\mu}\exp(\beta|k|)|h(k)|$$

is finite.

In the following lemmas we consider multiplication and integral operators which are shown to be continuous on the spaces $E_{(\nu,\beta,\mu)}$.

Lemma 1 *Let $K : \mathbb{R} \to \mathbb{R}$ be a continuous function such that there exists $M_1 \geq 0$ with $|K(k)| \leq M_0|k| + M_1$, for all $k \in \mathbb{R}$. Let $p(t,k) : \mathbb{C} \times \mathbb{R} \to \mathbb{C}$ be a polynomial with respect to t and bounded continuous with respect to k. Then, there exists $C_1 > 0$ (depending on p, ρ, M_1 and ν) such that*

$$||p(t,k)\exp(iK(k)t)f(k)||_{(\nu,\beta,\mu)} \leq C_1||f(k)||_{(\beta+M_0\rho+\varphi(0),\mu)} \tag{5}$$

for all $f \in E_{(\beta+M_0\rho+\varphi(0),\mu)}$.

Proof We write

$$\begin{aligned} &||p(t,k)\exp(iK(k)t)f(k)||_{(\nu,\beta,\mu)} \\ &\quad = \sup_{t \in \bar{D}(0,\rho)\cup\mathbb{R}, k \in \mathbb{R}} (1+|k|)^{\mu}\exp((\beta+M_0\rho+\varphi(0))|k|)|f(k)| \\ &\quad \times (1+|t|^2)\exp(-\nu|t|)|p(t,k)||\exp(itK(k))|\exp(\varphi(|t|)|k| - (M_0\rho+\varphi(0))|k|) \\ &\quad \leq \tilde{C}_1||f(k)||_{(\beta+M_0\rho+\varphi(0),\mu)} \end{aligned}$$

where

$$\tilde{C}_1 = \sup_{t\in\bar{D}(0,\rho)\cup\mathbb{R}} \sup_{k\in\mathbb{R}} (1+|t|^2)\exp(-\nu|t|)|p(t,k)| \times |\exp(itK(k))|\exp(\varphi(|t|)|k| - (M_0\rho+\varphi(0))|k|).$$

From the fact that φ is decreasing and the estimates on K, we get that

$$\begin{aligned} C_{1,1} &= \sup_{t\in\bar{D}(0,\rho),k\in\mathbb{R}} (1+|t|^2)\exp(-\nu|t|)|p(t,k)||\exp(itK(k))|\exp(\varphi(|t|)|k| - (M_0\rho+\varphi(0))|k|) \\ &\leq \sup_{t\in\bar{D}(0,\rho),k\in\mathbb{R}} (1+|t|^2)\exp(-\nu|t|)|p(t,k)| \times \exp(|K(k)||t| + \varphi(0)|k| - (M_0\rho+\varphi(0))|k|) \\ &\qquad\qquad \leq C_{1,1,1} = \sup_{t\in\bar{D}(0,\rho),k\in\mathbb{R}} (1+|t|^2)\exp(-\nu|t| + M_1|t|)|p(t,k)| \end{aligned}$$

which is finite because $p(t,k)$ is bounded with respect to $k \in \mathbb{R}$. On the other hand, from the fact that $K(k)$ is real valued and $\varphi(|t|) = 0$ for $|t| > \rho$, we get

$$\begin{aligned} C_{1,2} = \sup_{t\in(-\infty,\rho)\cup(\rho,+\infty),k\in\mathbb{R}} & (1+|t|^2)\exp(-\nu|t|)|p(t,k)||\exp(itK(k))| \\ & \times \exp(\varphi(|t|)|k| - (M_0\rho+\varphi(0))|k|) \\ & \leq C_{1,2,1} = \sup_{t\in(-\infty,\rho)\cup(\rho,+\infty),k\in\mathbb{R}} (1+|t|^2)\exp(-\nu|t|)|p(t,k)| \end{aligned}$$

which is finite from the fact that $p(t,k)$ is polynomial in t and bounded in k. From the inequality $\tilde{C}_1 \leq C_1 = \max(C_{1,1,1}, C_{1,2,1})$, we get the inequality (5). □

Lemma 2 *There exists $C_2 > 0$ (depending on ρ, ν), such that*

$$(6) \qquad ||\int_0^t f(s,k)ds||_{(\nu,\beta,\mu)} \leq C_2||f(t,k)||_{(\nu,\beta,\mu)}$$

for all $f \in E_{(\nu,\beta,\mu)}$.

Proof We write

$$\begin{aligned} ||\int_0^t & f(s,k)ds||_{(\nu,\beta,\mu)} \\ &= \sup_{t\in\bar{D}(0,\rho)\cup\mathbb{R},k\in\mathbb{R}} (1+|k|)^\mu(1+|t|^2)\exp(\beta|k| - \nu|t| + \varphi(|t|)|k|) \\ &\quad \times |\int_0^t \{(1+|s|^2)(1+|k|)^\mu \exp(\beta|k| - \nu|s| + \varphi(|s|)|k|)f(s,k)\}\mathcal{A}(t,s,k)ds| \end{aligned}$$

where

$$\mathcal{A}(t,s,k) = (1+|t-s|^2)\exp(-\nu|t-s|) \times \frac{\exp(-\beta|k| + \nu(|t-s|+|s|) - \varphi(|s|)|k|)}{(1+|k|)^\mu(1+|t-s|^2)(1+|s|^2)}$$

Using the parametrisation $s = th$, $h \in [0,1]$, we get that

$$||\int_0^t f(s,k)ds||_{(\nu,\beta,\mu)} \leq \sup_{x\in\mathbb{R}} (1+|x|^2)\exp(-\nu|x|)\tilde{C}_2||f(t,k)||_{(\nu,\beta,\mu)},$$

where

$$\tilde{C}_2 = \sup_{t \in \bar{D}(0,\rho) \cup \mathbb{R}, k \in \mathbb{R}} (1+|t|^2)|t| \int_0^1 \frac{\exp(-\nu|t| + \nu|t|(1-h) + \nu|t|h + \varphi(|t|)|k| - \varphi(|t|h))|k|)}{(1+|t|^2(1-h)^2)(1+|t|^2h^2)} dh$$

From the fact that φ is decreasing, we get that

$$\tilde{C}_2 \leq C_2' = \sup_{t \in \bar{D}(0,\rho) \cup \mathbb{R}} (1+|t|^2)|t| \int_0^1 \frac{1}{(1+|t|^2(1-h)^2)(1+|t|^2h^2)} dh.$$

This latter integral C_2' is finite from the corollary 4.9 of [10]. The inequality (6) follows. □

Lemma 3 *Let $K : \mathbb{R} \to \mathbb{R}$ be a continuous function such that there exists $M_1 \geq 0$ with $|K(k)| \leq M_0|k| + M_1$, for all $k \in \mathbb{R}$. Let $p(t,k) : \mathbb{C} \times \mathbb{R} \to \mathbb{C}$ be a polynomial in t and bounded continuous with respect to k. Then, there exists $C_3 > 0$ (depending on p, ρ, M_1 and ν) such that*

$$||\int_0^t p(t-s,k)\exp(i(t-s)K(k))f(s,k)ds||_{(\nu,\beta,\mu)} \leq C_3||f(t,k)||_{(\nu,\beta,\mu)} \tag{7}$$

for all $f \in E_{(\nu,\beta,\mu)}$.

Proof We write

$$||\int_0^t p(t-s,k)\exp(i(t-s)K(k))f(s,k)ds||_{(\nu,\beta,\mu)} = \sup_{t \in \bar{D}(0,\rho) \cup \mathbb{R}, k \in \mathbb{R}} (1+|k|)^{\mu}(1+|t|^2)$$

$$\exp(\beta|k| - \nu|t| + \varphi(|t|)|k|) \times |\int_0^t \{(1+|k|)^{\mu}(1+|s|^2)\exp(\beta|k| - \nu|s| + \varphi(|s|)|k|)f(s,k)\}\mathcal{B}(t,s,k)ds|$$

where

$$\mathcal{B}(t,s,k) = (1+|t-s|^2)\exp(-\nu|t-s|)p(t-s,k) \\ \times \frac{\exp(-\beta|k| + \nu|s| + \nu|t-s| + i(t-s)K(k) - \varphi(|s|)|k|)}{(1+|k|)^{\mu}(1+|s|^2)(1+|t-s|^2)}$$

