Renormalization group, hidden symmetries and approximate Ward identities in the XYZ model, II.

G. Benfatto*, **V. Mastropietro***
Dipartimento di Matematica, Università di Roma "Tor Vergata"
Via della Ricerca Scientifica, I-00133, Roma

ABSTRACT. *An expansion based on renormalization group methods for the spin correlation function in the z direction of the Heisenberg-Ising XYZ chain with an external magnetic field directed as the z axis is derived. Moreover, by using the hidden symmetries of the model, we show that the running coupling constants are small, if the coupling in the z direction is small enough, that a critical index appearing in the correlation function is exactly vanishing (because of an approximate Ward identity) and other properties, so obtaining a rather detailed description of the XYZ correlation function.*

1. Introduction

1.1 In a preceding paper [BeM1] we have derived an expansion, based on renormalization group methods, for the ground state energy and the effective potential of the Heisenberg-Ising XYZ chain, whose hamiltonian is written in terms of fermionic operators. The expansion is in terms of a set of *running coupling constants* and two *renormalization constants*, related with the spectral gap and the wave function renormalization; the running coupling constants have to be small enough to have convergence of the expansion.

In this paper we continue our analysis of the XYZ model by writing an expansion for the spin correlation function in the direction of the magnetic field, see §3. With respect to the ground state energy or the effective potential expansion, two new renormalization constants appear, related with the (fermionic) density renormalization.

In order to study the asymptotic behaviour of the spin correlation function, one has to face two main problems. The first one is to show that the running coupling constants indeed remain small if the coupling J_3 between spins in the direction of the magnetic field is small enough. The second problem is to prove that one of the renormalization constants corresponding to the density renormalization is almost equal to the square of the wave function renormalization. This last property is crucial to obtain the correct asymptotic behaviour of the correlation function, since it is related to the vanishing of a critical index.

Such properties are proved by writing the beta function governing the flow of the renormalization constants or their ratio as the sum of several terms. One has to prove that one of such terms is exactly vanishing at any order; once that this is proved, the above properties

* Supported by MURST, Italy, and EC HCM contract number CHRX-CT94-0460.
e-mail: benfatto@mat.uniroma2.it, mastropi@mat.uniroma2.it.

follow if the magnetic field is chosen properly, see §2. One recognizes that such contribution to the Beta function of the XYZ model is coinciding with the Beta function obtained by applying the same renormalization group analysis to the Luttinger model. For such model many symmetry properties are true, and in this sense we can speak of "hidden symmetries" for the XYZ model; they are not enjoyed by the XYZ hamiltonian, but the model is close, in a renormalization group sense, to a model enjoying them.

A crucial role is played in our analysis by the local Gauge invariance, see §5; note however that, despite the fact that the Luttinger model Hamiltonian is formally gauge invariant, the ultraviolet and infrared cutoffs introduced to perform our renormalization group analysis have the effect that gauge invariance is broken. Nevertheless we can derive an approximate Ward identity (approximate as the gauge invariance is only approximately true), which tells us that the ratio between the density renormalization and the square of the wave function renormalization in the Luttinger model is approximately one. Note that, if one uses the Ward identity formally obtained by neglecting the cutoffs, one obtains a ratio exactly equal to one. This means that the corresponding Luttinger model beta function is vanishing (but the XYZ beta function is not vanishing) and we can prove that the related critical index appearing in the correlation function asymptotic behaviour of the XYZ model is *exactly* vanishing.

We could proceed in a similar way and derive a suitable Ward identity to prove that the Beta function for the running coupling constants appearing in the Luttinger model is vanishing; this was done formally in [MD]. However we find simpler to prove this property by using the explicit expression of the Luttinger model correlation functions [BGM] based on the exact solution [ML]; this was done in [GS], [BGPS], [BM1].

Finally, in §4 other hidden symmetries are exploited in order to prove many properties about the correlation function.

The paper is not self-consistent; we use heavily the notations and the results of [BeM1], to which we refer also for the general introduction on the XYZ model. We will denote equation (x.y) of [BeM1] by (**I**x.y.).

2. The flow of the running coupling constants

2.1 The convergence of the expansion for the effective potential is proved by theorems **I**3.12, **I**3.17 under the hypothesis that, uniformly in $h \geq h^*$, the running coupling constants are small enough and the bounds (**I**2.98) and (**I**3.88) are satisfied. In this section we prove that, if $|\lambda|$ is small enough and ν is properly chosen, the above conditions are indeed verified.

Let us consider first the bounds in (**I**2.98). They immediately follow from (**I**3.91) and (**I**3.92), by a simple inductive argument, if the bounds (**I**3.88) are verified and

$$\varepsilon_h \leq \bar{\varepsilon}_0 \leq \bar{\varepsilon}\ , \quad \text{for } h > h^*\ , \tag{2.1}$$

with $\bar{\varepsilon}_0$ small enough.

Let us now consider the bounds (**I**3.88). By (**I**2.83), (**I**2.84), the first of (**I**2.89) and the third of (**I**2.98), we get

$$\frac{Z_{h-1}}{Z_h} = 1 + z_h \;, \tag{2.2}$$

$$\frac{\sigma_{h-1}}{\sigma_h} = 1 + \frac{s_h/\sigma_h - z_h}{1+z_h} \;. \tag{2.3}$$

By explicit calculation of the lower order non zero terms contributing to z_h and s_h/σ_h, one can prove that

$$\begin{aligned} z_h &= b_1 \lambda_h^2 + O(\varepsilon_h^3) \;, \quad b_1 > 0 \;, \\ s_h/\sigma_h &= -b_2\lambda_h + O(\varepsilon_h^2) \;, \quad b_2 > 0 \;, \end{aligned} \tag{2.4}$$

which imply (**I**3.88), if $\bar{\varepsilon}_0$ is small enough, with a suitable constant c_1 depending on the constant c_0 appearing in Theorem **I**3.12, since the value of c_0 is independent of c_1.

The equation (2.2) and the definitions (**I**2.109) allow to get the following representation of the Beta function in terms of the tree expansions (**I**3.71):

$$\lambda_h = \lambda_{h+1} + \left(\frac{1}{1+z_h}\right)^2 \left[-\lambda_{h+1}(z_h^2 + 2z_h) + \sum_{n=2}^{\infty} \sum_{\tau\in\mathcal{T}_{h,n}} l_h(\tau) \right] \;, \tag{2.5}$$

$$\delta_h = \delta_{h+1} + \frac{1}{1+z_h} \left[-\delta_{h+1} z_h + c_0^\delta \lambda_1 \delta_{h,0} + \sum_{n=2}^{\infty} \sum_{\tau\in\mathcal{T}_{h,n}} \Big(a_h(\tau) - z_h(\tau)\Big) \right] \;, \tag{2.6}$$

$$\nu_h = \gamma\nu_{h+1} + \frac{1}{1+z_h} \left[-\gamma\nu_{h+1} z_h + c_h^\nu \gamma^h \lambda_{h+1} + \sum_{n=2}^{\infty} \sum_{\tau\in\mathcal{T}_{h,n}} n_h(\tau) \right] \;, \tag{2.7}$$

where we have extracted the terms of first order in the running couplings and we have extended to $h = +1$ the definition of λ_h and δ_h, so that, see (**I**2.81),

$$\lambda_1 = 4\lambda \sin^2(p_F + \pi/L) \;, \quad \delta_1 = -v_0 \delta^* \;. \tag{2.8}$$

Note that the first order term proportional to λ_{h+1} in the equation for ν_h is of size γ^h, while the similar term in the equation for δ_h is equal to zero, if $h < 0$; moreover the constants c_0^ν and c_h^λ are bounded uniformly in L, β.

Hence, if we put $\vec{a}_h = (\delta_h, \lambda_h)$, the Beta function can be written, if condition (2.1) is satisfied, with $\bar{\varepsilon}_0$ small enough, in the form

$$\lambda_{h-1} = \lambda_h + \beta_h^\lambda(\vec{a}_h, \nu_h; \ldots; \vec{a}_1, \nu_1; u, \delta^*) \;, \tag{2.9}$$

$$\delta_{h-1} = \delta_h + \beta_h^\delta(\vec{a}_h, \nu_h; \ldots; \vec{a}_1, \nu_1; u, \delta^*) \;, \tag{2.10}$$

$$\nu_{h-1} = \gamma\nu_h + \beta_h^\nu(\vec{a}_h, \nu_h; \ldots; \vec{a}_1, \nu_1; u, \delta^*) \;, \tag{2.11}$$

where β_h^λ, β_h^δ and β_h^ν are functions of $\vec{a}_h, \nu_h, \ldots, \vec{a}_1, \nu_1, u$, which can be easily bounded, by using Theorem **I**3.12, if the condition (2.1) is verified. Note that these functions depend on

$\vec{a}_h, \nu_h, \ldots, \vec{a}_1, \nu_1, u$, directly trough the endpoints of the trees, indirectly trough z_h and the quantities $Z_{h'}/Z_{h'-1}$ and $\sigma_{h'-1}(\mathbf{k}')$, $h < h' \le 0$, appearing in the tree expansions.

Let us define

$$\mu_h = \sup_{k\ge h} \max\{|\lambda_k|, |\delta_k|\} \ , \quad \bar\lambda_h = \sup_{k\ge h} |\lambda_k| \ . \tag{2.12}$$

We want to prove the following Lemma.

2.2 LEMMA. *Suppose that u satisfies the condition (***I***2.117) and let us consider the equation (2.11) for fixed values of $\vec{a}_h$, Z_{h-1} and $\sigma_{h-1}(\mathbf{k}')$, $\tilde h \le h \le 1$, satisfying the conditions*

$$\mu_h \le \bar\varepsilon_1 \le \bar\varepsilon_0 \ , \tag{2.13}$$

$$a_0 \gamma^{h-1} \ge 4|\sigma_h| \ , \tag{2.14}$$

$$\gamma^{-c_0\mu_h} \le \frac{\sigma_{h-1}}{\sigma_h} \le \gamma^{+c_0\mu_h} \ , \tag{2.15}$$

$$\gamma^{-c_0\mu_h^2} \le \frac{Z_{h-1}}{Z_h} \le \gamma^{+c_0\mu_h^2} \ , \tag{2.16}$$

for some constant c_0.

Then, if $\bar\varepsilon_0$ is small enough, there exist some constants $\bar\varepsilon_1$, η, γ', c_1, B, and a family of intervals $I^{(\bar h)}$, $\tilde h \le \bar h \le 0$, such that $\bar\varepsilon_1 \le \bar\varepsilon_0$, $0 < \eta < 1$, $1 < \gamma' < \gamma$, $I^{(\bar h)} \subset I^{(\bar h+1)}$, $|I^{(\bar h)}| \le c_1 \bar\varepsilon_1 (\gamma')^{\bar h}$ and, if $\nu = \nu_1 \in I^{(\bar h)}$,

$$|\nu_h| \le B\bar\varepsilon_1 \left[\gamma^{-\frac12 (h-\bar h)} + \gamma^{\eta h}\right] \le \bar\varepsilon_0 \ , \quad \bar h \le h \le 1 \ . \tag{2.17}$$

2.3 *Proof.* Let us consider (2.11), for fixed values of $\vec{a}_h$, Z_h/Z_{h-1} (hence of z_h) and $\sigma_{h-1}(\mathbf{k}')$, $\tilde h \le h \le 1$, satisfying (2.13)-(2.16).

Note that, if $|\nu_h| \le \bar\varepsilon_0$ for $\bar h \le h \le 1$ and $\bar\varepsilon_0$ is small enough, the r.h.s. of (2.11) is well defined for $h = \bar h$ and we can write, by using (2.7),

$$\nu_{\bar h-1} = \gamma \nu_{\bar h} + b_{\bar h} + r_{\bar h} \ , \tag{2.18}$$

where $b_{\bar h} = c^{\nu}_{\bar h-1} \gamma^{\bar h-1} \lambda_{\bar h}$ and $r_{\bar h}$ collects all terms of second or higher order in $\bar\varepsilon_0$.

Note also that, in the tree expansion of $n_h(\tau)$, the dependence on $\nu_h, \ldots, \nu_1$ appears only in the endpoints of the trees and there is no contribution from the trees with $n \ge 2$ endpoints, which are only of type ν or δ, because of the support properties of the single scale propagators. It follows, by using (**I**3.91) and (2.14)-(2.16), that

$$|r_{\bar h}| \le c_2 \mu_{\bar h} \bar\varepsilon_0 \ . \tag{2.19}$$

Let us now fix a positive constant c, consider the intervals

$$J^{(h)} = [-\frac{b_h}{\gamma-1} - c\bar\varepsilon_1, -\frac{b_h}{\gamma-1} + c\bar\varepsilon_1] \ . \tag{2.20}$$

and suppose that there is an interval $I^{(\bar h)}$ such that, if ν_1 spans $I^{(\bar h)}$, then $\nu_{\bar h}$ spans the interval $J^{(\bar h+1)}$ and $|\nu_h|\le \bar\varepsilon_0$ for $\bar h\le h\le 1$. Let us call $\tilde J^{(\bar h)}$ the interval spanned by $\nu_{\bar h-1}$ when ν_1 spans $I^{(\bar h)}$. Equation (2.18) can be written in the form

$$\nu_{\bar h-1}+\frac{b_{\bar h}}{\gamma-1}=\gamma\Big(\nu_{\bar h}+\frac{b_{\bar h}}{\gamma-1}\Big)+r_{\bar h}\ . \tag{2.21}$$

Hence, by using also the definition of b_h and (2.19), we see that

$$\begin{aligned}&\min_{\nu_1\in I^{(\bar h)}}\left[\nu_{\bar h-1}+\frac{b_{\bar h}}{\gamma-1}\right]=\\ &=\gamma\min_{\nu_{\bar h}\in J^{(\bar h+1)}}\left[\nu_{\bar h}+\frac{b_{\bar h+1}}{\gamma-1}\right]+\min_{\nu_1\in I^{(\bar h)}}\left[r_{\bar h}+\frac{\gamma}{\gamma-1}(b_{\bar h}-b_{\bar h+1})\right]\le\\ &\le -\gamma c\bar\varepsilon_1+c_2\bar\varepsilon_1\bar\varepsilon_0+c_3\gamma^{\bar h}\bar\varepsilon_1\ ,\end{aligned} \tag{2.22}$$

for some constant c_3. In a similar way we can show that

$$\max_{\nu_1\in I^{(\bar h)}}\left[\nu_{\bar h-1}+\frac{b_{\bar h}}{\gamma-1}\right]\ge \gamma c\bar\varepsilon_1-c_2\bar\varepsilon_1\bar\varepsilon_0-c_3\gamma^{\bar h}\bar\varepsilon_1\ . \tag{2.23}$$

It follows that, if c is large enough and $\bar\varepsilon_0$ is small enough, $J^{(\bar h)}$ is strictly contained in $\tilde J^{(\bar h)}$. On the other hand, it is obvious that there is a one to one correspondence between ν_1 and the sequence ν_h, $\bar h-1\le h\le 1$. Hence there is an interval $I^{(\bar h-1)}\subset I^{(\bar h)}$, such that, if ν_1 spans $I^{(\bar h-1)}$, then $\nu_{\bar h-1}$ spans the interval $J^{(\bar h)}$ and, if $\bar\varepsilon_1$ is small enough, $|\nu_h|\le\bar\varepsilon_0$ for $\bar h-1\le h\le 1$.

The previous calculations also imply that the inductive hypothesis is verified for $\bar h=0$, so that we have proved that there exists a decreasing family of intervals $I^{(\bar h)}$, $\tilde h\le\bar h\le 0$, such that, if $\nu=\nu_1\in I^{(\bar h)}$, then the sequence ν_h is well defined for $h\ge\bar h$ and satisfies the bound $|\nu_h|\le\bar\varepsilon_0$.

The bound on the size of $I^{(\bar h)}$ easily follows (2.18) and (2.19). Let us denote by ν_h and ν'_h, $\bar h\le h\le 1$, the sequences corresponding to $\nu_1,\nu'_1\in I^{(\bar h)}$. We have

$$\nu_{h-1}-\nu'_{h-1}=\gamma(\nu_h-\nu'_h)+r_h-r'_h\ , \tag{2.24}$$

where r'_h is a shorthand for the value taken from r_h in correspondence of the sequence ν'_h. Let us now observe that $r_h-r'_h$ is equal to $\gamma z_{h-1}(1+z_{h-1})^{-1}(\nu'_h-\nu_h)$ plus a sum of terms, associated with trees, containing at least one endpoint of type ν, with a difference $\nu_k-\nu'_k$, $k\ge h$, in place of the corresponding running coupling, and one endpoint of type λ. Then, if $|\nu_k-\nu'_k|\le|\nu_h-\nu'_h|$, $k\ge h$, we have

$$|\nu_h-\nu'_h|\le\frac{|\nu_{h-1}-\nu'_{h-1}|}{\gamma}+C\bar\varepsilon_1|\nu_h-\nu'_h|\ . \tag{2.25}$$

On the other hand, if $h=1$, this bound implies that $|\nu_1-\nu'_1|\le|\nu_0-\nu'_0|$, if $\bar\varepsilon_1$ is small enough; hence it allows to show inductively that, given any γ', such that $1<\gamma'<\gamma$, if $\bar\varepsilon_1$ is small enough, then

$$|\nu_1-\nu'_1|\le\gamma'^{(\bar h-1)}|\nu_{\bar h}-\nu'_{\bar h}|\ . \tag{2.26}$$

Since, by definition, if ν_1 spans $I^{(\bar h)}$, then $\nu_{\bar h}$ spans the interval $J^{(\bar h+1)}$, of size $2c\bar\varepsilon_1$, the size of $I^{(\bar h)}$ is bounded by $2c\bar\varepsilon_1\gamma'^{(\bar h-1)}$.

In order to complete the proof of Lemma 2.2, we have still to prove the bound (2.17). Note that, if we iterate (2.11), we can write, if $\bar h \le h \le 0$ and $\nu_1 \in I^{(\bar h)}$,

$$\nu_h = \gamma^{-h+1}\left[\nu_1 + \sum_{k=h+1}^{1} \gamma^{k-2}\beta_k^\nu(\nu_k,\ldots,\nu_1)\right] , \tag{2.27}$$

where now the functions β_ν^k are thought as functions of $\nu_k,\ldots,\nu_1$ only.

If we put $h=\bar h$ in (2.27), we get the following identity:

$$\nu_1 = -\sum_{k=\bar h+1}^{1} \gamma^{k-2}\beta_k^\nu(\nu_k,\ldots,\nu_1) + \gamma^{\bar h-1}\nu_{\bar h} . \tag{2.28}$$

(2.27) and (2.28) are equivalent to

$$\nu_h = -\gamma^{-h}\sum_{k=\bar h+1}^{h} \gamma^{k-1}\beta_k^\nu(\nu_k,\ldots,\nu_1) + \gamma^{-(h-\bar h)}\nu_{\bar h} , \qquad \bar h < h \le 1 . \tag{2.29}$$

The discussion following (2.18) implies that

$$|\beta_k^\nu(\nu_k,\ldots,\nu_1)| \le C\mu_k , \tag{2.30}$$

if $\bar\varepsilon_0$ is small enough. However this bound it is not sufficient and we have to analyze in more detail the structure of the functions β_h^ν, by looking in particular to the trees in the expansion of $n_h(\tau)$, which have no endpoint of type ν. Let us suppose that, given a tree with this property, we decompose the propagators by using (**I**2.99); we get a family of C^n different contributions, which can be bounded as before, by using an argument similar to that used in §**I**3.13. However, the terms containing only the propagators $g_{L,\omega}^{(h')}$ cancel out, for simple parity properties. On the other hand, the terms containing at least one propagator $r_2^{(h_v)}$ or two propagators $g_{\omega,-\omega}^{(h_v)}$ (the number of such propagators has to be even) can be bounded by $(C\varepsilon_h)^n(|\sigma_h|/\gamma^h)^2$, by using (**I**2.101) and (**I**3.106). Analogously the terms with at least one propagator $r_1^{(h_v)}$ can be bounded by $(C\varepsilon_h)^n\gamma^{\eta h}$, with some positive $\eta<1$. In fact, for these terms, by using (**I**2.101), the bound can be improved by a factor $\gamma^{h_v} \le \gamma^{\eta h}\gamma^{\eta(h_v-h)}$, for any positive $\eta \le 1$, and the bad factor $\gamma^{\eta(h_v-h)}$ can be controlled by the sum over the scales, if η is small enough, thanks to (**I**3.111). Finally, the parity properties of the propagators imply that the only term linear in the running couplings, which contributes to ν_h, is of order γ^h. Hence, we can write

$$\beta_h^\nu = \mu_h\sum_{k=h}^{1}\nu_k\tilde\beta_{h,k}^\nu\gamma^{-2\eta(k-h)} + \mu_h\varepsilon_h\left(\frac{|\sigma_h|}{\gamma^h}\right)^2\hat\beta_h^\nu + \gamma^{\eta h}\mu_h R_h^\nu , \tag{2.31}$$

where $|R_h^\nu|, |\hat\beta_h^\nu|, |\tilde\beta_{h,k}^\nu| \le C$.

The factor $\gamma^{-2\eta(k-h)}$ in the r.h.s. of (2.31) follows from the simple remark that the bound over all the trees contributing to ν_h, which have at least one endpoint of fixed scale $k>h$, can be improved by a factor $\gamma^{-\eta'(k-h)}$, with η' positive but small enough. It is sufficient to use again (**I**3.111), which allows to extract such factor from the r.h.s. before performing the sum over the scale indices, and to choose $\eta' = 2\eta$, which is possible if η is small enough.

Let us now observe that the sequence ν_h, $\bar h < h \le 1$, satisfying (2.29) can be obtained as the limit as $n \to \infty$ of the sequence $\{\nu_h^{(n)}\}$, $\bar h < h \le 1$, $n \ge 0$, parameterized by $\nu_{\bar h} \in J^{(\bar h+1)}$ and defined recursively in the following way:

$$\begin{aligned} &\nu_h^{(0)} = 0 \;, \\ &\nu_h^{(n)} = -\gamma^{-h} \sum_{k=\bar h+1}^{h} \gamma^{k-1} \beta_k^\nu(\nu_k^{(n-1)}, \ldots, \nu_1^{(n-1)}) + \gamma^{-(h-\bar h)} \nu_{\bar h} \;, \qquad n \ge 1 \;. \end{aligned} \tag{2.32}$$

In fact, it is easy to show inductively, by using (2.30), that, if $\bar\varepsilon_1$ is small enough, $|\nu_h^{(n)}| \le C\bar\varepsilon_1 \le \bar\varepsilon_0$, so that (2.32) is meaningful, and

$$\max_{h^* < h \le 1} |\nu_h^{(n)} - \nu_h^{(n-1)}| \le (C\bar\varepsilon_1)^n \;. \tag{2.33}$$

In fact this is true for $n = 1$ by (2.30) and the fact that $\nu_h^{(0)} = 0$; for $n > 1$ it follows trivially by the fact that $\beta_k^\nu(\nu_k^{(n-1)}, \ldots, \nu_1^{(n-1)}) - \beta_k^\nu(\nu_k^{(n-2)}, \ldots, \nu_1^{(n-2)})$ can be written as a sum of terms in which there are at least one endpoint of type ν, with a difference $\nu_{h'}^{n-1} - \nu_{h'}^{n-2}$, $h' \ge k$, in place of the corresponding running coupling, and one endpoint of type λ. Then $\nu_h^{(n)}$ converges as $n \to \infty$, for $\bar h < h \le 1$, to a limit ν_h, satisfying (2.29) and the bound $|\nu_h| \le \bar\varepsilon_0$, if $\bar\varepsilon_1$ is small enough. Since the solution of the equations (2.29) is unique, it must coincide with the previous one.

Conditions (2.14) and (2.15) imply that

$$\frac{|\sigma_h|}{\gamma^h} = \frac{|\sigma_{\bar h}|}{\gamma^{\bar h}} \frac{|\sigma_h|}{|\sigma_{\bar h}|} \gamma^{\bar h - h} \le C\gamma^{-(h-\bar h)(1-c_0\bar\varepsilon_1)} \;. \tag{2.34}$$

Hence, if $\bar\varepsilon_1$ is small enough, by (2.31),

$$|\beta_k^\nu| \le C\bar\varepsilon_1 \left[\sum_{m=k}^{1} |\nu_m| \gamma^{-2\eta(m-k)} + \bar\varepsilon_0 \gamma^{-\frac12 (h-\bar h)} + \gamma^{\eta k} \right] \;. \tag{2.35}$$

Hence, it is easy to show that there exists a constant $\bar c$ such that

$$\begin{aligned} |\nu_h^{(n)}| \le \bar c \bar\varepsilon_1 \Big[&\sum_{m=\bar h+1}^{h} |\nu_m^{(n-1)}| \gamma^{-(h-m)} + \sum_{m=h+1}^{1} |\nu_m^{(n-1)}| \gamma^{-2\eta(m-h)} + \\ &+ \bar\varepsilon_0 \gamma^{-\frac12 (h-\bar h)} + \gamma^{\eta h} + \gamma^{-(h-\bar h)} \Big] \;. \end{aligned} \tag{2.36}$$

Let us now suppose that, for some constant c_{n-1},

$$|\nu_m^{(n-1)}| \le c_{n-1} \bar\varepsilon_1 \left[\gamma^{-\frac12 (h-\bar h)} + \gamma^{\eta h} \right] \le \bar\varepsilon_0 \;, \tag{2.37}$$

which is true for $n = 1$, since $\nu_m^{(0)} = 0$, if $\bar\varepsilon_1$ is small enough. Then, it is easy to verify that the same bound is verified by $\nu_m^{(n)}$, if c_{n-1} is substituted with

$$c_n = \bar c (1 + c_4 c_{n-1} \bar\varepsilon_1) \;, \tag{2.38}$$

where c_4 is a suitable constant. Hence, we can easily prove the bound (2.17) for $\nu_h = \lim_{n\to\infty} \nu_h^{(n)}$, for $\bar\varepsilon_1$ small enough.

