Weak graph-convergences of Coercive Operators and its Application to Fissured Medium Equations

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In this note, we find an equivalent relationship between the weak convergence of linear semi-group and that of their generators. To find this, we introduce a notion of weak graph convergence of unbounded operators, and find some equivalent relationships among weak-convergence of linear semi-groups, weak graph convergence of generators and G-convergence of resolvents. As a nontrivial application, we introduce the homogenized differential equation for fissured medium equations.

Key Words: Homogenization, Weak-graph convergence, G-convergence, Cauchy problems, Fissured medium equations
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1. INTRODUCTION

Even though the classical Trotter approximation theorem gives a good guide to understand the convergence relationship between resolvants of m-accretive operators and their semi-groups, it is restricted to strong (norm) convergence. That is to say, the Trotter Theorem is useful for some time-dependent problems that converge strongly. However, in some situations we need to deal with weak convergence, for example, in the case of the Cauchy problem;

$$\frac{\partial}{\partial t}u_\varepsilon + \mathcal{A}_\varepsilon u_\varepsilon = f^\varepsilon, \quad u_\varepsilon(0) = u_0$$

energy estimates justify only the weak convergence. The main purpose of this paper is to find the equivalent conditions for this weak convergence. The role of the operators $\mathcal{A}_\varepsilon$ in the above Cauchy problems looks very important. Once the notion of weak graph-convergence for unbounded operators is introduced, we prove that in the category of coercive operators, we can find an equivalent relationship between the weak convergence of linear semi-groups and weak graph-convergence of their generators. The notion of G-convergence has been introduced to deal with this weak convergence for elliptic and stationary problems, but it can not resolve some problems like a time-dependent Cauchy problem. In this note, we find some equivalent relations of G-convergence and weak graph convergence, and we present a weak version of Trotter approximation theorem which gives the equivalent relationship between weak-convergence of linear semi-groups and G-convergence of resolvents. This gives the relationshup between the weak convergence of Cauchy problems and that of the stationary problems.

A fissured medium consists of a matrix of porous and permeable blocks or cells which are separated from one another by a highly developed system of fissures or flow paths through which the majority of diffusion occurs ([7]). In Section 4, we make use of the weak graph convergence for the solutions of the system of fissured medium equations to get the homogenized (fissured medium) system. We observe that the homogenized system is the same as the ε-fissured medium equation.

2. PRELIMINARIES

We recall some notations. Let $1 \le p \le \infty$. Let Ω be a domain in $\mathbf{R}^n$ and B be a Banach space. We denote by $L^p(\Omega; B)$ the space of measurable functions $f : \Omega \to B$ such that for $1 \le p < \infty$, $\int_\Omega \|f(x)\|_B^p \, dx$ is finite or for $p = \infty$, $\operatorname{ess\,sup}_{x\in\Omega} \|f(x)\|_B$ is finite. Hence as a special case when $B = \mathbf{R}$, $L^2(\Omega) \equiv L^2(\Omega; \mathbf{R})$ is the Lebesgue space of (equivalence classes of) functions whose modulus squared is Lebesgue integrable on the domain Ω. Denote by $C^\infty(\bar{\Omega})$ the set of restrictions of functions to $\bar{\Omega}$ of infinitely differentiable functions in $\mathbf{R}^n$ with compact support. Then $H^m(\Omega)$, $m \ge 1$, is the Sobolev space obtained by the completion of $C^\infty(\bar{\Omega})$ in the norm $\|u\|_{H^m(\Omega)} = \left(\sum_{|\alpha| \le m} \|\nabla^\alpha u\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multiple index, $|\,\alpha\,| = \sum_{i=1}^n \alpha_i$, $\alpha_i \ge 0$. Also $H_0^m(\Omega)$ is the closure in $H^m(\Omega)$ of the set of those infinitely differentiable functions in Ω each of which has compact support in Ω.

Let $\mathbf{V}$ be a Hilbert space that is dense in another Hilbert space $\mathbf{H}$ and assume the identity $\mathbf{V} \to \mathbf{H}$ is continuous. Let $a(\cdot,\cdot)$ be a continuous bilinear form on $\mathbf{V}$. Then we define D to be the set of all $u \in \mathbf{V}$ such that the function $v \mapsto a(u,v)$ is continuous on $\mathbf{V}$ *with the* $\mathbf{H}$*-norm*. For each such $u \in D$ there is then a unique $\mathbf{A}u \in \mathbf{H}$ such that

$$a(u,v) = \langle \mathbf{A}u, v \rangle_{\mathbf{H}} , \qquad u \in D \, , v \in \mathbf{V} \, ,$$

and this defines a linear operator $\mathbf{A} : D \to \mathbf{H}$. A form $a : \mathbf{V} \times \mathbf{V} \to \mathbf{R}$ is $\mathbf{V}$*-coercive* (or $\mathbf{V}$-elliptic) if there exists a constant $c_0 > 0$ for which

$$c_0 \|u\|^2_{\mathbf{V}} \le a(u,u), \qquad u \in \mathbf{V}.$$

Equivalently, $\mathcal{A} \in \mathcal{L}(\mathbf{V}, \mathbf{V}')$ defined by $\mathcal{A}u(v) \equiv a(u,v)$, $(u, v \in \mathbf{V})$ is $\mathbf{V}$-coercive if $a(\cdot,\cdot)$ is. An unbounded operator $\mathbf{A} : D \to \mathbf{H}$ is *accretive* if