Using the parametrization $s = th$, $h \in [0,1]$, we get

$$||\int_0^t p(t-s,k)\exp(i(t-s)K(k))f(s,k)ds||_{(\nu,\beta,\mu)} \\ \leq \sup_{x \in \bar{D}(0,\rho) \cup \mathbb{R}, k \in \mathbb{R}} (1+|x|^2)\exp(-\nu|x|)|p(x,k)| \times \tilde{C}_3 \times ||f(s,k)||_{(\nu,\beta,\mu)}$$

where

$$\tilde{C}_3 = \sup_{t \in \bar{D}(0,\rho) \cup \mathbb{R}, k \in \mathbb{R}} |t|(1+|t|^2) \int_0^1 \frac{|\exp(it(1-h)K(k))|\exp((-\varphi(|t|h) + \varphi(|t|))|k|)}{(1+|t|^2h^2)(1+|t|^2(1-h)^2)} dh.$$

On the other hand, we have

$$C_{3,1} = \sup_{t \in \bar{D}(0,\rho), k \in \mathbb{R}} |t|(1+|t|^2) \int_0^1 \frac{|\exp(it(1-h)K(k))|\exp((-\varphi(|t|h) + \varphi(|t|))|k|)}{(1+|t|^2h^2)(1+|t|^2(1-h)^2)} dh \\ \leq \sup_{t \in \bar{D}(0,\rho), k \in \mathbb{R}} |t|(1+|t|^2) \int_0^1 \frac{\exp(|K(k)||t|(1-h) + (\varphi(|t|) - \varphi(|t|h))|k|)}{(1+|t|^2h^2)(1+|t|^2(1-h)^2)} dh$$

From the hypothesis on φ', we get by the mean value theorem that

$$\varphi(|t|) - \varphi(|t|h) \leq -M_0|t|(1-h)$$

for all $|t| \leq \rho$, $h \in [0,1]$. So that from the estimates on K, we get

$$C_{3,1} \leq C_{3,1,1} = \sup_{t \in \bar{D}(0,\rho)} |t|(1+|t|^2) \int_0^1 \frac{\exp(M_1|t|(1-h))}{(1+|t|^2h^2)(1+|t|^2(1-h)^2)} dh$$

which is finite. Moreover, from the fact that $K(k)$ is real valued and that $\varphi(|t|) = 0$, for $|t| > \rho$, we have

$$C_{3,2} = \sup_{t \in (-\infty,-\rho)\cup(\rho,+\infty), k \in \mathbb{R}} |t|(1+|t|^2) \int_0^1 \frac{|\exp(it(1-h)K(k))| \exp((-\varphi(|t|h) + \varphi(|t|))|k|)}{(1+|t|^2h^2)(1+|t|^2(1-h)^2)} dh$$

$$\leq C_{3,2,1} = \sup_{t \in (-\infty,-\rho)\cup(\rho,+\infty)} |t|(1+|t|^2) \int_0^1 \frac{1}{(1+|t|^2h^2)(1+|t|^2(1-h)^2)} dh$$

This latter integral $C_{3,2,1}$ is finite from the corollary 4.9 of [10]. From the inequality $\tilde{C}_3 \leq C'_3 = \max(C_{3,1,1}, C_{3,2,1})$, the inequality (7) follows. □

Lemma 4 *Let $s_1, s_2 \geq 0$ be non negative integers. Assume that $\mu > \max(s_1+1, s_2+1)$. We choose $s_{1,2} \leq \min(-s_1, -s_2)$. Then, there exists $C_4 > 0$ (depending on ρ, μ, s_1, s_2) such that*

$$(8) \quad ||(1+|k|)^{s_{1,2}} \int_0^t \int_{-\infty}^{+\infty} (k-k_1)^{s_1} k_1^{s_2} f(t-s, k-k_1) g(s,k_1) dk_1 ds||_{(\nu,\beta,\mu)} \leq C_4 ||f(t,k)||_{(\nu,\beta,\mu)} ||g(t,k)||_{(\nu,\beta,\mu)}$$

for all $f, g \in E_{(\nu,\beta,\mu)}$.

Proof We write

$$||(1+|k|)^{s_{1,2}} \int_0^t \int_{-\infty}^{+\infty} (k-k_1)^{s_1} k_1^{s_2} f(t-s, k-k_1) g(s,k_1) dk_1 ds||_{(\nu,\beta,\mu)}$$
$$= \sup_{t \in \bar{D}(0,\rho)\cup\mathbb{R}, k \in \mathbb{R}} (1+|k|)^{\mu+s_{1,2}} (1+|t|^2) \times \exp(\beta|k| - \nu|t| + \varphi(|t|)|k|)$$
$$\times |\int_0^t \int_{-\infty}^{+\infty} \{(1+|k-k_1|)^{\mu}(1+|t-s|^2)\exp(\beta|k-k_1| - \nu|t-s| + \varphi(|t-s|)|k-k_1|) f(t-s, k-k_1)\}$$
$$\times \{(1+|k_1|)^{\mu}(1+|s|^2)\exp(\beta|k_1| - \nu|s| + \varphi(|s|)|k_1|) g(s,k_1)\} \mathcal{C}(t,s,k,k_1) dk_1 ds|$$

where

$$\mathcal{C}(t,s,k,k_1) = (k-k_1)^{s_1} k_1^{s_2} \times \frac{\exp(-\beta|k-k_1| - \beta|k_1| + \nu(|t-s| + |s|) - \varphi(|s|)|k_1| - \varphi(|t-s|)|k-k_1|)}{(1+|k-k_1|)^{\mu}(1+|k_1|)^{\mu}(1+|t-s|^2)(1+|s|^2)}$$

Using the parametrization $s = th$, $h \in [0,1]$, we get

$$(9) \quad ||(1+|k|)^{s_{1,2}} \int_0^t \int_{-\infty}^{+\infty} (k-k_1)^{s_1} k_1^{s_2} f(t-s, k-k_1) g(s,k_1) dk_1 ds||_{(\nu,\beta,\mu)} \leq \tilde{C}_4 ||f(t,k)||_{(\nu,\beta,\mu)} ||g(t,k)||_{(\nu,\beta,\mu)}$$

where

$$\tilde{C}_4 = \sup_{t\in\bar{D}(0,\rho)\cup\mathbb{R},k\in\mathbb{R}} (1+|k|)^{\mu+s_{1,2}}(1+|t|^2)|t|$$
$$\times \int_0^1\int_{-\infty}^{+\infty} \frac{(1+|k-k_1|)^{s_1}(1+|k_1|)^{s_2}\exp(d(t,h,k,k_1))}{(1+|k-k_1|)^{\mu}(1+|k_1|)^{\mu}(1+|t|^2(1-h)^2)(1+|t|^2h^2)}dk_1 dh$$

where

$$d(t,h,k,k_1) = \beta|k| - \beta|k-k_1| - \beta|k_1| - \nu|t| + \nu|t|(1-h) + \nu|t|h$$
$$+ \varphi(|t|)|k| - \varphi(|t|h)|k_1| - \varphi(|t|(1-h))|k-k_1|$$

Now, we have the convexity inequality

$$\varphi(|t|)|k| \leq \varphi(|t|(1-h))|k-k_1| + \varphi(|t|h)|k_1| \tag{10}$$

for all $|t|\geq 0$, $k,k_1\in\mathbb{R}$. Indeed, if $k=0$ this inequality is satisfied. Let $k\neq 0$. From the triangular inequality $|k|\leq |k-k_1|+|k_1|$, we get that

$$\varphi(|t|) \leq \frac{\varphi(|t|)|k-k_1| + \varphi(|t|)|k_1|}{|k|}$$

and the fact that φ is decreasing gives $\varphi(|t|)\leq\varphi(|t|h)$ and $\varphi(|t|)\leq\varphi(|t|(1-h))$. This yields (10). From (10) and the triangular inequality, we get that $d(t,h,k,k_1)\leq 0$, for all $t\in\mathbb{C}$, $h\in[0,1]$ and $k,k_1\in\mathbb{R}$. So that

$$\tilde{C}_4 \leq C_4 = \sup_{t\in\bar{D}(0,\rho)\cup\mathbb{R}} (1+|t|^2)|t| \int_0^1 \frac{1}{(1+|t|^2(1-h)^2)(1+|t|^2h^2)}dh$$
$$\times \sup_{k\in\mathbb{R}}(1+|k|)^{\mu+s_{1,2}} \int_{-\infty}^{+\infty} \frac{1}{(1+|k-k_1|)^{\mu-s_1}(1+|k_1|)^{\mu-s_2}}dk_1.$$