2.4 Let us now consider the equations (2.9) and (2.10), for a fixed, arbitrary, sequence ν_h, $\bar h \le h \le 1$, satisfying the bound (2.17). In order to study the corresponding flow, we compare our model with an approximate model, obtained by putting $u = \nu = 0$ and by substituting all the propagators with the Luttinger propagator $g^{(k)}_{L,\omega}(\mathbf{x};\mathbf{y})$, see (**I**2.100). It is easy to see that, in this model, $\sigma_h(\mathbf{k}') = \nu_h = 0$, for any $h \le 1$, so that the flow of the running couplings is described only by the equations

$$\begin{aligned} \lambda^{(L)}_{h-1} &= \lambda^{(L)}_h + \beta^{\lambda,L}_h(\vec a^{(L)}_h, \ldots, \vec a^{(L)}_1; \delta^*) \;, \\ \delta^{(L)}_{h-1} &= \delta^{(L)}_h + \beta^{\delta,L}_h(\vec a^{(L)}_h, \ldots, \vec a^{(L)}_1; \delta^*) \;, \end{aligned} \tag{2.39}$$

where the functions $\beta^{\lambda,L}_h$ and $\beta^{\delta,L}_h$ can be represented as in (2.5) and (2.6), by suitably changing the definition of the trees and of the related quantities $l_h(\tau)$, $a_h(\tau)$, $z_h(\tau)$, which we shall distinguish by a superscript L. Of course Theorem **I**3.12 applies also to the new model, which differs from the well known Luttinger model only because the space variables are restricted to the unit lattice, instead of the real axis.

Let us define, for $\alpha = \lambda, \delta$,

$$r^\alpha_h(\vec a_h, \nu_h; \ldots; \vec a_1, \nu_1; u, \delta^*) = \beta^\alpha_h(\vec a_h, \nu_h; \ldots; \vec a_1, \nu_1; u, \delta^*) - \beta^{\alpha,L}_h(\vec a_h, \ldots, \vec a_1; \delta^*) \;. \tag{2.40}$$

Note that, in the r.h.s. of (2.40), the function $\beta^{\alpha,L}_h$ is calculated at the values of $\vec a_{h'}$, $h \le h' \le 1$, which are the solutions of the equations (2.9) and (2.10); these values are of course different from those satisfying the equations (2.39). We shall prove the following Lemma.

2.5 Lemma. *Suppose that u satisfies the condition (**I**2.117), the sequence ν_h, $\bar h \le h \le 1$, satisfies the bound (2.17) and δ^* satisfies the condition*

$$| - \delta^* v_0 + c^\delta_0 \lambda_1| \le |\lambda_1| \;, \tag{2.41}$$

c^δ_0 being the constant appearing in the r.h.s. of (2.6),

Then, if η is defined as in Lemma 2.2 and $\mu_h \le \bar\varepsilon_0$ (hence (2.1) is satisfied) and $\bar\varepsilon_0$ is small enough,

$$|r^\lambda_h| + |r^\delta_h| \le C \bar\lambda^2_h [\gamma^{-\frac12 (h-\bar h)} + \gamma^{\eta h}] \;, \quad \bar h \le h \le 0 \;; \tag{2.42}$$

$$|r^\lambda_1| \le C \lambda^2_1 \;, \quad |r^\delta_1| \le C |\lambda_1| \;. \tag{2.43}$$

2.6 *Sketch of the proof.* Note that all trees with $n \ge 2$ endpoints, contributing to the expansions in the r.h.s. of the equations (2.5)-(2.7), may have an endpoint of type ν or δ only if there are at least two endpoints of type λ; this claim follows from the definition of localization and the support properties of the single scale propagators. The bound (2.43) is an easy consequence of this remark, equations (2.5), (2.6), condition (2.41) and Theorem **I**3.12.

We then consider $h \le 0$ and we define

$$\Delta z_h = z_h - z_h^L = \frac{Z_{h-1}}{Z_h} - \frac{Z_{h-1}^L}{Z_h^L} \; . \tag{2.44}$$

Remember that all quantities in (2.44) have to be considered as functions of the same running couplings. Suppose now that

$$|\Delta z_k| \le c_0 \mu_k^2 [\gamma^{-\frac{1}{2}(k-\bar h)} + \gamma^{\eta k}] \; , \quad h < k \le 0 \; . \tag{2.45}$$

We want to prove that this bound is verified also for $k = h$, together with (2.42). Since the proof will also imply that (2.45) is verified for $k = 0$, we shall achieve the proof of Lemma 2.5.

By using the decomposition (**I**2.99) of the propagator, it is easy to see that

$$r_h^\alpha = \sum_{i=1}^3 r_h^{\alpha,i} \; , \tag{2.46}$$

where the quantities $r_h^{\alpha,i}$ are defined in the following way.

1) $r_h^{\alpha,1}$ is obtained from β_h^α by restricting the sum over the trees in the r.h.s. of (2.5) and (2.6) to those containing at least one endpoint of type ν.

2) $r_h^{\alpha,2}$ is obtained from β_h^α by restricting the sum over the trees to those containing no endpoint of type ν, and by substituting, in each term contributing to the expansions appearing in the r.h.s. of (2.5) and (2.6), at least one propagator with a propagator of type $r_1^{(h')}$ or $r_2^{(h')}$ (see (**I**2.99)), $h \le h' \le 1$. Note that z_h and all the ratios Z_k/Z_{k-1}, $k > h$, appearing in the expansions are left unchanged.

3) $r_h^{\alpha,3}$ is obtained by subtracting $\beta_h^{\alpha,L}$ from the expression we get, if we substitute all propagators appearing in the expansions contributing to β_h^α with Luttinger propagators and if we eliminate all trees containing endpoints of type ν.

By using (2.17), (**I**2.101) and (2.34), it is easy to prove that $r_h^{\alpha,1}$ and $r_h^{\alpha,2}$ satisfy a bound like (2.42). The main point is the remark, already used in the proof of Lemma 2.2, that there is an improvement of order $\gamma^{-\eta'(k-h)}$, $0 < \eta' < 1$, in the bound of the sum over the trees with a vertex of fixed scale $k > h$. One has also to use a trick similar to that of §**I**3.13, in order to keep the bound (**I**3.94) on the determinants, after the decomposition of the propagators. Finally, one has to use the remark made at the beginning of this section in order to justify the presence of $\bar\lambda_h^2$, instead of ε_h^2, in the r.h.s. of (2.42).

In order to prove that a bound like (2.42) is satisfied also by $r_h^{\alpha,3}$, one must first prove that the bound in (2.45) is valid for $k = h$, with the same constant c_0. This result can be achieved by decomposing Δz_h in a way similar to that used for r_h^α; let us call $\Delta_i z_h$ the three corresponding terms. By proceeding as before, we can show that

$$|\Delta_1 z_h| + |\Delta_2 z_h| \le C\bar\lambda_h^2 [\gamma^{-\frac{1}{2}(h-\bar h)} + \gamma^{\eta h}] \; . \tag{2.47}$$

Let us now consider $\Delta_3 z_h$; we can write $\Delta_3 z_h = \sum_{n=2}^\infty \sum_{\tau\in\mathcal{T}_{h,n}} \Delta_3 z_h(\tau)$, with $\Delta_3 z_h(\tau) = 0$, if τ contains endpoints of type ν, and $\Delta_3 z_h(\tau) = \sum_{v\in\tau} \bar z_h(\tau, v)$, where $\bar z_h(\tau, v) = 0$, if v is an endpoint, otherwise $\bar z_h(\tau, v)$ is obtained from $z_h(\tau)$ by selecting a family V vertices, which are not endpoints, containing v, and by substituting, for each $v' \in V$, the factor $Z_{h_{v'}}/Z_{h_{v'}-1}$ with $Z_{h_{v'}}/Z_{h_{v'}-1} - Z^L_{h_{v'}}/Z^L_{h_{v'}-1}$. By using (2.2), we have

$$|Z_{h_v}/Z_{h_v-1} - Z^L_{h_v}/Z^L_{h_v-1}| \le C|\Delta z_{h_v}|^2 \; ; \tag{2.48}$$

hence it is easy to show that $\Delta_3 z_h$ can be bounded as in the proof of Theorem **I**3.12, by adding a sum over the non trivial vertices (whose number is proportional to n) and, for each term of this sum, a factor

$$Cc_0\bar\lambda_h^2 [\gamma^{-\frac{1}{2}(h-\bar h)} + \gamma^{\eta h}]\gamma^{\eta(h_{\tilde v}-h)}(h_{\tilde v} - h_{\tilde v'}) \; , \tag{2.49}$$

where $\tilde v$ is the non trivial vertex corresponding to the selected term and $\tilde v'$ is the non trivial vertex immediately preceding $\tilde v$ or the root. Hence, we get

$$|\Delta_3 z_h| \le Cc_0\varepsilon_h^2 \bar\lambda_h^2 [\gamma^{-\frac{1}{2}(h-\bar h)} + \gamma^{\eta h}] \; , \tag{2.50}$$

implying, together with (2.47), the bound (2.45) for $k = h$, if $\bar\varepsilon_0$ is small enough and c_0 is large enough.

Given this result, it is possible to prove in the same manner that $r_h^{\alpha,3}$ satisfies a bound like (2.42). This completes the proof of Lemma 2.5.

2.7 Lemma 2.5 allows to reduce the study of running couplings flow to the same problem for the flow (2.39). This one, in its turn, can be reduced to the study of the beta function for the *Luttinger model*, see [BGM]. This model is exactly solvable, see [ML], and the Schwinger functions can be exactly computed, see [BGM]. It is then possible to show, see [BGM], [BGPS], [GS], [BM1], that there exists $\bar\varepsilon > 0$, such that, if $|\vec a_h| \le \bar\varepsilon$,

$$|\bar\beta_h^{\alpha,L}(\vec a_h, \ldots, \vec a_h)| \le C\mu_h^2 \gamma^{\eta' h} \; , \tag{2.51}$$

where $\bar\beta_h^{\alpha,L}(\vec a_h, ..., \vec a_1)$, $\alpha = \lambda, \delta$, denote the analogous of the functions $\beta_h^{\alpha,L}(\vec a_h, ..., \vec a_1)$ for this model and $0 < \eta' < 1$. Note that in the l.h.s. of (2.51) all running couplings $\vec a_k$, $h \le k \le 1$, are put equal to $\vec a_h$ and that $\vec a_h$ can take any value such that $|\vec a_h| \le \bar\varepsilon$, since $\vec a_h$ is a continuous function of $\vec a_0$ and $\vec a_h = \vec a_0 + O(\mu_h^2)$, see [BGPS].

We argue now that a bound like (2.51) is valid also for the functions $\beta_h^{\alpha,L}$. In fact the Luttinger model differs from our approximate model only because the space coordinates take values on the real axis, instead of the unit lattice. This implies, in particular, that we have to introduce a scale decomposition with a scale index h going up to $+\infty$. However, as it has been shown in [GS], the effective potential on scale $h=0$ is well defined; on the other hand, it differs from the effective potential on scale $h=0$ of our approximate model only for the non local part of the interaction. In terms of the representation (**I**2.61) of $\mathcal{V}^{(0)}(\psi^{(\le 0)})$, this difference is the same we would get, by changing the kernels of the non local terms (without qualitatively affecting their bounds) and the delta function, which in the Luttinger model is defined as $L\beta\delta_{k,0}\delta_{k_0,0}$, instead of as in (**I**2.62).

Note that the difference of the two delta functions has no effect on the local part of $\mathcal{V}^{(0)}(\psi^{(\le 0)})$, because of the support properties of $\hat\psi^{(\le 0)}$, but it slightly affects the non local terms on any scale, hence it affects the beta function; however, it is easy to show that this is a negligible phenomenon. Let us consider in fact a particular tree τ and a vertex $v\in\tau$ of scale h_v with $2n$ external fields of space momenta k'_r, $r=1,\ldots,2n$; the conservation of momentum implies that $\sum_{r=1}^{2n}\sigma_r k'_r = 2\pi m$, with $m=0$ in the continuous model, but m arbitrary integer for the lattice model. On the other hand, k'_r is of order γ^{h_v} for any r, hence m can be different from 0 only if n is of order γ^{h_v}. Since the number of endpoints following a vertex with $2n$ external fields is greater or equal to $n-1$ and there is a small factor (of order μ_h) associated with each endpoint, we get an improvement, in the bound of the terms with $|m|>0$, with respect to the others, of a factor $\exp(-C\gamma^{-h_v})$. Hence, by using the usual arguments, it is easy to show that the difference between the two beta functions is of order $\mu_h^2\gamma^{\eta h}$.

The previous considerations prove the following, very important, Lemma.

2.8 Lemma. *There are $\bar\varepsilon_0$ and $\eta'>0$, such that, if $|\mu_h|\le\bar\varepsilon_0$, $\alpha=\lambda,\delta$ and $h\le 0$,*

$$|\beta_h^{\alpha,L}(\vec a_h,\ldots,\vec a_h)|\le C\bar\lambda_h^2\gamma^{\eta' h}\ . \tag{2.52}$$

We are now ready to prove the following main Theorem on the running couplings flow.

2.9 Theorem. *If $u\ne 0$ satisfies the condition (**I**2.117) and δ^* satisfies the condition (2.41), there exist $\bar\varepsilon_3$ and a finite integer $h^*\le 0$, such that, if $|\lambda_1|\le\bar\varepsilon_3$ and ν belongs to a suitable interval $I^{(h^*)}$, of size smaller than $c|\lambda_1|\gamma'^{h^*}$ for some constants c and γ', $1<\gamma'<\gamma$, then the running coupling constants are well defined for $h^*-1\le h\le 0$ and h^* satisfies the definition (**I**2.116). Moreover, there exist positive constants c_i, $i=1,\ldots,5$, such that*

$$|\lambda_h-\lambda_1|\le c_1|\lambda_1|^{3/2}\ ,\qquad |\delta_h|\le c_1|\lambda_1|\ , \tag{2.53}$$

$$\gamma^{\lambda_1 c_2 h}<\frac{\sigma_h}{\sigma_0}<\gamma^{\lambda_1 c_3 h}\ , \tag{2.54}$$

$$\gamma^{-c_4\lambda_1^2 h}<Z_h<\gamma^{-c_5\lambda_1^2 h}\ , \tag{2.55}$$

$$\max\left\{h_{L,\beta},\frac{\log_\gamma\left(\frac{4\gamma a_0^{-1}}{1+\delta^*}|\sigma_0|\right)}{1-\lambda_1 c_2}\right\}\le h^*\le \max\left\{h_{L,\beta},\frac{\log_\gamma\left(\frac{4\gamma a_0^{-1}}{1+\delta^*}|\sigma_0|\right)+1-\lambda_1 c_3}{1-\lambda_1 c_3}\right\}\ . \tag{2.56}$$

Finally, it is possible to choose δ^* so that, for a suitable $\eta>0$,

$$|\delta_h|\le C|\lambda_1|^{3/2}[\gamma^{-\eta(h-h^*)}+\gamma^{\eta h}]\ . \tag{2.57}$$

2.10 *Proof.* We shall proceed by induction. Equations (2.5), (2.6) and Lemma 2.2 imply that, if λ_1 is small enough, there exists an interval $I^{(0)}$, whose size is of order λ_1, such that, if $\nu\in I^{(0)}$, then the bound (2.17) is satisfied, together with

$$|\lambda_0-\lambda_1|\le C|\lambda_1|^2\ ,\quad |\delta_0-\delta_1|=|\delta_0|\le C|\lambda_1|\ . \tag{2.58}$$

Let us now suppose that the solution of (2.9)-(2.11) is well defined for $\bar h\le h\le 0$ and satisfies the conditions (2.14)-(2.17), for any ν belonging to an interval $I^{(\bar h)}$, defined as in Lemma 2.2. This implies, in particular, that $h^*\le \bar h$, see (2.14) and (**I**2.116). Suppose also that there exists a constant c_0, such that

$$\bar\lambda_{\bar h}\le 2|\lambda_1|\ . \tag{2.59}$$

We want to prove that all these conditions are verified also if $\bar h$ is substituted with $\bar h-1$, if λ_1 is small enough. The induction will be stopped as soon as the condition (2.14) is violated for some $\nu\in I^{(\bar h)}$. We shall put ν equal to one of these values, so defining h^* as equal to $\bar h+1$.

The fact that the condition on ν_1 and the bound (2.17) are verified also if $\bar h-1$ takes the place of $\bar h$, follows from Lemma 2.2, since the condition (2.13) follows from (2.59), if λ_1 is small enough. There is apparently a problem in using this Lemma, since in its proof we used the hypothesis that the values of $\vec a_h$, Z_{h-1} and $\sigma_{h-1}(\mathbf{k}')$, $\bar h\le h\le 1$, are independent of ν_1. This is not true for the full flow, but the proof of Lemma 2.5 can be easily extended to cover this case. In fact, the only part of the proof, where we use the fact that $\vec a_h$ is constant, is the identity (2.24), which should be corrected by adding to the r.h.s. the difference $b_h-b'_h$. However, since λ_1 is independent of ν_1, it is not hard to prove that $|b_h-b'_h|\le C|\nu_h-\nu'_h|$ and that the bound on $r_h-r'_h$ does not change (qualitatively), if we take into account also the dependence on ν_1 of the various quantities, before considered as constant. Hence, the bound (2.25) is left unchanged.

The conditions (2.15) and (2.16) follow immediately from (2.59) and (2.2)-(2.4). Hence, we still have to show only that (2.59) is verified also if $\bar h$ is substituted with $\bar h-1$, if λ_1 is small enough.

By using (2.39) and (2.40), we have, if $\alpha=\lambda,\delta$,

$$\alpha_{\bar h-1}=\alpha_{\bar h}+\beta_{\bar h}^{\alpha,L}(\vec a_{\bar h},\ldots,\vec a_{\bar h})+\sum_{k=\bar h+1}^{1}D^\alpha_{\bar h,k}+r^\alpha_{\bar h}(\vec a_{\bar h},\nu_{\bar h};\ldots;\vec a_1,\nu_1;u)\ , \tag{2.60}$$

where

$$D^{\alpha}_{h,k} = \beta^{\alpha,L}_h(\vec{a}_h,\ldots,\vec{a}_h,\vec{a}_k,\vec{a}_{k+1},\ldots,\vec{a}_1) - \beta^{\alpha,L}_h(\vec{a}_h,\ldots,\vec{a}_h,\vec{a}_h,\vec{a}_{k+1},\ldots,\vec{a}_1)\;. \tag{2.61}$$

On the other hand, it is easy to see that $D^{\alpha}_{h,k}$ admits a tree expansion similar to that of $\beta^{\alpha,L}_h(\vec{a}_h,\ldots,\vec{a}_1)$, with the property that all trees giving a non zero contribution must have an endpoint of scale h, associated with a difference $\lambda_k-\lambda_h$ or $\delta_k-\delta_h$. Hence, if η is the same constant of Lemma 2.2 and Lemma 2.5 and $h\le 0$,

$$|D^{\alpha}_{h,k}| \le C|\bar\lambda_h|\gamma^{-\eta(k-h)}|\vec{a}_k-\vec{a}_h|\;. \tag{2.62}$$

Let us now suppose that $\bar h \le h \le 0$ and that there exists a constant c_0, such that

$$|\vec{a}_{k-1}-\vec{a}_k| \le c_0|\lambda_1|^{3/2}[\gamma^{-\frac12(k-\bar h)}+\gamma^{\vartheta k}]\;,\quad h<k\le 0\;. \tag{2.63}$$

where $\vartheta=\min\{\eta/2,\eta'\}$, η' being defined as in Lemma 2.8. (2.63) is certainly verified for $k=0$, thanks to (2.5), (2.6); we want to show that it is verified also if h is substituted with $h-1$, if λ_1 is small enough.

By using (2.60), (2.62), (2.42), (2.52) and (2.63), we get

$$\begin{aligned}|\vec{a}_{h-1}-\vec{a}_h| \le\; & C\bar\lambda_h^2\gamma^{\eta' h}+C|\bar\lambda_h|^2[\gamma^{-\frac12(h-\bar h)}+\gamma^{\eta h}]+\\ &+Cc_0|\bar\lambda_h|^{5/2}\sum_{k=h+1}^{1}\gamma^{-\eta(k-h)}\sum_{h'=h+1}^{k}[\gamma^{-\frac12(h'-h^*)}+\gamma^{\vartheta h'}]\;,\end{aligned} \tag{2.64}$$

which immediately implies (2.63) with $h\to h-1$ and (2.59) with $\bar h\to\bar h-1$.

The bound (2.64) implies also (2.53), while the bounds (2.54) and (2.55) are an immediate consequence of (2.15), (2.16) and an explicit calculation of the leading terms; (2.56) easily follows from (2.54) and the definition (**I**2.116) of h^*.

All previous results can be obtained uniformly in the value of δ^*, under the condition (2.41). However, by using (2.63) with $\bar h = h^*$, it is not hard to prove, by an implicit function theorem argument (we omit the details, which are of the same type of those used many times before), that one can choose δ^* so that

$$|\delta_0|\le C|\lambda_1|^2\;,\quad \delta_{h^*/2}=0\;, \tag{2.65}$$

which easily implies (2.57), for a suitable value of η.

3. The Correlation function

3.1 The correlation function $\Omega^3_{L,\beta,\mathbf{x}}$, in terms of fermionic operators, is given by

$$\Omega^3_{L,\beta,\mathbf{x}} = < a^+_{\mathbf{x}} a^-_{\mathbf{x}} a^+_0 a^-_0 >_{L,\beta} - < a^+_{\mathbf{x}} a^-_{\mathbf{x}} >_{L,\beta} < a^+_0 a^-_0 >_{L,\beta} = \frac{\partial^2 \mathcal{S}(\phi)}{\partial\phi(\mathbf{x})\phi(\mathbf{0})}|_{\phi=0} \; , \tag{3.1}$$

where $\phi(\mathbf{x})$ is a bosonic external field, periodic in x and x_0, of period L and β, respectively, and

$$e^{\mathcal{S}(\phi)} = \int P(d\psi^{(\leq 1)}) e^{-\mathcal{V}^{(1)}(\psi^{(\leq 1)}) + \int d\mathbf{x}\phi(\mathbf{x})\psi^{(\leq 1)+}_{\mathbf{x}}\psi^{(\leq 1)-}_{\mathbf{x}}} \; . \tag{3.2}$$

Note that, because of the discontinuity at $x_0 = 0$ of the scale 1 free measure propagator $\tilde g^{(1)}_{\omega,\omega}$ in the limit $M \to \infty$ (see §**I**2.3), the product $\psi^{(\leq 1)+}_{\mathbf{x}}\psi^{(\leq 1)-}_{\mathbf{x}}$ has to be understood as $\psi^{(\leq 0)+}_{\mathbf{x}}\psi^{(\leq 0)-}_{\mathbf{x}} + \lim_{\varepsilon\to 0^+} \psi^{(1)+}_{(x,x_0+\varepsilon)}\psi^{(\leq 1)-}_{(x,x_0)}$. Since this remark is important only in the explicit calculation of some physical quantities, but does not produce any problem in the analysis of this section, we shall in general forget it in the notation.