$$\langle \mathbf{A}x, x \rangle_{\mathbf{H}} \ge 0 \, , \qquad x \in D \, ,$$

and it is m*-accretive* if, in addition, $I + \mathbf{A}$ maps D onto $\mathbf{H}$. It is easy to see how the unbounded operator $\mathbf{A}$ with domain D in $\mathbf{H}$ constructed as above from the continuous bilinear form $a(\cdot,\cdot)$ on $\mathbf{V}$ is related to the continuous operator $\mathcal{A} \in \mathcal{L}(\mathbf{V}, \mathbf{V}')$ which is equivalent to $a(\cdot,\cdot)$. In fact, the graph of $\mathbf{A}$ is the restriction of the graph of $\mathcal{A}$ to $\mathbf{V} \times \mathbf{H}$. That is, note that $\mathbf{H}' \hookrightarrow \mathbf{V}'$ by the restriction to $\mathbf{V}$ of functionals on $\mathbf{H}$, so $D = \{u \in \mathbf{V} : \mathcal{A}u \in \mathbf{H}'\}$ and then $\mathbf{A}u \in \mathbf{H}$ is just that $\mathcal{A}u \in \mathbf{H}'$ which corresponds through the identification of $\mathbf{H}$ with $\mathbf{H}'$ by its Riesz map. Thus, with this identification $\mathbf{H} = \mathbf{H}'$, it is clear that $\mathbf{A}$ is the restriction of $\mathcal{A}$ to $\mathbf{H} \subset \mathbf{V}'$; that is, if $\mathcal{R}$ is the Riesz map of $\mathbf{H}$ onto $\mathbf{H}'$, then $\mathbf{A} = \mathcal{R}^{-1}\mathcal{A}|_D$. An operator $\mathcal{A} \in \mathcal{L}(\mathbf{V}, \mathbf{V}')$ is called *monotone* if

$$\mathcal{A}v(v) \ge 0, \qquad v \in \mathbf{V}.$$

The corresponding unbounded operator on $\mathbf{H}$, $\mathbf{A} = \mathcal{R}^{-1}\mathcal{A}|_D$, is then accretive and is called *regular accretive* when it is so determined by a triple $\{\mathcal{A}, \mathbf{V}, \mathbf{H}\}$. One can show that every accretive self-adjoint operator $\mathbf{A} : D \to \mathbf{H}$ can be extended to an operator $\mathcal{A} \in \mathcal{L}(\mathbf{V}, \mathbf{V}')$ with the identification $\mathbf{H} = \mathbf{H}'$, that is, it is regular m-accretive. For this, let $\mathbf{V}$ be the closure of D with the scalar product $\langle u, v\rangle_{\mathbf{V}} \equiv \langle u, v\rangle_{\mathbf{H}} + \langle \mathbf{A}u, v\rangle_{\mathbf{H}}$.

Remark 1. 1. (Lax-Milgram) Let $a(\cdot,\cdot)$ be bilinear, continuous and $|a(\cdot,\cdot)|$ be $\mathbf{V}$-coercive. Then for each $f \in \mathbf{V}'$ there is a unique $u \in V$ such that

$$a(u, v) = f(v), \qquad v \in \mathbf{V}.$$

2. (Hille-Yosida) A necessary and sufficient condition for $-\mathbf{A} : D \to \mathbf{H}$ to be the generator of a linear contraction semigroup is that $\mathbf{A}$ is m-accretive.
3. Let the operator $\mathbf{A}$ be m-accretive on the Hilbert space $\mathbf{H}$. Then for every $u_0 \in D$ and $f \in C^1([0,\infty), \mathbf{H})$ there is a unique solution $u \in C^1([0,\infty), \mathbf{H})$ of the initial-value problem

$$\begin{aligned} \frac{\partial}{\partial t}u(t) + \mathbf{A}u(t) = f(t)\ , \qquad t > 0\ , \\ u(0) = u_0\ . \end{aligned} \tag{1}$$

If, in addition, $\mathbf{A}$ is self-adjoint, then for each $u_0 \in \mathbf{H}$ and each Hölder continuous function $f \in C^\alpha([0,\infty), \mathbf{H})$, $0 < \alpha < 1$, there is a unique solution $u \in C([0,\infty), \mathbf{H}) \cap C^1((0,\infty), \mathbf{H})$ of (1).

The full discussion of this matter can be found in [4], [5].

3. WEAK GRAPH CONVERGENCE

In this Section, we assume that $\mathbf{V}$ is a Hilbert space that is *dense* in another Hilbert space $\mathbf{H}$ and that the identity map $\mathbf{V} \hookrightarrow \mathbf{H}$ is *compact.* For an example, take $\mathbf{H} = L^2(\Omega)$, and $\mathbf{V} = H^1(\Omega)$. We identify $\mathbf{H} = \mathbf{H}'$ as pointed out in Section 2, and I represents the identity operator (or Riesz map) from $\mathbf{V}$ to $\mathbf{V}(\cong \mathbf{V}')$. Hereafter, operators written by script letter (for example, $\mathcal{A}$) are operators from $\mathbf{V}$ to $\mathbf{V}'$, and operators written by bold letter are the restriction of the script operator into the range $\mathbf{H}$, for example, $\mathbf{A}$ is the restriction of $\mathcal{A}$. We now introduce the notion of weak-graph convergence.

