HARDY AND RELLICH TYPE INEQUALITIES WITH REMAINDERS FOR BAOUENDI-GRUSHIN VECTOR FIELDS

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Abstract. In this paper we study Hardy and Rellich type inequalities for Baouendi-Grushin vector fields : $\nabla_\gamma = (\nabla_x, |x|^{2\gamma}\nabla_y)$ where $\gamma > 0$, ∇_x and ∇_y are usual gradient operators in the variables $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^k$, respectively. In the first part of the paper, we prove some weighted Hardy type inequalities with remainder terms. In the second part, we prove two versions of weighted Rellich type inequality on the whole space. We find sharp constants for these inequalities. We also obtain their improved versions for bounded domains.

1. Introduction

This paper is concerned with Hardy and Rellich type inequalities with remainder terms for Baouendi-Grushin vector fields. Let $x \in \mathbb{R}^m$, $y \in \mathbb{R}^k$, $\gamma > 0$ and $n = m + k$, with $m, k \geq 1$. Then the following Hardy type inequality for Baouendi-Grushin vector fields has been proved by Garofalo [G]

$$\int_{\mathbb{R}^n} \big(|\nabla_x\phi|^2 + |x|^{2\gamma}|\nabla_y\phi|^2\big)dxdy \geq \Big(\frac{Q-2}{2}\Big)^2 \int_{\mathbb{R}^n} \Big(\frac{|x|^{2\gamma}}{|x|^{2+2\gamma} + (1+\gamma^2)^2|y|^2}\Big)\phi^2 dxdy \tag{1.1}$$

where $\phi \in C_0^\infty(\mathbb{R}^m \times \mathbb{R}^k \setminus \{(0,0)\})$ and $Q = m + (1+\gamma)k$. Here, $\nabla_x\phi$ and $\nabla_y\phi$ denotes the gradients of ϕ in the variables x and y, respectively. A similar inequality with the same sharp constant $(\frac{Q-2}{2})^2$ holds if $\mathbb{R}^n$ replaced by Ω and Ω contains the origin [D]. If $\gamma = 0$ then it is clear that the inequality (1.1) recovers the classical Hardy inequality in $\mathbb{R}^n$

$$\int_{\mathbb{R}^n} |\nabla\phi(z)|^2 dz \geq \Big(\frac{n-2}{2}\Big)^2 \int_{\mathbb{R}^n} \frac{|\phi(z)|^2}{|z|^2} dz \tag{1.2}$$

where $z = (x, y) \in \mathbb{R}^m \times \mathbb{R}^k$ and the constant $(\frac{n-2}{2})^2$ is sharp. There exists a large literature concerning with the Hardy inequalities and, in particular, sharp inequalities as well as their improved versions which have attracted a lot of attention because of their application to singular problems (See [BG], [PV], [BV], [GP], [CM], [VZ], [K1] and references therein).

A sharp improvement of the Hardy inequality (1.2) was discovered by Brezis and Vázquez [BV]. They proved that for a bounded domain $\Omega \subset \mathbb{R}^n$

$$\int_{\Omega} |\nabla\phi(z)|^2 dz \geq \Big(\frac{n-2}{2}\Big)^2 \int_{\Omega} \frac{|\phi(z)|^2}{|z|^2} dz + \mu\Big(\frac{\omega_n}{|\Omega|}\Big)^{2/n} \int_{\Omega} \phi^2 dz, \tag{1.3}$$

where $\phi \in C_0^\infty(\Omega)$, ω_n and $|\Omega|$ denote the n-dimensional Lebesgue measure of the unit ball $B \subset \mathbb{R}^n$ and the domain Ω respectively. Here μ is the first eigenvalue of the Laplace

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operator in the two dimensional unit disk and it is optimal when Ω is a ball centered at the origin. In a recent paper Abdelloui, Colorado and Peral [ACP] obtained, among other things, the following improved Caffarelli-Kohn-Nirenberg inequality

$$\int_\Omega |\nabla\phi(z)|^2|z|^{-2a}dz \geq \Big(\frac{n-2a-2}{2}\Big)^2\int_\Omega \frac{|\phi(z)|^2}{|z|^{2a+2}}dz + C\Big(\int_\Omega |\nabla\phi|^q|z|^{-aq}\Big)^{2/q}dz \tag{1.4}$$

where $\phi \in C_0^\infty(\Omega)$, $-\infty < a < \frac{n-2}{2}$, $1 < q < 2$ and $C = C(q, n, \Omega) > 0$. Motivated by these results, our first goal is to find improved weighted Hardy type inequalities for Baouendi-Grushin vector fields.

It is well known that an important extension of Hardy's inequality to higher-order derivatives is the following Rellich inequality

$$\int_{\mathbb{R}^n} |\Delta\phi(z)|^2dz \geq \frac{n^2(n-4)^2}{16}\int_{\mathbb{R}^n}\frac{|\phi(z)|^2}{|z|^4}dz \tag{1.5}$$

where $\phi \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$, $n \neq 2$ and the constant $\frac{n^2(n-4)^2}{16}$ is sharp. Davies and Hinz [DH], among other results, obtained sharp weighted Rellich inequalities of the form

$$\int_{\mathbb{R}^n}\frac{|\Delta\phi(z)|^2}{|z|^\alpha}dz \geq C\int_{\mathbb{R}^n}\frac{|\phi(z)|^2}{|z|^\beta}dz \tag{1.6}$$

for suitable values of α, β, p and $\phi \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$. In a recent paper, Tertikas and Zographopoulos [TZ], among other results, obtained the following new Rellich type inequalities that connects first to second order derivatives:

$$\int_{\mathbb{R}^n} |\Delta\phi|^2dz \geq \frac{n^2}{4}\int_{\mathbb{R}^n}\frac{|\nabla\phi|^2}{|z|^2}dz \tag{1.7}$$

where $\phi \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ and the constant $\frac{n^2}{4}$ is sharp. Recently, Kombe [K2] obtained analogues of (1.6) and (1.7), and their improved versions on Carnot groups. Motivated by the above results, our second goal is to find sharp weighted Rellich type inequalities and their improved versions for Baouendi-Grushin vector fields in that they do not arise from any Carnot group. We should also mention that Kombe and Özaydin [KÖ] obtained (under some geometric assumptions) improved Hardy and Rellich inequalities on a Riemannian manifold that does not recover our current results. Analogues inequalities for the Greiner vector fields will be given in a forthcoming paper [K2].