We shall evaluate the integral in the r.h.s. of (3.2) in a way which is very close to that used for the integration in (**I**2.13). We introduce the scale decomposition described in §**I**2.3 and we perform iteratively the integration of the single scale fields, starting from the field of scale 1. The main difference is of course the presence in the interaction of a new term, that we shall call $\mathcal{B}^{(1)}(\psi^{(\leq 1)},\phi)$; in terms of the fields $\psi^{(\leq 1)\sigma}_{\mathbf{x},\omega}$, it can be written as

$$\mathcal{B}^{(1)}(\psi^{(\leq 1)},\phi) = \sum_{\sigma_1,\sigma_2} \int d\mathbf{x} e^{i\mathbf{p}_F\mathbf{x}(\sigma_1+\sigma_2)}\phi(\mathbf{x})\psi^{(\leq 1)\sigma_1}_{\mathbf{x},\sigma_1}\psi^{(\leq 1)\sigma_2}_{\mathbf{x},-\sigma_2} \; . \tag{3.3}$$

After integrating the fields $\psi^{(1)},\ldots\psi^{(h+1)}$, $0 \geq h \geq h^*$, we find

$$e^{\mathcal{S}(\phi)} = e^{-L\beta E_h + S^{(h+1)}(\phi)} \int P_{Z_h,\sigma_h,C_h}(d\psi^{\leq h}) e^{-\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\leq h)}) + \mathcal{B}^{(h)}(\sqrt{Z_h}\psi^{(\leq h)},\phi)} \; , \tag{3.4}$$

where $P_{Z_h,\sigma_h,C_h}(d\psi^{(\leq h)})$ and $\mathcal{V}^h$ are given by (**I**2.66) and (**I**3.3), respectively, while $S^{(h+1)}$ (ϕ), which denotes the sum over all the terms dependent on ϕ but independent of the ψ field, and $\mathcal{B}^{(h)}(\psi^{(\leq h)},\phi)$, which denotes the sum over all the terms containing at least one ϕ field and two ψ fields, can be represented in the form

$$S^{(h+1)}(\phi) = \sum_{m=1}^\infty \int d\mathbf{x}_1 \cdots d\mathbf{x}_m S^{(h+1)}_m(\mathbf{x}_1,\ldots,\mathbf{x}_m)\Big[\prod_{i=1}^m \phi(\mathbf{x}_i)\Big] \tag{3.5}$$

$$\begin{aligned}\mathcal{B}^{(h)}(\psi^{(\leq h)},\phi) = \sum_{m=1}^\infty \sum_{n=1}^\infty \sum_{\underline\sigma,\underline\omega} \int d\mathbf{x}_1\cdots d\mathbf{x}_m d\mathbf{y}_1\cdots d\mathbf{y}_{2n} \cdot \\ \cdot\, B^{(h)}_{m,2n,\underline\sigma,\underline\omega}(\mathbf{x}_1,\ldots,\mathbf{x}_m;\mathbf{y}_1,\ldots,\mathbf{y}_{2n})\Big[\prod_{i=1}^m \phi(\mathbf{x}_i)\Big]\Big[\prod_{i=1}^{2n} \psi^{(\leq h)\sigma_i}_{\mathbf{y}_i,\omega_i}\Big] \; .\end{aligned} \tag{3.6}$$

Since the field ϕ is equivalent, from the point of view of dimensional considerations, to two ψ fields, the only terms in the r.h.s. of (3.6) which are not irrelevant are those with $m = 1$ and $n = 1$, which are marginal. However, if $\sum_{i=1}^2 \sigma_i\omega_i \neq 0$, also these terms are indeed irrelevant, since the dimensional bounds are improved by the presence of a non diagonal propagator, as for the analogous terms with no ϕ field and two ψ fields, see §**I**3.14. Hence we extend the definition of the localization operator $\mathcal{L}$, so that its action on $\mathcal{B}^{(h)}(\psi^{(\le h)},\phi)$ in described in the following way, by its action on the kernels $B^{(h)}_{m,2n,\underline{\sigma},\underline{\omega}}(\mathbf{x}_1,\ldots,\mathbf{x}_m;\mathbf{y}_1,\ldots,\mathbf{y}_{2n})$:

1) if $m=1$, $n=1$ and $\sum_{i=1}^2\sigma_i\omega_i=0$, then

$$\begin{aligned}\mathcal{L}B^{(h)}_{1,2,\underline{\sigma},\underline{\omega}}(\mathbf{x}_1;\mathbf{y}_1,\mathbf{y}_2) &= \sigma_1\omega_1\delta(\mathbf{y}_1-\mathbf{x}_1)\delta(\mathbf{y}_2-\mathbf{x}_1)\cdot\\ &\cdot\int d\mathbf{z}_1 d\mathbf{z}_2 c_\beta(2x_0-z_{10}-z_{20})c_L(z_1-z_2)B^{(h)}_{1,2,\underline{\sigma},\underline{\omega}}(\mathbf{x}_1;\mathbf{z}_1,\mathbf{z}_2)\;;\end{aligned}\tag{3.7}$$

2) in all the other cases

$$\mathcal{L}B^{(h)}_{m,2n,\underline{\sigma},\underline{\omega}}(\mathbf{x}_1,\ldots\mathbf{x}_m;\mathbf{y}_1,\ldots,\mathbf{y}_{2n})=0\;.\tag{3.8}$$

Let us define, in analogy to definition (**I**3.2), the Fourier transform of $B^{(h)}_{1,2,\underline{\sigma},\underline{\omega}}(\mathbf{x}_1;\mathbf{y}_1,\mathbf{y}_2)$ by the equation

$$\begin{aligned}&B^{(h)}_{1,2,\underline{\sigma},\underline{\omega}}(\mathbf{x}_1;\mathbf{y}_1,\mathbf{y}_2)=\\ &=\frac{1}{(L\beta)^3}\sum_{\mathbf{p},\mathbf{k}'_1,\mathbf{k}'_2} e^{i\mathbf{p}\mathbf{x}-i\sum_{r=1}^2\sigma_r\mathbf{k}'_r\mathbf{y}_r}\hat B^{(h)}_{1,2,\underline{\sigma},\underline{\omega}}(\mathbf{p},\mathbf{k}'_1)\delta(\sum_{r=1}^2\sigma_r(\mathbf{k}'_r+\mathbf{p}_F)-\mathbf{p})\;,\end{aligned}\tag{3.9}$$

where $\mathbf{p}=(p,p_0)$ is summed over momenta of the form $(2\pi n/L,2\pi m/\beta)$, with n,m integers. Hence (3.7) can be written in the form

$$\begin{aligned}\mathcal{L}B^{(h)}_{1,2,\underline{\sigma},\underline{\omega}}(\mathbf{x}_1;\mathbf{y}_1,\mathbf{y}_2)&=\sigma_1\omega_1\delta(\mathbf{y}_1-\mathbf{x}_1)\delta(\mathbf{y}_2-\mathbf{x}_1)e^{i\mathbf{p}_F\mathbf{x}(\sigma_1+\sigma_2)}\cdot\\ &\cdot\frac{1}{4}\sum_{\eta,\eta'=\pm1}\hat B^{(h)}_{1,2,\underline{\sigma},\underline{\omega}}(\bar{\mathbf{p}}_{\eta'}+2\mathbf{p}_F(\sigma_1+\sigma_2),\bar{\mathbf{k}}_{\eta,\eta'})\;,\end{aligned}\tag{3.10}$$

where $\bar{\mathbf{k}}_{\eta,\eta'}$ is defined as in (**I**2.73) and

$$\bar{\mathbf{p}}_{\eta'}=\left(0,\eta'\frac{2\pi}{\beta}\right)\;.\tag{3.11}$$

By using the symmetries of the interaction, as in §**I**2.4, it is easy to show that

$$\mathcal{L}\mathcal{B}^{(h)}(\psi^{(\le h)},\phi)=\frac{Z^{(1)}_h}{Z_h}F_1^{(\le h)}+\frac{Z^{(2)}_h}{Z_h}F_2^{(\le h)}\;,\tag{3.12}$$

where $Z_h^{(1)}$ and $Z_h^{(2)}$ are real numbers, such that $Z_1^{(1)}=Z_1^{(2)}=1$ and

$$F_1^{(\le h)}=\sum_{\sigma=\pm1}\int d\mathbf{x}\phi(\mathbf{x})e^{2i\sigma\mathbf{p}_F\mathbf{x}}\psi^{(\le h)\sigma}_{\mathbf{x},\sigma}\psi^{(\le h)\sigma}_{\mathbf{x},-\sigma}\;,\tag{3.13}$$

$$F_2^{(\le h)}=\sum_{\sigma=\pm1}\int d\mathbf{x}\phi(x)\psi^{(\le h)\sigma}_{\mathbf{x},\sigma}\psi^{(\le h)-\sigma}_{\mathbf{x},\sigma}\;.\tag{3.14}$$

By using the notation of §**I**2.5, we can write the integral in the r.h.s. of (3.4) as

$$\begin{aligned}&e^{-L\beta t_h}\int P_{\tilde Z_{h-1},\sigma_{h-1},C_h}(d\psi^{(\le h)})e^{-\tilde{\mathcal{V}}^{(h)}(\sqrt{Z_h}\psi^{(\le h)})+\mathcal{B}^{(h)}(\sqrt{Z_h}\psi^{(\le h)},\phi)}=\\ &=e^{-L\beta t_h}\int P_{Z_{h-1},\sigma_{h-1},C_{h-1}}(d\psi^{(\le h-1)})\cdot\\ &\cdot\int P_{Z_{h-1},\sigma_{h-1},\tilde f_h^{-1}}(d\psi^{(h)})e^{-\hat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)})+\hat{\mathcal{B}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)},\phi)}\;,\end{aligned}\tag{3.15}$$

where $\hat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)})$ is defined as in (**I**2.107) and

$$\hat{\mathcal{B}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)},\phi) = \mathcal{B}^{(h)}(\sqrt{Z_h}\psi^{(\le h)},\phi) \; . \tag{3.16}$$

$\mathcal{B}^{(h-1)}(\sqrt{Z_{h-1}}\psi^{(\le h-1)},\phi)$ and $S^{(h)}(\phi)$ are then defined through the analogous of (**I**2.110), that is

$$\begin{aligned} & e^{-\mathcal{V}^{(h-1)}(\sqrt{Z_{h-1}}\psi^{(\le h-1)})+\mathcal{B}^{(h-1)}(\sqrt{Z_{h-1}}\psi^{(\le h-1)},\phi)-L\beta\tilde{E}_h+\tilde{S}^{(h)}(\phi)} = \\ & = \int P_{Z_{h-1},\sigma_{h-1},\tilde{f}_h^{-1}}(d\psi^{(h)}) e^{-\hat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)})+\hat{\mathcal{B}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)},\phi)} \; . \end{aligned} \tag{3.17}$$

The definitions (3.16) and (3.12) easily imply that

$$\frac{Z_{h-1}^{(i)}}{Z_h^{(i)}} = 1 + z_h^{(i)} \; , \quad i = 1,2 \; , \tag{3.18}$$

where $z_h^{(1)}$ and $z_h^{(2)}$ are some quantities of order ε_h, which can be written in terms of a tree expansion similar to that described in §**I**3, as we shall explain below.

As in §**I**3, the fields of scale between h^* and $h_{L,\beta}$ are integrated in a single step, so we define, in analogy to (**I**3.125),

$$\begin{aligned} & e^{\tilde{S}^{(h^*)}(\phi)-L\beta\tilde{E}_{h^*}} = \\ & \int P_{Z_{h^*-1},\sigma_{h^*-1},C_{h^*}}(d\psi^{(\le h^*)}) e^{-\hat{\mathcal{V}}^{(h^*)}(\sqrt{Z_{h^*-1}}\psi^{(\le h^*)})+\hat{\mathcal{B}}^{(h^*)}(\sqrt{Z_{h^*-1}}\psi^{(\le h^*)},\phi)} \; . \end{aligned} \tag{3.19}$$

It follows, by using (**I**3.126), that

$$S(\phi) = -L\beta E_{L,\beta} + S^{(h)}(\phi) = -L\beta E_{L,\beta} + \sum_{h=h^*}^{1} \tilde{S}^{(h)}(\phi) \; ; \tag{3.20}$$

hence, by (3.1)

$$\Omega^3_{L,\beta,\mathbf{x}} = S_2^{(h)}(\mathbf{x},0) = \sum_{h=h^*}^{1} \tilde{S}_2^{(h)}(\mathbf{x},0) \; . \tag{3.21}$$

3.2 The functionals $\mathcal{B}^{(h)}(\sqrt{Z_h}\psi^{(\le h)},\phi)$ and $S^{(h)}(\phi)$ can be written in terms of a tree expansion similar to the one described in §(3.2). We introduce, for each $n \ge 0$ and each $m \ge 1$, a family $\mathcal{T}^m_{h,n}$ of trees, which are defined as in §(3.2), with some differences, that we shall explain.

1) First of all, if $\tau \in \mathcal{T}^m_{h,n}$, the tree has $n+m$ (instead of n) endpoints. Moreover, among the $n+m$ endpoints, there are n endpoints, which we call *normal endpoints*, which are associated with a contribution to the effective potential on scale $h_v - 1$. The m remaining endpoints, which we call *special endpoints*, are associated with a local term of the form (3.13) or (3.14); we shall say that they are of type $Z^{(1)}$ or $Z^{(2)}$, respectively.

2) We associate with each vertex v a new integer $l_v \in [0,m]$, which denotes the number of special endpoints following v, *i.e.* contained in L_v.

3) We introduce an *external field label* f^ϕ to distinguish the different ϕ fields appearing in the special endpoints. I_v^ϕ will denote the set of external field labels associated with the endpoints following the vertex v; of course $l_v = |I_v^\phi|$ and $m = |I_{v_0}^\phi|$.

These definitions allow to represent $\mathcal{B}^{(h)}(\sqrt{Z_h}\psi^{(\le h)}, \phi) + S^{(h+1)}(\phi)$ in a way similar to that described in detail in §**I**3.3-3.11. It is sufficient to extend in an obvious way some notations and some procedures, in order to take into account the presence of the new terms depending on the external field and the corresponding localization operation.

In particular, if $l_v \neq 0$, the $\mathcal{R}$ operation associated with the vertex v can be deduced from (3.7) and (3.8) and can be represented as acting on the kernels or on the fields in a way similar to what we did in §**I**3.1. We will not write it in detail; we only remark that such definition is chosen so that, when $\mathcal{R}$ is represented as acting on the fields, no derivative is applied to the ϕ field.

All the considerations in §**I**3.2, up to the modifications listed above, can be trivially repeated. The same is true for the definition of the labels $r_v(f)$, described in §**I**3.3. One has only to consider, in addition to the cases listed there, the case in which $|P_v| = 2$ and $l_v = 1$; in such a case, if there is no non trivial vertex v' such that $v_0 \le v' < v$, we make an arbitrary choice, otherwise we put $r_v(f) = 1$ for the ψ field which is an internal field in the nearest non trivial vertex preceding v. As in §**I**3.2, this is sufficient to avoid the proliferation of r_v indices.

Also the considerations in §**I**3.4-**I**3.7 can be adjusted without any difficulty. It is sufficient to add to the three items listed after (**I**3.69) the case $l_{v_0} = 1$, $P_{v_0} = (f_1, f_2)$, by noting that in this case the action of $\mathcal{R}$ consists in replacing one external ψ field with a $D^{11}_{\mathbf{y},\mathbf{x}}$ field.

3.3 Let us consider in more detail the representation we get for the constants $z_h^{(l)}$, $l = 1, 2$, defined in (3.18). We have

$$z_h^{(l)} = \sum_{n=1}^{\infty} \sum_{\substack{\tau \in \mathcal{T}^1_{h,n}, \mathbf{P} \in \mathcal{P}_\tau, \mathbf{r}: P_{v_0} = (f_1, f_2), \\ \sigma_1 = \omega_1 = (-1)^{l-1} \sigma_2 = (-1)^l \omega_2 = +1}} \sum_{T \in \mathbf{T}} \sum_{\substack{\alpha \in A_T \\ q_\alpha(P_{v_0}) = 0}} z_h^{(l)}(\tau, \mathbf{P}, \mathbf{r}, T, \alpha) , \tag{3.22}$$

where, if $\mathbf{x}$ is the space time point associated with the special endpoint,

$$\begin{aligned} z_h^{(l)}(\tau, \mathbf{P}, \mathbf{r}, T, \alpha) = &\Big[\prod_{v \text{ not e.p.}} \Big(Z_{h_v} / Z_{h_v - 1} \Big)^{|P_v|/2} \Big] \cdot \\ &\cdot \int d(\mathbf{x}_{v_0} \backslash \mathbf{x}) h_\alpha(\mathbf{x}_{v_0}) \Big[\prod_{i=1}^n d^{b_\alpha(v_i^*)}_{j_\alpha(v_i^*)}(\mathbf{x}_i, \mathbf{y}_i) K^{h_i}_{v_i^*}(\mathbf{x}_{v_i^*}) \Big] \Big\{ \prod_{v \text{ not e.p.}} \frac{1}{s_v!} \int dP_{T_v}(\mathbf{t}_v) \cdot \\ &\cdot \det G^{h_v, T_v}_\alpha(\mathbf{t}_v) \Big[\prod_{l \in T_v} \hat\partial^{q_\alpha(f_l^-)}_{j_\alpha(f_l^-)} \hat\partial^{q_\alpha(f_l^+)}_{j_\alpha(f_l^+)} [d^{b_\alpha(l)}_{j_\alpha(l)}(\mathbf{x}_l, \mathbf{y}_l) \bar\partial_1^{m_l} g^{(h_v)}_{\omega_l^-, \omega_l^+}(\mathbf{x}_l - \mathbf{y}_l)] \Big] \Big\} . \end{aligned} \tag{3.23}$$

The notations are the same as in §**I**3.10 and we can derive for $z_h^{(l)}(\tau, \mathbf{P}, \mathbf{r}, T, \alpha)$ a bound similar to (**I**3.110), without the volume factor $L\beta$ (the integration over x_{v_0} is done keeping $\mathbf{x}$ fixed). The only relevant difference is that the bounds (**I**3.83) and (**I**3.107) have to be modified, in order to take into account the properties of the extended localization operation, by substituting $z(P_v)$ and $\tilde z(P_v)$ with $z(P_v,l_v)$ and $\tilde z(P_v,l_v)$, respectively, with

$$z(P_v,l_v)=\begin{cases}1 & \text{if } |P_v|=4,\ l_v=0\\ 1 & \text{if } |P_v|=2,\ l_v=0 \text{ and } \sum_{f\in P_v}\omega(f)\neq 0\ ,\\ 2 & \text{if } |P_v|=2,\ l_v=0 \text{ and } \sum_{f\in P_v}\omega(f)=0\ ,\\ 1 & \text{if } |P_v|=2,\ l_v=1 \text{ and } \sum_{f\in P_v}\sigma(f)\omega(f)=0\ ,\\ 0 & \text{otherwise.}\end{cases} \tag{3.24}$$

$$\tilde z(P_v,l_v)=\begin{cases}1 & \text{if } |P_v|=2,\ l_v=0 \text{ and } \sum_{f\in P_v}\omega(f)\neq 0\ ,\\ 1 & \text{if } |P_v|=2,\ l_v=1 \text{ and } \sum_{f\in P_v}\sigma(f)\omega(f)\neq 0\ ,\\ 0 & \text{otherwise.}\end{cases} \tag{3.25}$$

It follows that

$$\begin{aligned}|z_h^{(l)}(\tau,\mathbf{P},\mathbf{r},T,\alpha)| \le C^n\varepsilon_h^n\gamma^{-h[D_0(P_{v_0})+l_{v_0}]}\prod_{v\ \text{not e.p.}}\Big\{C^{\sum_{i=1}^{s_v}|P_{v_i}|-|P_v|}\ \cdot\\ \cdot\ \frac{1}{s_v!}\Big(Z_{h_v}/Z_{h_v-1}\Big)^{|P_v|/2}\gamma^{-[-2+\frac{|P_v|}{2}+l_v+z(P_v,l_v)+\frac{\tilde z(P_v,l_v)}{2}]}\Big\}\ ,\end{aligned} \tag{3.26}$$

with

$$-2+\frac{|P_v|}{2}+l_v+z(P_v,l_v)+\frac{\tilde z(P_v,l_v)}{2}\ge\frac12\ ,\quad \forall v\ \text{not e.p.}\ . \tag{3.27}$$

Hence, we can proceed as in §**I**3.14 and, since $D_0(P_{v_0})+l_{v_0}=0$, we can easily prove the following Theorem.

3.4 THEOREM. *Suppose that $u\neq 0$ satisfies the condition (**I**2.117), δ^* satisfies the condition (2.41), $\bar\varepsilon_3$ is defined as in Theorem 2.9 and $\nu\in I^{(h^*)}$. Then, there exist two constants $\bar\varepsilon_4\le\bar\varepsilon_3$ and c, independent of u, L, β, such that, if $|\lambda_1|\le\bar\varepsilon_4$, then*

$$|z_h^{(l)}|\le c|\lambda_1|\ ,\quad 0\le h\le h^*\ . \tag{3.28}$$

3.5 Theorem 3.4, the bound (2.55) on Z_h, the definition (3.18) and an explicit first order calculation of $z_h^{(1)}$ imply that there exist two positive constants c_1 and c_2, such that

$$\gamma^{-c_2\lambda_1 h}\le\frac{Z_h^{(1)}}{Z_h}\le\gamma^{-c_1\lambda_1 h}\ . \tag{3.29}$$

A similar bound is in principle valid also for $Z_h^{(2)}/Z_h$, but we shall prove that a much stronger bound is verified, by comparing our model with the Luttinger model. First of all, we consider an approximated Luttinger model, which is similar to that introduced in §2.4. It is obtained from the original model by substituting the free measure and the potential with the following expressions, where we use the notation of §**I**2:

$$\begin{aligned}P^{(L)}(d\psi^{(\le 0)})=\prod_{\mathbf{k}':C_0^{-1}(\mathbf{k}')>0}\prod_{\omega=\pm1}\frac{d\hat\psi^{(\le0)+}_{\mathbf{k}',\omega}d\hat\psi^{(\le0)-}_{\mathbf{k}',\omega}}{\mathcal{N}_L(\mathbf{k}')}\ \cdot\\ \cdot\ \exp\Big\{-\frac{1}{L\beta}\sum_{\omega=\pm1}\sum_{\mathbf{k}':C_0^{-1}(\mathbf{k}')>0}C_0(\mathbf{k}')\big(-ik_0+\omega v_0^*k'\big)\hat\psi^{(\le0)+}_{\mathbf{k}',\omega}\hat\psi^{(\le0)-}_{\mathbf{k}',\omega}\Big\}\ ,\end{aligned} \tag{3.30}$$

$$
\begin{aligned}
V^{(L)}(\psi^{(\le 0)}) = \lambda_0^{(L)} \int_{\mathbb{T}_{L,\beta}} d\mathbf{x}\, \psi^{(\le 0)+}_{\mathbf{x},+1} \psi^{(\le 0)-}_{\mathbf{x},-1} \psi^{(\le 0)+}_{\mathbf{x},-1} \psi^{(\le 0)-}_{\mathbf{x},+1} + \\
+ \delta_0^{(L)} \sum_{\omega=\pm 1} i\omega \int_{\mathbb{T}_{L,\beta}} d\mathbf{x}\, \psi^{(\le h)+}_{\mathbf{x},\omega} \partial_x \psi^{(\le h)-}_{\mathbf{x},\omega} \ ,
\end{aligned}
\tag{3.31}
$$

where $\mathcal{N}_L(\mathbf{k}') = C_0(\mathbf{k}')(L\beta)^{-1}[k_0^2 + (v_0^* k')^2]^{1/2}$, $\lambda_0^{(L)}$ and $\delta_0^{(L)}$ have the role of the running couplings on scale 0 of the original model, but are not necessarily equal to them, $\mathbb{T}_{L,\beta}$ is the (continuous, as in §**I**3.15) torus $[0,L]\times[0,\beta]$ and $\psi^{(\le 0)}$ is the (continuous) Grassmanian field on $\mathbb{T}_{L,\beta}$ with antiperiodic boundary conditions. Moreover, the interaction with the external field $\mathcal{B}^{(1)}(\psi^{(\le 1)},\phi)$ is substituted with the corresponding expression on scale 0, deprived of the irrelevant terms, that is

$$
\mathcal{B}^{(0)}(\psi^{(\le 0)},\phi) = \sum_{\sigma=\pm 1} \int d\mathbf{x} \phi(\mathbf{x}) \left(e^{2i\sigma \mathbf{p}_F \mathbf{x}} \psi^{(\le h)\sigma}_{\mathbf{x},\sigma} \psi^{(\le h)\sigma}_{\mathbf{x},-\sigma} + \psi^{(\le h)\sigma}_{\mathbf{x},\sigma} \psi^{(\le h)-\sigma}_{\mathbf{x},\sigma} \right) \ . \tag{3.32}
$$

We shall call $Z_h^{(2,L)}$, $z_h^{(2,L)}$, $Z_h^{(L)}$ and $z_h^{(L)}$ the analogous of $Z_h^{(2)}$, $z_h^{(2)}$, Z_h and z_h for this approximate Luttinger model.

We want to compare the flow of $Z_h^{(2,L)}/Z_h^{(L)}$ with the flow of $Z_h^{(2)}/Z_h$; hence we write

$$
\frac{Z_{h-1}^{(2)}}{Z_{h-1}} = \frac{Z_h^{(2)}}{Z_h} \left[1 + \beta^{(2)}(\vec a_h, \nu_h; \ldots; \vec a_1, \nu_1; u, \delta^*) \right] , \tag{3.33}
$$

$$
\frac{Z_{h-1}^{(2,L)}}{Z_{h-1}^{(L)}} = \frac{Z_h^{(2,L)}}{Z_h^{(L)}} \left[1 + \beta^{(2,L)}(\vec a_h^{(L)}, \ldots; \vec a_0^{(L)}, \delta^*) \right] , \tag{3.34}
$$

where $a_h^{(L)}$ are the running couplings in the approximated Luttinger model (by symmetry $\nu_h^{(L)} = 0$, since $\nu = 0$, see §2.4), $1+\beta^{(2)} = (1+z_h^{(2)})/(1+z_h)$ and $1+\beta^{(2,L)} = (1+z_h^{(2,L)})/(1+z_h^{(L)})$.