Definition 1. Let $\mathcal{A}_\varepsilon$ ($\varepsilon \geq 0$) be operators. Then $\{\mathcal{A}_\varepsilon\}$ is *graph-convergent* to $\mathcal{A}_0$ if for every $u \in \mathbf{V}$, $f \in \mathbf{V}'$ with $\mathcal{A}_0 u = f$ there is a pair $(u_\varepsilon, f_\varepsilon) \in \mathbf{V} \times \mathbf{V}'$ such that $\mathcal{A}_\varepsilon u_\varepsilon = f_\varepsilon$, and that $(u_\varepsilon, f_\varepsilon)$ converges strongly to (u, f) in $\mathbf{V} \times \mathbf{V}'$. We also say that $\{\mathcal{A}_\varepsilon\}$ is *weakly graph-convergent* to $\mathcal{A}_0$ if for every $u \in \mathbf{V}$, $f \in \mathbf{V}'$ with $\mathcal{A}_0 u = f$ there are $u_\varepsilon \in \mathbf{V}$, $f_\varepsilon \in \mathbf{V}'$ with $\mathcal{A}_\varepsilon u_\varepsilon = f_\varepsilon$ satisfying $u_\varepsilon \rightharpoonup u$ in $\mathbf{V}$, and $f_\varepsilon \rightharpoonup f$ in $\mathbf{V}'$.

We say that a linear operator $\mathcal{B}$ obeys the *elliptic condition* if it is a $\mathbf{V}$-coercive with respect to $\mathbf{H}$-inner product and continuous operator, that is,

$$c_0\|u\|_{\mathbf{V}}^2 \leq \langle \mathbf{B}u, u\rangle_{\mathbf{H}} \quad \text{and} \quad \|\mathcal{B}u\|_{\mathbf{V}'} \leq c_1\|u\|_{\mathbf{V}}, \qquad u \in \mathbf{V}, \tag{2}$$

for some positive constants c_0, c_1. Here $\mathbf{B}$ is the restriction of $\mathcal{B}$ on $D = \{u \in \mathbf{V} : \mathcal{B}u \in \mathbf{H}'\}$; that is, $\mathbf{B} = \mathcal{R}^{-1}\mathcal{B}|_D$. Linear operators $\mathcal{B}_\varepsilon$ obey an *uniform* elliptic condition if there are c_0, c_1 in (2) uniform in ε. For an accretive self-adjoint operator

$\mathbf{B}$, this condition means that the inner product $\langle \mathbf{B}u, u\rangle_{\mathbf{H}}$ produces the equivalent norm with $\mathbf{V}$. An example of this is $\mathbf{B} = I - \Delta$.

The main purpose of this section is to find equivalent conditions between the weak-convergence of linear semi-groups and weak graph convergence of their generators. First, we present our main theorem.

THEOREM 1 (**Main Theorem**). *Let $I + \mathcal{A}_\varepsilon$ $(\varepsilon \geq 0)$ satisfy the uniform elliptic condition. Then the followings are equivalent:*
(g) ...... $\{\mathcal{A}_\varepsilon\}$ is weakly graph-convergent to $\mathcal{A}_0$,
(S) ...... for any $f \in \mathbf{V}$ and any $\phi \in \mathbf{V}$

$$\langle e^{-t\mathcal{A}_\varepsilon} f, \phi\rangle_{\mathbf{V}} \longrightarrow \langle e^{-t\mathcal{A}_0} f, \phi\rangle_{\mathbf{V}},$$

uniformly in $t \in [0, T]$.

Remark 2. The uniform elliptic condition on $I + \mathcal{A}_\varepsilon$ $(\varepsilon \geq 0)$ is not essential, and that can be replaced by the condition that $\lambda_0 I + \mathcal{A}_\varepsilon$ satisfyes the uniform elliptic condition for some $\lambda_0 \in \mathbf{R}$.

The main Theorem can be immediately applied to the convergence of Cauchy problems.

COROLLARY 1. *Let $I + \mathcal{A}_\varepsilon$ $(\varepsilon \geq 0)$ satisfy the uniform elliptic condition and let u_ε be the solutions of the differential equations*

$$\frac{\partial}{\partial t} u_\varepsilon + \mathcal{A}_\varepsilon u_\varepsilon = f, \quad (\varepsilon \geq 0),$$

where $u(0) \in \mathbf{V}$, $f \in L^2([0,T]; \mathbf{V})$. Suppose $\{\mathcal{A}_\varepsilon\}$ is weakly graph-convergent to $\mathcal{A}_0$. Then for any $\phi \in \mathbf{V}$, $\langle u_\varepsilon, \phi\rangle_{\mathbf{V}} \longrightarrow \langle u_0, \phi\rangle_{\mathbf{V}}$, uniformly in $t \in [0, T]$.

Proof. By the dominated convergence theorem, we notice that for any $\phi \in \mathbf{V}$,

$$\int_0^t \langle e^{-(t-s)\mathcal{A}_\varepsilon} f(s), \phi\rangle_{\mathbf{V}} ds \longrightarrow \int_0^t \langle e^{-(t-s)\mathcal{A}_0} f(s), \phi\rangle_{\mathbf{V}} ds$$

uniformly in $t \in [0, T]$. Therefore from the semi-group representations of u_ε;

$$u_\varepsilon(t) = e^{-t\mathcal{A}_\varepsilon} u(0) + \int_0^t e^{-(t-s)\mathcal{A}_\varepsilon} f(s) ds$$

we get for any $\phi \in \mathbf{V}$, $\langle u_\varepsilon, \phi\rangle_{\mathbf{V}} \longrightarrow \langle u_0, \phi\rangle_{\mathbf{V}}$, uniformly in $t \in [0, T]$.

The notion of G-convergence is introduced to prove the main Theorem.