2. Notations and Back ground material

In this section, we shall collect some notations, definitions and preliminary facts which will be used throughout the article. The generic point is $z = (x_1, ..., x_m, y_1, ..., y_k) = (x, y) \in \mathbb{R}^m \times \mathbb{R}^k$ with $m, k \geq 1, m + k = n$. The sub-elliptic gradient is the n dimensional vector field given by

$$\nabla_\gamma = (X_1, \cdots, X_m, Y_1, \cdots, Y_k) \tag{2.1}$$

where

$$X_j = \frac{\partial}{\partial x_j}, \quad j = 1, \cdots, m, \quad Y_j = |x|^\gamma\frac{\partial}{\partial y_j}, \quad j = 1, \cdots, k. \tag{2.2}$$

The Baouendi-Grushin operator on $\mathbb{R}^{m+k}$ is the operator

$$\Delta_\gamma = \nabla_\gamma \cdot \nabla_\gamma = \Delta_x + |x|^{2\gamma}\Delta_y, \tag{2.3}$$

where Δ_x and Δ_y are Laplace operators in the variables $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^k$, respectively (see [B], [G1], [G2]). If γ is an even positive integer then Δ_γ is a sum of squares of C^∞ vector fields satisfying Hörmander finite rank condition: *rank Lie* $[\ X_1, \cdots, X_m, Y_1, \cdots, Y_k]\ = n$. The anisotropic dilation attached to Δ_γ is given by

$$\delta_\lambda(z) = (\lambda x, \lambda^{\gamma+1} y), \quad \lambda > 0, \quad z = (x, y) \in \mathbb{R}^{m+k}.$$

The change of variable formula for the Lebesgue measure gives that

$$d \circ \delta_\lambda(x, y) = \lambda^Q dxdy,$$

where

$$Q = m + (1 + \gamma)k$$

is the homogeneous dimension with respect to dilation δ_λ. For $z = (x, y) \in \mathbb{R}^m \times \mathbb{R}^k$, let

$$\rho = \rho(z) := \Big(|x|^{2(1+\gamma)} + (1 + \gamma)^2 |y|^2\Big)^{\frac{1}{2(1+\gamma)}}. \tag{2.4}$$

By direct computation we get

$$|\nabla_\gamma \rho| = \frac{|x|^\gamma}{\rho^\gamma}.$$

Let $f \in C^2(0, \infty)$ and define $u = f(\rho)$ then we have the following useful formula

$$\Delta_\gamma u = \frac{|x|^{2\gamma}}{\rho^{2\gamma}} \Big(f'' + \frac{Q-1}{\rho} f'\Big). \tag{2.5}$$

We let $B_\rho = \{z \in \mathbb{R}^n \mid \rho(z) < r\}$, $B_{\tilde\rho} = \{z \in \mathbb{R}^n \mid \tilde\rho(z, 0) < r\}$ and call these sets, respectively, ρ-ball and Carnot-Carathéodory metric ball centered at the origin with radius r. The Carnot-Carathéodory distance $\tilde\rho$ between the points z andz_0 is defined by

$$\tilde\rho(z, z_0) = \inf\{\text{length}(\eta) \,|\, \eta \in \mathcal{K}\}$$

where the set $\mathcal{K}$ is the set of all curves η such that $\eta(0) = z$, $\eta(1) = z_0$ and $\dot\eta(t)$ is in $span\{X_1(\eta(t)), ..., X_m(\eta(t)), Y_1(\eta(t)), ..., Y_k(\eta(t))\}$. If γ is a positive even integer then Carnot-Carathéodory distance of z from the origin $\tilde\rho(z, 0)$ is comparable to $\rho(z)$. (See [FGW] and [Be] for further details.)

It is well known that Sobolev and Poincaré type inequalities are important in the study of partial differential equations, especially in the study of those arising from geometry and physics. In[FGW], Franchi, Gutierrez and Wheeden obtained the following Sobolev-Poincaré inequality for metric balls associated with Baouendi-Grushin type operators:

$$\Big(\frac{1}{w_1(B)} \int_B |\nabla_\gamma \phi|^p w_1(z) dz\Big)^{1/p} \geq \frac{1}{cr} \Big(\frac{1}{w_2(B)} \int_B |\phi(z)|^q w_2(z) dz\Big)^{1/q} \tag{2.6}$$

where $\phi \in C_0^\infty(B)$ and the weight functions w_1 and w_2 satisfies some certain conditions. Here, c is independent of ϕ and B, $1 \leq p \leq q < \infty$ and $w(B) = \int_B w(z) dz$. If $w_1 = w_2 = 1$ then Monti [M] obtained the following sharp Sobolev inequality

$$\Big(\int_{\mathbb{R}^n} (|\nabla_x \phi|^2 + |x|^{2\gamma} |\nabla_y \phi|^2) dxdy\Big)^{1/2} \geq C\Big(\int_{\mathbb{R}^n} |\phi|^{\frac{2Q}{Q-2}} dxdy\Big)^{\frac{Q-2}{2Q}} \tag{2.7}$$

where $C = C(m, k, \alpha) > 0$.

3. Improved Hardy-type inequalities

In this section we study improved Hardy type inequalities. These inequalities plays key role in establishing improved Rellich type inequalities. In the various integral inequalities below (Section 3 and Section 4), we allow the values of the integrals on the left-hand sides to be $+\infty$. The following theorem is the first result of this section.

Theorem 3.1. *Let γ be an even positive integer, $\alpha \in \mathbb{R}$, $-\frac{m}{\gamma} < t < \frac{m}{\gamma}$, and $Q+\alpha-2>0$. Then the following inequality is valid*

$$\begin{aligned}\int_{B_\rho} \rho^\alpha |\nabla_\gamma \rho|^t |\nabla_\gamma \phi|^2 dz \geq \Big(\frac{Q+\alpha-2}{2}\Big)^2 \int_{B_\rho} \rho^\alpha \frac{|\nabla_\gamma \rho|^{t+2}}{\rho^2}\phi^2 dz \\ + \frac{1}{C^2 r^2}\int_{B_\rho} \rho^\alpha |\nabla_\gamma \rho|^t \phi^2 dz\end{aligned} \tag{3.1}$$

for all compactly supported smooth function $\phi \in C_0^\infty(B_\rho)$.