We now proceed as in Lemma 2.2 of [11]. We write

$$(1+|k|)^{\mu+s_{1,2}} \int_{-\infty}^{+\infty} \frac{1}{(1+|k-k_1|)^{\mu-s_1}(1+|k_1|)^{\mu-s_2}}dk_1 = (1+|k|)^{\mu+s_{1,2}}(\mathcal{A}_k+\mathcal{B}_k)$$

where

$$\mathcal{A}_k = \int_{|k_1|\leq|k|/2} \frac{1}{(1+|k-k_1|)^{\mu-s_1}(1+|k_1|)^{\mu-s_2}}dk_1$$

and

$$\mathcal{B}_k = \int_{|k_1|>|k|/2} \frac{1}{(1+|k-k_1|)^{\mu-s_1}(1+|k_1|)^{\mu-s_2}}dk_1,$$

for all $k\in\mathbb{R}$. From the inequality $1+|k-k_1|\geq\frac{1}{2}(|k|+1)$, when $|k_1|\leq|k|/2$, we get

$$\mathcal{A}_k \leq \frac{2^{\mu-s_1}}{(1+|k|)^{\mu-s_1}} \int_{-\infty}^{+\infty} \frac{1}{(1+|k_1|)^{\mu-s_2}}dk_1, \tag{11}$$

for all $k\in\mathbb{R}$. So that $\mathcal{A}_k$ is finite provided that $\mu > s_2+1$. From the inequality $1+|k_1| > \frac{1}{2}(1+|k|)$, when $|k_1|>|k|/2$, we get

$$\mathcal{B}_k \leq \frac{2^{\mu-s_2}}{(1+|k|)^{\mu-s_2}} \int_{-\infty}^{+\infty} \frac{1}{(1+|k-k_1|)^{\mu-s_1}}dk_1 = \frac{2^{\mu-s_2}}{(1+|k|)^{\mu-s_2}} \int_{-\infty}^{+\infty} \frac{1}{(1+|k_1|)^{\mu-s_1}}dk_1, \tag{12}$$

for all $k \in \mathbb{R}$. So that $\mathcal{B}_k$ is finite if $\mu > s_1 + 1$. From the inequalities (11), (12), we deduce that

$$\sup_{k \in \mathbb{R}} (1+|k|)^{\mu+s_{1,2}} (\mathcal{A}_k + \mathcal{B}_k)$$

is finite if $s_{1,2} \leq \min(-s_1, -s_2)$. Moreover, from Corollary 4.9 of [10], we know that the integral C_2' is finite. Finally, the product of integrals C_4 is finite. □

The next estimates follow directly from the definitions.

Lemma 5 *Let $\mu_1 \in \mathbb{R}$ with $\mu_1 > -\mu$. Let $h \in E_{(0,\mu_1)}$. We have*

$$||h(k)f(t,k)||_{(\nu,\beta,\mu+\mu_1)} \leq ||h(k)||_{(0,\mu_1)} ||f(t,k)||_{(\nu,\beta,\mu)} \tag{13}$$

for all $f \in E_{(\nu,\beta,\mu)}$.

3 Cauchy problems with a real parameter

Let $P(T,X), R(X) \in \mathbb{C}[T,X]$ be polynomials and $s_0, s_1, s_2 \geq 0$ be non negative integers. We put

$$\deg_T P(T,X) = n \geq 1 \quad , \quad \deg_X(R(X)) + \deg_X P(T,X) = \sigma \geq 0.$$

In this section, we will study the following problem

$$\begin{aligned} (14) \quad & R(ik)P(\partial_\tau^2, ik)w(\tau,k) \\ &= \frac{1}{(2\pi)^{1/2}} \int_0^\tau \int_0^{\tau_1} \int_{-\infty}^{+\infty} (ik_1)^{s_0} a(\tau_1 - \tau_2, k - k_1) w(\tau_2, k_1) \\ &\qquad + (i(k-k_1))^{s_1} (ik_1)^{s_2} w(\tau_1 - \tau_2, k - k_1) w(\tau_2, k_1) dk_1 d\tau_2 d\tau_1 \end{aligned}$$

for initial conditions

$$w(0,k) = w_0(k) \quad , \quad (\partial_\tau^j w)(0,k) \equiv 0, \tag{15}$$

for all $1 \leq j \leq 2n-1$.

We now make assumptions on the polynomial $R(X)$, on the family of polynomial $T \mapsto Q(T,k)$, defined by

$$Q(T,k) = P(T^2, ik) = \sum_{r=0}^{2n} a_r(ik) T^r$$

for all $k \in \mathbb{R}$, on the initial condition $w_0(k)$ and on the coefficient $a(\tau,k)$.
(H0) The polynomial $k \mapsto R(ik)$ has no real roots.
(H1) $a_{2n}(ik) = 1$, for all $k \in \mathbb{R}$.
(H2) The roots of $T \mapsto Q(T,k)$ are of the form $iK_j(k)$ with multiplicity $n_j \geq 1$, not depending on k, for all $1 \leq j \leq p$, with $1 \leq p \leq n$, such that $n_1 + \ldots + n_p = 2n$. The functions $K_j(k)$, $1 \leq j \leq p$, are supposed to be continuous functions from $\mathbb{R}$ to $\mathbb{R}$ such that there exists $M_1 \geq 0$ with

$$|K_j(k)| \leq M_0|k| + M_1$$

for all $k \in \mathbb{R}$, all $1 \leq j \leq p$, where M_0 was introduced in section 2.

Consider the companion matrix associated to $Q(T,k)$, defined as $A(k) = A_1(k) + J(k)$, where $A_1(k)$ is the $(2n,2n)-$matrix whose line number $2n$ is $(-a_0(ik), \cdots, -a_{2n-1}(ik))$ and with all other coefficients be equal to 0 and $J(k)$ is the upper triangular Jordan $(2n,2n)-$matrix with eigenvalue equal to 0.

We denote by $\mathbb{F}=\mathbb{C}(k,(K_j(k))_{1\leq j\leq p})$ the fields of rational functions in k and in the variables $K_j(k)$, $1\leq j\leq p$ with coefficients in $\mathbb{C}$. From classical linear algebra, we know that the matrix $A(k)$ has a Dunford unique decomposition of the form

$$A(k)=D(k)+N(k) \tag{16}$$

whith $D(k)=\sum_{j=1}^{p}(iK_j(k))P_j(A(k))$, is a diagonable matrix, and $P_j(A(k))$ are the projectors on the characteristic subspaces, where $P_j(T)\in\mathbb{F}[T]$, for $1\leq j\leq p$. Recall that the polynomial $P_j(T)$ are constructed as follows. Let $Q_l(T)=\prod_{j\neq l}(T-iK_j(k))^{n_j}$, for $1\leq l\leq p$. From the Bezout theorem, there exist $U_l(T)\in\mathbb{F}[T]$, $1\leq l\leq p$, such that

$$1=\sum_{l=1}^{p}U_l(T)Q_l(T).$$

Then, the polynomial $P_j(T)$ is defined by $P_j(T)=U_j(T)Q_j(T)$, for $1\leq j\leq p$. The matrix $N(k)$ is nilpotent and can be written as $P_N(A(k))$, where $P_N(T)\in\mathbb{F}[T]$ is defined by

$$P_N(T)=\sum_{j=1}^{p}(T-iK_j(k))P_j(T).$$

Moreover, the matrices $D(k)$ and $N(k)$ are commuting. So that the matrix $\mathbb{K}(\tau,k)=\exp(\tau A(k))$ has coefficients of the form

$$\mathbb{K}_{l,m}(\tau,k)=\sum_{j=1}^{p}\exp(i\tau K_j(k))p_{l,m,j}(\tau,k) \tag{17}$$

where $p_{l,m,j}(\tau,k)$ are polynomials in τ, with coefficients in $\mathbb{F}$, of degree less than n_j-1, for all $1\leq j\leq p$, $1\leq l,m\leq 2n$.

(H3) We make the assumption that the functions $p_{1,1,j}(\tau,k)$, $1\leq j\leq p$, are polynomials in τ with coefficients in $E_{(0,-\mu_{1,1})}$ for $\mu_{1,1}\geq 0$ and that the functions $p_{1,2n,j}(\tau,k)$, $1\leq j\leq p$, are polynomials in τ with coefficients in $E_{(0,-\deg(R)-s_{0,1,2})}$, where $s_{0,1,2}\leq\min(-s_0,-s_1,-s_2)$.
(H4) The initial condition satisfies $w_0(k)\in E_{(\tilde{\beta},\mu+\mu_{1,1})}$, for given $\tilde{\beta}>0$ and $\mu>2n+\sigma+\max(1+s_0,1+s_1,1+s_2)$.
(H5) Let $\beta=\tilde{\beta}-M_0\rho-\varphi(0)$ and let $\nu>0$. We assume that the coefficient $a(\tau,k)$ belongs to $E_{(\nu,\beta,\mu)}$ and that $a(\tau,k)=a(-\tau,k)$ for all $(\tau,k)\in\mathbb{R}^2$. Moreover, we assume that the function $\tau\mapsto a(\tau,k)$ is $2n$-times differentiable with respect to $\tau\in\mathbb{R}$, for all $k\in\mathbb{R}$ and that $\partial_\tau^l a(\tau,k)\in E^{\mathbb{R}}_{(\nu,\beta,\mu-l)}$, for all $0\leq l\leq 2n$.