DEFINITION 2. Let $\mathcal{B}_\varepsilon$ $(\varepsilon \geq 0)$ be linear $\mathbf{V}$-coercive operators. Then $\{\mathcal{B}_\varepsilon\}$ is G-convergent to $\mathcal{B}_0$ as $\varepsilon \to 0$ (we write it as $\mathcal{B}_\varepsilon \stackrel{\mathrm{G}}{\rightharpoonup} \mathcal{B}_0$) if

$${\mathbf{B}_\varepsilon}^{-1} f \rightharpoonup \mathbf{B}_0^{-1} f \quad \text{in } \mathbf{V}$$

for every $f \in \mathbf{H}$. We say it is *strong* G-convergence if the convergence is the strong norm convergence in $\mathbf{V}$ (we write it as $\mathcal{B}_\varepsilon \stackrel{\mathrm{G}}{\longrightarrow} \mathcal{B}_0$).

The above definition is justified by the following Remark.

Remark 3. For a linear $\mathbf{V}$-coercive operator $\mathcal{B} : \mathbf{V} \to \mathbf{V}'$, the range $\mathcal{R}(\mathcal{B})$ of $\mathcal{B}$ contains $\mathbf{H} = \mathbf{H}'$;

$$\mathbf{H} \subseteq \mathcal{R}(\mathcal{B}).$$

Hence $\mathcal{B}^{-1} : \mathbf{H} \to \mathbf{H}$ is well-defined.

Proof. If we assume that the image does not contain $\mathbf{H}$, then in view of the reflexivity of $\mathbf{H}$ and Hahn-Banach theorem we can find a non-zero element $z \in \mathbf{H}$ such that $\langle \mathbf{B}u, z\rangle_{\mathbf{H}} = 0$ for any $u \in \mathbf{V}$. Putting $u = z$ in this equation we see that $z = 0$ by the coercive condition of $\mathcal{B}$. This contradicts the assumption.

PROPOSITION 1. *Let $I + \mathcal{A}_\varepsilon$ $(\varepsilon \geq 0)$ be uniformly $\mathbf{V}$-coercive operators. The followings are equivalent:*
(g) ...... $\{\mathcal{A}_\varepsilon\}$ is weakly graph-convergent to $\mathcal{A}_0$,
(G) ..... $(I + \mathcal{A}_\varepsilon) \overset{\mathrm{G}}{\rightharpoonup} (I + \mathcal{A}_0)$ in $\mathbf{V}$.

Proof. We assume the condition (g). For $f \in \mathbf{H}$, let $u_\varepsilon \equiv (I + \mathcal{A}_\varepsilon)^{-1} f$ $(\varepsilon \geq 0)$, so we have $\mathcal{A}_0 u_0 = f - u_0$. Then there exist $\bar{u}_\varepsilon$, and g_ε such that $\mathcal{A}_\varepsilon \bar{u}_\varepsilon = g_\varepsilon$, $\bar{u}_\varepsilon \rightharpoonup u_0$ in $\mathbf{V}$, $g_\varepsilon \rightharpoonup f - u_0$ in $\mathbf{V}'$. We notice that $\bar{u}_\varepsilon = (I + \mathcal{A}_\varepsilon)^{-1}(\bar{u}_\varepsilon + g_\varepsilon)$. Since $\|u\|_{\mathbf{H}} \leq C\|u\|_{\mathbf{V}}$ for some constant $C > 0$, we have

$$\frac{c_0}{C^2}\|u\|_{\mathbf{H}}^2 \leq c_0\|u\|_{\mathbf{V}}^2 \leq \langle (I + \mathbf{A}_\varepsilon)u, u\rangle_{\mathbf{H}} \leq \|(I + \mathbf{A}_\varepsilon)u\|_{\mathbf{H}}\|u\|_{\mathbf{H}}$$

for some constant $c_0 > 0$. Hence we have $\|(I+\mathbf{A}_\varepsilon)^{-1}u\|_{\mathbf{H}} \leq C_1\|u\|_{\mathbf{H}}$, with the choice of $C_1 \equiv C^2/c_0$. Since $\mathbf{V}$ is compactly imbedded to $\mathbf{H}$, a subsequence $\bar{u}_{\varepsilon_j} + g_{\varepsilon_j} - f$ converges to 0 as $\varepsilon_j \to 0$ in $\mathbf{H}$. Therefore we have

$$\begin{aligned} \|\bar{u}_{\varepsilon_j} - u_{\varepsilon_j}\|_{\mathbf{H}} &= \|(I + \mathbf{A}_{\varepsilon_j})^{-1}(\bar{u}_{\varepsilon_j} + g_{\varepsilon_j}) - (I + \mathbf{A}_{\varepsilon_j})^{-1} f\|_{\mathbf{H}} \\ &\leq C_1\|\bar{u}_{\varepsilon_j} + g_{\varepsilon_j} - f\|_{\mathbf{H}} \\ &\to 0, \end{aligned} \tag{3}$$

as $\varepsilon_j \to 0$. Similarly, from the fact that

$$c_0\|u\|_{\mathbf{V}}^2 \leq \langle (I + \mathbf{A}_\varepsilon)u, u\rangle_{\mathbf{H}} \leq C^2\|(I + \mathbf{A}_\varepsilon)u\|_{\mathbf{V}}\|u\|_{\mathbf{V}},$$