The Luttinger model has a special symmetry, the *local gauge invariance*, which allows to prove many *Ward identities*. As we shall prove in §5, the approximate Luttinger model satisfies some approximate version of these identities and one of them implies that, if $|\delta_* + (\delta_0^{(L)}/v_0)| \le 1/2$,

$$
\gamma^{-C|\lambda_0^{(L)}|} \le \frac{Z_h^{(2,L)}}{Z_h^{(L)}} \le \gamma^{C|\lambda_0^{(L)}|} \ . \tag{3.35}
$$

By proceeding as in the proof of (2.51) (see [BGPS], §7), one can show that (3.35) implies that there exists $\bar\varepsilon > 0$ and $\eta' < 1$, such that, if $|\vec a_h| \le \bar\varepsilon$,

$$
|\beta_h^{(2,L)}(\vec a_h, \ldots, \vec a_h, \delta^*)| \le C \mu_h^2 \gamma^{\eta' h} \ . \tag{3.36}
$$

Remark - The analogous bound (2.51) was obtained in [BGPS] by a comparison with the exact solution of the Luttinger model; this was possible, thanks to the proof given in [GS] that the effective potential on scale 0 is well defined also in the Luttinger model, a non trivial result because of the ultraviolet problem. This procedure would be much harder in the case of the bound (3.36), because the density is not well defined in the Luttinger model, see §**I**1.3. In any case, the bound (3.35), whose proof is relatively simple, allows to get very easily the same result.

One can also show, as in the proof of Lemma 2.5, that

$$|\beta^{(2)}(\vec a_h,\nu_h;\ldots;\vec a_1,\nu_1;u,\delta^*)-\beta^{(2,L)}(\vec a_h,,\ldots;\vec a_0,\delta^*)|\le C\bar\lambda_h^2[\gamma^{-\frac12(h-h^*)}+\gamma^{\eta h}]\;, \tag{3.37}$$

for any $h\ge h^*$ and for some $\eta<1$.

Note that, in (3.37), $\beta^{(2,L)}$ is evaluated at the values of the running couplings $\vec a_h$ of the original model; this is meaningful, since in (3.36) $\vec a_h$ can take any value such that $|\vec a_h|\le\bar\varepsilon$; this follows from the remark, already used in §2.7, that $\vec a_h^{(L)}$ is a continuous function of $\vec a_0^{(L)}$ and $\vec a_h^{(L)}=\vec a_0^{(L)}+O(\mu_h^2)$, see also [BGPS].

By using (3.36) and (3.37) and proceeding as in the proof of Theorem 2.9, one can easily prove the following Theorem.

3.6 THEOREM. *If the hypotheses of Theorem 3.4 are verified, there exists a positive constant c_1, independent of u, L, β, such that*

$$\gamma^{-c_1|\lambda_1|}\le\frac{Z_h^{(2)}}{Z_h}\le\gamma^{c_1|\lambda_1|}\;. \tag{3.38}$$

3.7 We are now ready to study the expansion of the correlation function $\Omega^3_{L,\beta}(\mathbf{x})$, which follows from (3.21) and the considerations of §3.2. We have to consider the trees with two special endpoints, whose space-points we shall denote $\mathbf{x}$ and $\mathbf{y}=\mathbf{0}$; moreover, we shall denote by $h_\mathbf{x}$ and $h_\mathbf{y}$ the scales of the two special endpoints and by $h_{\mathbf{x},\mathbf{y}}$ the scale of the smallest cluster containing both special endpoints. Finally $\mathcal{T}^2_{h,n,l}$ will denote the family of all trees belonging to $\mathcal{T}^2_{h,n}$, such that the two special endpoints are both of type $Z^{(1)}$, if $l=1$, both of type $Z^{(2)}$, if $l=2$, one of type $Z^{(1)}$ and the other of type $Z^{(2)}$, if $l=3$.

If we extract from the expansion the contribution of the trees with one special endpoint and no normal endpoints, we can write

$$\begin{aligned}\Omega^3_{L,\beta}(\mathbf{x})=&\sum_{h,h'=h^*}^{1}\sum_{\sigma=\pm1}\Big\{e^{2i\sigma p_F x}\cdot\\
&\cdot\frac{(Z^{(1)}_{h\vee h'})^2}{Z_{h-1}Z_{h'-1}}[g^{(h)}_{\sigma,\sigma}(-\sigma\mathbf{x})g^{(h')}_{-\sigma,-\sigma}(-\sigma\mathbf{x})-g^{(h)}_{+1,-1}(-\sigma\mathbf{x})g^{(h')}_{-1,+1}(-\sigma\mathbf{x})]+\\
&+\frac{(Z^{(2)}_{h\vee h'})^2}{Z_{h-1}Z_{h'-1}}[-g^{(h)}_{\sigma,\sigma}(-\sigma\mathbf{x})g^{(h')}_{\sigma,\sigma}(\sigma\mathbf{x})+g^{(h)}_{-1,+1}(-\sigma\mathbf{x})g^{(h')}_{+1,-1}(\sigma\mathbf{x})]\Big\}+\\
&+\sum_{h=h^*}^{1}\left\{\left(\frac{Z_h^{(1)}}{Z_h}\right)^2G^{(h)}_{1,L,\beta}(\mathbf{x})+\left(\frac{Z_h^{(2)}}{Z_h}\right)^2G^{(h)}_{2,L,\beta}(\mathbf{x})+\frac{Z_h^{(1)}Z_h^{(2)}}{Z_h^2}G^{(h)}_{3,L,\beta}(\mathbf{x})\right\}\;,\end{aligned} \tag{3.39}$$

where $h\vee h'=\max\{h,h'\}$ and $g^{(h^*)}_{\omega_1,\omega_2}(\mathbf{x})$ has to be understood as $g^{(\le h^*)}_{\omega_1,\omega_2}(\mathbf{x})$; moreover,

$$G^{(h)}_{l,L,\beta}(\mathbf{x})=\sum_{n=1}^\infty\sum_{h_r=h^*-1}^{h-1}\sum_{\substack{\tau\in\mathcal{T}^2_{h_r,n,l}\\ h_{\mathbf{x},\mathbf{y}}=h}}\sum_{\substack{\mathbf{P}\in\mathcal{P}_{\tau,\mathbf{r}}\\ P_{v_0}=\emptyset}}\sum_{T\in\mathbf{T}}\sum_{\alpha\in A_T}G^{(h,h_r)}_{l,L,\beta}(\mathbf{x},\tau,\mathbf{P},\mathbf{r},T,\alpha)\;, \tag{3.40}$$

where, if $\hat{\mathbf{x}}_{v_0}$ denotes the set of space-time points associated with the normal endpoints and $i_{\mathbf{x}} = i$, if the corresponding special endpoint is of type $Z^{(i)}$,

$$
\begin{aligned}
&G^{(h,h_r)}_{l,L,\beta}(\mathbf{x},\tau,\mathbf{P},\mathbf{r},T,\alpha) = \\
&= \left(\frac{Z^{(i_\mathbf{x})}_{h_\mathbf{x}} Z_h}{Z_{h_\mathbf{x}-1} Z^{(i_\mathbf{x})}_h} \right) \left(\frac{Z^{(i_\mathbf{y})}_{h_\mathbf{y}} Z_h}{Z_{h_\mathbf{y}-1} Z^{(i_\mathbf{y})}_h} \right) \Big[\prod_{v \text{ not e.p.}} \Big(Z_{h_v}/Z_{h_v-1} \Big)^{|P_v|/2} \Big] \cdot \\
&\cdot \int d\hat{\mathbf{x}}_{v_0} h_\alpha(\hat{\mathbf{x}}_{v_0}) \Big[\prod_{i=1}^n d^{b_\alpha(v_i^*)}_{j_\alpha(v_i^*)}(\mathbf{x}_i,\mathbf{y}_i) K^{(h_i)}_{v_i^*}(\mathbf{x}_{v_i^*}) \Big] \Big\{ \prod_{v \text{ not e.p.}} \frac{1}{s_v!} \int dP_{T_v}(\mathbf{t}_v) \cdot \\
&\cdot \det G^{h_v,T_v}_\alpha(\mathbf{t}_v) \Big[\prod_{l\in T_v} \hat{\partial}^{q_\alpha(f_l^-)}_{j_\alpha(f_l^-)} \hat{\partial}^{q_\alpha(f_l^+)}_{j_\alpha(f_l^+)} [d^{b_\alpha(l)}_{j_\alpha(l)}(\mathbf{x}_l,\mathbf{y}_l) \bar{\partial}_1^{m_l} g^{(h_v)}_{\omega_l^-,\omega_l^+}(\mathbf{x}_l-\mathbf{y}_l)] \Big] \Big\} \; .
\end{aligned}
\tag{3.41}
$$

In the r.h.s. of (3.41) all quantities are defined as in §**I3**, except the kernels $K^{(h_i)}_{v_i^*}(\mathbf{x}_{v_i^*})$ associated with the special endpoints. If v is one of these endpoints, $\mathbf{x}_v$ is always a single point and

$$
K^{(h_v)}_v(\mathbf{x}_v) = e^{i\mathbf{p}_F \mathbf{x}_v \sum_{f\in I_v} \sigma(f)} \; . \tag{3.42}
$$

We want to prove the following Theorem.

3.8 THEOREM. *Suppose that the conditions of Theorem 3.4 are verified, that $\bar{\varepsilon}_4$ is defined as in that theorem and that δ^* is chosen so that condition (2.57) is satisfied. Then, there exist positive constants $\vartheta < 1$ and $\bar{\varepsilon}_5 \le \bar{\varepsilon}_4$, independent of u, L, β, such that, if $|\lambda_1| \le \bar{\varepsilon}_5$ and $\gamma \ge 1+\sqrt{2}$, given any integer $N \ge 0$,*

$$
|G^{(h)}_{1,L,\beta}(\mathbf{x})| + |G^{(h)}_{2,L,\beta}(\mathbf{x})| + \gamma^{-\vartheta h} |G^{(h)}_{3,L,\beta}(\mathbf{x})| \le C_N |\lambda_1| \frac{\gamma^{2h}}{1+[\gamma^h |\mathbf{d}(\mathbf{x})|]^N} \; , \tag{3.43}
$$

for a suitable constant C_N.

Moreover, if $h \le 0$, we can write

$$
\begin{aligned}
G^{(h)}_{1,L,\beta}(\mathbf{x}) &= \cos(2p_F x) \bar{G}^{(h)}_{1,L,\beta}(\mathbf{x}) + \sum_{\sigma=\pm 1} e^{ip_F \sigma x} s^{(h)}_{1,\sigma,L,\beta}(\mathbf{x}) + r^{(h)}_{1,L,\beta}(\mathbf{x}) \; , \\
G^{(h)}_{2,L,\beta}(\mathbf{x}) &= \bar{G}^{(h)}_{2,L,\beta}(\mathbf{x}) + s^{(h)}_{2,L,\beta}(\mathbf{x}) + r^{(h)}_{2,L,\beta}(\mathbf{x}) \; ,
\end{aligned}
\tag{3.44}
$$

so that

$$
\bar{G}^{(h)}_{l,L,\beta}(\mathbf{x}) = \bar{G}^{(h)}_{l,L,\beta}(-\mathbf{x}) \; , \quad l = 1,2 \; , \tag{3.45}
$$

$$
|r^{(h)}_{1,L,\beta}(\mathbf{x})| + |r^{(h)}_{2,L,\beta}(\mathbf{x})| \le C_N |\lambda_1| \gamma^{2h} \frac{\gamma^{\vartheta h}}{1+[\gamma^h |\mathbf{d}(\mathbf{x})|]^N} \; , \tag{3.46}
$$

and, if we define $D_{m_0,m_1} = \partial_0^{m_0} \bar{\partial}_1^{m_1}$, given any integers $m_0, m_1 \ge 0$, there exists a constant C_{N,m_0,m_1}, such that

$$
\sum_{l=1,2} |D_{m_0,m_1} \bar{G}^{(h)}_{l,L,\beta}(\mathbf{x})| \le C_{N,m_0,m_1} |\lambda_1| \frac{\gamma^{2h} \gamma^{h(m_0+m_1)}}{1+[\gamma^h |\mathbf{d}(\mathbf{x})|]^N} \; , \tag{3.47}
$$

$$
\begin{aligned}
&\sum_{\sigma=\pm 1} |D_{m_0,m_1} s^{(h)}_{1,\sigma,L,\beta}(\mathbf{x})| + |D_{m_0,m_1} s^{(h)}_{2,L,\beta}(\mathbf{x})| \le \\
&\le C_{N,m_0,m_1} |\lambda_1| \frac{\gamma^{2h} \gamma^{h(m_0+m_1)}}{1+[\gamma^h |\mathbf{d}(\mathbf{x})|]^N} [\gamma^{-\vartheta(h-h^*)} + \gamma^{\vartheta h}] \; .
\end{aligned}
\tag{3.48}
$$

$\Omega^3_{L,\beta}(\mathbf{x})$, as well as the functions $\bar G^{(h)}_{l,L,\beta}(\mathbf{x})$, $r^{(h)}_{l,L,\beta}(\mathbf{x})$, $s^{(h)}_{1,\sigma,L,\beta}(\mathbf{x})$ and $s^{(h)}_{2,L,\beta}(\mathbf{x})$ converge, as $L,\beta\to\infty$, to continuous bounded functions on $\mathbb{Z}\times\mathbb{R}$, that we shall denote $\Omega^3(\mathbf{x})$, $\bar G^{(h)}_l(\mathbf{x})$, $r^{(h)}_l(\mathbf{x})$, $s^{(h)}_{1,\sigma}(\mathbf{x})$ and $s^{(h)}_2(\mathbf{x})$, respectively. $\bar G^{(h)}_1(\mathbf{x})$ and $\bar G^{(h)}_2(\mathbf{x})$ are the restrictions to $\mathbb{Z}\times\mathbb{R}$ of two even functions on $\mathbb{R}^2$ satisfying the bound (3.47) with the continuous derivative ∂_1 in place of the discrete one and $|\mathbf{x}|$ in place of $|\mathbf{d}(\mathbf{x})|$.

Finally, $\bar G^{(h)}_1(\mathbf{x})$, as a function on $\mathbb{R}^2$, satisfies the symmetry relation

$$\bar G^{(h)}_1(x,x_0)=\bar G^{(h)}_1(x_0v_0^*,\frac{x}{v_0^*})\;. \tag{3.49}$$

3.9 *Proof.* As in the proof of Theorem 3.4, we shall try to mimic as much as possible the proof of the bound (**I**3.110), by only remarking the relevant differences. Since $D_0(P_{v_0})+l_{v_0}=0$, if the integral in the r.h.s. of (3.41) were over the set of variables $x_{v_0}\backslash\mathbf{x}$, we should get for $G^{(h,h_r)}_{l,L,\beta}(\mathbf{x},\tau,\mathbf{P},\mathbf{r},T,\alpha)$ the same bound we derived in §3.3 for $z^{(l)}_h(\tau,\mathbf{P},\mathbf{r},T,\alpha)$. However, in this case, we have to perform the integration over the set x_{v_0} by keeping fixed two points ($\mathbf{x}$ and $\mathbf{y}$), instead of one; hence we have to modify the bound (**I**3.102) in a way different from what we did in the proof of Theorem 3.4.

Let us call $\bar v_0$ the higher vertex $v\in\tau$, such that both $\mathbf{x}$ and $\mathbf{y}$ belong to $\mathbf{x}_v$; by the definition of h, it is a non trivial vertex and its scale is equal to h. Moreover, given the tree graph T on x_{v_0}, let us call $T_{\mathbf{x},\mathbf{y}}$ its subtree connecting the points of $\mathbf{x}_{\bar v_0}$ and $\tilde T_{\mathbf{x},\mathbf{y}}=\cup_{v\ge\bar v_0}\tilde T_v$, $\tilde T_v$ being defined as §**I**3.15, after (**I**3.118). We want to bound $\mathbf{d}(\mathbf{x}-\mathbf{y})$ in terms of the distances between the points connected by the lines $l\in\tilde T_{\mathbf{x},\mathbf{y}}$.

Let us call $\bar v^{(i)}$, $i=1,\ldots,s_{\bar v_0}$ the non trivial vertices or endpoints following $\bar v_0$. The definition of $\bar v_0$ implies that $s_{\bar v_0}>1$ and that $\mathbf{x}$ and $\mathbf{y}$ belong to two different sets $\mathbf{x}_{\bar v^{(i)}}$; note also that $\tilde T_{\bar v_0}$ is an anchored tree graph between the sets of points $\mathbf{x}_{\bar v^{(i)}}$. Hence there is an integer r, a family $l_1,\ldots,l_r$ of lines belonging to $\tilde T_{\bar v_0}$ and a family $v^{(1)},\ldots,v^{(r+1)}$ of vertices to be chosen among $\bar v^{(1)},\ldots,\bar v^{(s_{\bar v_0})}$, such that $1\le r\le s_{\bar v_0}-1$ and

$$\begin{aligned}|\mathbf{d}(\mathbf{x}-\mathbf{y})|&\le\sum_{j=1}^r|\mathbf{d}(\mathbf{x}'_{l_j}-\mathbf{y}'_{l_j})|+\sum_{j=1}^{r+1}|\mathbf{d}(\mathbf{x}^{(j)}-\mathbf{y}^{(j)})|\;\le\\&\le\sum_{l\in\tilde T_{\bar v_0}}|\mathbf{d}(\mathbf{x}'_l-\mathbf{y}'_l)|+\sum_{j=1}^{r+1}|\mathbf{d}(\mathbf{x}^{(j)}-\mathbf{y}^{(j)})|\;,\end{aligned} \tag{3.50}$$

where $\mathbf{x}^{(1)}=\mathbf{x}$, $\mathbf{y}^{(r+1)}=\mathbf{y}$, $\mathbf{x}'_{l_j}$ and $\mathbf{y}'_{l_j}$ are defined as in (**I**3.114) and, finally, the couple of points $(\mathbf{x}'_{l_j},\mathbf{y}'_{l_j})$ coincide, up to the order, with the couple $(\mathbf{y}^{(j)},\mathbf{x}^{j+1})$.

If no propagator associated with a line $l\in\tilde T_{\mathbf{x},\mathbf{y}}$ is affected by the regularization, we can iterate in an obvious way the previous considerations, so getting the bound

$$|\mathbf{d}(\mathbf{x}-\mathbf{y})|\le\sum_{l\in\tilde T_{\mathbf{x},\mathbf{y}}}|\mathbf{d}(\mathbf{x}'_l-\mathbf{y}'_l)|\;. \tag{3.51}$$

However, this is not in general true and we have to consider in more detail the subsequent steps of the iteration.

Let us consider one of the vertices $\mathbf{x}_{v^{(j)}}$; if $\mathbf{x}^{(j)}=\mathbf{y}^{(j)}$, there is nothing to do. Hence we shall suppose that $\mathbf{x}^{(j)}\neq\mathbf{y}^{(j)}$ and we shall say that the propagators associated with the lines l_j, if $1\le j\le r$, and l_{j-1}, if $2\le j\le r+1$, are *linked* to $v^{(j)}$. There are two different cases to consider.

1) $\mathbf{x}^{(j)}$ and $\mathbf{y}^{(j)}$ belong to two different non trivial vertices or endpoints following $v^{(j)}$ and the propagators linked to $v^{(j)}$ are not affected by action of $\mathcal{R}$ on the vertex $v^{(j)}$ or some trivial vertex v, such that $\bar v_0<v<v^{(j)}$. In this case, we iterate the previous procedure without any change.

2) One of the propagators linked to $v^{(j)}$ is affected by action of $\mathcal{R}$ on the vertex $v^{(j)}$ or some trivial vertex v, such that $\bar v_0<v<v^{(j)}$; note that, if there are two linked propagators, only one may have this property, as a consequence of the regularization procedure described in §**I**3. This means that $\mathbf{x}^{(j)}$ or $\mathbf{y}^{(j)}$, let us say $\mathbf{x}^{(j)}$, is of the form (**I**3.115), with $t_l\neq 1$, that is there are two points $\tilde{\mathbf{x}}_l,\mathbf{x}_l\in\mathbf{x}_{v^{(j)}}$ and a point $\bar{\mathbf{x}}_l\in\mathbb{R}^2$, coinciding with $\mathbf{x}_l$ modulo (L,β), such that

$$\mathbf{x}^{(j)}=\tilde{\mathbf{x}}_l+t_l(\bar{\mathbf{x}}_l-\tilde{\mathbf{x}}_l)\;,\quad |\bar x_l-x_l|\le 3L/4\;,|\bar x_{l,0}-x_{l,0}|\le 3\beta/4\;. \tag{3.52}$$

By using (**I**2.96), (3.52), the fact that $0\le|t_l|\le1$ and the remark that $\mathbf{d}(\bar{\mathbf{x}}_l-\tilde{\mathbf{x}}_l)=\mathbf{d}(\mathbf{x}_l-\tilde{\mathbf{x}}_l)$, we get

$$|\mathbf{d}(\mathbf{x}^{(j)}-\mathbf{y}^{(j)})|\le|\mathbf{d}(\tilde{\mathbf{x}}_l-\mathbf{y}^{(j)})|+\sqrt{2}\,|\mathbf{d}(\mathbf{x}_l-\tilde{\mathbf{x}}_l)|\;. \tag{3.53}$$

We can now bound $|\mathbf{d}(\tilde{\mathbf{x}}_l-\mathbf{y}^{(j)})|$ and $|\mathbf{d}(\mathbf{x}_l-\tilde{\mathbf{x}}_l)|$, by proceeding as in the proof of (3.50), since the points $\tilde{\mathbf{x}}_l$, $\mathbf{x}_l$ and $\mathbf{y}^{(j)}$ all belong to $v^{(j)}$. We get

$$|\mathbf{d}(\mathbf{x}^{(j)}-\mathbf{y}^{(j)})|\le(1+\sqrt{2})\left[\sum_{l\in\tilde T_{v^{(j)}}}|\mathbf{d}(\mathbf{x}_l'-\mathbf{y}_l')|+\sum_{m=1}^{r_j}|\mathbf{d}(\mathbf{x}'^{(m)}-\mathbf{y}'^{(m)})|\right]\;, \tag{3.54}$$

where $2\le r_j\le s_{v^{(j)}}$ and the points $\mathbf{x}'^{(m)}$, $\mathbf{y}'^{(m)}$ are endpoints of propagators linked to some non trivial vertex or endpoint following $v^{(j)}$.

By iterating the previous procedure we get, instead of (3.51), the bound

$$|\mathbf{d}(\mathbf{x}-\mathbf{y})|\le\sum_{l\in\tilde T_{\mathbf{x},\mathbf{y}}}(1+\sqrt{2})^{p_l}|\mathbf{d}(\mathbf{x}_l'-\mathbf{y}_l')|\;, \tag{3.55}$$

where, if $l\in T_{v_l}$, p_l is an integer less or equal to the number of non trivial vertices v such that $\bar v_0\le v<v_l$; note that

$$p_l\le h_{v_l}-h\;. \tag{3.56}$$

Let us now suppose that

$$\gamma\ge1+\sqrt{2}\;. \tag{3.57}$$

Since there are at most $2n+1$ lines in T, (3.55), (3.56) and (3.57) imply that there exists at least one line $l \in T_{\mathbf{x},\mathbf{y}}$, such that

$$\gamma^{h_{v_l}} |\mathbf{d}(\mathbf{x}'_l - \mathbf{y}'_l)| \geq \frac{\gamma^h |\mathbf{d}(\mathbf{x}-\mathbf{y})|}{2n+1} \; . \tag{3.58}$$

It follows that, given any $N \geq 0$, for the corresponding propagator we can use, instead of the bound (**I**3.116), the following one:

$$\begin{aligned} &\left| \tilde\partial^{q_\alpha(f_l^-)}_{j_\alpha(f_l^-)} \tilde\partial^{q_\alpha(f_l^+)}_{j_\alpha(f_l^+)} [d^{b_\alpha(l)}_{j_\alpha(l)}(\mathbf{x}'_l(t_l), \mathbf{y}'_l(s_l)) \bar\partial_1^{m_l} g^{(h_v)}_{\omega_l^-,\omega_l^+}(\mathbf{x}'_l(t_l) - \mathbf{y}'_l(s_l))] \right| \leq \\ &\leq \frac{\gamma^{h_v[1+q_\alpha(f_l^+)+q_\alpha(f_l^-)+m(f_l^-)+m(f_l^+)-b_\alpha(l)]}}{1+[\gamma^{h_v}|\mathbf{d}(\mathbf{x}'_l(t_l)-\mathbf{y}'_l(s_l))|]^3} \Big(\frac{|\sigma_{h_v}|}{\gamma^{h_v}}\Big)^{\rho_l} \frac{C_N (2n+1)^N}{1+[\gamma^h |\mathbf{d}(\mathbf{x}-\mathbf{y})|]^N} \; . \end{aligned} \tag{3.59}$$

For all others propagators we use again the bound (**I**3.116) with $N=3$ and we proceed as in §**I**3.15, recalling that we have to substitute in (**I**3.118) $d(\mathbf{x}_{v_0} \backslash \bar{\mathbf{x}})$ with $d\hat{\mathbf{x}}_{v_0}$. This implies that, in the r.h.s. of (**I**3.119), one has to eliminate one $d\mathbf{r}_l$ factor and, of course, this can be done in an arbitrary way. We choose to eliminate the integration over the $\mathbf{r}_l$ corresponding to a propagator of scale h (there is at least one of them), so that the bound (**I**3.118) is improved by a factor γ^{2h}.