Remark: Notice that if the polynomials $P_j(T)$, $1\leq j\leq p$, are well defined for all $k\in\mathbb{R}$, the assumption **(H2)** implies that the assumption **(H3)** holds for some $\mu_{1,1}\geq 0$ and $\deg(R)\geq 0$. For instance, if we make the assumption that the roots $K_j(k)$ are *holomorphic functions* on some domain Ω in $\mathbb{C}$ containing $\mathbb{R}$, then the results of the chapter 2 of [17] shows that the Dunford decomposition (16) holds on Ω with holomorphic projectors $P_j(A(k))$, $1\leq j\leq p$ and holomorphic matrix $N(k)$. In that case, we deduce that the assumption **(H3)** holds for some $\mu_{1,1},\deg(R)\geq 0$.

Example: Let $P(T,X) = T - X^2 + 1$ and $R(X) = 1$. We have $Q(T,k) = T^2 + k^2 + 1$. In that case, we get that

$$A(k) = \begin{pmatrix} 0 & 1 \\ -(k^2+1) & 0 \end{pmatrix},$$

which satisfies $A(k) = P(k)S(k)P^{-1}(k)$, where

$$P(k) = \begin{pmatrix} -i(1+k^2)^{-1/2} & i(1+k^2)^{-1/2} \\ 1 & 1 \end{pmatrix}, S(k) = \begin{pmatrix} i(k^2+1)^{1/2} & 0 \\ 0 & -i(k^2+1)^{1/2} \end{pmatrix},$$

for all $k \in \mathbb{R}$. So that

$$\mathbb{K}_{1,1}(\tau,k) = \frac{1}{2}(\psi(\tau,k) + \psi(-\tau,k)) \quad , \quad \mathbb{K}_{1,2}(\tau,k) = \frac{i}{2(k^2+1)^{1/2}}(\psi(-\tau,k) - \psi(\tau,k))$$

where $\psi(\tau,k) = \exp(i\tau(k^2+1)^{1/2})$. All the assumptions **(H0)**,**(H1)**,**(H2)**,**(H3)** are fulfilled in this situation.

Lemma 6 *Let $w_0 : \mathbb{R} \to \mathbb{R}$ and $F : \mathbb{R}^2 \to \mathbb{C}$ be continuous functions. Then, the unique solution of the linear problem*

$$R(ik)P(\partial_\tau^2, ik)w(\tau,k) = F(\tau,k) \tag{18}$$

for initial conditions

$$w(0,k) = w_0(k) \quad , \quad (\partial_\tau^j w)(0,k) \equiv 0, \tag{19}$$

for all $1 \leq j \leq 2n-1$, is

$$w(\tau,k) = \mathbb{K}_{11}(\tau,k)w_0(k) + \int_0^\tau \frac{\mathbb{K}_{1,2n}(\tau-\tau_1,k)}{R(ik)}F(\tau_1,k)d\tau_1$$

for all $\tau \in \mathbb{R}$ and $k \in \mathbb{R}$.

Proof If we put

$$Y(\tau,k) = \begin{pmatrix} w(\tau,k) \\ \vdots \\ w^{(2n-1)}(\tau,k) \end{pmatrix}, Y_0(k) = \begin{pmatrix} w_0(k) \\ 0 \\ \vdots \\ 0 \end{pmatrix}, B(\tau,k) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ F(\tau,k)/R(ik) \end{pmatrix}$$

Then, the problem (18), (19) can be written into the differential system

$$Y'(\tau,k) = A(k)Y(\tau,k) + B(\tau,k),$$

with initial condition $Y(0,k) = Y_0(k)$. A classical result shows that the unique solution of this differential system is

$$Y(\tau,k) = \exp(\tau A(k))Y_0(k) + \int_0^\tau \exp((\tau-\tau_1)A(k))B(\tau_1,k)d\tau_1,$$

for all $\tau, k \in \mathbb{R}$. The lemma 6 follows. $\square$

Lemma 7 *Let φ be the function defined in section 2. We put $\beta = \tilde{\beta} - M_0\rho - \varphi(0)$ and assume that $\beta > 0$ (for ρ and $\varphi(0)$ small enough). Then, there exist $R, \delta > 0$ (depending only on $s_0, s_1, s_2, \rho, \mu, \nu, P(T,X), R(X)$) such that if the functions $w_0 \in E_{(\tilde{\beta},\mu+\mu_{1,1})}$ and $a(\tau,k) \in E_{(\nu,\beta,\mu)}$ satisfies*

$$||w_0(k)||_{(\tilde{\beta},\mu+\mu_{1,1})} \leq \delta \quad , \quad ||a(\tau,k)||_{(\nu,\beta,\mu)} \leq \delta,$$

the map defined by

$$\mathcal{H}(w(\tau,k)) := \mathbb{K}_{11}(\tau,k)w_0(k) + \int_0^\tau \frac{\mathbb{K}_{1,2n}(\tau-\tau_1,k)}{(2\pi)^{1/2}R(ik)} \times \int_0^{\tau_1}\int_0^{\tau_2}\int_{-\infty}^{+\infty} (ik_1)^{s_0}a(\tau_2-\tau_3,k-k_1)w(\tau_3,k_1) + (i(k-k_1))^{s_1}(ik_1)^{s_2}w(\tau_2-\tau_3,k-k_1)w(\tau_3,k_1)dk_1d\tau_3d\tau_2d\tau_1$$

satisfies the following properties.
i) *$\mathcal{H}(\bar{D}(0,R)) \subset \bar{D}(0,R)$, where $\bar{D}(0,R)$ is the closed ball centered at 0 with radius $R > 0$ in $E_{(\nu,\beta,\mu)}$.*
ii) *There exists $0 < M < 1$ such that*

$$||\mathcal{H}(w_1) - \mathcal{H}(w_2)||_{(\nu,\beta,\mu)} \leq M||w_1 - w_2||_{(\nu,\beta,\mu)}$$

for all $w_1, w_2 \in \bar{D}(0,R)$.

Proof We first check the property **i)**. Let $w(\tau,k) \in E_{(\nu,\beta,\mu)}$ such that $||w(\tau,k)||_{(\nu,\beta,\mu)} \leq R$, with $R > 0$. From Lemma 1 and Lemma 5, there exists a constant C_1 (depending on M_1, ν, ρ, $p_{1,1,j}(\tau,k)$, for $1 \leq j \leq p$) such that

$$(20)\quad ||\mathbb{K}_{11}(\tau,k)w_0(k)||_{(\nu,\beta,\mu)} = ||\frac{\mathbb{K}_{11}(\tau,k)}{(1+|k|)^{\mu_{1,1}}}(1+|k|)^{\mu_{1,1}}w_0(k)||_{(\nu,\beta,\mu)} \leq C_1||(1+|k|)^{\mu_{1,1}}w_0(k)||_{(\tilde{\beta},\mu)} \leq C_1||w_0(k)||_{(\tilde{\beta},\mu+\mu_{1,1})} \leq C_1\delta.$$

From Lemma 4, there exists $C_4 > 0$ (depending on ρ,μ,s_0,s_1,s_2) such that

$$(21)\quad ||(1+|k|)^{s_{0,1,2}}\int_0^{\tau_2}\int_{-\infty}^{+\infty}(ik_1)^{s_0}a(\tau_2-\tau_3,k-k_1)w(\tau_3,k_1) + (i(k-k_1))^{s_1}(ik_1)^{s_2}w(\tau_2-\tau_3,k-k_1)w(\tau_3,k_1)dk_1d\tau_3||_{(\nu,\beta,\mu)} \leq C_4(||a(\tau,k)||_{(\nu,\beta,\mu)}||w(\tau,k)||_{(\nu,\beta,\mu)} + ||w(\tau,k)||^2_{(\nu,\beta,\mu)}) \leq C_4(\delta R + R^2).$$

From Lemma 2, there exists $C_2 > 0$ (depending on ρ, ν) such that

$$(22)\quad ||(1+|k|)^{s_{0,1,2}}\int_0^{\tau_1}\int_0^{\tau_2}\int_{-\infty}^{+\infty}(ik_1)^{s_0}a(\tau_2-\tau_3,k-k_1)w(\tau_3,k_1) + (i(k-k_1))^{s_1}(ik_1)^{s_2}w(\tau_2-\tau_3,k-k_1)w(\tau_3,k_1)dk_1d\tau_3d\tau_2||_{(\nu,\beta,\mu)} \leq C_2||(1+|k|)^{s_{0,1,2}}\int_0^{\tau_2}\int_{-\infty}^{+\infty}(ik_1)^{s_0}a(\tau_2-\tau_3,k-k_1)w(\tau_3,k_1) + (i(k-k_1))^{s_1}(ik_1)^{s_2}w(\tau_2-\tau_3,k-k_1)w(\tau_3,k_1)dk_1d\tau_3||_{(\nu,\beta,\mu)}.$$