we have $\|(I+\mathbf{A}_\varepsilon)^{-1}u\|_{\mathbf{V}} \leq C_1\|u\|_{\mathbf{V}}$. Therefore we notice that $\|\bar{u}_\varepsilon - u_\varepsilon\|_{\mathbf{V}}$ is bounded because $\|\bar{u}_\varepsilon - u_\varepsilon\|_{\mathbf{V}} \leq \|(I+\mathbf{A}_\varepsilon)^{-1}(\bar{u}_\varepsilon + g_\varepsilon - f)\|_{\mathbf{V}} \leq C_1\|\bar{u}_\varepsilon + g_\varepsilon - f\|_{\mathbf{V}}$ and $\bar{u}_\varepsilon + g_\varepsilon \rightharpoonup f$. Taking (3) into consideration, we get $\bar{u}_\varepsilon - u_\varepsilon \rightharpoonup 0$ in $\mathbf{V}$ as $\varepsilon \to 0$. Therefore since $\bar{u}_\varepsilon \rightharpoonup u_0$, we have $u_\varepsilon \rightharpoonup u_0$ in $\mathbf{V}$ as $\varepsilon \to 0$.
Assuming the condition (G), we let $u \in \mathbf{V}$, $f \in \mathbf{H}$ with $\mathcal{A}_0 u = f$. Hence we have $u = (I + \mathcal{A}_0)^{-1}(u + f)$. Set $u_\varepsilon \equiv (I + \mathcal{A}_\varepsilon)^{-1}(u + f)$. Then $u_\varepsilon \rightharpoonup u$ in $\mathbf{V}$, $f_\varepsilon \equiv u + f - u_\varepsilon \rightharpoonup f$, in $\mathbf{V}' = \mathbf{V}$ as $\varepsilon \to 0$, and we also observe $\mathcal{A}_\varepsilon u_\varepsilon = f_\varepsilon$. Since $\mathbf{H} = \mathbf{H}'$ is dense in $\mathbf{V}'$, the above holds for all $f \in \mathbf{V}'$.

In order to prove the other part of the main Theorem, we need some lemmas. First, we notice that suppose $\lambda_0 I + \mathcal{A}$ is monotone, then by Hill-Yosida Theorem, we have the semi-group $e^{-t(\lambda_0 I + \mathcal{A})}$, and also the semi-group $e^{-t\mathcal{A}} (= e^{t\lambda_0 I} e^{-t(\lambda_0 I + \mathcal{A})})$ exists.

LEMMA 1. *Let $\lambda_0 I + \mathcal{A}$ be monotone and $S(t)$ be the semigroup associated with $\mathcal{A}$. Then for $\lambda > \lambda_0$,*

$$(\lambda I + \mathcal{A})^{-1} = \int_0^\infty e^{-\lambda t} S(t)\,dt.$$

Proof. See [5] for the proof. Particularly, we refer to the proof of the Hille-Yosida Theorem in Chapter 1.

LEMMA 2. *Let $\lambda_0 I+\mathcal{A}$, $\lambda_0 I+\mathcal{B}$ be monotone and $e^{-t\mathcal{A}}$, $e^{-t\mathcal{B}}$ be the semigroups associated with $\mathcal{A}$ and $\mathcal{B}$, respectively . Then for $\lambda > \lambda_0$,*

$$(\lambda I+\mathcal{A})^{-1}(e^{-t\mathcal{B}}-e^{-t\mathcal{A}})(\lambda I+\mathcal{B})^{-1} = \int_0^t e^{-(t-s)\mathcal{A}}[(\lambda I+\mathcal{B})^{-1}-(\lambda I+\mathcal{A})^{-1}]e^{-s\mathcal{B}}ds.$$

Proof. Note that

$$\frac{\partial}{\partial s}[e^{-(t-s)\mathcal{A}}(\lambda I+\mathcal{A})^{-1}e^{-s\mathcal{B}}(\lambda I+\mathcal{B})^{-1}] = e^{-(t-s)\mathcal{A}}[(\lambda I+\mathcal{B})^{-1}-(\lambda I+\mathcal{A})^{-1}]e^{-s\mathcal{B}}.$$

Now integrate both sides with respect to s on $(0,t)$ to get the result.

LEMMA 3. *Let B_ε $(\varepsilon \geq 0)$ be continuous linear functional on $\mathbf{V}$. Then if $\{B_\varepsilon\}$ weak*-converges to B_0, that is, $B_\varepsilon f \to B_0 f$ for all $f \in \mathbf{V}$, then $B_\varepsilon f \to B_0 f$ uniformly for all f of compact subset K of $\mathbf{V}$.*

Proof. We may assume that $B_0 \equiv 0$. For any $\delta > 0$, there are $f_1, f_2, \ldots, f_n$ in K such that for each $f \in K$, there exists k, $1 \leq k \leq n$ with $\| f - f_k \|_{\mathbf{V}} < \delta$, by the compactness of K. By virtue of the uniform bounded principle, we have that $\| B_\varepsilon \|_{\mathbf{V}'} \leq M$ for some $M > 0$. Also $| B_\varepsilon f_k | \leq \delta$, $k = 1, 2, \ldots, n$ for sufficiently small ε. Hence

$$| B_\varepsilon f | \leq | B_\varepsilon (f - f_k) | + | B_\varepsilon f_k | \leq (M+1)\delta.$$

Now, we prove a weak-version of Trotter Convergence Theorem.

PROPOSITION 2. *Let $I + \mathcal{A}_\varepsilon$ $(\varepsilon \geq 0)$ obey the uniform elliptic condition. The following two conditions are equivalent:*
(G) ..... $(I+\mathcal{A}_\varepsilon) \xrightarrow{\mathrm{G}} (I+\mathcal{A}_0)$ in $\mathbf{V}$,
(S) ...... for any $f \in \mathbf{V}$ and any $\phi \in \mathbf{V}$

$$\langle e^{-t\mathcal{A}_\varepsilon} f, \phi\rangle_{\mathbf{V}} \longrightarrow \langle e^{-t\mathcal{A}_0} f, \phi\rangle_{\mathbf{V}},$$

uniformly in $t \in [0,T]$.