At the end, we get

$$\begin{aligned} &|G^{(h,h_r)}_{l,L,\beta}(\mathbf{x},\tau,\mathbf{P},\mathbf{r},T,\alpha)| \leq (C\varepsilon_h)^n C_N (2n+1)^N \frac{\gamma^{2h}}{1+[\gamma^h \mathbf{d}(\mathbf{x})]^N} \cdot \\ &\cdot \left(\frac{Z^{(i_\mathbf{x})}_{h_\mathbf{x}} Z_h}{Z_{h_\mathbf{x}-1} Z^{(i_\mathbf{x})}_h} \right) \left(\frac{Z^{(i_\mathbf{y})}_{h_\mathbf{y}} Z_h}{Z_{h_\mathbf{y}-1} Z^{(i_\mathbf{y})}_h} \right) \prod_{v \text{ not e.p.}} \Big\{ \frac{1}{s_v!} C^{\sum_{i=1}^{s_v} |P_{v_i}| - |P_v|} \cdot \\ &\cdot \Big(Z_{h_v}/Z_{h_v-1} \Big)^{|P_v|/2} \gamma^{-[-2+\frac{|P_v|}{2}+l_v+z(P_v,l_v)+\frac{\tilde z(P_v,l_v)}{2}]} \, . \Big\} \end{aligned} \tag{3.60}$$

We can now perform as in §**I**3.14 the various sums in the r.h.s. of (3.40). There are some differences in the sum over the scale labels, but they can be easily treated. First of all, one has to take care of the factors $(Z^{(i_\mathbf{x})}_{h_\mathbf{x}} Z_h)/(Z_{h_\mathbf{x}-1} Z^{(i_\mathbf{x})}_h)$ and $(Z^{(i_\mathbf{y})}_{h_\mathbf{y}} Z_h)/(Z_{h_\mathbf{y}-1} Z^{(i_\mathbf{y})}_h)$. However, by using (3.29) and (3.38), it is easy to see that these factors have the only effect to add to the final bound a factor $\gamma^{C|\lambda_1|(h_v - h_{v'})}$ for each non trivial vertex v containing one of the special endpoints and strictly following the vertex $v_{\mathbf{x},\mathbf{y}}$; this has a negligible effect, thanks to analogous of the bound (**I**3.111), valid in this case. The other difference is in the fact that, instead of fixing the scale of the root, we have now to fix the scale of $v_{\mathbf{x},\mathbf{y}}$. However, this has no effect, since we bound the sum over the scales with the sum over the the differences $h_v - h_{v'}$.

The previous considerations are sufficient to get the bound (3.43) for $G^{(h)}_{1,L,\beta}(\mathbf{x})$ and $G^{(h)}_{2,L,\beta}(\mathbf{x})$. In order to explain the factor $\gamma^{\vartheta h}$ multiplying $G^{(h)}_{3,L,\beta}(\mathbf{x})$, one has to note that the trees whose normal endpoints are all of scale lower than 2 give no contribution to $G^{(h)}_{3,L,\beta}(\mathbf{x})$. In fact, these endpoints have the property that $\sum_{f \in P_v} \sigma(f) = 0$, while this condition is satisfied from one of the special endpoints but not from the other, in any tree contributing to $G^{(h)}_{3,L,\beta}(\mathbf{x})$. It follows, since any propagator couples two fields with different σ indices, that it is possible to produce a non zero contribution to $G^{(h)}_{3,L,\beta}(\mathbf{x})$, only if there is at least one endpoint of scale 2; this allows to extract from the bound a factor $\gamma^{\vartheta h}$, with $0<\vartheta<1$, as remarked many times before.

We now want to show that $G^{(h)}_{1,L,\beta}(\mathbf{x})$ and $G^{(h)}_{2,L,\beta}(\mathbf{x})$ can be decomposed as in (3.44), so that the bounds (3.46), (3.47) and (3.45) are satisfied. To begin with, we define $r^{(h)}_{i,L,\beta}(\mathbf{x})$, $i=1,2$, by using the definition (3.40) of $G^{(h)}_{i,L,\beta}(\mathbf{x})$, with the constraint that the sum is restricted to the trees, which contain at least one endpoint of scale $h_v=2$; this implies, in particular, that $G^{(+1)}_{i,L,\beta}(\mathbf{x})-r^{(+1)}_{i,L,\beta}(\mathbf{x})=0$. Moreover, in the remaining trees, we decompose the propagators in the following way:

$$g^{(h)}_{\omega,\omega'}(\mathbf{x})=\bar g^{(h)}_{\omega,\omega'}(\mathbf{x})+\delta g^{(h)}_{\omega,\omega'}(\mathbf{x})\;, \tag{3.61}$$

where $\bar g^{(h)}_{\omega,\omega'}(\mathbf{x})$ is defined by putting, in the r.h.s. of (**I**2.94), $(v_0^* k')$ in place of $E(k')$, and we absorb in $r^{(h)}_{i,L,\beta}(\mathbf{x})$ the terms containing at least one propagator $\delta g^{(h)}_{\omega,\omega'}(\mathbf{x})$, which is of size γ^{2h}. The substitution of $(v_0^* k')$ in place of $E(k')$ is done also in the definition of the $\mathcal{R}$ operator, so producing other "corrections", to be added to $r^{(h)}_{i,L,\beta}(\mathbf{x})$. An argument similar to that used for $G^{(h)}_{3,L,\beta}(\mathbf{x})$ easily allows to prove the bound (3.46).

$\sum_{\sigma=\pm1}\exp(i\sigma p_F x)s^{(h)}_{1,\sigma,L,\beta}(\mathbf{x})$ and $s^{(h)}_{2,L,\beta}(\mathbf{x})$ will denote the sum of the trees contributing to $G^{(h)}_{1,L,\beta}(\mathbf{x})-r^{(h)}_{1,L,\beta}(\mathbf{x})$ and $G^{(h)}_{2,L,\beta}(\mathbf{x})-r^{(h)}_{2,L,\beta}(\mathbf{x})$, respectively, which have at least one endpoint of type ν or δ.

Let us now consider the "leading" contribution to $G^{(h)}_{2,L,\beta}(\mathbf{x})$, which is defined by the second of the equations (3.44) as $\bar G^{(h)}_{2,L,\beta}(\mathbf{x})$ and is obtained by using again (3.40), but with the constraint that the sum over the trees is restricted to those having only endpoints with scale $h_v\le 1$ and only normal endpoints of type λ. Moreover we have to use everywhere the propagator $\bar g^{(h)}_{\omega,\omega'}(\mathbf{x})$, which has well defined parity properties in the $\mathbf{x}$ variables; it is odd, if $\omega=\omega'$, and even, if $\omega=-\omega'$.

Note that all the normal endpoints with $h_v\le 1$ are such that $\sum_{f\in I_v}\sigma(f)=0$ and that this property is true also for the special endpoints, which have to be of type $Z^{(2)}$; hence there is no oscillating factor in the kernels associated with the endpoints, which are suitable constants (the associated effective potential terms are local). It follows that any graph contributing to $\bar G^{(h)}_{2,L,\beta}(\mathbf{x})$ is given, up to a constant, by an integral over the product of an even number of propagators (we are using here the fact that there is no endpoint of type ν or δ). Moreover, since all the endpoints satisfy also the condition $\sum_{f\in I_v}\sigma(f)\omega(f)=0$, which is violated by the set of two lines connected by a non diagonal propagator, the number of non diagonal propagators has to be even. These remarks immediately imply that $\bar G^{(h)}_{2,L,\beta}(\mathbf{x})=\bar G^{(h)}_{2,L,\beta}(-\mathbf{x})$.

In order to prove the bound (3.47) for $\bar G^{(h)}_{2,L,\beta}(\mathbf{x})$, we observe that, since the propagators only couple fields with different σ indices and $\sum_{f\in I_v}\sigma(f)=0$, given any tree τ contributing to $\bar G^{(h)}_{2,L,\beta}(\mathbf{x})$ and any $v\in\tau$, we must have

$$\sum_{f\in P_v}\sigma(f)=0\ . \tag{3.62}$$

Let us now consider the vertex $\bar v_0$, defined as in §3.9, that is the higher vertex $v\in\tau$, such that both $\mathbf{x}$ and $\mathbf{y}=\mathbf{0}$ belong to $\mathbf{x}_v$, and let $v_{\mathbf{x}}$ be the vertex immediately following $\bar v_0$, such that $\mathbf{x}\in v_{\mathbf{x}}$. We can associate with $v_{\mathbf{x}}$ a contribution to $\mathcal{B}^h(\psi^{(\le h)},\phi)$ (recall that h is the scale of $\bar v_0$ and hence the scale of the external fields of $v_{\mathbf{x}}$), with $m=1$ and $2n=P_{v_{\mathbf{x}}}$ (see (3.6)), whose kernel is of the form, thanks to (3.62)

$$\begin{aligned}B(\mathbf{x};\mathbf{y}_1,\ldots,\mathbf{y}_{2n})=&\frac{1}{(L\beta)^{2n+1}}\sum_{\mathbf{p},\mathbf{k}'_1,\ldots,\mathbf{k}'_{2n}}e^{i\mathbf{p}\mathbf{x}-i\sum_{r=1}^{2n}\sigma_r\mathbf{k}'_r\mathbf{y}_r}\ \cdot\\ &\cdot\ \hat B(\mathbf{p};\mathbf{k}'_1,\ldots,\mathbf{k}'_{2n-1})\delta(\sum_{r=1}^{2n}\sigma_r\mathbf{k}'_r-\mathbf{p})\ .\end{aligned}\tag{3.63}$$

If we apply the differential operator $\partial_0^{m_0}$ to $\bar G^{(h)}_{2,L,\beta}(\mathbf{x})$, this operator acts on $B(\mathbf{x};\mathbf{y}_1,\ldots,\mathbf{y}_{2n})$, so that its Fourier transform is multiplied by $(ip_0)^{m_0}$; since $p_0=\sum_{r=1}^{2n}\sigma_r k_{r0}$ and the external fields of $v_{\mathbf{x}}$ are contracted on a scale smaller or equal to h, it is easy to see that there is an improvement on the bound of $\partial_0\bar G^{(h)}_{2,L,\beta}(\mathbf{x})$, with respect to the bound of $\bar G^{(h)}_{2,L,\beta}(\mathbf{x})$, of a factor $c_{m_0}\gamma^{hm_0}$, for a suitable constant c_{m_0}. We are using here the fact that $\bar G^{(+1)}_{i,L,\beta}(\mathbf{x})=0$, so that we can suppose $h\le 0$, otherwise we would be involved with the singularity of the scale 1 propagator $g^{(1)}_{\omega_l^-,\omega_l^+}(\mathbf{x}_l-\mathbf{y}_l)$ at $x_l-y_l=0$, which allows to get uniform bounds on the derivatives only for $|x_l-y_l|$ bounded below, a condition not verified in general.

Let us now consider $\bar\partial_1^{m_1}\bar G^{(h)}_{2,L,\beta}(\mathbf{x})$ (see (**I**3.6) for the definition of $\bar\partial_1$). By using (**I**2.62) and the conservation of the spatial momentum, we find that $\bar\partial_1^{m_1}$ acts on $B(\mathbf{x};\mathbf{y}_1,\ldots,\mathbf{y}_{2n})$, so that its Fourier transform is multiplied by $\sin(px)^{m_1}$, with $p=\sum_{r=1}^{2n}\sigma_r k'_r+2\pi m$, where m is an arbitrary integer and p is chosen so that $|p|\le\pi$. If $m=0$, we proceed as in the case of the time derivative, otherwise we note that the support properties of the external fields, see §**I**2.2, implies that $|\sum_{r=1}^{2n}\sigma_r k'_r|\le 2na_0\gamma^h$; hence, if $|m|>0$, $2n\ge(\pi/a_0)\gamma^{-h}$. Since the number of endpoints following $v_{\mathbf{x}}$ is proportional to $2n$ and each endpoint carries a small factor of order λ_1, it is clear that, if λ_1 is small enough, we get an improvement in the bound of the terms with $|m|>0$, with respect to the corresponding contributions to $\bar G^{(h)}_{2,L,\beta}(\mathbf{x})$, of a factor $\exp(-C\gamma^{-h})\le c_{m_1}\gamma^{hm_1}$, for some constant c_{m_1}. In the same manner, we can treat the operator D_{m_0,m_1}, so proving the bound (3.47) for $D_{m_0,m_1}\bar G^{(h)}_{2,L,\beta}(\mathbf{x})$.

Let us now consider $G^{(h)}_{1,L,\beta}(\mathbf{x}-\mathbf{y})-r^{(h)}_{1,L,\beta}(\mathbf{x}-\mathbf{y})$. In this case the kernels of the two special endpoints $\mathbf{x}$ and $\mathbf{y}$ are equal to $\exp(2i\sigma_{\mathbf{x}}p_F x)$ and $\exp(2i\sigma_{\mathbf{y}}p_F y)$, respectively. However, since the propagators couple fields with different σ indices and all the other endpoints satisfy the condition $\sum_{f\in I_v}\sigma(f)=0$, $\sigma_{\mathbf{x}}=-\sigma_{\mathbf{y}}$ and we can write

$$G^{(h)}_{1,L,\beta}(\mathbf{x}-\mathbf{y})-r^{(h)}_{1,L,\beta}(\mathbf{x}-\mathbf{y})=\frac12\sum_{\sigma=\pm1}e^{2i\sigma p_F(x-y)}\left[\bar G^{(h)}_{1,\sigma}(\mathbf{x}-\mathbf{y})+2s^{(h)}_{1,\sigma,L,\beta}(\mathbf{x}-\mathbf{y})\right]\ , \tag{3.64}$$

with $\bar G^{(h)}_{1,\sigma}(\mathbf{x})$ having the same properties as $\bar G^{(h)}_2(\mathbf{x})$; in particular it is an even function of $\mathbf{x}$ and satisfies the bound (3.47). Moreover, it is easy to see that $\bar G^{(h)}_{1,+}(\mathbf{x}-\mathbf{y})$ is equal to $\bar G^{(h)}_{1,-}(\mathbf{y}-\mathbf{x}) = \bar G^{(h)}_{1,-}(\mathbf{x}-\mathbf{y})$, hence $\bar G^{(h)}_{1,\sigma}(\mathbf{y}-\mathbf{x})$ is independent of σ and we get the decomposition in the first line of (3.44), with $\bar G^{(h)}_1(\mathbf{x}-\mathbf{y})$ satisfying (3.47) and (3.45).

The bound (3.48) is proved in the same way as the bound (3.47). The factor $[\gamma^{-\vartheta(h-h^*)}+\gamma^{\vartheta h}]$ in the r.h.s. comes from the fact that the trees contributing to $s^{(h)}_{1,\sigma,L,\beta}(\mathbf{x})$ and $s^{(h)}_{2,L,\beta}(\mathbf{x})$ have at least one vertex of type ν or δ, whose running constants satisfy (2.17) and (2.57).

Note that $s^{(h)}_{1,\sigma,L,\beta}(\mathbf{x})$ and $s^{(h)}_{2,L,\beta}(\mathbf{x})$ are not even functions of $\mathbf{x}$ and that $s^{(h)}_{1,\sigma,L,\beta}(\mathbf{x})$ is not independent of σ.

In order to complete the proof of Theorem 3.8, we observe that all the functions appearing in the r.h.s. of (3.39), as well as those defined in (3.44), clearly converge, as $L,\beta\to\infty$, and that their limits can be represented in the same way as the finite L and β quantities, by substituting all the propagators with the corresponding limits. This follows from the tree structure of our expansions and some straightforward but lengthy standard arguments; we shall omit the details.

Let us consider, in particular, the limits $G^{(h)}_i(\mathbf{x})$ of the functions $G^{(h)}_{i,L,\beta}(\mathbf{x})$. Their tree expansions contain only trees with endpoints of scale $h_v\le 1$, which are associated with local terms of type λ or of the form (3.13) and (3.14), whose ψ fields are of scale less or equal to 0. The support properties of the field Fourier transform imply that the local terms of type λ can be rewritten by substituting the sum over the corresponding lattice space point with a continuous integral over $\mathbb{R}^1$. We can of course use these new expressions to build the expansions, since the propagators of scale $h\le 0$, in the limit $L,\beta\to\infty$, are well defined smooth functions on $\mathbb{R}^2$. For the same reason, the tree expansions are well defined also if the space points associated with the special endpoints vary over $\mathbb{R}^1$, instead of $\mathbb{Z}^1$; therefore there is a natural way to extend to $\mathbb{R}^2$ the functions $G^{(h)}_i(\mathbf{x})$, which of course satisfy the bound (3.47), with the continuous derivative ∂_1 in place of the discrete one and $|\mathbf{x}|$ in place of $|\mathbf{d}(\mathbf{x})|$, as well as the analogous of identity (3.45).

The function $G^{(h)}_{1,L,\beta}(\mathbf{x})$ satisfies also another symmetry relation, related with a remarkable property of the propagators $\bar g^{(h)}_{\omega,\omega'}$, see (3.61), appearing in its expansion, that is

$$\begin{aligned} \bar g^{(h)}_{\omega,\omega}(x,x_0) &= -i\omega \bar g^{(h)}_{-\omega,-\omega}\left(v_0^* x_0, \frac{x}{v_0^*}\right) ,\\ \bar g^{(h)}_{\omega,-\omega}(x,x_0) &= -\bar g^{(h)}_{-\omega,+\omega}\left(v_0^* x_0, \frac{x}{v_0^*}\right) . \end{aligned} \tag{3.65}$$

On the other hand, each tree contributing to $G^{(h)}_{1,L,\beta}(\mathbf{x})$ with n normal endpoints (which are all of type λ) can be written as a sum of Feynman graphs (if we use the representation of the regularization operator as acting on the kernels, see §**I3**), built by using $4n+4$ ψ fields, $2n+2$ with $\omega=+1$ and $2n+2$ with $\omega=-1$, hence containing the same number of propagators $\bar g^{(h)}_{+1,+1}$ and $\bar g^{(h)}_{-1,-1}$ and, by the argument used in the proof of (3.45), an even number of non diagonal propagators. Then, by using (3.65), we can easily show that the value of any graph, calculated at (x, x_0), is equal to the value at $(v_0^* x_0, x/v_0^*)$ of the graph with the same structure but opposite values for the ω-indices of all propagators, which implies (3.49).

4. Proof of Theorem I1.5

4.1 Theorem **I**3.12 and the analysis performed in §2 and §3 imply immediately the statements in item a) of Theorem **I**1.5, except the continuity of $\Omega^3_{L,\beta}(\mathbf{x})$ in $x_0 = 0$, which will be briefly discussed below. Hence, from now on we shall suppose that all parameters are chosen as in item a).

Let us define

$$\eta = \log_\gamma(1+z^*) \;, \qquad z^* = z_{[h^*/2]} \;, \tag{4.1}$$

z_h being defined as in (2.2). The analysis performed in §2 allows to show (we omit the details) that there exists a positive $\vartheta < 1$, such that

$$|z_h - z_{h+1}| \le C\lambda_1^2[\gamma^{-\vartheta(h-h^*)} + \gamma^{\vartheta h}] \;, \quad h^* \le h \le 0 \;. \tag{4.2}$$

We can write

$$\log_\gamma Z_h = \sum_{h'=h+1}^{0} \log_\gamma[1+z^* + (z_{h'} - z^*)] = -\eta h + \sum_{h'=h+1}^{0} r_{h'} \;. \tag{4.3}$$

On the other hand, if $h > [h^*/2]$, thanks to (4.2), $|r_h| \le C\sum_{h'=[h^*/2]}^{h-1} |z_{h'} - z_{h'+1}| \le C\lambda_1^2\gamma^{\vartheta h}$ and, if $h \le [h^*/2]$, $|r_h| \le C\lambda_1^2\gamma^{-\vartheta(h-h^*)}$; it follows that

$$|r_h| \le C\lambda_1^2[\gamma^{-\vartheta(h-h^*)} + \gamma^{\vartheta h}] \;. \tag{4.4}$$

Hence, if we define

$$c_h = \frac{\gamma^{-\eta h}}{Z_{h-1}} \;, \tag{4.5}$$

we get immediately the bound

$$|c_h - 1| \le C\lambda_1^2 \;. \tag{4.6}$$

In a similar way, if we define

$$\tilde\eta_1 = \log_\gamma(1+z^{(1)}_{[h^*/2]}) \;, \quad c^{(1)}_h = \frac{\gamma^{-\tilde\eta_1 h}}{Z^{(1)}_h} \;, \tag{4.7}$$

$z^{(1)}_h$ being defined by (3.18), we get the bound

$$|c^{(1)}_h - 1| \le C|\lambda_1| \;. \tag{4.8}$$

Bounds similar to (4.7) and (4.8) are valid also for the constants $Z^{(2)}_h$, but in this case Theorem 3.6 implies a stronger result; if we define

$$c^{(2)}_h = \frac{Z^{(2)}_h}{Z_{h-1}} \;, \tag{4.9}$$

then

$$|c^{(2)}_h - 1| \le C|\lambda_1| \;. \tag{4.10}$$

Let us now consider the terms in the first three lines of the r.h.s. of (3.39) and let us call $\Omega^{3,0}_{L,\beta}$ their sum; we can write

$$\Omega^{3,0}_{L,\beta}(\mathbf{x}) = \bar\Omega^{3,0}_{L,\beta}(\mathbf{x}) + \delta\Omega^{3,0}_{L,\beta}(\mathbf{x}) \;, \tag{4.11}$$

where $\bar\Omega^{3,0}_{L,\beta}$ is obtained from $\Omega^{3,0}_{L,\beta}$ by restricting the sums over h and h' to the values ≤ 0 and by substituting the propagators $g^{(h)}_{\omega,\omega'}$ with the propagators $\bar g^{(h)}_{\omega,\omega'}$, defined in (3.61). By using the symmetry relations

$$\begin{aligned} \bar g^{(h)}_{\omega,\omega}(x,x_0) &= -\bar g^{(h)}_{\omega,\omega}(-x,-x_0) = \bar g^{(h)}_{+,+}(\omega x, x_0) \;, \\ \bar g^{(h)}_{\omega,-\omega}(\mathbf{x}) &= \bar g^{(h)}_{\omega,-\omega}(-\mathbf{x}) = \omega \bar g^{(h)}_{+,-}(\mathbf{x}) \;, \end{aligned} \tag{4.12}$$

it is easy to show that we can write

$$\bar\Omega^{3,0}_{L,\beta}(\mathbf{x}) = \cos(2p_F x)\bar\Omega_{1,L,\beta}(\mathbf{x}) + \bar\Omega_{2,L,\beta}(\mathbf{x}) \;, \tag{4.13}$$

$$\begin{aligned} \bar\Omega_{1,L,\beta}(\mathbf{x}) = 2 \sum_{h^*\le h,h'\le 0} \frac{(Z^{(1)}_{h\vee h'})^2}{Z_{h-1}Z_{h'-1}} \Big[&\bar g^{(h)}_{+,+}(x,x_0)\bar g^{(h')}_{+,+}(-x,x_0) + \\ &+ \bar g^{(h)}_{+,-}(x,x_0)\bar g^{(h')}_{+,-}(x,x_0)\Big] \;, \end{aligned} \tag{4.14}$$

$$\begin{aligned} \bar\Omega_{2,L,\beta}(\mathbf{x}) = \sum_{h,h'\le 0} \frac{(Z^{(2)}_{h\vee h'})^2}{Z_{h-1}Z_{h'-1}} \Big[&\sum_\omega \bar g^{(h)}_{+,+}(\omega x,x_0)\bar g^{(h')}_{+,+}(\omega x,x_0) - \\ &- 2\bar g^{(h)}_{+,-}(x,x_0)\bar g^{(h')}_{+,-}(x,x_0)\Big] \;. \end{aligned} \tag{4.15}$$

By using (3.39), (3.44), (4.13) and the fact that $G^{(+1)}_{i,L,\beta}(\mathbf{x}) - r^{(+1)}_{i,L,\beta}(\mathbf{x}) = 0$ for $i = 1,2$, we can decompose $\Omega^3_{L,\beta}$ as in (**I**1.13), by defining

$$\Omega^{3,a}_{L,\beta}(\mathbf{x}) = \bar\Omega_{1,L,\beta}(\mathbf{x}) + \sum_{h=h^*}^{0} \left(\frac{Z^{(1)}_h}{Z_h}\right)^2 \bar G^{(h)}_{1,L,\beta}(\mathbf{x}) \;, \tag{4.16}$$

$$\Omega^{3,b}_{L,\beta}(\mathbf{x}) = \bar\Omega_{2,L,\beta}(\mathbf{x}) + \sum_{h=h^*}^{0} \left(\frac{Z^{(2)}_h}{Z_h}\right)^2 \bar G^{(h)}_{2,L,\beta}(\mathbf{x}) \;, \tag{4.17}$$

$$\begin{aligned} \Omega^{3,c}_{L,\beta}(\mathbf{x}) = \delta\Omega^{3,0}_{L,\beta}(\mathbf{x}) + \sum_{h=h^*}^{1} \Bigg\{ &\left(\frac{Z^{(1)}_h}{Z_h}\right)^2 r^{(h)}_{1,L,\beta}(\mathbf{x}) + \left(\frac{Z^{(2)}_h}{Z_h}\right)^2 r^{(h)}_{2,L,\beta}(\mathbf{x}) + \\ &+ \frac{Z^{(1)}_h Z^{(2)}_h}{Z_h^2} G^{(h)}_{3,L,\beta}(\mathbf{x}) \Bigg\} + s_{L,\beta}(\mathbf{x}) \;, \end{aligned} \tag{4.18}$$

$$s_{L,\beta}(\mathbf{x}) = \sum_{h=h^*}^{0} \left\{ \sum_{\sigma=\pm1} e^{2i\sigma p_F x} \left(\frac{Z^{(1)}_h}{Z_h}\right)^2 s^{(h)}_{1,\sigma,L,\beta}(\mathbf{x}) + \left(\frac{Z^{(2)}_h}{Z_h}\right)^2 s^{(h)}_{2,L,\beta}(\mathbf{x}) \right\} \;. \tag{4.19}$$

Theorem 3.8 implies that $\Omega^{3,a}_{L,\beta}(\mathbf{x})$, $\Omega^{3,b}_{L,\beta}(\mathbf{x})$ and $s_{L,\beta}(\mathbf{x})$ are smooth functions of x_0, essentially because their expansions do not contain any graph with a propagator of scale $+1$ (this propagator has a discontinuity at $x_0 = 0$). The function $\Omega^{3,c}_{L,\beta}(\mathbf{x})$ is not differentiable at $x_0 = 0$, but it is in any case continuous, since all graphs contributing to it have a Fourier transform decaying at least as k_0^{-2} as $k_0 \to \infty$.