Proof. Let $\phi = \rho^\beta \psi \in C_0^\infty(B_\rho)$ and $\beta \in \mathbb{R}\setminus\{0\}$. A direct calculation shows that

$$\begin{aligned}\int_{B_\rho} \rho^\alpha |\nabla_\gamma \rho|^t |\nabla_\gamma \phi|^2 dz = \beta^2 \int_{B_\rho} \rho^{\alpha+2\beta-2} |\nabla_\gamma \rho|^{t+2} \psi^2 dz \\ + 2\beta \int_{B_\rho} \rho^{\alpha+2\beta-1} |\nabla_\gamma \rho|^t \psi \nabla_\gamma \rho \cdot \nabla_\gamma \psi dz \\ + \int_{B_\rho} \rho^{\alpha+2\beta} |\nabla_\gamma \rho|^t |\nabla_\gamma \psi|^2 dz.\end{aligned} \tag{3.2}$$

Applying integration by parts to the middle term and using the following fact

$$\nabla_\gamma \cdot \big(\rho^{\alpha+2\beta-1}|\nabla_\gamma \rho|^t \nabla_\gamma \rho\big) = (Q+\alpha+2\beta-2)\rho^{\alpha+2\beta-2}|\nabla_\gamma\rho|^{t+2}$$

yields

$$\int_{B_\rho} \rho^\alpha |\nabla_\gamma \rho|^t |\nabla_\gamma \phi|^2 dz = f(\beta)\int_{B_\rho} \rho^{\alpha+2\beta-2}|\nabla_\gamma\rho|^{t+2}\psi^2 dz + \int_{B_\rho} \rho^{\alpha+2\beta}|\nabla_\gamma\rho|^t|\nabla_\gamma\psi|^2 dz \tag{3.3}$$

where $f(\beta) = -\beta^2 - \beta(\alpha+Q-2)$. Note that $f(\beta)$ attains the maximum for $\beta = \frac{2-\alpha-Q}{2}$, and this maximum is equal to $C_H = (\frac{Q+\alpha-2}{2})^2$. Therefore we have the following

$$\int_{B_\rho} \rho^\alpha |\nabla_\gamma \rho|^t |\nabla_\gamma \phi|^2 dz = C_H \int_{B_\rho} \rho^{\alpha-2}|\nabla_\gamma\rho|^{t+2}\phi^2 dz + \int_{B_\rho} \rho^{2-Q}|\nabla_\gamma\rho|^t|\nabla_\gamma\psi|^2 dz. \tag{3.4}$$

It is easy to show that the weight functions $w_1 = w_2 = \rho^{2-Q}|\nabla_\gamma\rho|^t$ satisfies the Muckenhoupt A_2 condition for $-\frac{m}{\gamma} < t < \frac{m}{\gamma}$. Therefore weighted Poincaré inequality holds (see [FGW], [Lu], [FGaW]) and we have

$$\begin{aligned}\int_{B_\rho} \rho^{2-Q}|\nabla_\gamma\rho|^t|\nabla_\gamma\psi|^2 dz &\geq \frac{1}{C^2r^2}\int_{B_\rho} \rho^{2-Q}|\nabla_\gamma\rho|^t\psi^2 dz \\ &= \frac{1}{C^2r^2}\int_{B_\rho} \rho^{\alpha}|\nabla_\gamma\rho|^t\phi^2 dz\end{aligned}$$

where C is a positive constant and r^2 is the radius of the ball B_ρ.

We now obtain the desired inequality

$$\int_{B_\rho} \rho^\alpha |\nabla_\gamma \rho|^t |\nabla_\gamma \phi|^2 dz \geq C_H \int_{B_\rho} \rho^{\alpha-2} |\nabla_\gamma \rho|^{t+2} \phi^2 dz + \frac{1}{C^2 r^2} \int_{B_\rho} \rho^\alpha |\nabla_\gamma \rho|^t \phi^2 dz. \tag{3.5}$$

□

Using the same method, we have the following weighted Hardy inequality which has a logarithmic remainder term. Similar results in the Euclidean setting can be found in [FT], [AR], [WW], [ACP].

Theorem 3.2. *Let $\alpha \in \mathbb{R}$, $t \in \mathbb{R}$, $Q + \alpha - 2 > 0$. Then the following inequality is valid*

$$\int_{B_\rho} \rho^\alpha |\nabla_\gamma \rho|^t |\nabla_\gamma \phi|^2 dz \geq C_H \int_{B_\rho} \rho^{\alpha-2} |\nabla_\gamma \rho|^{t+2} \phi^2 dz + \frac{1}{4} \int_{B_\rho} \rho^{\alpha-2} |\nabla_\gamma \rho|^{t+2} \frac{\phi^2}{(\ln \frac{r}{\rho})^2} dz \tag{3.6}$$

for all compactly supported smooth function $\phi \in C_0^\infty(B_\rho)$.

Proof. We have the following result from (3.4):

$$\int_{B_\rho} \rho^\alpha |\nabla_\gamma \rho|^t |\nabla_\gamma \phi|^2 dz = C_H \int_{B_\rho} \rho^{\alpha-2} |\nabla_\gamma \rho|^{t+2} \phi^2 dz + \int_{B_\rho} \rho^{2-Q} |\nabla_\gamma \rho|^t |\nabla_\gamma \psi|^2 dz. \tag{3.7}$$

Let $\varphi \in C_0^\infty(B_\rho)$ and set $\psi(z) = (\ln \frac{r}{\rho})^{1/2} \varphi(z)$. A direct computation shows that

$$\begin{aligned} \int_{B_\rho} \rho^{2-Q} |\nabla_\gamma \rho|^t |\nabla_\gamma \psi|^2 dz &\geq \frac{1}{4} \int_{B_\rho} \rho^{-Q} |\nabla_\gamma \rho|^{t+2} \frac{\psi^2}{(\ln \frac{r}{\rho})^2} dz \\ &= \frac{1}{4} \int_{B_\rho} \rho^{\alpha-2} |\nabla_\gamma \rho|^{t+2} \frac{\phi^2}{(\ln \frac{r}{\rho})^2} dz. \end{aligned} \tag{3.8}$$

Substituting (3.8) into (3.7) which yields the desired inequality (3.6). □

We now first prove the following weighted L^p-Hardy inequality which plays an important role in the proof of Theorem 3.3, Theorem 4.1 and Theorem 4.5.