Proof. ($(G) \Longleftarrow (S)$) For $\phi \in \mathbf{V}$, $f \in \mathbf{H}$, by the dominated convergence theorem, we get

$$\int_0^\infty e^{-t}\langle (e^{-t\mathcal{A}_\varepsilon} - e^{-t\mathcal{A}_0})f,\ \phi\rangle_{\mathbf{V}} dt \to 0.$$

Hence by Lemma 1, this gives $(I+\mathcal{A}_\varepsilon) \xrightarrow{\mathrm{G}} (I+\mathcal{A}_0)$ in $\mathbf{V}$.
($(G) \Longrightarrow (S)$) We first note that since $I + \mathcal{A}_\varepsilon$ is uniformly bounded with respect to $\mathbf{V}$-norm, it produces a group $e^{-t\mathcal{A}_\varepsilon} = \sum_{n=0}^\infty 1/n! \mathcal{A}_\varepsilon^n$ and $\|e^{-t(I+\mathcal{A}_\varepsilon)}\|_{\mathcal{L}(\mathbf{V},\mathbf{V}')} \leq e^{t\|I+\mathcal{A}_\varepsilon\|_{\mathcal{L}(\mathbf{V},\mathbf{V}')}} \leq e^{Tc}$, that is $\|e^{-t\mathcal{A}_\varepsilon}\|_{\mathcal{L}(\mathbf{V},\mathbf{V}')} \leq e^{T(c+1)}$ for some constant c. Hence for each $f \in \mathbf{V}$ and fixed $t \in [0,T]$, (taking a subsequence, if necessarily,) we notice

that $e^{-t\mathcal{A}_\varepsilon} f \rightharpoonup S(t)f$ in $\mathbf{V}$ for some $S(t)f \in \mathbf{V}$. We will show that $S(t)f = e^{-t\mathcal{A}_0} f$. Hence the original sequence converges. We have from Lemma 2 that

$$\begin{aligned}
\|(I+\mathcal{A}_\varepsilon)^{-1}(e^{-t\mathcal{A}_\varepsilon}- &\ e^{-t\mathcal{A}_0})(I+\mathcal{A}_0)^{-1} f\|_{\mathbf{H}} \\
&\le \int_0^t \left\| e^{-(t-s)\mathcal{A}_\varepsilon} \left[(I+\mathcal{A}_\varepsilon)^{-1} - (I+\mathcal{A}_0)^{-1}\right] e^{-s\mathcal{A}_0} f \right\|_{\mathbf{H}} ds. \\
&\le \int_0^t e^{(t-s)\lambda_0} \left\| [(I+\mathcal{A}_\varepsilon)^{-1} - (I+\mathcal{A}_0)^{-1}] e^{-s\mathcal{A}_0} f \right\|_{\mathbf{H}} ds.
\end{aligned}$$

since $\|e^{-(t-s)\mathcal{A}_\varepsilon}\|_{\mathcal{L}(\mathbf{H})} \le e^{(t-s)\lambda_0}$ for some λ_0. The integral tends to zero by the dominated convergence theorem. In fact, the last integrand tends to zero for each fixed s and is bounded by $2e^{2\lambda_0 T} C_1 \|f\|$ (C_1 is the constant found in the proof of Proposition 1) which is independent of ε. Thus we have shown that

$$(I+\mathcal{A}_\varepsilon)^{-1}(e^{-t\mathcal{A}_\varepsilon} - e^{-t\mathcal{A}_0}) g \to 0 \ \text{ in } \mathbf{H}$$

for all $g \in \mathbf{V}$. On the other hand, from the fact that

$$\|(I+\mathcal{A}_\varepsilon)^{-1}(e^{-t\mathcal{A}_\varepsilon} - e^{-t\mathcal{A}_0})\|_{\mathcal{L}(\mathbf{V},\mathbf{V}')} \le 2C_1 e^{(c+1)T},$$

the above implies that for all $f \in \mathbf{V}$ and any $\phi \in \mathbf{V}$,

$$\langle (I+\mathcal{A}_\varepsilon)^{-1}(e^{-t\mathcal{A}_\varepsilon} - e^{-t\mathcal{A}_0}) f, \phi\rangle_{\mathbf{V}} \to 0 \quad \text{in } \mathbf{V} \qquad \text{uniformly in } t \in [0,T].$$

In the following identity

$$\begin{aligned}
(e^{-t\mathcal{A}_\varepsilon} - e^{-t\mathcal{A}_0})(I+\mathcal{A}_0)^{-1} f = &\ e^{-t\mathcal{A}_\varepsilon}\left[(I+\mathcal{A}_0)^{-1} - (I+\mathcal{A}_\varepsilon)^{-1}\right] f \\
&+ (I+\mathcal{A}_\varepsilon)^{-1}\left[e^{-t\mathcal{A}_\varepsilon} - e^{-t\mathcal{A}_0}\right] f \\
&+ \left[(I+\mathcal{A}_\varepsilon)^{-1} - (I+\mathcal{A}_0)^{-1}\right] e^{-t\mathcal{A}_0} f,
\end{aligned}$$

we know that each term of the right-hand side of the identity weakly converges to zero. This means that $(e^{-t\mathcal{A}_\varepsilon} - e^{-t\mathcal{A}_0}) g \rightharpoonup 0$ in $\mathbf{V}$ for all $g \in \mathbf{V}$. This convergence is uniform in t. Indeed, the convergences of the first and the second terms are uniform in t and the last one is uniform in t by Lemma 3.