From Lemma 3, there exists $C_3 > 0$ (depending on s_0,s_1,s_2,M_1,ν,ρ,$R(ik)$, $p_{1,2n,j}(\tau,k)$, for $1 \leq j \leq p$) such that

$$(23)\quad ||\int_0^{\tau} \frac{\mathbb{K}_{1,2n}(\tau-\tau_1,k)}{(2\pi)^{1/2}R(ik)(1+|k|)^{s_{0,1,2}}} \times (1+|k|)^{s_{0,1,2}} \int_0^{\tau_1}\int_0^{\tau_2}\int_{-\infty}^{+\infty} (ik_1)^{s_0} a(\tau_2-\tau_3,k-k_1)w(\tau_3,k_1) + (i(k-k_1))^{s_1}(ik_1)^{s_2} w(\tau_2-\tau_3,k-k_1)w(\tau_3,k_1) dk_1 d\tau_3 d\tau_2 d\tau_1 ||_{(\nu,\beta,\mu)}$$
$$\leq C_3 ||(1+|k|)^{s_{0,1,2}} \int_0^{\tau_1}\int_0^{\tau_2}\int_{-\infty}^{+\infty} (ik_1)^{s_0} a(\tau_2-\tau_3,k-k_1)w(\tau_3,k_1) + (i(k-k_1))^{s_1}(ik_1)^{s_2} w(\tau_2-\tau_3,k-k_1)w(\tau_3,k_1) dk_1 d\tau_3 d\tau_2 ||_{(\nu,\beta,\mu)}.$$

From the inequalities (20), (21), (22), (23), we get that

$$||\mathcal{H}(w(\tau,k))||_{(\nu,\beta,\mu)} \leq C_1\delta + C_2C_3C_4(\delta R + R^2)$$

Now, we choose δ, R small enough, such that $C_1\delta + C_2C_3C_4(\delta R + R^2) \leq R$. This yields the property **i)**.

We check the property **ii)**. Let w_1, w_2 be in $E_{(\nu,\beta,\mu)}$ such that $||w_j(\tau,k)||_{(\nu,\beta,\mu)} \leq R$, $j = 1,2$. We write

$$\mathcal{H}(w_1(\tau,k)) - \mathcal{H}(w_2(\tau,k)) = \int_0^{\tau} \frac{\mathbb{K}_{1,2n}(\tau-\tau_1,k)}{(2\pi)^{1/2}R(ik)(1+|k|)^{s_{0,1,2}}} \times (1+|k|)^{s_{0,1,2}} \int_0^{\tau_1}\int_0^{\tau_2}\int_{-\infty}^{+\infty} (ik_1)^{s_0} a(\tau_2-\tau_3,k-k_1)(w_1(\tau_3,k_1) - w_2(\tau_3,k_1)) + (i(k-k_1))^{s_1}(ik_1)^{s_2} w_{1,2}(\tau_2,\tau_3,k,k_1) dk_1 d\tau_3 d\tau_2 d\tau_1$$

where

$$w_{1,2}(\tau_2,\tau_3,k,k_1) = (w_1(\tau_2-\tau_3,k-k_1) - w_2(\tau_2-\tau_3,k-k_1)) \times w_1(\tau_3,k_1) + w_2(\tau_2-\tau_3,k-k_1) \times (w_1(\tau_3,k_1) - w_2(\tau_3,k_1))$$

Using the same estimates as (21), (22), (23), we get that

$$||\mathcal{H}(w_1(\tau,k)) - \mathcal{H}(w_2(\tau,k))||_{(\nu,\beta,\mu)} \leq (\delta + 2R)C_2C_3C_4||w_1(\tau,k) - w_2(\tau,k)||_{(\nu,\beta,\mu)}.$$

Now, we choose R,δ small enough such that $M = (\delta+2R)C_2C_3C_4 < 1$. This yields the property **ii)**. □

Proposition 1 *Let φ be the function defined in section 2. We put $\beta = \tilde{\beta} - M_0\rho - \varphi(0)$ and assume that $\beta > 0$ (for ρ and $\varphi(0)$ small enough). Then, there exist $R, \delta > 0$ (depending only on $s_0, s_1, s_2, \rho, \mu, \nu, P(T,X), R(X)$) such that if the functions $w_0 \in E_{(\tilde{\beta},\mu+\mu_{1,1})}$ and $a(\tau,k) \in E_{(\nu,\beta,\mu)}$ satisfies*

$$||w_0(k)||_{(\tilde{\beta},\mu+\mu_{1,1})} \leq \delta \quad , \quad ||a(\tau,k)||_{(\nu,\beta,\mu)} \leq \delta,$$

then the problem

$$(24)\quad w(\tau,k) = \mathbb{K}_{11}(\tau,k)w_0(k) + \int_0^{\tau} \frac{\mathbb{K}_{1,2n}(\tau-\tau_1,k)}{(2\pi)^{1/2}R(ik)} \times \int_0^{\tau_1}\int_0^{\tau_2}\int_{-\infty}^{+\infty} (ik_1)^{s_0} a(\tau_2-\tau_3,k-k_1)w(\tau_3,k_1) + (i(k-k_1))^{s_1}(ik_1)^{s_2} w(\tau_2-\tau_3,k-k_1)w(\tau_3,k_1) dk_1 d\tau_3 d\tau_2 d\tau_1$$

has a unique solution $w(\tau,k)$ *in the space* $E_{(\nu,\beta,\mu)}$ *such that* $||w(\tau,k)||_{(\nu,\beta,\mu)} \leq R$.

Proof The ball $\bar{D}(0,R)$ is a complete metric space for the norm $||.||_{(\nu,\beta,\mu)}$. From Lemma 7, there exists $R > 0$ such that $\mathcal{H}$ is a shrinking map from $\bar{D}(0,R)$ into itself. From the classical shrinking map theorem on complete metric spaces, the map $\mathcal{H}$ has a unique fixed point $w(\tau,k)$ in $\bar{D}(0,R)$ (i.e. $\mathcal{H}(w) = w$). This yields the proposition. □

Proposition 2 *Let* $w(\tau,k) \in E_{(\nu,\beta,\mu)}$ *be the solution of the problem (24). Then, for all* $k \in \mathbb{R}$, *the function* $\tau \mapsto w(\tau,k)$ *is the unique solution of the problem (14), (15), for all* $\tau, k \in \mathbb{R}$.

Proof We put

$$F(\tau,k) = \frac{(1+|k|)^{s_{0,1,2}}}{(2\pi)^{1/2}} \int_0^{\tau} \int_0^{\tau_1} \int_{-\infty}^{+\infty} (ik_1)^{s_0} a(\tau_1 - \tau_2, k - k_1) w(\tau_2, k_1) + (i(k-k_1))^{s_1} (ik_1)^{s_2} w(\tau_1 - \tau_2, k - k_1) w(\tau_2, k_1) dk_1 d\tau_2 d\tau_1$$

for all $\tau, k \in \mathbb{R}$. From the fact that $w(\tau,k) \in E_{(\nu,\beta,\mu)}$, we get from (21), (22) that $F(\tau,k)$ is continuous with respect to $\tau, k \in \mathbb{R}$. From Lemma 6, the unique solution of the problem

$$R(ik)P(\partial_\tau^2, ik)W(\tau,k) = F(\tau,k)/(1+|k|)^{s_{0,1,2}}$$

for initial conditions

$$W(0,k) = w_0(k) \quad , \quad (\partial_\tau^j W)(0,k) \equiv 0,$$

for all $1 \leq j \leq 2n-1$, is

$$W(\tau,k) = \mathbb{K}_{11}(\tau,k) w_0(k) + \int_0^{\tau} \frac{\mathbb{K}_{1,2n}(\tau - \tau_1, k)}{R(ik)(1+|k|)^{s_{0,1,2}}} F(\tau_1, k) d\tau_1$$

for all $\tau \in \mathbb{R}$ and $k \in \mathbb{R}$. But the equation (24) implies that $W(\tau,k) = w(\tau,k)$ for all $\tau, k \in \mathbb{R}$. So that $\tau \mapsto w(\tau,k)$ is the unique solution of (14), (15), for all $\tau, k \in \mathbb{R}$. □

Lemma 8 *Let* $w(\tau,k) \in E_{(\nu,\beta,\mu)}$ *be the solution of the problem (24). Then, we have that* $(\tau,k) \mapsto \partial_\tau^l w(\tau,k) \in E^{\mathbb{R}}_{(\nu,\beta,\mu-l)}$ *for all* $0 \leq l \leq 2n$.