Theorem 3.3. *Let Ω be either bounded or unbounded domain with smooth boundary which contains origin, or $\mathbb{R}^n$. Let $\alpha \in \mathbb{R}$, $t \in \mathbb{R}$, $1 \leq p < \infty$ and $Q + \alpha - p > 0$. Then the following inequality holds*

$$\int_\Omega \rho^\alpha |\nabla_\gamma \rho|^t |\nabla_\gamma \phi|^p dz \geq \Big(\frac{Q + \alpha - p}{p}\Big)^p \int_\Omega \rho^\alpha |\nabla_\gamma \rho|^t \frac{|\nabla_\gamma \rho|^p}{\rho^p} |\phi|^p dz \tag{3.9}$$

for all compactly supported smooth functions $\phi \in C_0^\infty(\Omega)$.

Proof. Let $\phi = \rho^\beta \psi \in C_0^\infty(\Omega)$ and $\beta \in \mathbb{R} - \{0\}$. We have

$$|\nabla_\gamma(\rho^\beta \psi)| = |\beta \rho^{\beta-1} \psi \nabla_\gamma \rho + \rho^\beta \nabla_\gamma \psi|.$$

We now use the following inequality which is valid for any $a, b \in \mathbb{R}^n$ and $p > 2$,

$$|a + b|^p - |a|^p \geq c(p)|b|^p + p|a|^{p-2} a \cdot b$$

where $c(p) > 0$. This yields

$$\rho^\alpha |\nabla_\gamma \rho|^t |\nabla \phi|^p \geq |\beta|^p \rho^{\beta p - p + \alpha} |\nabla_\gamma \rho|^{p+t} |\psi|^p + p|\beta|^{p-2} \beta \rho^{\alpha + \beta p + 1 - p} |\nabla_\gamma \rho|^{p+t-2} |\psi|^{p-2} \psi \nabla \rho \cdot \nabla \psi.$$

Integrating over the domain Ω gives

$$\begin{aligned}\int_\Omega \rho^\alpha |\nabla_\gamma \rho|^t |\nabla \phi|^p dx \geq |\beta|^p &\int_\Omega \rho^{\beta p - p + \alpha} |\nabla_\gamma \rho|^t |\psi|^p dz \\ &- p \int_\Omega |\beta|^{p-2} \beta \rho^{\alpha + \beta p + 1 - p} |\nabla_\gamma \rho|^{p+t-2} |\psi|^{p-2} \psi \nabla \rho \cdot \nabla \psi dz.\end{aligned} \tag{3.10}$$

Applying integration by parts to second integral on the right-hand side of (3.10) and using the fact that $\nabla_\gamma(|\nabla_\gamma \rho|) \cdot \nabla_\gamma \rho = 0$ then we get

$$\int_\Omega \rho^\alpha |\nabla_\gamma \rho|^t |\nabla \phi|^p dx \geq \Big(|\beta|^p - |\beta|^{p-2}\beta(\beta p - p + \alpha + Q)\Big) \int_\Omega \rho^{\beta p - p + \alpha} |\nabla_\gamma \rho|^{p+t} |\psi|^p dz.$$

We now choose $\beta = \frac{p-Q-\alpha}{p}$ to get the desired inequality

$$\int_\Omega \rho^\alpha |\nabla_\gamma \rho|^t |\nabla \phi|^p dz \geq \Big(\frac{Q+\alpha-p}{p}\Big)^p \int_\Omega \rho^\alpha |\nabla_\gamma \rho|^t \frac{|\nabla_\gamma \rho|^p}{\rho^p} |\phi|^p dz. \tag{3.11}$$

Theorem (3.3) also holds for $1 < p < 2$ and in this case we use the following inequality

$$|a+b|^p - |a|^p \geq c(p) \frac{|b|^2}{(|a|+|b|)^{2-p}} + p|a|^{p-2} a \cdot b$$

where $c(p) > 0$ (see [L]). □

We now have the following improved Hardy inequality which is inspired by recent result of Abdellaoui, Colorado and Peral [ACP]. It is clear that if $\gamma = t = 0$ then our result recovers the inequality (1.4).

Theorem 3.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary which contains origin, $1 < q < 2$, $Q + \alpha - 2 > 0$, $Q = m + (1+\gamma)k$ and $\phi \in C_0^\infty(\Omega)$ then there exists a positive constant $C = C(Q, q, \Omega)$ such that the following inequality is valid*

$$\int_\Omega \rho^\alpha |\nabla_\gamma \rho|^t |\nabla_\gamma \phi|^2 dz \geq C_H \int_\Omega \rho^\alpha \frac{|\nabla_\gamma \rho|^{t+2}}{\rho^2} \phi^2 dz + C\Big(\int_\Omega |\nabla_\gamma \phi|^q \big(|\nabla_\gamma \rho|^t \rho^\alpha\big)^{\frac{q}{2}} dz\Big)^{2/q} \tag{3.12}$$

where $C_H = \big(\frac{Q+\alpha-2}{2}\big)^2$.

Proof. Let $\phi \in C_0^\infty(\Omega)$ and $\psi = \rho^\beta$ where $\beta \in \mathbb{R} \setminus \{0\}$. Then straightforward computation shows that

$$|\nabla_\gamma \phi|^2 - \nabla_\gamma\Big(\frac{\phi^2}{\psi}\Big) \cdot \nabla_\gamma \psi = \Big|\nabla_\gamma \phi - \frac{\phi}{\psi} \nabla_\gamma \psi\Big|^2.$$