4.2 We want now to prove the bounds in item b) of Theorem **I**1.5. To start with, we consider the function $\bar\Omega_{1,L,\beta}(\mathbf{x})$ defined in (4.14) and note that it can be written in the form

$$\bar\Omega_{1,L,\beta}(\mathbf{x}) = \sum_{h=h^*}^{0} \left(\frac{Z_h^{(1)}}{Z_h} \right)^2 \bar\Omega_{1,L,\beta}^{(h)}(\mathbf{x}) \; , \tag{4.20}$$

with $\bar\Omega^{(h)}_{1,L,\beta}(\mathbf{x})$ satisfying a bound similar to that proved for $\bar G^{(h)}_{1,L,\beta}(\mathbf{x})$, see (3.47), that is

$$|D_{m_0,m_1} \bar\Omega^{(h)}_{1,L,\beta}(\mathbf{x})| \le C_{N,m_0,m_1} \frac{\gamma^{2h}\gamma^{h(m_0+m_1)}}{1+[\gamma^h|\mathbf{d}(\mathbf{x})|]^N} \; . \tag{4.21}$$

This claim easily follows from Lemma **I**2.6, together with (4.5) and (4.6). Hence we can write, by using (3.47), (4.6), (4.8) and (4.21), given any positive integers n_0, n_1 and putting $n = n_0 + n_1$,

$$\begin{aligned} |\partial_{x_0}^{n_0} \bar\partial_x^{n_1} \Omega^{3,a}_{L,\beta}(\mathbf{x})| &\le C_{N,n} \sum_{h=h^*}^{0} \frac{\gamma^{(2+2\eta_1+n)h}}{[1+(\gamma^h|\mathbf{d}(\mathbf{x})|)^N]} \le \\ &\le \frac{C_{N,n}}{|\mathbf{d}(\mathbf{x})|^{2+2\eta_1+n}} H_{N,2+2\eta_1+n}(|\mathbf{d}(\mathbf{x})|) \; , \end{aligned} \tag{4.22}$$

where

$$\eta_1 = \eta - \tilde\eta_1 \; , \tag{4.23}$$

$$H_{N,\alpha}(r) = \sum_{h=h^*}^{0} \frac{(\gamma^h r)^\alpha}{1+(\gamma^h r)^N} \; . \tag{4.24}$$

By using the second of the definitions (**I**2.2), the definition (2.8) and the bounds (2.16), (3.29), one can see that the constant η_1 can be represented as in (**I**1.14).

On the other hand, it is easy to see that, if $\alpha \ge 1/2$ and $N-\alpha \ge 1$, there exists a constant $C_{N,\alpha}$ such that

$$H_{N,\alpha}(r) \le \frac{C_{N,\alpha}}{1+(\Delta r)^{N-\alpha}} \; , \quad \Delta = \gamma^{h^*} \; . \tag{4.25}$$

The definition (**I**2.40), the first of definitions (**I**2.33), the second bound in (**I**2.34) and the bound (2.56) easily imply that Δ can be represented as in (**I**1.19), with η_2 satisfying the second of equations (**I**1.14).

By using (4.22) and (4.25), one immediately gets the bound (**I**1.16). A similar procedure allows to get also the bound (**I**1.17), by using (4.10).

Let us now consider $\Omega^{3,c}_{L,\beta}(\mathbf{x})$. By using (3.43) and (3.46), as well as the remark that one gains a factor γ^h in the bound of $g^{(h)}_{\omega,\omega'}(\mathbf{x}) - \bar g^{(h)}_{\omega,\omega'}(\mathbf{x})$ with respect to the bound of $\bar g^{(h)}_{\omega,\omega'}(\mathbf{x})$, we get

$$|\Omega^{3,c}_{L,\beta}(\mathbf{x}) - s_{L,\beta}(\mathbf{x})| \le \frac{C_N}{|\mathbf{d}(\mathbf{x})|^2} \left[\frac{H_{N,2+2\eta_1+\vartheta}(|\mathbf{d}(\mathbf{x})|)}{|\mathbf{d}(\mathbf{x})|^{\vartheta+2\eta_1}} + \frac{H_{N,2+\vartheta}(|\mathbf{d}(\mathbf{x})|)}{|\mathbf{d}(\mathbf{x})|^{\vartheta}} \right] \; , \tag{4.26}$$

for some positive $\vartheta < 1$.

The bound of $s_{L,\beta}(\mathbf{x})$ is slightly different, because of the $\gamma^{-\vartheta(h-h^*)}$ in the r.h.s. of (3.48). We get, in addition to a term of the same form as the r.h.s. of (4.26), another term of the form

$$\frac{C_N}{|\mathbf{d}(\mathbf{x})|^2}(\Delta|\mathbf{d}(\mathbf{x})|)^\vartheta \left[\frac{H_{N,2+2\eta_1-\vartheta}(|\mathbf{d}(\mathbf{x})|)}{|\mathbf{d}(\mathbf{x})|^{2\eta_1}} + H_{N,2-\vartheta}(|\mathbf{d}(\mathbf{x})|)\right] . \tag{4.27}$$

The bounds (4.26) and (4.27) immediately imply (**I**1.18), if λ is so small that, for example, $2|\eta_1| \le \vartheta/2$.

4.3 We want now to prove the statements in item c) of Theorem **I**1.5. The existence of the limit as $L,\beta \to \infty$ of all functions follows from Theorem 3.8. The claim that $\Omega^{3,a}(\mathbf{x})$ and $\Omega^{3,b}(\mathbf{x})$ are even as functions of $\mathbf{x}$ follows from (3.45) and (4.14)-(4.18). Moreover $\Omega^{3,a}(\mathbf{x})$ and $\Omega^{3,b}(\mathbf{x})$ are the restriction to $\mathbb{Z}\times\mathbb{R}$ of two functions on $\mathbb{R}^2$, that we shall denote by the same symbols, and $\Omega^{3,a}(\mathbf{x})$ satisfies the symmetry relation (**I**1.22), since this is true for $\lim_{L,\beta\to\infty}\bar\Omega_{1,L,\beta}(\mathbf{x})$, as it is easy to check by using (3.65), and for $\bar G_1^{(h)}(\mathbf{x})$, see (3.49).

In order to prove (**I**1.20), we suppose that $|\mathbf{x}|\ge 1$ and we put $\bar\Omega_i(\mathbf{x}) = \lim_{L,\beta\to\infty}\bar\Omega_{i,L,\beta}(\mathbf{x})$; then we define $\tilde\Omega_i(\mathbf{x})$, $i=1,2$, as the functions which are obtained by making in the r.h.s. of (4.14) and (4.15), evaluated in the limit $L,\beta\to\infty$, the substitutions

$$\frac{(Z^{(1)}_{h\vee h'})^2}{Z_{h-1}Z_{h'-1}} \to [x^2+(v_0^*x_0)^2]^{-\eta_1} , \qquad \frac{(Z^{(2)}_{h\vee h'})^2}{Z_{h-1}Z_{h'-1}} \to 1 . \tag{4.28}$$

Note that the choice of $x^2+(v_0^*x_0)^2$, instead of $x^2+x_0^2$, which is equivalent for what concerns the following arguments, was done only in order to have a function $\tilde\Omega_1(\mathbf{x})$ satisfying the same symmetry relation as $\bar\Omega_1(\mathbf{x})$ in the exchange of (x,x_0) with $(v_0^*x_0, x/v_0^*)$.

It is easy to see that

$$\begin{aligned}|\bar\Omega_1(\mathbf{x})-\tilde\Omega_1(\mathbf{x})| \le \frac{C_N}{|\mathbf{x}|^{2+2\eta_1}} \sum_{h^*\le h,h'\le 0} \frac{\gamma^h|\mathbf{x}|}{1+(\gamma^h|\mathbf{x}|)^N}\frac{\gamma^{h'}|\mathbf{x}|}{1+(\gamma^{h'}|\mathbf{x}|)^N}\cdot \\ \cdot\left|\left(\frac{x^2+x_0^2}{x^2+(v_0^*x_0)^2}\right)^{\eta_1}(\gamma^h|\mathbf{x}|)^\eta(\gamma^{h'}|\mathbf{x}|)^\eta(\gamma^{h\vee h'}|\mathbf{x}|)^{-2\tilde\eta_1}\frac{c_hc_{h'}}{(c^{(1)}_{h\vee h'})^2}-1\right| .\end{aligned} \tag{4.29}$$

Note that, if $r>0$ and $\alpha\in\mathbb{R}$

$$|r^\alpha - 1| \le |\alpha\log r|\left(r^\alpha + r^{-\alpha}\right) ; \tag{4.30}$$

Hence, by using (4.6), (4.8), (4.25) and (**I**1.14), we get

$$|\bar\Omega_1(\mathbf{x})-\tilde\Omega_1(\mathbf{x})| \le \frac{|J_3|}{|\mathbf{x}|^{2+2\eta_1}}\frac{C_N}{1+(\Delta|\mathbf{x}|)^N} . \tag{4.31}$$

In the same way, one can show that

$$|\bar\Omega_2(\mathbf{x})-\tilde\Omega_2(\mathbf{x})| \le \frac{|J_3|}{|\mathbf{x}|^2}\frac{C_N}{1+(\Delta|\mathbf{x}|)^N} . \tag{4.32}$$

Let us now define

$$\Omega_1^*(\mathbf{x}) = \frac{2}{[x^2+(v_0^*x_0)^2]^{\eta_1}}\frac{1}{(v_0^*)^2}g_{\mathcal{L}}(x/v_0^*,x_0)g_{\mathcal{L}}(-x/v_0^*,x_0) , \tag{4.33}$$

$$\Omega_2^*(\mathbf{x}) = \frac{1}{(v_0^*)^2} \sum_{\omega=\pm 1} g_{\mathcal{L}}(\omega x/v_0^*, x_0) g_{\mathcal{L}}(\omega x/v_0^*, x_0) \;, \tag{4.34}$$

where

$$g_{\mathcal{L}}(\mathbf{x}) = \frac{1}{(2\pi)^2} \int d\mathbf{k} e^{i\mathbf{k}\mathbf{x}} \frac{\chi_0(\mathbf{k})}{-ik_0 + k} \;, \tag{4.35}$$

$\chi_0(\mathbf{k})$ being a smooth function of $\mathbf{k}$, which is equal to 1, if $|\mathbf{k}| \le t_0$, and equal to 0, if $|\mathbf{k}| \ge \gamma t_0$ (see §**I**2.3 for the definition of t_0).

It is easy to check that $\Omega_i^*(\mathbf{x})$, $i = 1, 2$, is obtained from $\tilde\Omega_i(\mathbf{x})$ by making in the $L, \beta = \infty$ expression of the propagators $\bar g^{(h)}_{\omega,\omega'}(\mathbf{x})$, which are evaluated from (**I**2.92), if $h^* < h \le 0$, and (**I**2.120), if $h = h^*$, the following substitutions:

$$\sigma_{h-1}(\mathbf{k}') \to 0 \;, \qquad \tilde f_h(\mathbf{k}') \to f_h(\mathbf{k}') \;. \tag{4.36}$$

Hence, by using also the remark that, by (**I**2.116) and (2.54), $|\sigma_h/\gamma^h| \le C\gamma^{-(h-h^*)/2}$, it is easy to show that

$$|\Omega_1^*(\mathbf{x}) - \tilde\Omega_1(\mathbf{x})| \le \frac{C_N}{|\mathbf{x}|^{2+2\eta_1}} H_{N,1}(\Delta|\mathbf{x}|) \left[\lambda_1^2 H_{N,1}(\Delta|\mathbf{x}|) + (\Delta|\mathbf{x}|)^{1/2} H_{N,1/2}(\Delta|\mathbf{x}|)\right] \;. \tag{4.37}$$

In a similar way, one can show also that

$$|\Omega_2^*(\mathbf{x}) - \tilde\Omega_2(\mathbf{x})| \le \frac{C_N}{|\mathbf{x}|^2} H_{N,1}(\Delta|\mathbf{x}|) \left[\lambda_1^2 H_{N,1}(\Delta|\mathbf{x}|) + (\Delta|\mathbf{x}|)^{1/2} H_{N,1/2}(\Delta|\mathbf{x}|)\right] \;. \tag{4.38}$$

Moreover, by an explicit calculation, one finds that, if $|\mathbf{x}| \ge 1$,

$$g_{\mathcal{L}}(\mathbf{x}) = \frac{x_0 - ix}{2\pi|\mathbf{x}|^2} F(\mathbf{x}) \;, \tag{4.39}$$

where $F(\mathbf{x})$ is a smooth function of $\mathbf{x}$, satisfying the bound

$$|F(\mathbf{x}) - 1| \le \frac{C_N}{1 + |\mathbf{x}|^N} \;. \tag{4.40}$$

The bounds (4.31) and (4.32), the similar bounds satisfied by $|\Omega^{3,a}(\mathbf{x}) - \bar\Omega_1(\mathbf{x})|$ and $|\Omega^{3,b}(\mathbf{x}) - \bar\Omega_2(\mathbf{x})|$ and the equations (4.37)-(4.40) allow to prove very easily (**I**1.20) and (**I**1.21).

4.4 We still have to prove the statements in items d) and e) of Theorem **I**1.5. By using **I**(1.13), (4.18) and (4.19), we see that

$$\begin{aligned} \hat\Omega^3(\mathbf{k}) = \sum_{\sigma=\pm 1} &\left[\frac{1}{2}\hat\Omega^{3,a}(k + 2\sigma p_F, k_0) + \hat s_{1,\sigma}(k + 2\sigma p_F, k_0)\right] + \\ &+ \hat\Omega^{3,b}(\mathbf{k}) + \hat s_2(\mathbf{k}) + \delta\hat\Omega^{3,c}(\mathbf{k}) \;, \end{aligned} \tag{4.41}$$

where we used the definitions

$$s_{1,\sigma}(\mathbf{x}) = \sum_{h=h^*}^{0} \left(\frac{Z_h^{(1)}}{Z_h}\right)^2 s_{1,\sigma}^{(h)}(\mathbf{x}) \;, \quad s_2(\mathbf{x}) = \sum_{h=h^*}^{0} \left(\frac{Z_h^{(2)}}{Z_h}\right)^2 s_2(h)(\mathbf{x}) \;, \tag{4.42}$$

$$\delta\Omega^{3,c}(\mathbf{x}) = \Omega^{3,c}(\mathbf{x}) - s(\mathbf{x}) \;. \tag{4.43}$$

Since any graph contributing to the expansion of $\Omega^{3,a}(\mathbf{x}-\mathbf{y})$ has only two propagators of scale ≤ 0 connected to $\mathbf{x}$ or $\mathbf{y}$, $\hat\Omega^{3,a}(\mathbf{k})$ has support on a set of value of $\mathbf{k}$ such that $|k| \le 2\gamma t_0 < \pi$; hence we can calculate $\hat\Omega^{3,a}(\mathbf{k})$ by thinking $\Omega^{3,a}(\mathbf{x})$ as a function on $\mathbb{R}^2$. Let us suppose that $|\mathbf{k}| > 0$ and $|k| \ge |\mathbf{k}|/2$; then

$$\hat\Omega^{3,a}(\mathbf{k}) = \int d\mathbf{x} e^{i\mathbf{k}\mathbf{x}} \Omega^{3,a}(\mathbf{x}) = \frac{i}{k}\int d\mathbf{x}\left[e^{i\mathbf{k}\mathbf{x}} - 1\right]\partial_x \Omega^{3,a}(\mathbf{x}) \;, \tag{4.44}$$

since $\Omega^{3,a}(\mathbf{x})$, by (**I**1.16), is a smooth function of fast decrease as $|\mathbf{x}| \to \infty$. If $|k| < |\mathbf{k}|/2$, it has to be true that $|k_0| \ge |\mathbf{k}|/2$ and we write a similar identity, with k_0 in place of k and ∂_{x_0} in place of ∂_x. In both case we can write, by using (**I**1.16),

$$|\hat\Omega^{3,a}(\mathbf{k})| \le \frac{C}{|\mathbf{k}|}\int_{|\mathbf{x}|\ge|\mathbf{k}|^{-1}} \frac{d\mathbf{x}}{1+|\mathbf{x}|^{3+2\eta_1}} + C\int_{|\mathbf{x}|\le|\mathbf{k}|^{-1}} d\mathbf{x}\frac{|\mathbf{x}|}{1+|\mathbf{x}|^{3+2\eta_1}} \;. \tag{4.45}$$

A even better bound can be proved for $|\hat s_{1,\sigma}(\mathbf{k})|$, $\sigma = \pm 1$, by using (3.48). Hence, uniformly for $u \to 0$, $|\hat\Omega^{3,a}(\mathbf{k})| + |\hat s_{1,\sigma}(\mathbf{k})| \le C|\mathbf{k}|^{-1}$ for $|\mathbf{k}| \ge 1$ and

$$\frac{1}{2}|\hat\Omega^{3,a}(\mathbf{k})| + |\hat s_{1,\sigma}(\mathbf{k})| \le C\left[1 + \frac{1-|\mathbf{k}|^{2\eta_1}}{2\eta_1}\right] \;, \quad 0 < |\mathbf{k}| \le 1 \;. \tag{4.46}$$

This bound is divergent for $|\mathbf{k}| \to 0$, if $\eta_1 < 0$, that is if $J_3 < 0$; however, if $u \ne 0$ and $|\mathbf{k}| \le \Delta$, we easily get from (**I**1.16) (with $n = 0$) the better bound

$$\frac{1}{2}|\hat\Omega^{3,a}(\mathbf{k})| + |\hat s_{1,\sigma}(\mathbf{k})| \le C\left[1 + \frac{1-\Delta^{2\eta_1}}{2\eta_1}\right] \;. \tag{4.47}$$

In a similar way, by using (**I**1.17), one can prove that

$$|\hat\Omega^{3,b}(\mathbf{k})| + |\hat s_2(\mathbf{k})| \le C\left[1 + \log|\mathbf{k}|^{-1}\right] \;, \quad 0 < |\mathbf{k}| \le 1 \;, \tag{4.48}$$

$$|\hat\Omega^{3,b}(\mathbf{k})| + |\hat s_2(\mathbf{k})| \le C\left[1 + \log\Delta^{-1}\right] \;. \tag{4.49}$$

However, a more careful analysis of the Fourier transform of the leading contribution to $\Omega^{3,b}(\mathbf{x})$, given by $\Omega_2^*(\mathbf{x})$ (see (4.34)), which takes into account the oddness in the exchange $(x, x_0) \to (x_0 v_0^*, x/v_0^*)$, shows that $|\hat\Omega_2^*(\mathbf{k})| \le C$. One can show that a similar bound is satisfied by the Fourier transform of the terms contributing to $\tilde\Omega_2(\mathbf{x})$ and proportional to σ_h/γ^h. Therefore, in the bounds (4.48) and (4.49), we can multiply by J_3 both $\log|\mathbf{k}|^{-1}$ and $\log\Delta^{-1}$.

Let us now consider $\delta\hat\Omega^{3,c}(\mathbf{k})$. By using (4.26), we see immediately that, uniformly in $\mathbf{k}$ and u,

$$|\delta\hat\Omega^{3,c}(\mathbf{k}) - \hat s(\mathbf{k})| \le C \;. \tag{4.50}$$

The bounds (4.46)-(4.50), together with the positivity of the leading term in (**I**1.20) and the remark after (4.49), immediately imply all the claims in item d) of Theorem **I**1.5.

Let us now consider $G(x) \equiv \Omega^3(x, 0)$, $x \in \mathbb{Z}$. It is easy to see, by using the previous results and the fact that also $s_{1,\sigma}(\mathbf{x})$ and $s_2(\mathbf{x})$ are even functions of $\mathbf{x}$, that $G(x)$ can be written in the form

$$G(x) = \sum_{\sigma=\pm1} e^{2i\sigma p_F x} G_{1,\sigma}(x) + G_2(x) + \delta G(x) \;, \tag{4.51}$$

where $G_{1,\sigma}(x)$ and $G_2(x)$ are the restrictions to $\mathbb{Z}$ of some even smooth functions on $\mathbb{R}$, satisfying, for any integers $n, N \ge 0$, the bounds

$$|\partial_x^n G_{1,\sigma}(x)| \le \frac{C_{n,N}}{[1+|x|^{2+n+2\eta_1}][1+(\Delta|x|)^N]} , \tag{4.52}$$

$$|\partial_x^n G_2(x)| \le \frac{C_{n,N}}{1+|x|^{2+n}[1+(\Delta|x|)^N]} , \tag{4.53}$$

while $\delta G(x)$ satisfies the bound

$$|\delta G(x)| \le \frac{C}{[1+|x|^{2+\vartheta}][1+(\Delta|x|)^N]} , \tag{4.54}$$

with some $\vartheta > 0$.

These properties immediately imply that, uniformly in k and u,

$$|\hat{G}(k)| + |\partial_k \delta\hat{G}(k)| \le C . \tag{4.55}$$

Let us now consider $\partial_k \hat{G}_{1,\sigma}(k)$ and note that, if $|k| > 0$,

$$\begin{aligned} \partial_k \hat{G}_{1,\sigma}(k) &= -\frac{1}{k}\int dx[e^{ikx}-1]\partial_x[xG_{1,\sigma}(x)] = \\ &= -\frac{1}{k}\int_{|x|\ge|k|^{-1}} dx[e^{ikx}-1]\partial_x[xG_{1,\sigma}(x)] - \\ &\quad -\frac{1}{k}\int_{|x|\le|k|^{-1}} dx[e^{ikx}-1-ikx]\partial_x[xG_{1,\sigma}(x)] , \end{aligned} \tag{4.56}$$

where we used the fact that $\partial_x[xG_{1,\sigma}(x)]$ is an even function of x, since $G_{1,\sigma}(x)$ is even, see (3.45). Hence, if $|k| \ge 1$, $|\partial_k \hat{G}_{1,\sigma}(k)| \le C|k|^{-1}$, while, if $0 < |k| \le 1$, uniformly in u,

$$|\partial_k \hat{G}_{1,\sigma}(k)| \le C[1+|k|^{2\eta_1}] . \tag{4.57}$$

In a similar way, we can prove that, uniformly in k and u,

$$|\partial_k \hat{G}_2(k)| \le C . \tag{4.58}$$

The bound (4.57) is divergent for $k \to 0$, if $J_3 < 0$; however, if $|u| > 0$ and $|k| \le \Delta$, one can get a better bound, by using the identity

$$\partial_k \hat{G}_{1,\sigma}(k) = i\int_{|x|\ge\Delta^{-1}} dx e^{ikx}[xG_{1,\sigma}(x)] + i\int_{|x|\le\Delta^{-1}} dx[e^{ikx}-1][xG_{1,\sigma}(x)] , \tag{4.59}$$

together with (4.52). One finds

$$|\partial_k \hat{G}_{1,\sigma}(k)| \le C[1+\Delta^{2\eta_1}] . \tag{4.60}$$

The bounds (4.55), (4.58) and (4.60), together with the identity (4.51), imply (**I**1.24). The statements about the discontinuities of $\partial_k \hat{G}(k)$ at $u = 0$ and $k = 0, \pm 2p_F$ follow from an explicit calculation involving the leading contribution, obtained by putting $A_1(\mathbf{x}) = A_2(\mathbf{x}) = 0$ in (**I**1.20).

5. Proof of the approximate Ward identity (3.35)

5.1 In this section we prove the relation (3.35) between the quantities $Z_h^{(L)}$ and $Z_h^{(2,L)}$, related to the approximate Luttinger model defined by (3.30) and (3.31).