4. APPLICATION TO FISSURED MEDIUM EQUATIONS

We consider the homogenization of the small temperature variation in a system consisting of two components which are introduced in [5] , [7];

$$\begin{aligned}
\frac{\partial}{\partial t} u_\varepsilon(t) - \operatorname{div} B(\frac{x}{\varepsilon})\nabla u_\varepsilon(t) + \frac{1}{\delta}(u_\varepsilon - v_\varepsilon) &= f(t), \\
-\operatorname{div} C(\frac{x}{\varepsilon})\nabla v_\varepsilon(t) + \frac{1}{\delta}(v_\varepsilon - u_\varepsilon) &= 0.
\end{aligned}$$

We assume that the coefficients $B \equiv (B_{ij})$, $C \equiv (C_{ij})$ are $n \times n$-matrices satisfying

$$\begin{aligned}
&c_0 \,|\xi|^2 \le B_{ij}(x)\xi_i\xi_j \le c_1 \,|\xi|^2 && \text{and} \\
&c_0 \,|\xi|^2 \le C_{ij}(x)\xi_i\xi_j \le c_1 \,|\xi|^2 && \text{for all } \xi \in \mathbf{R}^n,
\end{aligned}$$

for some constants $c_0 > 0$, $c_1 > 0$, and assume $\delta > 0$ which represents the resistance of the medium to this exchange. Let Ω be the given open subset of $\mathbf{R}^n$. If there is a boundary, one poses Dirichlet boundary condition $u = 0$ on $\partial\Omega$, so we have the function space $\mathbf{V} \equiv H_0^1(\Omega)$. We also assume that the external source $f : [0,T] \to L^2(\Omega)$ is Hölder continuous, that is, for some $0 < \alpha < 1$,

$$\|f(t_1) - f(t_2)\|_{L^2(S)} \le C|t_1 - t_2|^\alpha \quad for\ t_1, t_2 \in [0,T].$$

Define bilinear forms $\mathcal{B}_\varepsilon, \mathcal{C}_\varepsilon \in \mathcal{L}(\mathbf{V}, \mathbf{V}')$ as follows:

$$\mathcal{B}_\varepsilon u_1(u_2) \equiv \int_\Omega B(\frac{x}{\varepsilon})\nabla u_1 \cdot \nabla u_2\, dx, \qquad u_1, u_2 \in \mathbf{V},$$

$$\mathcal{C}_\varepsilon v_1(v_2) \equiv \int_\Omega C(\frac{x}{\varepsilon})\nabla v_1 \cdot \nabla v_2\, dx, \qquad v_1, v_2 \in \mathbf{V}.$$

Then we have a variational formulation;

$$u_\varepsilon'(t) + \mathcal{B}_\varepsilon u_\varepsilon(t) + \frac{1}{\delta}(u_\varepsilon - v_\varepsilon) = f(t)\ , \tag{4}$$

$$\mathcal{C}_\varepsilon v_\varepsilon(t) + \frac{1}{\delta}(v_\varepsilon - u_\varepsilon) = 0 \qquad \text{in } \mathbf{V}'. \tag{5}$$

From equation (5), we have

$$v_\varepsilon = \left(\mathcal{C}_\varepsilon + \frac{1}{\delta}\right)^{-1} \frac{1}{\delta} u_\varepsilon. \tag{6}$$

This makes sense because the linear operator $\mathcal{C}_\varepsilon + \frac{1}{\delta} : \mathbf{V} \longrightarrow \mathbf{V}'$ is uniformly coercive. Substituting (6) into equation (5), we get

$$\frac{\partial u_\varepsilon}{\partial t} + \left\{ \frac{1}{\delta}\left(I - \frac{1}{\delta}\left(\mathcal{C}_\varepsilon + \frac{1}{\delta}\right)^{-1}\right) + \mathcal{B}_\varepsilon \right\} u_\varepsilon = f \quad \text{in } \mathbf{V}'. \tag{7}$$

Clearly, we have $u_\varepsilon(t) \in \mathbf{V}$ at each $t \in [0,T]$. Define the corresponding unbounded operator $\mathbf{A}_\varepsilon : D(\mathbf{A}_\varepsilon) \longrightarrow \mathbf{H}$ on the Hilbert space $\mathbf{H} \equiv L^2(\Omega)$. The domain of $\mathbf{A}_\varepsilon$ is $D(\mathbf{A}_\varepsilon) \equiv \{v \in \mathbf{V} : \mathbf{A}_\varepsilon v \in \mathbf{H}\}$, and it is defined on this domain by

$$\mathbf{A}_\varepsilon \equiv \frac{1}{\delta}\left(I - \frac{1}{\delta}\left(\mathcal{C}_\varepsilon + \frac{1}{\delta}\right)^{-1}\right) + \mathcal{B}_\varepsilon.$$

Then equation (7) reads

$$\frac{\partial u_\varepsilon}{\partial t} + \mathbf{A}_\varepsilon u_\varepsilon = f \quad \text{in } \mathbf{V}'. \tag{8}$$

Note that each operator $\mathbf{A}_\varepsilon : D(\mathbf{A}_\varepsilon) \longrightarrow \mathbf{H}$ is self-adjoint and (regular) m-accretive. In fact, the first part is the Yosida approximations of $\mathcal{B}_\varepsilon$, so it is m-accretive ([5]). Hence by Remark 1, there exists the unique solution of initial value problem for (8). Also, we define $v_\varepsilon(t)$ by (6). Thereby we obtain the unique solution $u_\varepsilon \in C([0,T]; L^2(\Omega)) \cap C^1((0,T); H_0^1(\Omega) \cap H^2(\Omega))$, and $v_\varepsilon \in C([0,T]; \mathbf{V})$ to the system of equations (4) and (5). We also note that $I + \mathcal{A}_\varepsilon$ satisfies the uniform elliptic condition.