Therefore

$$\begin{aligned}\int_\Omega \Big(|\nabla_\gamma \phi|^2 - \nabla_\gamma\Big(\frac{\phi^2}{\psi}\Big) \cdot \nabla_\gamma \psi\Big) \rho^\alpha |\nabla_\gamma \rho|^t dz &= \int_\Omega \Big|\nabla_\gamma \phi - \frac{\phi}{\psi} \nabla_\gamma \psi\Big|^2 \rho^\alpha |\nabla_\gamma \rho|^t dz \\ &\geq c\Big(\int_\Omega \Big|\nabla_\gamma \phi - \frac{\phi}{\psi} \nabla_\gamma \psi\Big|^q \rho^{\frac{q\alpha}{2}} |\nabla_\gamma \rho|^{\frac{qt}{2}} dz\Big)^{2/q}\end{aligned}$$

where we used the Jensen's inequality in the last step. Applying integration by parts, we obtain

$$\begin{aligned}\int_\Omega \Big(|\nabla_\gamma\phi|^2-\nabla_\gamma(\frac{\phi^2}{\psi})\cdot\nabla_\gamma\psi\Big)\rho^\alpha|\nabla_\gamma\rho|^t dz &= \int_\Omega |\nabla_\gamma\phi|^2\rho^\alpha|\nabla_\gamma\rho|^t dz\\ &+\frac{\beta}{\alpha+\beta}\int_\Omega\Big(\frac{\Delta_\gamma(\rho^{\alpha+\beta})}{\rho^\beta}\Big)|\nabla_\gamma\rho|^t\phi^2 dz\\ &=\int_\Omega \rho^\alpha|\nabla_\gamma\rho|^t|\nabla_\gamma\phi|^2 dz\\ &+\beta(\alpha+\beta+Q-2)\int_\Omega\rho^\alpha\frac{|\nabla_\gamma\rho|^{t+2}}{\rho^2}\phi^2 dz.\end{aligned}$$

Therefore we have

$$\begin{aligned}\int_\Omega \rho^\alpha|\nabla_\gamma\rho|^t|\nabla_\gamma\phi|^2 dz &\geq -\beta(\alpha+\beta+Q-2)\int_\Omega\rho^\alpha\frac{|\nabla_\gamma\rho|^{t+}2}{\rho^2}\phi^2 dz\\ &+c\Big(\int_\Omega\Big|\nabla_\gamma\phi-\frac{\phi}{\psi}\nabla_\gamma\psi\Big|^q\rho^{\frac{q\alpha}{2}}|\nabla_\gamma\rho|^{\frac{qt}{2}}dz\Big)^{2/q}.\end{aligned}\tag{3.13}$$

We can use the following inequality which is valid for any $w_1,w_2\in\mathbb{R}^n$ and $1<q<2$

$$c(q)|w_2|^q\geq|w_1+w_2|^q-|w_1|^q-q|w_1|^{q-2}\langle w_1,w_2\rangle.\tag{3.14}$$

Using the inequality (3.14), Young's inequality and the weighted L^p-Hardy inequality (3.9), we get

$$\int_\Omega\Big|\nabla_\gamma\phi-\frac{\phi}{\psi}\nabla_\gamma\psi\Big|^q\rho^{\frac{q\alpha}{2}}|\nabla_\gamma\rho|^{\frac{qt}{2}}dz\geq C\int_\Omega|\nabla_\gamma\phi|^q\rho^{\frac{q\alpha}{2}}|\nabla_\gamma\rho|^{\frac{qt}{2}}dz\tag{3.15}$$

where $C>0$. Substituting (3.15) into (3.13) then we obtain

$$\int_\Omega \rho^\alpha|\nabla_\gamma\rho|^t|\nabla_\gamma\phi|^2 dz\geq-\beta(\alpha+\beta+Q-2)\int_\Omega\rho^\alpha\frac{|\nabla_\gamma\rho|^{t+2}}{\rho^2}\phi^2dz+C\Big(\int_\Omega|\nabla_\gamma\phi|^q\rho^{\frac{q\alpha}{2}}|\nabla_\gamma\rho|^{\frac{qt}{2}}dz\Big)^{2/q}.$$

Now choosing $\beta=\frac{2-\alpha-Q}{2}$ then we have the following inequality

$$\int_\Omega \rho^\alpha|\nabla_\gamma\rho|^t|\nabla_\gamma\phi|^2 dz\geq\Big(\frac{Q+\alpha-2}{2}\Big)^2\int_\Omega\rho^\alpha\frac{|\nabla_\gamma\rho|^{t+2}}{\rho^2}\phi^2dz+C\Big(\int_\Omega|\nabla_\gamma\phi|^q\rho^{\frac{q\alpha}{2}}|\nabla_\gamma\rho|^{\frac{qt}{2}}dz\Big)^{2/q}.$$

□

4. Sharp Weighted Rellich-type inequalities

The main goal of this section is to find sharp analogues of (1.6) and (1.7) for Baouendi-Grushin vector fields. We then obtain their improved versions for bounded domains. The proofs are mainly based on Hardy type inequalities. The following is the first result of this section.

Theorem 4.1. *(Rellich type inequality I) Let $\phi\in C_0^\infty(\mathbb{R}^{m+k}\setminus\{(0,0)\})$, $Q=m+(1+\gamma)k$ and $\alpha>2$. Then the following inequality is valid*

$$\int_{\mathbb{R}^n}\frac{\rho^\alpha}{|\nabla_\gamma\rho|^2}|\Delta_\gamma\phi|^2dz\geq\frac{(Q+\alpha-4)^2(Q-\alpha)^2}{16}\int_{\mathbb{R}^n}\rho^\alpha\frac{|\nabla_\gamma\rho|^2}{\rho^4}\phi^2dz.\tag{4.1}$$

Moreover, the constant $\frac{(Q+\alpha-4)^2(Q-\alpha)^2}{16}$ is sharp.

Proof. A straightforward computation shows that

$$\Delta_\gamma \rho^{\alpha-2} = (Q+\alpha-4)(\alpha-2)\rho^{\alpha-4}|\nabla_\gamma \rho|^2. \tag{4.2}$$

Multiplying both sides of (4.2) by ϕ^2 and integrating over $\mathbb{R}^n$, we obtain

$$\int_{\mathbb{R}^n} \phi^2 \Delta_\gamma \rho^{\alpha-2} dz = \int_{\mathbb{R}^n} \rho^{\alpha-2}(2\phi\Delta_\gamma\phi + 2|\nabla_\gamma\phi|^2)dz.$$

Since

$$\int_{\mathbb{R}^n} \phi^2 \Delta_\gamma \rho^{\alpha-2} dz = (Q+\alpha-4)(\alpha-2)\int_{\mathbb{R}^n} \rho^{\alpha-4}|\nabla_\gamma\rho|^2\phi^2 dz.$$

Therefore

$$(Q+\alpha-4)(\alpha-2)\int_{\mathbb{R}^n} \rho^{\alpha-4}|\nabla_\gamma\rho|^2\phi^2 dz - 2\int_{\mathbb{R}^n}\rho^{\alpha-2}\phi\Delta_\gamma\phi dx = 2\int_{\mathbb{R}^n}\rho^{\alpha-2}|\nabla_\gamma\phi|^2 dz. \tag{4.3}$$