Proof From the expressions (17) and the assumption **(H2)**, we get that

$$(1+|k|)^{-l-\mu_{1,1}} \partial_\tau^l \mathbb{K}_{11}(\tau,k) = \sum_{j=1}^{p} \exp(i\tau K_j(k)) p^l_{1,1,j}(\tau,k), \tag{25}$$

where $p^l_{1,1,j}(\tau,k)$ are polynomials in τ and bounded continuous with respect to k on $\mathbb{R}$, and

$$(1+|k|)^{-l} \frac{\partial_\tau^l \mathbb{K}_{1,2n}(\tau,k)}{R(ik)(1+|k|)^{s_{0,1,2}}} = \sum_{j=1}^{p} \exp(i\tau K_j(k)) p^l_{1,2n,j}(\tau,k), \tag{26}$$

where $p^l_{1,2n,j}(\tau,k)$ are polynomials in τ and bounded continuous with respect to k on $\mathbb{R}$, for all $l \geq 0$. By construction, we have that $w(\tau,k) \in E^{\mathbb{R}}_{(\nu,\beta,\mu)}$. Let us assume in the following that

$$\partial_\tau^l w(\tau,k) \in E^{\mathbb{R}}_{(\nu,\beta,\mu-l)}, \tag{27}$$

for $0 \leq l \leq q$, with $q < 2n$. Then, we have to show that $\partial_{\tau}^{q+1} w(\tau,k) \in E^{\mathbb{R}}_{(\nu,\beta,\mu-(q+1))}$. We put

$$I(\tau_1,k) = (1+|k|)^{s_{0,1,2}} \int_0^{\tau_1} \int_0^{\tau_2} \int_{-\infty}^{+\infty} (ik_1)^{s_0} a(\tau_2-\tau_3,k-k_1) w(\tau_3,k_1) + (i(k-k_1))^{s_1} (ik_1)^{s_2} w(\tau_2-\tau_3,k-k_1) w(\tau_3,k_1) dk_1 d\tau_3 d\tau_2$$

for all $\tau_1, k \in \mathbb{R}$. From the hypothesis (27) and the assumption **(H5)** we deduce that

$$\partial_{\tau_1} I(\tau_1,k) = (1+|k|)^{s_{0,1,2}} \int_{-\infty}^{+\infty} \int_0^{\tau_1} (ik_1)^{s_0} a(\tau_1-\tau_3,k-k_1) w(\tau_3,k_1) + (i(k-k_1))^{s_1} (ik_1)^{s_2} w(\tau_1-\tau_3,k-k_1) w(\tau_3,k_1) dk_1 d\tau_3,$$

for all $\tau_1, k \in \mathbb{R}$ and for all $j \geq 0$,

$$\begin{aligned} \partial_{\tau_1}^{j+2} I(\tau_1,k) &= (1+|k|)^{s_{0,1,2}} \int_{-\infty}^{+\infty} \Big(\sum_{l_1+l_2=j} (ik_1)^{s_0} (\partial_{\tau_1}^{l_1} a)(0,k-k_1)(\partial_{\tau_1}^{l_2} w)(\tau_1,k_1) \\ &\quad + (i(k-k_1))^{s_1} (ik_1)^{s_2} (\partial_{\tau_1}^{l_1} w)(0,k-k_1)(\partial_{\tau_1}^{l_2} w)(\tau_1,k_1)\Big) dk_1 \\ &\quad + (1+|k|)^{s_{0,1,2}} \int_{-\infty}^{+\infty} \int_0^{\tau_1} (ik_1)^{s_0} (\partial_{\tau_1}^{j+1} a)(\tau_1-\tau_3,k-k_1) w(\tau_3,k_1) \\ &\quad + (i(k-k_1))^{s_1} (ik_1)^{s_2} (\partial_{\tau_1}^{j+1} w)(\tau_1-\tau_3,k-k_1) w(\tau_3,k_1) d\tau_3 dk_1 \end{aligned}$$

for all $\tau_1, k \in \mathbb{R}$, all $0 \leq j \leq q-1$. From Lemma 4 and the continuous inclusion

$$E_{(\nu,\beta,\mu-m_1)} \hookrightarrow E_{(\nu,\beta,\mu-m_2)},$$

if $0 \leq m_1 \leq m_2 < \mu$, we get that

$$\partial_{\tau_1}^{l} I(\tau_1,k) \in E^{\mathbb{R}}_{(\nu,\beta,\mu-l)}, \tag{28}$$

for all $0 \leq l \leq q$. From (25), (26), (28) and Lemma 3 and 5, we get that

$$\begin{aligned} (1+|k|)^{-(q+1)} \partial_{\tau}^{q+1} w(\tau,k) &= (1+|k|)^{-(q+1)} \partial_{\tau}^{q+1} \mathbb{K}_{1,1}(\tau,k) w_0(k) \\ &\quad + \sum_{l_1+l_2=q} \frac{(\partial_{\tau}^{l_1} \mathbb{K}_{1,2n})(0,k)}{(2\pi)^{1/2} R(ik)(1+|k|)^{s_{0,1,2}} (1+|k|)^{l_1}} \frac{\partial_{\tau}^{l_2} I(\tau,k)}{(1+|k|)^{l_2+1}} \\ &\quad + \int_0^{\tau} \frac{\partial_{\tau}^{q+1} \mathbb{K}_{1,2n}(\tau-\tau_1,k)}{(2\pi)^{1/2} R(ik)(1+|k|)^{s_{0,1,2}} (1+|k|)^{q+1}} I(\tau_1,k) d\tau_1 \end{aligned}$$

for all $\tau, k \in \mathbb{R}$. We also deduce that $(1+|k|)^{-(q+1)} \partial_{\tau}^{q+1} w(\tau,k) \in E^{\mathbb{R}}_{(\nu,\beta,\mu)}$. From Lemma 5, we finally get that $\partial_{\tau}^{q+1} w(\tau,k) \in E^{\mathbb{R}}_{(\nu,\beta,\mu-(q+1))}$. □

Lemma 9 *Let $w(\tau,k) \in E_{(\nu,\beta,\mu)}$ be the unique solution of the problem (14), (15). Then, we get that $w(-\tau,k) = w(\tau,k)$ for all $\tau, k \in \mathbb{R}$.*

Proof We put $\tilde{w}(\tau,k) := w(-\tau,k)$ for all $\tau, k \in \mathbb{R}$. From the equation (14), transforming τ into $-\tau$ yields

$$\begin{aligned} &R(ik)(P(\partial_{\tau}^2, ik)w)(-\tau,k) \\ &= \frac{1}{(2\pi)^{1/2}} \int_0^{-\tau} \int_0^{\tau_1} \int_{-\infty}^{+\infty} (ik_1)^{s_0} a(\tau_1-\tau_2,k-k_1) w(\tau_2,k_1) \\ &\quad + (i(k-k_1))^{s_1} (ik_1)^{s_2} w(\tau_1-\tau_2,k-k_1) w(\tau_2,k_1) dk_1 d\tau_2 d\tau_1 \end{aligned}$$

for all $\tau, k \in \mathbb{R}$. But we have that

$$(P(\partial_\tau^2, ik)w)(-\tau, k) = P(\partial_\tau^2, ik)(\tilde{w}(\tau, k))$$

for all $\tau, k \in \mathbb{R}$. From the assumption **(H5)** and using the change of variable $\tau_2 = -u_2$ and $\tau_1 = -u_1$, we get that

$$\begin{aligned}\int_0^{-\tau}\int_0^{\tau_1}\int_{-\infty}^{+\infty} & (ik_1)^{s_0} a(\tau_1 - \tau_2, k - k_1) w(\tau_2, k_1) \\ & + (i(k-k_1))^{s_1}(ik_1)^{s_2} w(\tau_1 - \tau_2, k - k_1) w(\tau_2, k_1) dk_1 d\tau_2 d\tau_1 \\ & = \int_0^{\tau}\int_0^{u_1}\int_{-\infty}^{+\infty} (ik_1)^{s_0} a(u_1 - u_2, k - k_1)\tilde{w}(u_2, k_1) \\ & \qquad + (i(k-k_1))^{s_1}(ik_1)^{s_2}\tilde{w}(u_1 - u_2, k - k_1)\tilde{w}(u_2, k_1) dk_1 du_2 du_1.\end{aligned}$$

for all $\tau, k \in \mathbb{R}$. We deduce that $\tilde{w}(\tau, k)$ satisfies the equation (14). Moreover, by construction, we also have

$$\tilde{w}(0, k) = w_0(k) \quad , \quad (\partial_\tau^j \tilde{w})(0, k) \equiv 0,$$

for all $1 \leq j \leq 2n - 1$. Finally, $(\tau, k) \mapsto \tilde{w}(\tau, k)$ satisfies the problem (14), (15) for all $\tau, k \in \mathbb{R}$. From the unicity of the solution of (14), (15), we get that $\tilde{w}(\tau, k) = w(\tau, k)$ for all $\tau, k \in \mathbb{R}$. □