First of all, we move from the interaction to the free measure (2.30) the term proportional to $\delta_0^{(L)}$ and we redefine correspondingly the interaction. This can be realized by slightly changing the free measure normalization (which has no effect on the problem we are studying), by putting $\delta_0^{(L)} = 0$ in (2.31) and by substituting, in (2.30), v_0^* with $\bar v_0(\mathbf{k}') = v_0^* + \delta_0^{(L)} C_0^{-1}(\mathbf{k}')$. However, since $C_0^{-1}(\mathbf{k}') = 1$ on all scales $h<0$, $Z_h^{(2,L)}$ and $Z_h^{(L)}$ may be modified only by a factor $\gamma^{C|\lambda_0^{(L)}|}$, if we substitute $\bar v_0(\mathbf{k}')$ with $\bar v_0 \equiv \bar v_0(\mathbf{0})$. It follows that it is sufficient to prove the bound (3.35) for the corresponding free measure

$$
\begin{aligned}
P^{(L)}(d\psi^{(\le 0)}) = & \prod_{\mathbf{k}': C_0^{-1}(\mathbf{k}')>0} \prod_{\omega=\pm 1} \frac{d\hat\psi^{(\le 0)+}_{\mathbf{k}',\omega} d\hat\psi^{(\le 0)-}_{\mathbf{k}',\omega}}{\mathcal{N}_L(\mathbf{k}')} \cdot \\
& \cdot \exp\left\{ -\frac{1}{L\beta} \sum_{\omega=\pm1} \sum_{\mathbf{k}': C_0^{-1}(\mathbf{k}')>0} C_0(\mathbf{k}')\big(-ik_0 + \omega \bar v_0 k'\big) \hat\psi^{(\le 0)+}_{\mathbf{k}',\omega} \hat\psi^{(\le 0)-}_{\mathbf{k}',\omega} \right\} ,
\end{aligned} \tag{5.1}
$$

by using as interaction the function

$$
V^{(L)}(\psi^{(\le 0)}) = \lambda_0^{(L)} \int_{\mathbb{T}_{L,\beta}} d\mathbf{x}\, \psi^{(\le 0)+}_{\mathbf{x},+1} \psi^{(\le 0)-}_{\mathbf{x},-1} \psi^{(\le 0)+}_{\mathbf{x},-1} \psi^{(\le 0)-}_{\mathbf{x},+1} . \tag{5.2}
$$

Let us consider, instead of the free measure (5.1), the corresponding measure with *infrared cutoff on scale h*, $h \le 0$, given by

$$
\begin{aligned}
P^{(L,h)}(d\psi^{[h,0]}) = & \prod_{\mathbf{k}': C_{h,0}^{-1}(\mathbf{k}')>0} \prod_{\omega=\pm 1} \frac{d\hat\psi^{[h,0]+}_{\mathbf{k}',\omega} d\hat\psi^{[h,0]-}_{\mathbf{k}',\omega}}{\mathcal{N}_L(\mathbf{k}')} \cdot \\
& \cdot \exp\left\{ -\frac{1}{L\beta} \sum_{\omega=\pm1} \sum_{\mathbf{k}': C_{h,0}^{-1}(\mathbf{k}')>0} C_{h,0}(\mathbf{k}')\big(-ik_0 + \omega \bar v_0 k'\big) \hat\psi^{[h,0]+}_{\mathbf{k}',\omega} \hat\psi^{[h,0]-}_{\mathbf{k}',\omega} \right\} ,
\end{aligned} \tag{5.3}
$$

where $C_{h,0}^{-1} = \sum_{k=h}^0 f_k$.

We will find convenient to write the above integration in terms of the space-time field variables; if we put

$$
\mathcal{D}\psi^{[h,0]} = \prod_{\mathbf{k}': C_{h,0}^{-1}(\mathbf{k}')>0} \prod_{\omega=\pm 1} \frac{d\hat\psi^{[h,0]+}_{\mathbf{k}',\omega} d\hat\psi^{[h,0]-}_{\mathbf{k}',\omega}}{\mathcal{N}_L(\mathbf{k}')} , \tag{5.4}
$$

we can rewrite (5.3) as

$$
P^{(L,h)}(d\psi^{[h,0]}) = \mathcal{D}\psi^{[h,0]} \exp\Big[-\sum_\omega \int_{\mathbb{T}_{L,\beta}} d\mathbf{x}\, \psi^{[h,0]+}_{\mathbf{x},\omega} D_\omega^{[h,0]} \psi^{[h,0]-}_{\mathbf{x},\omega} \Big] , \tag{5.5}
$$

where

$$
D_\omega^{[h,0]} \psi^{[h,0]\sigma}_{\mathbf{x},\omega} = \frac{1}{L\beta} \sum_{\mathbf{k}': C_{h,0}^{-1}(\mathbf{k}')>0} e^{i\sigma\mathbf{k}'\mathbf{x}} C_{h,0}(\mathbf{k}')(i\sigma k_0 - \omega\sigma \bar v_0 k') \hat\psi^{[h,0]\sigma}_{\mathbf{k}',\omega} . \tag{5.6}
$$

$D^{[h,0]}_\omega$ has to be thought as a "regularization" of the linear differential operator

$$D_\omega = \frac{\partial}{\partial x_0} + i\omega \bar v_0 \frac{\partial}{\partial x} \; . \tag{5.7}$$

Let us now introduce the external field variables $\phi^\sigma_{\mathbf{x},\omega}$, $\mathbf{x}\in\mathbb{T}_{L,\beta}$, $\omega=\pm 1$, antiperiodic in x_0 and x and anticommuting with themselves and $\psi^{[h,0]\sigma}_{\mathbf{x},\omega}$, and let us define

$$U(\phi) = -\log \int P^{(L,h)}(d\psi^{[h,0]}) e^{-V^{(L)}(\psi^{[h,0]}+\phi)} \; . \tag{5.8}$$

If we perform the *gauge transformation*

$$\psi^{[h,0]\sigma}_{\mathbf{x},\omega} \to e^{i\sigma\alpha_{\mathbf{x}}} \psi^{[h,0]\sigma}_{\mathbf{x},\omega} \; , \tag{5.9}$$

and we define $(e^{-i\alpha}\phi)^\sigma_{\mathbf{x},\omega} = e^{-i\sigma\alpha_{\mathbf{x}}}\phi^\sigma_{\mathbf{x},\omega}$, we get

$$\begin{aligned} U(\phi) = -\log \int P^{(L,h)}(d\psi^{[h,0]}) \exp\Big\{ -V^{(L)}(\psi^{[h,0]} + e^{-i\alpha}\phi) - \\ -\sum_\omega \int d\mathbf{x}\, \psi^{[h,0]+}_{\mathbf{x},\omega} \Big(e^{i\alpha_{\mathbf{x}}} D^{[h,0]}_\omega e^{-i\alpha_{\mathbf{x}}} - D^{[h,0]}_\omega \Big) \psi^{[h,0]-}_{\mathbf{x},\omega} \Big\} \; . \end{aligned} \tag{5.10}$$

Since $U(\phi)$ is independent of α, the functional derivative of the r.h.s. of (5.10) w.r.t. $\alpha_{\mathbf{x}}$ is equal to 0 for any $\mathbf{x}\in\mathbb{T}_{L,\beta}$. Hence, we find the following identity:

$$\sum_\omega \left[-\phi^+_{\mathbf{x},\omega} \frac{\partial U}{\partial \phi^+_{\mathbf{x},\omega}} + \frac{\partial U}{\partial \phi^-_{\mathbf{x},\omega}} \phi^-_{\mathbf{x},\omega} + \frac{1}{Z(\phi)} \int P^{(L,h)}(d\psi^{[h,0]}) T_{\mathbf{x},\omega}\, e^{-V^{(L)}(\psi^{[h,0]}+\phi)} \right] = 0 \; , \tag{5.11}$$

where

$$Z(\phi) = \int P^{(L,h)}(d\psi^{[h,0]}) e^{-V^{(L)}(\psi^{[h,0]}+\phi)} \; , \tag{5.12}$$

$$\begin{aligned} T_{\mathbf{x},\omega} &= \psi^{[h,0]+}_{\mathbf{x},\omega}[D^{[h,0]}_\omega \psi^{[h,0]-}_{\mathbf{x},\omega}] + [D^{[h,0]}_\omega \psi^{[h,0]+}_{\mathbf{x},\omega}]\psi^{[h,0]-}_{\mathbf{x},\omega} = \\ &= \frac{1}{(L\beta)^2} \sum_{\mathbf{p},\mathbf{k}} e^{-i\mathbf{p}\mathbf{x}} \hat\psi^{[h,0],+}_{\mathbf{k},\omega} [C_{h,0}(\mathbf{p}+\mathbf{k}) D_\omega(\mathbf{p}+\mathbf{k}) - C_{h,0}(\mathbf{k}) D_\omega(\mathbf{k})] \hat\psi^{[h,0],-}_{\mathbf{p}+\mathbf{k},\omega} \; , \end{aligned} \tag{5.13}$$

$$D_\omega(\mathbf{k}) = -ik_0 + \omega \bar v_0 k \; . \tag{5.14}$$

Moreover, the sum over $\mathbf{p}$ and $\mathbf{k}$ in (5.13) is restricted to the momenta of the form $\mathbf{p} = (2\pi n/L, 2\pi m/\beta)$ and $\mathbf{k} = (2\pi(n+1/2)/L, 2\pi(m+1/2)/\beta)$, with n and m integers, such that $|p|$, $|p_0|$, $|k_0|$, $|k|$ are all smaller or equal to π and satisfy the constraints $C^{-1}_{h,0}(\mathbf{p}+\mathbf{k}) > 0$, $C^{-1}_{h,0}(\mathbf{k}) > 0$.

Note that (5.13) can be rewritten as

$$T_{\mathbf{x},\omega} = D_\omega[\psi^{[h,0]+}_{\mathbf{x},\omega} \psi^{[h,0]-}_{\mathbf{x},\omega}] + \delta T_{\mathbf{x},\omega} \; , \tag{5.15}$$

where

$$\begin{aligned} \delta T_{\mathbf{x},\omega} = \frac{1}{(L\beta)^2} \sum_{\mathbf{p},\mathbf{k}} e^{-i\mathbf{p}\mathbf{x}} \hat\psi^{[h,0],+}_{\mathbf{k},\omega} \cdot \\ \cdot \big\{ [C_{h,0}(\mathbf{p}+\mathbf{k}) - 1] D_\omega(\mathbf{p}+\mathbf{k}) - [C_{h,0}(\mathbf{k}) - 1] D_\omega(\mathbf{k}) \big\} \hat\psi^{[h,0],-}_{\mathbf{p}+\mathbf{k},\omega} \; . \end{aligned} \tag{5.16}$$

It follows that, if $C_{h,0}$ is substituted with 1, that is if we consider the formal theory without any ultraviolet and infrared cutoff, $T_{\mathbf{x},\omega} = D_\omega[\psi^{[h,0]+}_{\mathbf{x},\omega}\psi^{[h,0]-}_{\mathbf{x},\omega}]$ and we would get the usual Ward identities. As we shall see, the presence of the cutoffs make the analysis a bit more involved and adds some corrections to the Ward identities, which however, for λ_0 small enough, can be controlled by the same type of multiscale analysis, that we used in §3.

5.2 Let us introduce a new external field $J_\mathbf{x}$, $\mathbf{x} \in \mathbb{T}_{L,\beta}$, periodic in x_0 and x and commuting with the fields ϕ^σ and $\psi^{[h,0]\sigma}$, and let us consider the functional

$$\mathcal{W}(\phi, J) = -\log \int P^{(L,h)}(d\psi^{[h,0]}) e^{-V^{(L)}(\psi^{[h,0]}+\phi)+\int d\mathbf{x} J_\mathbf{x} \sum_\omega \psi^{[h,0]+}_{\mathbf{x},\omega}\psi^{[h,0]-}_{\mathbf{x},\omega}} . \tag{5.17}$$

We also define the functions

$$\Sigma_{h,\omega}(\mathbf{x}-\mathbf{y}) = \frac{\partial^2}{\partial\phi^+_{\mathbf{x},\omega}\partial\phi^-_{\mathbf{y},\omega}} U(\phi)\Big|_{\phi=0} = \frac{\partial^2}{\partial\phi^+_{\mathbf{x},\omega}\partial\phi^-_{\mathbf{y},\omega}} \mathcal{W}(\phi,J)\Big|_{\phi=J=0} , \tag{5.18}$$

$$\Gamma_{h,\omega}(\mathbf{x};\mathbf{y},\mathbf{z}) = \frac{\partial}{\partial J_\mathbf{x}} \frac{\partial^2}{\partial\phi^+_{\mathbf{y},\omega}\partial\phi^-_{\mathbf{z},\omega}} \mathcal{W}(\phi,J)|_{\phi=J=0} . \tag{5.19}$$

These functions have here the role of the *self-energy* and the *vertex part* in the usual treatment of the Ward identities. However, they do not coincide with them, because the corresponding Feynman graphs expansions are not restricted to the one particle irreducible graphs. However, their Fourier transforms at zero external momenta, which are the interesting quantities in the limit $L,\beta\to\infty$, are the same; in fact, because of the support properties of the fermion fields, the propagators vanish at zero momentum, hence the one particle reducible graphs give no contribution at that quantities.

In the language of this paper, if we did not perform any free measure regularization, $\Sigma_{h,\omega}(\mathbf{x}-\mathbf{y})$ would coincide with the kernel of the contribution to the effective potential on scale $h-1$ with two external fields, that is the function $W^{(h-1)}_{2,(+,-),(\omega,\omega)}$ of equation (**I**3.3). Analogously, $1+\Gamma_{h,\omega}(\mathbf{x};\mathbf{y},\mathbf{z})$ would coincide with the kernel $B^{(h-1)}_{1,2,(+,-),(\omega,\omega)}$ of equation (3.6).

Note that

$$\Gamma_{h,\omega}(\mathbf{x};\mathbf{y},\mathbf{z}) = \sum_{\tilde\omega} \Gamma_{h,\omega,\tilde\omega}(\mathbf{x};\mathbf{y},\mathbf{z}) , \tag{5.20}$$

where $\Gamma_{h,\omega,\tilde\omega}(\mathbf{x};\mathbf{y},\mathbf{z})$ is defined as in (5.17), by substituting $J_\mathbf{x}\sum_\omega \psi^{[h,0]+}_{\mathbf{x},\omega}\psi^{[h,0]-}_{\mathbf{x},\omega}$ with $J_\mathbf{x}$ $\psi^{[h,0]+}_{\mathbf{x},\tilde\omega}\ \psi^{[h,0]-}_{\mathbf{x},\tilde\omega}$.

If we derive the l.h.s. of (5.11) with respect to $\phi^+_{\mathbf{y},\omega}$ and to $\phi^-_{\mathbf{z},\omega}$ and we put $\phi=0$, we get

$$\begin{aligned} 0 = &-\delta(\mathbf{x}-\mathbf{y})\Sigma_{h,\omega}(\mathbf{x}-\mathbf{z}) + \delta(\mathbf{x}-\mathbf{z})\Sigma_{h,\omega}(\mathbf{y}-\mathbf{x}) - \\ &< \left[\frac{\partial^2 V}{\partial\psi^{[h,0]+}_{\mathbf{y},\omega}\partial\psi^{[h,0]-}_{\mathbf{z},\omega}} - \frac{\partial V}{\partial\psi^{[h,0]+}_{\mathbf{y},\omega}}\frac{\partial V}{\partial\psi^{[h,0]-}_{\mathbf{z},\omega}}\right] ; \sum_{\tilde\omega}\left[D_{\tilde\omega}(\psi^{[h,0]+}_{\mathbf{x},\tilde\omega}\psi^{[h,0]-}_{\mathbf{x},\tilde\omega}) + \delta T_{\mathbf{x},\tilde\omega}\right] >^T , \end{aligned} \tag{5.21}$$

where $<\cdot;\cdot>^T$ denotes the truncated expectation w.r.t. the measure $Z(0)^{-1}P^{(L,h)}(d\psi^{[h,0]})$ $e^{-V^{(L)}(\psi^{[h,0]})}$.

By using the definitions (5.18) and (5.19), equation (5.21) can be rewritten as

$$
\begin{aligned}
0 = &-\delta(\mathbf{x}-\mathbf{y})\Sigma_{h,\omega}(\mathbf{x}-\mathbf{z}) + \delta(\mathbf{x}-\mathbf{z})\Sigma_{h,\omega}(\mathbf{y}-\mathbf{x}) - \\
&- \sum_{\tilde\omega} D_{\mathbf{x},\tilde\omega}\Gamma_{h,\omega,\tilde\omega}(\mathbf{x};\mathbf{y},\mathbf{z}) - \Delta_{h,\omega}(\mathbf{x};\mathbf{y},\mathbf{z}) \;,
\end{aligned}
\tag{5.22}
$$

where

$$
\Delta_{h,\omega}(\mathbf{x};\mathbf{y},\mathbf{z}) =< \left[\frac{\partial^2 V}{\partial\psi^+_{\mathbf{y},\omega}\partial\psi^-_{\mathbf{z},\omega}} - \frac{\partial V}{\partial\psi^+_{\mathbf{y},\omega}}\frac{\partial V}{\partial\psi^-_{\mathbf{z},\omega}} \right] \;;\; \sum_{\tilde\omega} \delta T_{\mathbf{x},\tilde\omega} >^T \;. \tag{5.23}
$$

In terms of the Fourier transforms, defined so that, in agreement with (**I**3.2) and (3.9),

$$
\Sigma_{h,\omega}(\mathbf{x}-\mathbf{y}) = \frac{1}{L\beta}\sum_{\mathbf{k}} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}\hat\Sigma_{h,\omega}(\mathbf{k}) \;, \tag{5.24}
$$

$$
\Gamma_{h,\omega,\tilde\omega}(\mathbf{x};\mathbf{y},\mathbf{z}) = \frac{1}{(L\beta)^2}\sum_{\mathbf{p},\mathbf{k}} e^{i\mathbf{p}(\mathbf{x}-\mathbf{z})}e^{-i\mathbf{k}(\mathbf{y}-\mathbf{z})}\hat\Gamma_{h,\omega,\tilde\omega}(\mathbf{p},\mathbf{k}) \;, \tag{5.25}
$$

$$
\Delta_{h,\omega}(\mathbf{x};\mathbf{y},\mathbf{z}) = \frac{1}{(L\beta)^2}\sum_{\mathbf{p},\mathbf{k}} e^{i\mathbf{p}(\mathbf{x}-\mathbf{z})}e^{-i\mathbf{k}(\mathbf{y}-\mathbf{z})}\hat\Delta_{h,\omega}(\mathbf{p},\mathbf{k}) \;, \tag{5.26}
$$

(5.22) can be written as

$$
0 = \hat\Sigma_{h,\omega}(\mathbf{k}-\mathbf{p}) - \hat\Sigma_{h,\omega}(\mathbf{k}) + \sum_{\tilde\omega}(-ip_0 + \tilde\omega p)\hat\Gamma_{h,\omega,\tilde\omega}(\mathbf{p},\mathbf{k}) + \hat\Delta_{h,\omega}(\mathbf{p},\mathbf{k}) \;. \tag{5.27}
$$

Let us now define

$$
\tilde Z_h^{(2)} = 1 + \frac{1}{4}\sum_{\eta,\eta'=\pm1} \hat\Gamma_{h,\omega}(\bar{\mathbf{p}}_{\eta'},\bar{\mathbf{k}}_{\eta,\eta'}) \;, \tag{5.28}
$$

$$
\tilde Z_h = 1 + \frac{i}{4}\sum_{\eta,\eta'=\pm1} \eta'\frac{\beta}{\pi}\hat\Sigma_{h,\omega}(\bar{\mathbf{k}}_{\eta,\eta'}) \;, \tag{5.29}
$$

where $\bar{\mathbf{p}}_{\eta'}$ is defined as in (3.11) and $\bar{\mathbf{k}}_{\eta,\eta'}$ as in (**I**2.73).

If we put in (5.27) $\mathbf{p} = \bar{\mathbf{p}}_{\eta'}$ and $\mathbf{k} = \bar{\mathbf{k}}_{\eta,\eta'}$, multiply both sides by $(i\eta'\beta)/(2\pi)$ and sum over η,η', we get

$$
\tilde Z_h = \tilde Z_h^{(2)} + \delta\tilde Z_h^{(2)} \;, \tag{5.30}
$$

where

$$
\delta\tilde Z_h^{(2)} = \frac{1}{4}\sum_{\eta,\eta'=\pm1} \frac{\hat\Delta_{h,\omega}(\bar{\mathbf{p}}_{\eta'},\bar{\mathbf{k}}_{\eta,\eta'})}{-i\bar p_{\eta' 0}} \;. \tag{5.31}
$$

5.3 The considerations preceding (5.21) suggest that $\tilde Z_h$ and $\tilde Z_h^{(2)}$ are "almost equal" to the quantities $Z_h^{(L)}$ and $Z_h^{(2,L)}$, related to the full approximate Luttinger model and defined analogously to Z_h and $Z_h^{(2)}$ for the original model, on the base of a multiscale analysis. In order to clarify this point, we consider the measure $P^{(L,h)}(d\psi^{[h,0]})e^{-V^{(L)}(\psi^{[h,0]}+\psi^{(<h)})}$, where $\psi^{(<h)}$ is fixed and has the same role of the external field ϕ in (5.17), and define $\bar E_{h-1}$ and $\bar{\mathcal V}^{(h-1)}(\psi^{(<h)})$, the *one step effective potential* on scale $h-1$, so that $\bar{\mathcal V}^{(h-1)}(0) = 0$ and

$$
e^{-\bar{\mathcal V}^{(h-1)}(\psi^{(<h)}) - L\beta\bar E_{h-1}} = \int P^{(L,h)}(d\psi^{[h,0]})e^{-V^{(L)}(\psi^{[h,0]}+\psi^{(<h)})} \;. \tag{5.32}
$$

We want to calculate this quantity, by extending to it the definitions of effective potentials and running couplings, given in §**I**2 for the original model.

We start from the scale 0 with potential $\mathcal{V}'^{(0)}(\psi^{[h,0]},\psi^{(<h)}) = V^{(L)}(\psi^{[h,0]}+\psi^{(<h)})$ and we introduce, in analogy to the procedure described in §**I**2.5, for each $\tilde h$ such that $h\le\tilde h\le 0$, two constants $Z'_{\tilde h}$, $E'_{\tilde h}$ and an effective potential $\mathcal{V}'^{(\tilde h)}(\psi,\psi^{(<h)})$, so that $Z'_0=1$, $E'_0=0$ and

$$e^{-\bar{\mathcal{V}}^{(h-1)}(\psi^{(<h)})-L\beta\bar E_{h-1}} = \int P_{Z'_{\tilde h},C_{h,\tilde h}}(d\psi^{[h,\tilde h]})e^{-\mathcal{V}'^{(\tilde h)}(\sqrt{Z'_{\tilde h}}\psi^{[h,\tilde h]},\psi^{(<h)})-E'_{\tilde h}}\;, \tag{5.33}$$

where $P_{Z'_{\tilde h},C_{h,\tilde h}}(d\psi^{[h,\tilde h]})$ is obtained from the analogous definition (**I**2.66), by putting $\sigma_{\tilde h}(\mathbf{k}')=0$, $E(\mathbf{k}')=\bar v_0\sin k'$, and by substituting $C_{\tilde h}^{-1}$ with $C_{h,\tilde h}^{-1}=\sum_{k=h}^{\tilde h}f_k$. Moreover, we suppose that the localization procedure is applied also to the field $\psi^{(<h)}$, even if it does appear in the integration measure and, therefore, can not be involved in the free measure renormalization.

We want to compare these effective potentials with the potentials $\mathcal{V}^{(\tilde h)}(\psi^{(\le\tilde h)})$, related to the approximate Luttinger model without any infrared cutoff and defined following again the procedure described in §**I**2. We shall use for the various objects related to this model the same notation of §**I**2, while the corresponding objects of the model with infrared cutoff will be distinguished with a superscript $'$. The definitions are such that $\mathcal{V}^{(0)}(\psi^{(\le0)}) = \mathcal{V}'^{(0)}(\psi^{[h,0]},\psi^{(<0)})$, $Z_0=1$ and

$$\int P_{Z_0,C_0}(d\psi^{(\le0)})\,e^{-\mathcal{V}^{(0)}(\sqrt{Z_0}\psi^{(\le0)})} = \int P_{Z_{\tilde h},C_{\tilde h}}(d\psi^{(\le\tilde h)})\,e^{-\mathcal{V}^{(\tilde h)}(\sqrt{Z_{\tilde h}}\psi^{(\le\tilde h)})-L\beta E_{\tilde h}}\;. \tag{5.34}$$

Note that the single scale propagators involved in the calculation of $\mathcal{V}^{(\tilde h)}(\sqrt{Z_{\tilde h}}\psi^{(\le\tilde h)})$ and $\mathcal{V}'^{(\tilde h)}(\sqrt{Z'_{\tilde h}}\psi^{[h,\tilde h]},\psi^{(<h)})$, that is those with scale $\bar h\ge\tilde h+1$, may differ only if $Z_{\bar h}\ne Z'_{\bar h}$ or $z_{\bar h}\ne z'_{\bar h}$. This immediately follows from the observation that, if $h+1\le\tilde h\le0$, the identity (**I**2.90) is satisfied even if we substitute in (**I**2.89) $C_{\tilde h}$ with $C_{h,\tilde h}$. This implies, in particular, since $z_0=z'_0=0$, that (see (**I**2.110) and (**I**2.107))

$$\mathcal{V}'^{(-1)}(\sqrt{Z'_{-1}}\psi^{[h,-1]},\psi^{(<h)}) = \mathcal{V}^{(-1)}[\sqrt{Z_{-1}}(\psi^{[h,-1]}+\psi^{(<h)})]\;, \tag{5.35}$$

with $Z'_{-1}=Z_{-1}=1$, and that $z_{-1}=z'_{-1}$, $\delta_{-1}=\delta'_{-1}$, $\lambda_{-1}=\lambda'_{-1}$, $Z'_{-2}=Z_{-2}$.