Applying the weighted Hardy inequality (3.9) to the right hand side of (4.3), we get

$$-\int_{\mathbb{R}^n}\rho^{\alpha-2}\phi\Delta_\gamma\phi dz \geq (\frac{Q+\alpha-4}{2})(\frac{Q-\alpha}{2})\int_{\mathbb{R}^n}\rho^{\alpha-4}|\nabla_\gamma\rho|^2\phi^2 dz. \tag{4.4}$$

We now apply the Cauchy-Schwarz inequality to obtain

$$-\int_{\mathbb{R}^n}\rho^{\alpha-2}\phi\Delta_\gamma\phi dz \leq \Big(\int_{\mathbb{R}^n}\rho^{\alpha-4}|\nabla_\gamma\rho|^2\phi^2 dz\Big)^{1/2}\Big(\int_{\mathbb{R}^n}\frac{\rho^\alpha}{|\nabla_\gamma\rho|^2}|\Delta_\gamma\phi|^2 dz\Big)^{1/2}. \tag{4.5}$$

Substituting (4.5) into (4.4) yields the desired inequality

$$\int_{\mathbb{R}^n}\frac{\rho^\alpha}{|\nabla_\gamma\rho|^2}|\Delta_\gamma\phi|^2 dz \geq \frac{(Q+\alpha-4)^2(Q-\alpha)^2}{16}\int_{\mathbb{R}^n}\rho^\alpha\frac{|\nabla_\gamma\rho|^2}{\rho^4}\phi^2 dz. \tag{4.6}$$

It only remains to show that the constant $C(Q,\alpha) = \frac{(Q+\alpha-4)^2(Q-\alpha)^2}{16}$ is the best constant for the Rellich inequality (4.1), that is

$$\frac{(Q+\alpha-4)^2(Q-\alpha)^2}{16} = \inf\Big\{\frac{\int_{\mathbb{R}^n}\rho^\alpha\frac{|\Delta_\gamma f|^2}{|\nabla_\gamma\rho|^2}dz}{\int_{\mathbb{R}^n}\rho^\alpha\frac{|\nabla_\gamma\rho|^2}{\rho^4}f^2 dz}, f\in C_0^\infty(\mathbb{R}^n), f\neq 0\Big\}.$$

Given $\epsilon > 0$, take the radial function

$$\phi_\epsilon(\rho) = \begin{cases}(\frac{Q+\alpha-4}{2}+\epsilon)\big(\rho-1\big)+1 & \text{if} \quad \rho\in[0,1],\\ \rho^{-(\frac{Q+\alpha-4}{2}+\epsilon)} & \text{if} \quad \rho>1,\end{cases} \tag{4.7}$$

where $\epsilon > 0$. In the sequel we indicate $B_1 = \{\rho(z) : \rho(z) \leq 1\}$ ρ-ball centered at the origin in $\mathbb{R}^n$ with radius 1.

By direct computation we get

$$\begin{aligned}\int_{B_\rho}\rho^\alpha\frac{|\Delta_\gamma\phi_\epsilon|^2}{|\nabla_\gamma\rho|^2}dz &= \int_{B_1}\rho^\alpha\frac{|\Delta_\gamma\phi_\epsilon|^2}{|\nabla_\gamma\rho|^2}dz + \int_{B_\rho\setminus B_1}\rho^\alpha\frac{|\Delta_\gamma\phi_\epsilon|^2}{|\nabla_\gamma\rho|^2}dz,\\ &= A(Q,\alpha,\epsilon) + B(Q,\alpha,\epsilon)\int_{B_\rho\setminus B_1}\rho^{-Q-2\epsilon}|\nabla_\gamma\rho|^2 dz\end{aligned} \tag{4.8}$$

where

$$B(Q,\alpha,\epsilon)=(\frac{Q+\alpha-4}{2}+\epsilon)^2(\frac{Q-\alpha}{2}-\epsilon)^2.$$

$$\begin{aligned}\int_{B_\rho}\rho^\alpha\frac{|\nabla_\gamma\rho|^2}{\rho^4}\phi^2dz&=\int_{B_1}\rho^\alpha\frac{|\nabla_\gamma\rho|^2}{\rho^4}\phi^2dz+\int_{B_\rho\setminus B_1}\rho^\alpha\frac{|\nabla_\gamma\rho|^2}{\rho^4}\phi^2dz\\&=C(Q,\alpha,\epsilon)+\int_{B_\rho\setminus B_1}\rho^{-Q-2\epsilon}dz.\end{aligned}\tag{4.9}$$

Since $Q+\alpha-4>0$ then $A(Q,\alpha,\epsilon)$, and $C(Q,\alpha,\epsilon)$ are bounded and we conclude by letting $\epsilon\longrightarrow 0$. □

Using the same argument as above and improved Hardy inequality (3.1), we obtain the following improved Rellich type inequality.

Theorem 4.2. *Let $\phi\in C_0^\infty(B_\rho)$, $Q=m+(1+\gamma)k$ and $4-Q<\alpha<Q$. Then the following inequality is valid*

$$\begin{aligned}\int_{B_\rho}\frac{\rho^\alpha}{|\nabla_\gamma\rho|^2}|\Delta_\gamma\phi|^2dz\geq&\frac{(Q+\alpha-4)^2(Q-\alpha)^2}{16}\int_{B_\rho}\rho^\alpha\frac{|\nabla_\gamma\rho|^2}{\rho^4}\phi^2dz\\&+\frac{(Q+\alpha-4)(Q-\alpha)}{2C^2r^2}\int_{B_\rho}\rho^{\alpha-2}\phi^2dz.\end{aligned}\tag{4.10}$$

Proof. We have the following fact from (4.3):

$$(Q+\alpha-4)(\alpha-2)\int_{B_\rho}\rho^{\alpha-4}|\nabla_\gamma\rho|^2\phi^2dz-2\int_{B_\rho}\rho^{\alpha-2}\phi\Delta_\gamma\phi dx=2\int_{B_\rho}\rho^{\alpha-2}|\nabla_\gamma\phi|^2dz.\tag{4.11}$$