4 Cauchy problems with a real parameter and Laplace transforms

Let $P(T, X), R(X) \in \mathbb{C}[T, X]$ be polynomials with $\deg_T P(T, X) = n \geq 1$ and $s_0, s_1, s_2 \geq 0$ be non negative integers. In this section, we will consider the following problem

$$(29) \quad \begin{aligned} R(ik)P(\partial_t, ik)X(t, k) = \frac{1}{(2\pi)^{1/2}} \int_0^t \int_{-\infty}^{+\infty} & (ik_1)^{s_0} A(t - s, k - k_1) X(s, k_1) \\ & + (i(k-k_1))^{s_1}(ik_1)^{s_2} X(t - s, k - k_1) X(s, k_1) dk_1 ds \end{aligned}$$

with initial conditions

$$(30) \qquad X(0, k) = w_0(k) \quad , \quad (\partial_t^j X)(0, k) \equiv 0,$$

for all $1 \leq j \leq n - 1$ and where $A(t, k) = X_a(t, k)$ is the modified Laplace transform of order 1 (defined in Proposition 3) of a function $a(\tau, k)$ which satisfies the assumption **(H5)**.

In this section, we make the assumptions that the polynomials P, R, the initial condition $w_0(k)$ satisfy the conditions **(Hj)**, for $0 \leq j \leq 4$. Moreover we assume that

$$||w_0||_{(\tilde{\beta}, \mu + \mu_{1,1})} \leq \delta \quad , \quad ||a(\tau, k)||_{(\nu, \beta, \mu)} \leq \delta,$$

where $\delta > 0$ is defined in the proposition 1.

4.1 Definition and properties of the Laplace transform

In this subsection, we define the Laplace transform of functions that belong to the Banach spaces introduced in the section 2 and we give some relations that will be useful in the sequel. The next proposition is a consequence of Theorem 58 in [3] applied to the acceleration operator $\mathcal{A}_{2,1}$ introduced by J. Écalle whose integral representation involves the heat kernel.

Proposition 3 *We put* $e(x) = \frac{1}{2\pi^{1/2}} x^{1/2} \exp(-x/4)$, *for all* $x \geq 0$. *Let* $w(\tau,k) \in E_{(\nu,\beta,\mu)}$ *with* $\beta > 0$, $\mu > 2$. *We define by*

$$X_w(t,k) = 2\int_0^{+\infty} \frac{e(\tau^2/t)}{\tau} w(\tau,k)d\tau \tag{31}$$

the modified Laplace transform of order 1 with kernel $e(x)$ *of* $\tau \mapsto w(\tau,k)$. *The function* $(t,k) \mapsto X_w(t,k)$ *is a holomorphic function with respect to* t *on*

$$S_{0,\pi/2} = \{t \in \mathbb{C}^* / \arg(t) \in (-\pi/2, \pi/2)\},$$

and continuous with respect to k *on* $\mathbb{R}$. *Moreover, for all* $t \in S_{0,\pi/2}$, *we have that* $||k \mapsto X_w(t,k)||_{(\beta,\mu)}$ *is finite.*
If we assume that $\partial_\tau^l w(\tau,k) \in E^{\mathbb{R}}_{(\nu,\beta,\mu-l)}$, *for* $l = 1,2$ *and that* $(\partial_\tau w)(0,k) \equiv 0$, *for* $k \in \mathbb{R}$, *the following relation hold*

$$\partial_t X_w(t,k) = 2\int_0^{+\infty} \frac{e(\tau^2/t)}{\tau} \partial_\tau^2 w(\tau,k)d\tau. \tag{32}$$

Moreover, let $a(\tau,k), w(\tau,k) \in E_{(\nu,\beta,\mu)}$. *Then,*

$$\begin{aligned} \int_0^t X_a(t-s,k-k_1)X_w(s,k_1)ds \\ = 2\int_0^{+\infty} \frac{e(\tau^2/t)}{\tau} \int_0^\tau \int_0^{\tau_1} a(\tau_1-\tau_2,k-k_1)w(\tau_2,k_1)d\tau_2 d\tau_1 d\tau \end{aligned} \tag{33}$$

for all $t \in S_{0,\pi/2}$, $k, k_1 \in \mathbb{R}$.

In the following proposition, we analyse the asymptotic expansion of the Laplace transform on a sector near the origin. This proposition follows from Theorem 2.4 in [21].

Proposition 4 *Let* $w(\tau,k) \in E_{(\nu,\beta,\mu)}$, *with* $\beta, \mu > 0$, *such that* $w(-\tau,k) = w(\tau,k)$ *for all* $\tau, k \in \mathbb{R}$. *In particular,* $w(\tau,k)$ *has the following Taylor expansion*

$$w(\tau,k) = \sum_{l\geq 0} w_l(k) \frac{\tau^{2l}}{(2l)!} \tag{34}$$

where the coefficients $w_l(k)$ *belong to* $E_{(\beta,\mu)}$, *for* $l \geq 0$ *(from the Cauchy formula) and which is convergent for all* $|\tau| < \rho$. *Then, the Laplace transform* $X_w(t,k)$ *satisfies the following estimates. For all* $0 < \theta < \pi/2$, *there exist* $C, K, R > 0$ *such that*

$$\sup_{k\in\mathbb{R}} (1+|k|)^\mu \exp(\beta|k|)|X_w(t,k) - \sum_{l=0}^{N-1} w_l(k)\frac{t^l}{l!}| \leq CK^N|t|^N N! \tag{35}$$

for all $t \in S_{0,\theta,R} = \{t \in \mathbb{C}^* / |\arg(t)| \leq \theta, |t| \leq R\}$, *all* $N \geq 1$.

4.2 Formal power series solutions of the Cauchy problem and Gevrey asymptotic expansion

In the next proposition, we construct an actual sectorial holomorphic solution with values in the Banach space $E_{(\beta,\mu)}$ of the Cauchy problem (29), (30) and we analyse it's asymptotic expansion near the origin in $\mathbb{C}$.

From the assumption **(H5)**, we have the following expansion $a(\tau,k) = \sum_{l\geq 0} a_l(k)\tau^{2l}/(2l)!$ where the coefficients $a_l(k)$, $l \geq 0$, belong to $E_{(\beta,\mu)}$. In the following, we put $\hat{A}(t,k) = \sum_{l\geq 0} a_l(k)t^l/l!$.

Proposition 5 *There exists a unique formal power series*

$$\hat{X}(t,k) = \sum_{l\geq 0} w_l(k)\frac{t^l}{l!} \tag{36}$$

with coefficients $w_l \in E_{(\beta,\mu)}$, for all $l \geq 0$, which is a solution of the problem

$$\begin{aligned} R(ik)P(\partial_t, ik)\hat{X}(t,k) = \frac{1}{(2\pi)^{1/2}}\int_0^t \int_{-\infty}^{+\infty} &(ik_1)^{s_0}\hat{A}(t-s,k-k_1)\hat{X}(s,k_1) \\ &+ (i(k-k_1))^{s_1}(ik_1)^{s_2}\hat{X}(t-s,k-k_1)\hat{X}(s,k_1)dk_1 ds \end{aligned} \tag{37}$$

with initial conditions

$$X(0,k) = w_0(k) \quad , \quad (\partial_t^j X)(0,k) \equiv 0. \tag{38}$$

Moreover, there exists an actual solution $t \mapsto X(t,k)$ of (29), (30), which is holomorphic with respect to t on the sector $S_{0,\pi/2}$ with values in $E_{(\beta,\mu)}$ satisfying the following property: For all $0 < \theta < \pi/2$, there exist $R, C, K > 0$ such that

$$\sup_{k\in\mathbb{R}}(1+|k|)^{\mu}\exp(\beta|k|)|X(t,k) - \sum_{l=0}^{N-1} w_l(k)\frac{t^l}{l!}| \leq CK^N|t|^N N! \tag{39}$$

for all $t \in S_{0,\theta,R}$, all $N \geq 1$.