We show that $\{\mathbf{A}_\varepsilon\}$ weak-graph converges to $\mathbf{A}$, which is defined as follows. In order to describe the limit operator $\mathbf{A}$, we define the operators $\mathcal{B}, \mathcal{C}$ as

$$\mathcal{B}u_1(u_2) \equiv \int_\Omega \bar{B}\nabla u_1 \cdot \nabla u_2\, dx, \qquad \mathcal{C}u_1(u_2) \equiv \int_\Omega \bar{C}\nabla u_1 \cdot \nabla u_2\, dx,$$

for $u_1, u_2 \in \mathbf{V}$, where

$$\bar{B} = \inf_{u \in H^1([0,1]^n)} \langle (I + \nabla u) \cdot A(I + \nabla u) \rangle,$$
$$\bar{C} = \inf_{u \in H^1([0,1]^n)} \langle (I + \nabla u) \cdot B(I + \nabla u) \rangle.$$

Then the operator $\mathbf{A} : D(\mathbf{A}) \longrightarrow \mathbf{H}$ defined by

$$\mathbf{A} \equiv \frac{1}{\delta}\left(I - \frac{1}{\delta}\left(\mathcal{C} + \frac{1}{\delta}\right)^{-1}\right) + \mathcal{B}$$

is self-adjoint and $\mathbf{V}$-coercive. So there exists the unique solution $u \in D(\mathbf{A})$ of an elliptic problem:

$$\mathbf{A}u = f. \tag{9}$$

We show that for a given $f \in \mathbf{V}'$, $\{u_\varepsilon\}$ converges weakly to u in $\mathbf{V}$, where u_ε, u are the solutions of the elliptic problems $\mathcal{A}_\varepsilon u_\varepsilon = f$ and the limit elliptic problem $\mathcal{A}u = f$, respectively (here the operators $\mathcal{A}_\varepsilon$, $\mathcal{A}$ are the extensions to $\mathbf{V}$ of $\mathbf{A}_\varepsilon$, $\mathbf{A}$, respectively, that is; $\mathcal{A}_\varepsilon, \mathcal{A} \in \mathcal{L}(\mathbf{V}, \mathbf{V}')$).

Define $v_\varepsilon : v_\varepsilon \equiv \frac{1}{\delta}\left(\mathcal{C}_\varepsilon + \frac{1}{\delta}\right)^{-1} u_\varepsilon$. Then we have

$$\mathcal{B}_\varepsilon u_\varepsilon + \frac{1}{\delta}(u_\varepsilon - v_\varepsilon) = f, \tag{10}$$
$$\mathcal{C}_\varepsilon v_\varepsilon + \frac{1}{\delta}(v_\varepsilon - u_\varepsilon) = 0. \tag{11}$$

We apply u_ε to (10) and v_ε to (11), and add the two equations together to get

$$c_0 \|\nabla u_\varepsilon\|^2_{L^2(\Omega)} + c_0 \|\nabla v_\varepsilon\|^2_{L^2(\Omega)} + \frac{1}{\delta}\|u_\varepsilon - v_\varepsilon\|^2_{L^2(\Omega)} \leq \int_\Omega f(x) u_\varepsilon(x) dx.$$

This shows that $\|\nabla u_\varepsilon\|_{L^2(\Omega)}$, $\|\nabla v_\varepsilon\|_{L^2(\Omega)}$ are bounded. It follows that a subsequence $\{u_{\varepsilon_j}\}$ converges weakly to some u in $\mathbf{V}$ and $v_{\varepsilon_j} \rightharpoonup v$ in $\mathbf{V}$. By heuristic homogenization rule presenting in [8] we can find that the corresponding flows converge weakly in $L^2(\Omega)$:

$$\mathcal{B}_{\varepsilon_j} u_{\varepsilon_j} \rightharpoonup \mathcal{B}u, \qquad \mathcal{C}_{\varepsilon_j} v_{\varepsilon_j} \rightharpoonup \mathcal{C}v.$$

Hence taking weak limits in (10) and (11) yields the limit system

$$\mathcal{B}u + \frac{1}{\delta}(u - v) = f,$$
$$\mathcal{C}v + \frac{1}{\delta}(v - u) = 0.$$

We have shown that for a given $f \in \mathbf{V}'$, by the uniqueness of the solution, we have the original sequence $\{u_\varepsilon\}$ converges weakly to u in $\mathbf{V}$ and $\{v_\varepsilon\}$ converges weakly to v in $\mathbf{V}$. Also note that the system above is the same as (9).

In all, by Corollary 1 we get the limit equation for (7)

$$\frac{\partial u}{\partial t} + \left\{\frac{1}{\delta}\left(I - \frac{1}{\delta}\left(\mathcal{C} + \frac{1}{\delta}\right)^{-1}\right) + \mathcal{B}\right\} u = f,$$

which is equivalent to the *homogenized fissured medium equation*;

$$\frac{\partial}{\partial t}u(t) - \mathrm{div}\bar{B}\nabla u(t) + \frac{1}{\delta}(u - v) = f(t) \ ,$$
$$-\mathrm{div}\bar{C}\nabla v(t) + \frac{1}{\delta}(v - u) = 0.$$

Remark 4. 1. We note that the homogenized fissured medium equation has *the same form* as the ε-fissured medium equations.
2. The convergence in homogenizational process is *uniform in time* on the interval $[0, T]$ by the main Theorem.

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