Applying the improved Hardy inequality (3.1) on the right hand side of (4.11), we get

$$\begin{aligned}&(Q+\alpha-4)(\alpha-2)\int_{B_\rho}\rho^{\alpha-4}|\nabla_\gamma\rho|^2\phi^2dz-2\int_{B_\rho}\rho^{\alpha-2}\phi\Delta_\gamma\phi dz\\&\geq 2(\frac{Q+\alpha-4}{2})^2\int_{B_\rho}\rho^{\alpha-4}|\nabla_\gamma\rho|^2\phi^2dz+\frac{2}{C^2r^2}\int_{B_\rho}\rho^{\alpha-2}\phi^2dz\end{aligned}$$

Now it is clear that,

$$\begin{aligned}-\int_{B_\rho}\rho^{\alpha-2}\phi\Delta_\gamma\phi dz\geq&(\frac{Q+\alpha-4}{2})(\frac{Q-\alpha}{2})\int_{B_\rho}\rho^{\alpha-4}|\nabla_\gamma\rho|^2\phi^2dz\\&+\frac{1}{C^2r^2}\int_{B_\rho}\rho^{\alpha-2}\phi^2dz.\end{aligned}\tag{4.12}$$

Next, we apply the Young's inequality to the expression $-\int_{B_\rho}\rho^{\alpha-2}\phi\Delta\phi dz$ and we obtain

$$-\int_{B_\rho}\rho^{\alpha-2}\phi\Delta_\gamma\phi dz\leq\epsilon\int_{B_\rho}\rho^{\alpha-4}|\nabla_\gamma\rho|^2\phi^2dz+\frac{1}{4\epsilon}\int_{B_\rho}\rho^\alpha\frac{|\Delta_\gamma\phi|^2}{|\nabla_\gamma\rho|^2}dz\tag{4.13}$$

where $\epsilon>0$. Combining (4.13) and (4.12), we obtain

$$\int_{B_\rho}\rho^\alpha\frac{|\Delta_\gamma\phi|^2}{|\nabla_\gamma\rho|^2}dz\geq\big(-4\epsilon^2-(Q+\alpha-4)(Q-\alpha)\epsilon\big)\int_{B_\rho}\rho^{\alpha-4}|\nabla_\gamma\rho|^2\phi^2dz+\frac{4\epsilon}{C^2r^2}\int_{B_\rho}\rho^{\alpha-2}\phi^2dz.$$

Note that the quadratic function $-4\epsilon^2 - (Q+\alpha-4)(Q-\alpha)\epsilon$ attains the maximum for $\epsilon = \frac{(Q+\alpha-4)(Q-\alpha)}{8}$ and this maximum is equal to $\frac{(Q+\alpha-4)^2(Q-\alpha)^2}{16}$. Therefore we obtain the desired inequality

$$\begin{aligned}\int_{B_\rho} \frac{\rho^\alpha}{|\nabla_\gamma \rho|^2}|\Delta_\gamma \phi|^2 dz \geq \frac{(Q+\alpha-4)^2(Q-\alpha)^2}{16}\int_{B_\rho} \rho^\alpha \frac{|\nabla_\gamma \rho|^2}{\rho^4}\phi^2 dz \\ + \frac{(Q+\alpha-4)(Q-\alpha)}{2C^2 r^2}\int_{B_\rho} \rho^{\alpha-2}\phi^2 dz.\end{aligned} \tag{4.14}$$

□

Arguing as above, and using the improved Hardy inequalities (3.2) and (3.4) we obtain the following Rellich type inequalities.

Theorem 4.3. *Let $\phi \in C_0^\infty(\mathbb{R}^{m+k} \setminus \{(0,0)\})$, $Q = m + (1+\gamma)k$ and $4 - Q < \alpha < Q$. Then the following inequality is valid*

$$\begin{aligned}\int_{B_\rho} \frac{\rho^\alpha}{|\nabla_\gamma \rho|^2}|\Delta_\gamma \phi|^2 dz \geq \frac{(Q+\alpha-4)^2(Q-\alpha)^2}{16}\int_{B_\rho} \rho^{\alpha-4}|\nabla_\gamma \rho|^2\phi^2 dz \\ + \frac{(Q+\alpha-4)(Q-\alpha)}{8}\int_{B_\rho} \rho^{\alpha-4}|\nabla_\gamma \rho|^2 \frac{\phi^2}{\ln(\frac{r}{\rho})^2} dz.\end{aligned} \tag{4.15}$$

Theorem 4.4. *Let $\phi \in C_0^\infty(\mathbb{R}^{m+k} \setminus \{(0,0)\})$, $Q = m + (1+\gamma)k$ and $4 - Q < \alpha < Q$. Then the following inequality is valid*

$$\begin{aligned}\int_{B_\rho} \frac{\rho^\alpha}{|\nabla_\gamma \rho|^2}|\Delta_\gamma \phi|^2 dz \geq \frac{(Q+\alpha-4)^2(Q-\alpha)^2}{16}\int_{B_\rho} \rho^\alpha \frac{|\nabla_\gamma \rho|^2}{\rho^4}\phi^2 dz \\ + \frac{C(Q-\alpha)(Q+3\alpha-8)}{4}\Big(\int_\Omega |\nabla_\gamma \phi|^q \rho^{\frac{q\alpha}{2}} dz\Big)^{2/q}\end{aligned} \tag{4.16}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary.

We now have the following Rellich type inequality that connects first to second order derivatives. It is clear that if $\alpha = \gamma = 0$ then our result covers the inequality (1.7).

Theorem 4.5. *(Rellich type inequality II) Let $\phi \in C_0^\infty(\mathbb{R}^{m+k} \setminus \{(0,0)\})$, $Q = m + (1+\gamma)k$ and $2 < \alpha < Q$. Then the following inequality is valid*

$$\int_{\mathbb{R}^n} \rho^\alpha \frac{|\Delta_\gamma \phi|^2}{|\nabla_\gamma \rho|^2} dz \geq \frac{(Q-\alpha)^2}{4}\int_{\mathbb{R}^n} \rho^\alpha \frac{|\nabla_\gamma \phi|^2}{\rho^2} dz. \tag{4.17}$$

Furthermore, the constant $C(Q,\alpha) = \big(\frac{Q-\alpha}{2}\big)^2$ is sharp.