Proof From Proposition 2, let $w(\tau,k)$ be the unique solution of the problem (14), (15) in the space $E_{(\nu,\beta,\mu)}$. From Lemma 9, the function w satisfies $w(-\tau,k) = w(\tau,k)$, for all $\tau, k \in \mathbb{R}$. We deduce that w has a convergent Taylor expansion of the form

$$w(\tau,k) = \sum_{l\geq 0} w_l(k)\frac{\tau^{2l}}{(2l)!}$$

where the coefficients $w_l \in E_{(\beta,\mu)}$, for $l \geq 0$, for all $\tau \in D(0,\rho)$. We consider now the formal series (36) constructed with the coefficients w_l, $l \geq 0$. A direct computation shows that $\hat{X}(t,k)$ is the unique formal solution of (37), (38). Let $X(t,k)$ be defined by the Laplace transform $X_w(t,k)$. From Lemma 8, we know that $\partial_\tau^l w(\tau,k) \in E^{\mathbb{R}}_{(\nu,\beta,\mu-l)}$, for all $0 \leq l \leq 2n$. From Lemma 9, we get that $(\partial_\tau^{2l+1}w)(0,k) \equiv 0$, for $0 \leq l \leq n-1$. We deduce from the formula (32) and (33) that if $w(\tau,k)$ satisfies the equation (14), then $X_w(t,k)$ satisfies the equation (29). From the proposition 4, we get that the estimates (39) hold. As a consequence, we get also that $X_w(t,k)$ satisfies the initial conditions (30). Finally, $X(t,k)$ is an actual solution of (29), (30) satisfying (39). □

5 Cauchy problems in holomorphic functions spaces

In the following lemma, we recall some properties of the inverse Fourier transform

Lemma 10 *Let $f \in E_{(\beta,\mu)}$ with $\beta > 0$, $\mu > 1$. The inverse Fourier transform of f is defined by*

$$\mathcal{F}^{-1}(f)(x) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} f(k) \exp(ixk) dk$$

for all $x \in \mathbb{R}$. The function $\mathcal{F}^{-1}(f)$ extends to an analytic function on the strip

$$H_\beta = \{z \in \mathbb{C}/|\mathrm{Im}(z)| < \beta\}.$$

Let $\phi(k) = ikf(k) \in E_{(\beta,\mu-1)}$. Then, we have

$$\partial_z \mathcal{F}^{-1}(f)(z) = \mathcal{F}^{-1}(\phi)(z) \tag{40}$$

for all $z \in H_\beta$. Let $g \in E_{(\beta,\mu)}$ and let $\psi(k) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} f(k-k_1)g(k_1)dk_1$, for all $k \in \mathbb{R}$. From Lemma 4, we know that $\psi \in E_{(\beta,\mu)}$. Then, we have

$$\mathcal{F}^{-1}(f)(z)\mathcal{F}^{-1}(g)(z) = \mathcal{F}^{-1}(\psi)(z) \tag{41}$$

for all $z \in H_\beta$.

Proof The fact that $\mathcal{F}^{-1}(f)$ extends to an analytic function on the strip H_β follows from the next inequality. There exists $C > 0$ such that

$$|f(k)||\exp(izk)| \leq \frac{C}{(1+|k|)^\mu} \exp((\beta' - \beta)|k|)$$

for all $k \in \mathbb{R}$, $z \in H_{\beta'}$, with $\beta' < \beta$. The relations (40), (41) are classical and can be found for instance in [22]. □

We can now state the main result of the paper.

Theorem 1 *Let $P(T,X), R(X) \in \mathbb{C}[T,X]$ be polynomials with $\deg_T P(T,X) = n \geq 1$ and $s_0, s_1, s_2 \geq 0$ be non negative integers. We consider the following Cauchy problem*

$$R(\partial_z)P(\partial_t,\partial_z)u(t,z) = \int_0^t b(t-s,z)\partial_z^{s_0}u(s,z)ds + \int_0^t \partial_z^{s_1}u(t-s,z)\partial_z^{s_2}u(s,z)ds \tag{42}$$

with initial conditions

$$u(0,z) = u_0(z) \quad , \quad (\partial_t^j u)(0,z) \equiv 0, \tag{43}$$

for all $1 \leq j \leq n-1$.
1) *Assume that the conditions* **(Hj)**, *$0 \leq j \leq 3$, hold for the polynomials P, R.*
2) *Let $w_0 \in E_{(\tilde{\beta},\mu+\mu_{1,1})}$, where $\tilde{\beta} > 0$, $\mu > 2n + \sigma + \max(1+s_0, 1+s_1, 1+s_2)$. We put $u_0(z) = \mathcal{F}^{-1}(w_0)(z)$, which is a holomorphic function on the strip $H_{\tilde{\beta}}$.*
3) *Let $M_0 > 0$ be a constant and φ be the function defined in Section 2. We choose $\rho, \varphi(0) > 0$ in such a way that $\beta = \tilde{\beta} - M_0\rho - \varphi(0) > 0$. Let $\nu > 0$ and consider a function $a(\tau,k) \in E_{(\nu,\beta,\mu)}$*

which satisfies the assumption **(H5)**. *In particular, we have the Taylor expansion* $a(\tau,k) = \sum_{l\geq 0} a_l(k)\tau^{2l}/(2l)!$ *where the coefficients* $a_l(k)$, $l \geq 0$, *belong to* $E_{(\beta,\mu)}$. *We put*

$$\hat{b}(t,z) = \sum_{l\geq 0} b_l(z)\frac{t^l}{l!},$$

where $b_l(z) = \mathcal{F}^{-1}(a_l)(z)$, $l \geq 0$, *are holomorphic functions on the strip* H_β. *Let* $A(t,k)$ *be the modified Laplace transform of order 1 of* $a(\tau,k)$. *We put* $b(t,z) = \mathcal{F}^{-1}(k \mapsto A(t,k))(z)$.
4) *Assume that*

$$||w_0||_{(\tilde{\beta},\mu+\mu_{1,1})} \leq \delta \quad , \quad ||a(\tau,k)||_{(\nu,\beta,\mu)} \leq \delta,$$

where $\delta > 0$ *is defined in the proposition 1.*
Then, there exists a formal power series

$$\hat{u}(t,z) = \sum_{l\geq 0} u_l(z)\frac{t^l}{l!},$$

solution of the equation

$$R(\partial_z)P(\partial_t,\partial_z)\hat{u}(t,z) = \int_0^t \hat{b}(t-s,z)\partial_z^{s_0}\hat{u}(s,z)ds + \int_0^t \partial_z^{s_1}\hat{u}(t-s,z)\partial_z^{s_2}\hat{u}(s,z)ds \tag{44}$$

with initial condition (43), where $u_l(z)$, $l \geq 0$, *are holomorphic functions on* H_β. *Moreover, there exists an actual solution* $u(t,z)$ *of (42), (43), which is holomorphic with respect to* t *on* $S_{0,\pi/2}$ *and with respect to* z *on* H_β *satisfying the following property : for all* $0 < \theta < \pi/2$, *all* $\beta' < \beta$, *there exist* $C, K, R > 0$ *such that*

$$\sup_{z\in H_{\beta'}} |u(t,z) - \sum_{l=0}^{N-1} u_l(z)\frac{t^l}{l!}| \leq CK^N|t|^N N! \tag{45}$$

for all $t \in S_{0,\theta,R}$, *all* $N \geq 1$.

Proof Let $\hat{A}(t,k) = \sum_{l\geq 0} a_l(k)t^l/l!$. From Proposition 5, let $\hat{X}(t,k) = \sum_{l\geq 0} w_l(k)t^l/l!$ be the unique formal power series satisfying the Cauchy problem (37), (38), with $w_l(k) \in E_{(\beta,\mu)}$, for all $l \geq 0$. We define by

$$\mathcal{F}^{-1}(\hat{X}(t,k))(z) := \sum_{l\geq 0} \mathcal{F}^{-1}(w_l)(z)\frac{t^l}{l!},$$

for all $z \in H_\beta$. From Lemma 10, we deduce that the formal power series $\hat{u}(t,z) = \mathcal{F}^{-1}(\hat{X}(t,k))(z)$ satisfies the problem (44), (43).

Let $X(t,k)$ be the actual solution of the problem (29), (30) constructed in Proposition 5. Consider now the function $u(t,z) = \mathcal{F}^{-1}(k \mapsto X(t,k))(z)$ which is by construction holomorphic with respect to t on $S_{0,\pi/2}$ and with respect to z on H_β. Again from Lemma 10, we also deduce that $u(t,z)$ is a solution of the problem (42), (43) on $S_{0,\pi/2} \times H_\beta$.

From the estimates (39), we deduce that for all $0 < \theta < \pi/2$, there exist $C, K, R > 0$ such that

$$|u(t,z) - \sum_{l=0}^{N-1} u_l(z)\frac{t^l}{l!}| = |\mathcal{F}^{-1}(X(t,k) - \sum_{l=0}^{N-1} w_l(k)\frac{t^l}{l!})(z)|$$
$$\leq CK^N|t|^N N! \int_{-\infty}^{+\infty} \frac{\exp((\beta'-\beta)|k|)}{(1+|k|)^\mu}dk$$

for all $z \in H_{\beta'}$, for all $\beta' < \beta$, for all $t \in S_{0,\theta,R}$, all $N \geq 1$. This yields the estimates (45). □

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