Let us now compare the effective potentials on scale -2. The fact that the free measure in (5.33) does not depend on the fields with scale less than h implies that the free measure renormalization does not use all the local part of $\mathcal{V}'^{(-1)}$ proportional to z_{-1}. Therefore, the analogous of the potential $\hat{\mathcal{V}}^{(-1)}(\sqrt{Z_{-2}}\,\psi^{(\le-1)})$ for the model with infrared cutoff has to be defined so that (see (**I**2.107))

$$\begin{aligned}&\hat{\mathcal{V}}'^{(-1)}(\sqrt{Z_{-2}}\psi^{[h,-1]},\psi^{(<h)}) = \hat{\mathcal{V}}^{(-1)}[\sqrt{Z_{-2}}(\psi^{[h,-1]}+\psi^{(<h)})]+\\ &+z_{-1}Z_{-1}\sum_{\omega=\pm1}\int d\mathbf{x}\;\Big[-\Big(D_\omega\psi^{[h,-1]+}_{\mathbf{x},\omega}\Big)\psi^{(<h)-}_{\mathbf{x},\omega}+\psi^{(<h)+}_{\mathbf{x},\omega}\Big(D_\omega\psi^{[h,-1]-}_{\mathbf{x},\omega}\Big)\Big]+\\ &+z_{-1}Z_{-1}\sum_{\omega=\pm1}\int d\mathbf{x}\;\psi^{(<h)+}_{\mathbf{x},\omega}D_\omega\psi^{(<h)-}_{\mathbf{x},\omega}\;.\end{aligned} \tag{5.36}$$

It follows, by using also the remark on the single scale propagators following (5.34), that $\mathcal{V}'^{(-2)}(\sqrt{Z_{-2}}\psi^{[h,-2]},\psi^{(<h)})$, calculated through the analogous of (**I**2.110), can be obtained from $\mathcal{V}^{(-2)}[\sqrt{Z_{-2}}(\psi^{[h,-2]}+\psi^{(<h)})]$ by adding some new terms. First of all, there is the term in the third line of (5.36), which is independent of the integration variables, and the two terms in the second line with $\psi^{[h,-2]\sigma}_{\mathbf{x},\omega}$ in place of $\psi^{[h,-1]\sigma}_{\mathbf{x},\omega}$. Moreover, in the Feynman graph expansion, we have to add the graphs which are obtained by inserting, in the external lines of a graph contributing to $\mathcal{V}^{(-2)}$, one or more vertices corresponding to the two terms in the second line of (5.36). These new terms are not irrelevant, if the number of external lines is 2 or 4; hence one could worry about the need of new running couplings in order to regularize the expansion. However, because of the support properties of the propagators, these new terms do not give any contribution to the local part (which is calculated by putting equal to $\bar{\mathbf{k}}_{\eta,\eta'}$ the external momenta, hence also the momenta of the internal line propagators of the insertions in the external lines), so that the only running couplings to consider are those related with $\mathcal{V}^{(-2)}$ and their values are the same, that is $z_{-2}=z'_{-2}$, $\delta_{-2}=\delta'_{-2}$, $\lambda_{-2}=\lambda'_{-2}$, $Z'_{-3}=Z_{-3}$.

By iterating the previous considerations, it is easy to show that, if $h\le\tilde{h}\le-2$, one can calculate $\mathcal{V}'^{(\tilde{h})}(\sqrt{Z_{\tilde{h}}}\psi^{[h,\tilde{h}]},\psi^{(<h)})$ by adding to $\mathcal{V}^{(\tilde{h})}[\sqrt{Z_{\tilde{h}}}(\psi^{[h,\tilde{h}]}+\psi^{(<h)})]$ some new terms. First of all, there are the local terms of the form of that in the second and the third line of (5.36), with $\psi^{[h,\tilde{h}]\sigma}_{\mathbf{x},\omega}$ in place of $\psi^{[h,-1]\sigma}_{\mathbf{x},\omega}$ and $z_{\bar{h}}Z_{\bar{h}}$, $\tilde{h}\le\bar{h}\le-1$ in place of $z_{-1}Z_{-1}$. Moreover, in the Feynman graph expansion, we have to add the graphs, which are obtained by inserting, in the external lines of a graph contributing to $\mathcal{V}^{(\tilde{h})}$, one or more vertices corresponding to terms similar to those in the second line of (5.36), with $\psi^{[h,\tilde{h}]\sigma}_{\mathbf{x},\omega}$ in place of $\psi^{[h,-1]\sigma}_{\mathbf{x},\omega}$ and $z_{\bar{h}}Z_{\bar{h}}$, $\tilde{h}\le\bar{h}\le-1$ in place of $z_{-1}Z_{-1}$. Finally

$$
\begin{aligned}
&\mathcal{L}\mathcal{V}'^{(\tilde{h})}(\sqrt{Z_{\tilde{h}}}\psi^{[h,\tilde{h}]},\psi^{(<h)})=\mathcal{L}\mathcal{V}^{(\tilde{h})}[\sqrt{Z_{\tilde{h}}}(\psi^{[h,\tilde{h}]}+\psi^{(<h)})]+\\
&+\sum_{\bar{h}=\tilde{h}+1}^{-1} z_{\bar{h}}Z_{\bar{h}}\sum_{\omega=\pm1}\int d\mathbf{x}\ \left[-\left(D_\omega\psi^{[h,\tilde{h}]+}_{\mathbf{x},\omega}\right)\psi^{(<h)-}_{\mathbf{x},\omega}+\psi^{(<h)+}_{\mathbf{x},\omega}\left(D_\omega\psi^{[h,\tilde{h}]-}_{\mathbf{x},\omega}\right)\right]+\\
&+\sum_{\bar{h}=\tilde{h}+1}^{-1} z_{\bar{h}}Z_{\bar{h}}\sum_{\omega=\pm1}\int d\mathbf{x}\ \psi^{(<h)+}_{\mathbf{x},\omega}D_\omega\psi^{(<h)-}_{\mathbf{x},\omega}\ ,
\end{aligned}
\tag{5.37}
$$

and all the running couplings, as well as the renormalization constants, are the same as those defined through $\mathcal{V}^{(\tilde{h})}(\sqrt{Z_{\tilde{h}}}\psi^{(\le\tilde{h})})$.

Equations (5.33) and (5.37) also imply that

$$
\begin{aligned}
&\mathcal{L}\bar{\mathcal{V}}^{(h-1)}(\psi^{(<h)})=\mathcal{L}\mathcal{V}'^{(h-1)}(\psi^{(<h)})+\\
&+\sum_{\bar{h}=h+1}^{-1} z_{\bar{h}}Z_{\bar{h}}\sum_{\omega=\pm1}\int d\mathbf{x}\ \psi^{(<h)+}_{\mathbf{x},\omega}D_\omega\psi^{(<h)-}_{\mathbf{x},\omega}+z'_hZ_h\sum_{\omega=\pm1}\int d\mathbf{x}\ \psi^{(<h)+}_{\mathbf{x},\omega}D_\omega\psi^{(<h)-}_{\mathbf{x},\omega}\ ,
\end{aligned}
\tag{5.38}
$$

where $\mathcal{V}'^{(h-1)}(\psi^{(<h)})$ is obtained from $\mathcal{V}^{(h-1)}(\psi^{(<h)})$ "almost" as before. We still have to add some new graphs with suitable insertions on the external lines, which do not affect the

local part, but we also have to change the propagators of scale h, since the function $\tilde f'_h(\mathbf{k}')$, calculated as $\tilde f_h(\mathbf{k}')$, see (**I**2.90), with $C^{-1}_{h,h}=f_h$ in place of C_h^{-1}, is different from $\tilde f_h(\mathbf{k}')$.

The definition (5.29) of $\tilde Z_h$ and the definition of $\mathcal{L}$, together with (5.38), imply that

$$\tilde Z_h = 1+\sum_{\bar h=h+1}^{-1} z_{\bar h} Z_{\bar h} + z'_h Z_h = Z_h(1+z'_h)\;. \tag{5.39}$$

Since $Z_h^{(L)}=Z_h$ and $|z'_h|\le C|\lambda_0|^2$, if λ_0 is small enough, as one can show by using the arguments of §2, we get the bound

$$\left|\frac{\tilde Z_h}{Z_h^{(L)}}-1\right|\le C|\lambda_0|\;. \tag{5.40}$$

A similar argument can be used for $Z_h^{(2,L)}$, by using the results of §3, and we get the similar bound

$$\left|\frac{\tilde Z_h^{(2)}}{Z_h^{(2,L)}}-1\right|\le C|\lambda_0|\;. \tag{5.41}$$

We will prove in §5.3 that

$$|\delta\tilde Z_h^{(2)}|\le CZ_h^{(2,L)}|\lambda_0|\;, \tag{5.42}$$

so that we finally get

$$|\frac{Z_h^{(L)}}{Z_h^{(2,L)}}-1|\le C|\lambda_0|\;, \tag{5.43}$$

implying (3.35).

Remark (5.42) shows that the corrections to the *exact* Ward identity $Z_h^{(L)}=Z_h^{(2,L)}$ could diverge as $h\to-\infty$. This is not important in our proof, since we are only interested in the ratio $Z_h^{(L)}/Z_h^{(2,L)}$, which is near to 1, but suggests that it would be difficult to prove the approximate Ward identity, by directly looking at the cancellations in presence of the cutoffs.

5.4 In order to prove (5.42), we note that

$$\begin{aligned}&[C_{h,0}(\mathbf{p}+\mathbf{k})-1]D_\omega(\mathbf{p}+\mathbf{k})-[C_{h,0}(\mathbf{k})-1]D_\omega(\mathbf{k})=\\&D_\omega(\mathbf{p})[C_{h,0}(\mathbf{p}+\mathbf{k})-1]+C_{h,0}(\mathbf{p}+\mathbf{k})D_\omega(\mathbf{k})C_{h,0}(\mathbf{k})[C^{-1}_{h,0}(\mathbf{k})-C^{-1}_{h,0}(\mathbf{p}+\mathbf{k})]\;,\end{aligned} \tag{5.44}$$

and that

$$\begin{aligned}C_{h,0}(\bar{\mathbf{p}}_{\eta'}+\mathbf{k})\frac{[C^{-1}_{h,0}(\mathbf{k})-C^{-1}_{h,0}(\bar{\mathbf{p}}_{\eta'}+\mathbf{k})]}{-i\bar p_{\eta'0}}&=\\ C_{h,0}(\mathbf{p}+\mathbf{k})\frac{[C^{-1}_{h,0}(\mathbf{k})-C^{-1}_{h,0}(\bar{\mathbf{p}}_{\eta'}+\mathbf{k})]}{-i\bar p_{\eta'0}}\Bigg|_{\mathbf{p}=\bar{\mathbf{p}}_{\eta'}}&\;,\end{aligned} \tag{5.45}$$

$$C_{h,0}(\mathbf{p}+\mathbf{k})=1+[C_{h,0}(\mathbf{p}+\mathbf{k})-1]\;. \tag{5.46}$$

Hence, by using (5.16) and (5.23), we can write

$$\frac{\hat\Delta_{h,\omega}(\bar{\mathbf{p}}_{\eta'},\bar{\mathbf{k}}_{\eta,\eta'})}{-i\bar p_{\eta'0}}=\hat\Delta^{(1)}_{h,\omega,\eta'}(\bar{\mathbf{p}}_{\eta'},\bar{\mathbf{k}}_{\eta,\eta'})\;, \tag{5.47}$$

where

$$\Delta^{(1)}_{h,\omega,\eta'}(\mathbf{x};\mathbf{y},\mathbf{z}) =< \left[\frac{\partial^2 V}{\partial\psi^+_{\mathbf{y},\omega}\partial\psi^-_{\mathbf{z},\omega}} - \frac{\partial V}{\partial\psi^+_{\mathbf{y},\omega}}\frac{\partial V}{\partial\psi^-_{\mathbf{z},\omega}}\right] \; ; \; \sum_{\tilde\omega} \delta^{(1)} T_{\mathbf{x},\tilde\omega,\eta'} >^T \;, \tag{5.48}$$

with

$$\delta^{(1)} T_{\mathbf{x},\omega,\eta'} = \psi^{[h,0]+}_{\mathbf{x},\omega}\delta\psi^{[h,0]-}_{\mathbf{x},\omega} + \delta\tilde\psi^{[h,0]+}_{\mathbf{x},\omega,\eta'}\delta\psi^{[h,0]-}_{\mathbf{x},\omega} + \delta\tilde\psi^{[h,0]+}_{\mathbf{x},\omega,\eta'}\psi^{[h,0]-}_{\mathbf{x},\omega} \;, \tag{5.49}$$

$$\delta\psi^{[h,0]-}_{\mathbf{x},\omega} = \frac{1}{L\beta}\sum_{\mathbf{k}:C^{-1}_{h,0}(\mathbf{k})>0} e^{-i\mathbf{k}\mathbf{x}} C_{h,0}(\mathbf{k})(1 - C^{-1}_{h,0}(\mathbf{k}))\hat\psi^{[h,0],-}_{\mathbf{k},\omega} \;, \tag{5.50}$$

$$\delta\tilde\psi^{[h,0]+}_{\mathbf{x},\omega,\eta'} = \frac{1}{L\beta}\sum_{\mathbf{k}:C^{-1}_{h,0}(\mathbf{k})>0} e^{i\mathbf{k}\mathbf{x}} D_\omega(\mathbf{k}) C_{h,0}(\mathbf{k}) \frac{[C^{-1}_{h,0}(\mathbf{k}) - C^{-1}_{h,0}(\bar{\mathbf{p}}_{\eta'} + \mathbf{k})]}{-i\bar p_{\eta' 0}} \hat\psi^{[h,0],+}_{\mathbf{k},\omega} \;. \tag{5.51}$$

Note that there is no divergence, in the limit $L,\beta\to\infty$, associated with the fields $\delta\psi^{[h,0]-}$ and $\delta\tilde\psi^{[h,0]+}$, even if the function $C_{h,0}(\mathbf{k})$ diverges on the boundary of the set $\{\mathbf{k} : C^{-1}_{h,0}(\mathbf{k}) > 0\}$. In fact, the integration of these fields on scale $\bar h$, with $h\le\bar h\le 0$, yields a factor $\tilde f'_{\bar h}(\mathbf{k})$ proportional to $f_{\bar h}(\mathbf{k})$ (see (**I**2.90) and the considerations after (5.38)), and the functions $f_{\bar h}(\mathbf{k})$ are non negative, if we suitably choose the function (**I**2.30); therefore $C_{h,0}(\mathbf{k})\tilde f'_{\bar h}(\mathbf{k})$ is bounded.

Note also that, $[C^{-1}_{h,0}(\mathbf{k}) - C^{-1}_{h,0}(\bar{\mathbf{p}}_{\eta'} + \mathbf{k})]/-i\bar p_{\eta' 0}$ is bounded, uniformly in β, and is equal to 0, at least if $|\mathbf{k}|$ belongs to the interval $[a_0\gamma^h + 2\pi/\beta, a_0 - 2\pi/\beta]$ (see §**I**2.3). However, the interval where this function vanishes can contain the interval $[a_0\gamma^h, a_0]$, if the function (**I**2.30) is suitably chosen (by slightly broadening the regions where it has to be equal to 1 or 0) and β is large enough, which is not of course an important restriction (the real problem is the uniformity of the bounds in the limit $\beta\to\infty$, and in any case the following arguments could be easily generalized to cover the general case). Hence, it is easy to show that

$$1 - C^{-1}_{h,0}(\mathbf{k}) = \frac{C^{-1}_{h,0}(\mathbf{k}) - C^{-1}_{h,0}(\bar{\mathbf{p}}_{\eta'} + \mathbf{k})}{-i\bar p_{\eta' 0}} = 0 \;, \quad \text{if } \tilde f'_{\bar h}(\mathbf{k}) \ne 0 \quad h < \bar h < 0 \;, \tag{5.52}$$

so that we can write

$$\delta\psi^{[h,0]-}_{\mathbf{x},\omega} = \delta\psi^{(0)-}_{\mathbf{x},\omega} + \delta\psi^{(h)-}_{\mathbf{x},\omega} \;, \quad \delta\tilde\psi^{[h,0]+}_{\mathbf{x},\omega,\eta'} = \delta\tilde\psi^{(0)+}_{\mathbf{x},\omega,\eta'} + \delta\tilde\psi^{(h)+}_{\mathbf{x},\omega,\eta'} \;, \tag{5.53}$$

where the fields $\delta\psi^{(h')-}_{\mathbf{x},\omega}$ and $\delta\tilde\psi^{(h')+}_{\mathbf{x},\omega,\eta'}$ are defined by substituting, in (5.50) and (5.51), $\hat\psi^{[h,0],+}_{\mathbf{k},\omega}$ with $\hat\psi^{(h'),+}_{\mathbf{k},\omega}$.

Let us now consider the functional

$$e^{\mathcal{S}_{h,\eta'}(\psi^{(<h)},J)} = \int P^{(L,h)}(d\psi^{[h,0]}) e^{-V^{(L)}(\psi^{[h,0]}+\psi^{(<h)}) + \sum_{\tilde\omega}\int d\mathbf{x} J_\mathbf{x} \delta^{(1)} T_{\mathbf{x},\tilde\omega,\eta'}} \;. \tag{5.54}$$

We can write for $\mathcal{S}_{h,\eta'}(\psi^{(<h)},J)$ an expansion similar to that used in §3 to study the correlation function of the original model. We introduce, for any $\tilde h$ such that $h\le\tilde h\le -1$, an effective potential $\mathcal{V}'^{(\tilde h)}(\psi,\psi^{(<h)})$, defined as in §5.3, and two functionals $S'^{(\tilde h+1)}(J)$, $\mathcal{B}'^{(\tilde h)}(\psi,\psi^{(<h)},J)$, so that, by using the notation of §5.4,

$$\begin{aligned} e^{\mathcal{S}_{h,\eta'}(\psi^{(<h)},J)} = e^{-L\beta E'_{\tilde h} + S'^{(\tilde h+1)}(J)} \int P_{Z'_{\tilde h}, C_{h,\tilde h}}(d\psi^{[h,\tilde h]}) \cdot \\ \cdot\, e^{-\mathcal{V}'^{(\tilde h)}(\sqrt{Z'_{\tilde h}}\psi^{[h,\tilde h]},\psi^{(<h)}) + \mathcal{B}'^{(\tilde h)}(\sqrt{Z'_{\tilde h}}\psi^{[h,\tilde h]},\psi^{(<h)},J)} \;. \end{aligned} \tag{5.55}$$

We introduce also the functionals $S^{'(h)}(J)$, $\mathcal{V}^{'(h-1)}(\psi^{(<h)})$ and $\mathcal{B}^{'(h-1)}(\psi^{(<h)},J)$, such that

$$\mathcal{S}_{h,\eta'}(\psi^{(<h)},J) = S^{'(h)}(J) - \mathcal{V}^{'(h-1)}(\psi^{(<h)}) + \mathcal{B}^{'(h-1)}(\psi^{(<h)},J)\ . \tag{5.56}$$

We can write for $\mathcal{B}^{'(h-1)}(\psi^{(<h)},J)$ a representation similar to (3.6), with J in place of ϕ and $\psi^{(<h)}$ in place of $\psi^{(\le h)}$. By (5.48)

$$\Delta^{(1)}_{h,\omega,\eta'}(\mathbf{x};\mathbf{y},\mathbf{z}) = B^{'(h-1)}_{1,2,(+,-),(\omega,\omega)}(\mathbf{x};\mathbf{y},\mathbf{z})\ ; \tag{5.57}$$

hence, in order to prove (5.42), we have to study the flow of the local part of $\mathcal{B}^{'(\tilde h)}(Z^{'-1/2}_{\tilde h}$ $\psi^{[h,\tilde h]},\psi^{(<h)},J)$.

To start with, let us consider $\mathcal{B}^{'(-1)}(Z^{'-1/2}_{-1}\psi^{[h,-1]},\psi^{(<h)},J)$. By (5.53), the graphs contributing to it may have an external line of type $\delta\psi$ or $\delta\tilde\psi$ only if that line is of scale h and $h<-1$. Moreover, if the graph has an external line of this type and it is not trivial, that is if it has more than one vertex, the corresponding local part, defined as in §3, is 0, even if there are only two external lines, because of the support properties of the propagators, since there is at least one internal line with momentum equal to one of the external momenta, which are of order β^{-1} for the local part. It follows that these graphs do not participate in any manner to the flow of $\mathcal{L}\mathcal{B}^{'(-1)}(Z^{'-1/2}_{-1}\psi^{[h,-1]},\psi^{(<h)},J)$, up to the scale h; therefore we modify the definition of $\mathcal{L}$, so that they are not included.

This modification of the definition of $\mathcal{L}$ allows to study the flow of $\mathcal{L}\mathcal{B}^{'(\tilde h)}(Z'_{\tilde h-1}\psi^{[h,\tilde h]}$, $\psi^{(<h)},J)$ essentially as in §3, since, as we have explained in §5.3, the infrared cutoff has no influence on the other local terms, except on the last scale, so that, if $h\le\tilde h\le -1$,

$$\mathcal{L}\mathcal{B}^{'(\tilde h)}(\sqrt{Z'_{\tilde h}}\psi^{[h,\tilde h]},\psi^{(<h)},J) = \mathcal{L}\mathcal{B}^{(\tilde h)}(\sqrt{Z_{\tilde h}}(\psi^{[h,\tilde h]}+\psi^{(<h)}),J)\ , \tag{5.58}$$

where $\mathcal{B}^{(\tilde h)}(\sqrt{Z_{\tilde h}}\psi^{(\le\tilde h)},J)$ is the expression we should get in absence of infrared cutoff and we used the fact, proved in §5.3, that $Z'_{\tilde h}=Z_{\tilde h}$. We can write

$$\mathcal{L}\mathcal{B}^{(\tilde h)}(\sqrt{Z_{\tilde h}}(\psi^{(\le\tilde h)}),J) = \frac{Z^{(3)}_{\tilde h}}{Z_{\tilde h}}\sum_{\omega=\pm1}\int d\mathbf{x}J_{\mathbf{x}}\psi^{(\le\tilde h)+}_{\mathbf{x},\omega}\psi^{(\le\tilde h)-}_{\mathbf{x},\omega}\ . \tag{5.59}$$

The flow of $Z^{(3)}_{\tilde h}$ can be studied, starting from the scale $h=-1$, as the flow of the renormalization constants $Z^{(2)}_{\tilde h}$ related to the analogous of the functional (3.2) for the model defined by (5.1) and (5.2), that is

$$e^{\mathcal{S}(J)} = \int P^{(L)}(d\psi^{(\le0)})e^{-V^{(L)}(\psi^{(\le0)})+\sum_{\omega=\pm1}\int d\mathbf{x}J_{\mathbf{x}}\psi^{(\le0)+}_{\mathbf{x},\omega}\psi^{(\le0)-}_{\mathbf{x},\omega}}\ . \tag{5.60}$$

Note that the values of $Z^{(3)}_{-1}$ and $Z^{(2)}_{-1}$ are very different; in fact, the previous considerations imply that

$$|Z^{(2)}_{-1}-1|\le C|\lambda_0|\ .\quad |Z^{(3)}_{-1}|\le C|\lambda_0|\ . \tag{5.61}$$

However, since the local part on scale -1 is of the same form and the contribution of the non local terms on scale -1 to $Z^{(3)}_{\tilde h}/Z^{(3)}_{\tilde h+1}$ or $Z^{(2)}_{\tilde h}/Z^{(2)}_{\tilde h+1}$ is exponentially depressed, as $\tilde h$ decreases, it is easy to show, by using the arguments of §2.4-§2.7, that

$$Z_h^{(3)} = \frac{Z_h^{(3)}}{Z_{-1}^{(3)}} Z_{-1}^{(3)} = \frac{Z_h^{(2)}}{Z_{-1}^{(2)}} [1+O(\lambda_0)] Z_{-1}^{(3)} \; . \tag{5.62}$$

The integration of the fields of scale h can only change this identity by a factor $[1+O(\lambda_0)]$, hence (5.61) and (5.62) imply that

$$\left| \frac{Z_{h-1}^{(3)}}{Z_h^{(2)}} - 1 \right| \le C|\lambda_0| \; . \tag{5.63}$$

If $\Delta^{(1)}_{h,\omega,\eta'}(\mathbf{x};\mathbf{y},\mathbf{z})$ were independent of η', $\delta\tilde Z_h^{(2)}$ would be exactly equal to $Z^{(3)}_{h-1}$ and (5.42) would have been proved. Since this is true only in the limit $\beta\to\infty$, we have to bound $\hat\Delta^{(1)}_{h,\omega,\eta'}(\bar{\mathbf{p}}_{\eta'},\bar{\mathbf{k}}_{\eta,\eta'})$ for each η,η'. This means that we have to bound even the Fourier transform at momenta of order β^{-1} of $\mathcal{R}B^{'(h-1)}_{1,2,(+,-),(\omega,\omega)}(\mathbf{x};\mathbf{y},\mathbf{z})$, see (5.57). However, it is easy to see that we still get the bound (5.42), on the base of a simple dimensional argument (we skip the details, which should be by now obvious). In fact, if we consider a term contributing to the expansion of $\mathcal{R}B^{'(h-1)}_{1,2,(+,-),(\omega,\omega)}(\mathbf{x};\mathbf{y},\mathbf{z})$ described in §3, whose external fields are affected by the regularization so that some derivative acts on them, the corresponding bound differs from the bound of a generic term contributing to $\mathcal{L}B^{'(h-1)}_{1,2,(+,-),(\omega,\omega)}(\mathbf{x};\mathbf{y},\mathbf{z})$ in the following way. One has to add a factor γ^{-h_v}, for each "zero" produced by the regularization and, at the same time, a factor β^{-1} produced by the corresponding derivative on the external momenta. Since $\beta^{-1}\gamma^{-h_v}\le 1$, we get the same result.

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