Proof. The proof of this theorem is similar to the proof Theorem (4.1). Using the same argument as above, we have the following from (4.3)

$$-\int_{\mathbb{R}^n} \rho^{\alpha-2}\phi \Delta_\gamma \phi dx = \int_{\mathbb{R}^n} \rho^{\alpha-2}|\nabla_\gamma \phi|^2 dz - \frac{(Q+\alpha-4)(\alpha-2)}{2}\int_{\mathbb{R}^n} \rho^{\alpha-4}|\nabla_\gamma \rho|^2 \phi^2 dz. \tag{4.18}$$

It is clear that $(Q+\alpha-4)(\alpha-2) > 0$ and using the Hardy inequality (3.9) $(p = 2, t = 0)$ we get

$$-\int_{\mathbb{R}^n}\rho^{\alpha-2}\phi\Delta_\gamma\phi dz \geq \frac{Q-\alpha}{Q+\alpha-4}\int_{\mathbb{R}^n}\rho^{\alpha-2}|\nabla_\gamma\phi|^2dz. \tag{4.19}$$

Let us apply Young's inequality to expression $-\int_{\mathbb{R}^n}\rho^{\alpha-2}\phi\Delta_\gamma\phi\, dz$ and we obtain

$$\begin{aligned}-\int_{\mathbb{R}^n}\rho^{\alpha-2}\phi\Delta_\gamma\phi dz &\leq \epsilon\int_{\mathbb{R}^n}\rho^{\alpha-4}|\nabla_\gamma\rho|^2\phi^2dz + \frac{1}{4\epsilon}\int_{\mathbb{R}^n}\rho^\alpha\frac{|\Delta_\gamma\phi|^2}{|\nabla_\gamma\rho|^2}dz\\ &\leq \epsilon(\frac{2}{Q+\alpha-4})^2\int_{\mathbb{R}^n}\rho^{\alpha-2}|\nabla_\gamma\phi|^2dz + \frac{1}{4\epsilon}\int_{\mathbb{R}^n}\rho^\alpha\frac{|\Delta_\gamma\phi|^2}{|\nabla_\gamma\rho|^2}dz\end{aligned} \tag{4.20}$$

where $\epsilon > 0$ and will be chosen later. Substituting (4.20) into (4.19) and rearranging terms, we get

$$\int_{\mathbb{R}^n}\rho^\alpha\frac{|\Delta_\gamma\phi|^2}{|\nabla_\gamma\rho|^2}dz \geq \frac{-16\epsilon^2}{(Q+\alpha-4)^2} + 4(\frac{Q-\alpha}{Q+\alpha-4})\epsilon\int_{\mathbb{R}^n}\rho^\alpha\frac{|\nabla_\gamma\phi|^2}{\rho^2}dz. \tag{4.21}$$

Choosing $\epsilon = \frac{1}{8}(Q-\alpha)(Q+\alpha-4)$ which yields the desired inequality

$$\int_{\mathbb{R}^n}\rho^\alpha\frac{|\Delta_\gamma\phi|^2}{|\nabla_\gamma\rho|^2}dz \geq \frac{(Q-\alpha)^2}{4}\int_{\mathbb{R}^n}\rho^\alpha\frac{|\nabla_\gamma\phi|^2}{\rho^2}dz. \tag{4.22}$$

To show that constant $\left(\frac{Q-\alpha}{2}\right)^2$ is sharp, we use the same sequence of functions (4.7) and we get

$$\frac{\int_{\mathbb{R}^n}\rho^\alpha\frac{|\Delta_\gamma\phi_\epsilon|^2}{|\nabla_\gamma\rho|^2}dz}{\int_{\mathbb{R}^n}\rho^\alpha\frac{|\nabla_\gamma\phi_\epsilon|^2}{\rho^2}dz} \longrightarrow (\frac{Q-\alpha}{2})^2$$

as $\epsilon \longrightarrow 0$.

Now, using the same argument as above and improved Hardy inequalities (3.1), (3.6) and (3.7) we obtain the following improved Rellich type inequalities.

Theorem 4.6. *Let $\phi \in C_0^\infty(B_\rho)$, $Q = m + (1+\gamma)k$ and $2 < \alpha < Q$. Then the following inequality is valid*

$$\int_{B_\rho}\rho^\alpha\frac{|\Delta_\gamma\phi|^2}{|\nabla_\gamma\rho|^2}dz \geq \frac{(Q-\alpha)^2}{4}\int_{B_\rho}\rho^\alpha\frac{|\nabla_\gamma\phi|^2}{\rho^2}dz + \frac{(Q-\alpha)(Q+3\alpha-8)}{4C^2r^2}\int_{B_\rho}\rho^\alpha\frac{\phi^2}{\rho^2}dz \tag{4.23}$$

where $C > 0$ and r is the radius of the ball B_ρ.

Theorem 4.7. *Let Ω be a bounded domain with smooth boundary $\partial\Omega$. Let $\phi \in C_0^\infty(\Omega)$, $Q = m + (1+\gamma)k$ and $2 < \alpha < Q$. Then the following inequality is valid*

$$\int_\Omega\rho^\alpha\frac{|\Delta_\gamma\phi|^2}{|\nabla_\gamma\rho|^2}dz \geq (\frac{Q-\alpha}{2})^2\int_\Omega\rho^\alpha\frac{|\nabla_\gamma\phi|^2}{\rho^2}dz + \tilde{C}\Big(\int_\Omega|\nabla_\gamma\phi|^q\rho^{\frac{q(\alpha-2)}{2}}dz\Big)^{2/q} \tag{4.24}$$

where $\tilde{C} = \frac{C(Q-\alpha)(Q+3\alpha-8)}{4}$ and $C > 0$.

Theorem 4.8. *Let $\phi \in C_0^\infty(B_\rho)$, $Q = m + (1+\gamma)k$ and $2 < \alpha < Q$. Then the following inequality is valid*

$$\int_{B_\rho}\rho^\alpha\frac{|\Delta_\gamma\phi|^2}{|\nabla_\gamma\rho|^2}dz \geq \frac{(Q-\alpha)^2}{4}\int_{B_\rho}\rho^\alpha\frac{|\nabla_\gamma\phi|^2}{\rho^2}dz + C(Q,\alpha)\int_{B_\rho}\rho^{\alpha-4}|\nabla_\gamma\rho|^2\frac{\phi^2}{(\ln\frac{r}{\rho})^2}dz \tag{4.25}$$

where $C(Q,\alpha) = \frac{(Q-\alpha)(Q+3\alpha-8)}{16}$.

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