# DIFFERENTIABILITY OF MINIMAL GEODESICS IN METRICS OF LOW REGULARITY

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**Abstract.** In Riemannian metrics that are only Hölder continuous of order $\alpha$, $0 \le \alpha \le 1$, let minimal geodesics be the continuous curves realizing the shortest distance between two points. It is shown that for $0 < \alpha \le 1$, minimal geodesics are differentiable with a derivative which is at least Hölder continuous of order $\alpha/2$ for $0 < \alpha < 1$ and which is Hölder continuous of order 1 for $\alpha = 1$.

## 1. **Introduction**

In this paper we study the regularity of curves that realize the shortest distance between two points, in a metric that has only low regularity. Let $M$ be finite-dimensional, smooth, i.e., $C^\infty$ manifold with a symmetric, positive definite metric $g$. In local coordinates $g$ can be written $\sum g_{ij}(x)dx^i dx^j$. If all the coefficient $g_{ij}$ are in $C^\infty$, the metric is conventionally called Riemannian and we will use the terminology $C^k$-Riemannian or even $C^{k,\alpha}$-Riemannian for metrics that have coefficients in $C^k$ and $C^{k,\alpha}$ respectively. Here $C^{k,\alpha}$ denotes $k$ times differentiable functions whose derivatives of order $k$ all are Hölder continuous of order $\alpha$. A curve is a continuous mapping from a closed interval on the real axis to $M$. If the curve $\gamma(t)$, $a \le t \le b$ is piecewise $C^1$ its arc-length is defined by

$$\int_a^b g(d\gamma(t)/dt, d\gamma(t)/dt)^{1/2}\, dt.$$

If $g(\gamma', \gamma') = 1$ is the curve said to be parametrized by arc-length. Corresponding to $g$ is the distance function $d : M \times M \to \mathbb{R}_+ \cup \{0\}$ defined by

$$d(x,y) = \inf \int_0^1 g(d\eta(t)/dt, d\eta(t)/dt)^{1/2}\, dt$$

where the infimum is taken over all piecewise smooth curves $\eta$ with $\eta(0) = x$ and $\eta(1) = y$. In Riemannian geometry a minimal geodesic is a curve with shortest arc-length between two points $x$, $y$. A minimal geodesic is smooth and the arc-length is equal to the distance $d(x,y)$. If the manifold is complete in the metric $g$ then any two points can be joined by a minimal geodesic according to the Hopf-Rinow theorem.

By inspecting the proof in standard Riemannian geometry, cf. for example [1], it is easily seen that it is not required that the metric is smooth. That $g$ is $C^1$-Riemannian is sufficient. For example, a geodesic $\gamma$ has vanishing covariant derivative, which can be written

$$d^2\gamma^k(t)/dt^2 + \sum_{i,j=1}^{\dim M} \Gamma^k_{ij}(\gamma(t))d\gamma^i(t)/dt\, d\gamma^j(t)/dt = 0.$$

Here $\Gamma^k_{ij}$ are the Christoffel symbols which in local coordinates can be written

$$\Gamma^k_{ij} = \sum_{l=1}^{\dim M} g^{kl}(\partial g_{lj}/\partial x_i + \partial g_{il}/\partial x_j - \partial g_{ij}/\partial x_l)$$

where $(g^{ij})$ is the inverse matrix of $(g_{ij})$. Hence, for any $k \geq 1$, the symbols $\Gamma^k_{ij}$ belong to $C^{k-1}$ if all $g_{ij} \in C^k$, and by standard regularity theorems of ordinary differential equations the solution $\gamma(t)$ is in $C^{k+1}$. Thus we have the simple observation

PROPOSITION 1.1. *A minimal geodesic parametrized by arc-length in a $C^k$-Riemannian metric with $k \geq 1$ is in $C^{k+1}$.*

That this result is optimal generally is shown by the following simple example.

EXAMPLE 1.2. Let $M = \mathbb{R}^2$ with Euclidean metric $g = dx^2 + dy^2$ and consider the line $\gamma : [-1/2, 1/2] \ni t \to (t, 0)$ which is a minimal geodesic going through the origin. Using polar coordinates centred at the origin the metric can be written $dr^2 + r^2 d\varphi^2$. Then the change of coordinates $(r, \varphi) \mapsto (s, \theta)$ given by $s = r$ and $\theta = \varphi - a(r)$ for $r < 1/2$ gives us the metric $ds^2 + (sd\theta + sa'(s)ds)^2$. With $a(r) = -|r|^k / \log|r|$, both the change of coordinates and the new metric is $C^k$. The minimal geodesic $\gamma$ is mapped to $t \to t(\cos a(t), \sin(a(t)))$ which is precisely $C^{k+1}$ at the origin.

The remainder of this paper is devoted to the study of the regularity of a minimal geodesic $\gamma$ parametrized by arc-length if $g$ is a $C^{0,\alpha}$-Riemannian for $0 \leq \alpha \leq 1$. Since it, a priori, is not clear that $\gamma$ is piecewise $C^1$, we will use following definition of a minimal geodesic parametrized by arc-length that takes into account that a minimal geodesic realizes the distance between two points. (See also [3] for other definitions of geodesics when the metric has low regularity.)

DEFINITION 1.3. A minimal geodesic parametrized by arc-length is a continuous curve $\gamma(t)$, $a \leq t \leq b$, such that

$$d(\gamma(s), \gamma(t)) = t - s, \qquad t_0 \leq s \leq t \leq t_1.$$

Then the result of this paper can be summarized in

THEOREM 1.4. *If $g$ is a $C^{0,\alpha}$-Riemannian metric on a smooth manifold and $\gamma$ is a minimal geodesic parametrized by arc-length, then for $\alpha = 1$, $\gamma \in C^{1,1}$, for $0 < \alpha < 1$, $\gamma \in C^{1,\alpha/2}$, and for $\alpha = 0$ then there are examples when $\gamma$ has not a continuous derivative provided that $\dim M > 1$.*

Hence it is sufficient that the metric is Hölder continuous of any positive order for a minimal geodesic to have a continuous tangent, and therefore a well-defined arc-length. The assumption that the geodesic is minimal crucial for the theorem and the proof will exploit this fact repeatedly.

The motivation for this study comes from the theory of the Schrödinger equation. There an important tool is the Agmon-Lithner metric $g = \sqrt{\max(v(x), E)}\, dx^2$ where $v$ is a potential function and the non-negative $E$ is the energy, and important spectral properties depend of the behaviour of a solution near a minimal geodesic.

The results of this paper are stated for general metrics. Naturally, better results can be obtained if one has more precise knowledge of the metric. For example, in the Agmon-Lithner metric above, let $v = \min(v_1, v_2)$ with $v_1$, $v_2$ smooth and positive. Then the metric $g$ is in $C^{0,1}$ but a more precise statement is that g is in $C^{0,1}(M) \cap C^\infty(M \setminus \Lambda)$ where $\Lambda$ is the submanifold where $v_1 = v_2$. If one additionally assumes that $\gamma$ only intersects $\Lambda$ transversally, then it can be shown that even the Jacobi field exists along $\gamma$. See [2, Section 2] for details.

## 2. **Regularity of geodesics in $C^{0,\alpha}$-Riemannian metrics**

We start with a lemma that estimates how much a minimal geodesic in $\mathbb{R}^n$ can deviate from a line if the metric is close to the Euclidean metric.

LEMMA 2.1. *In $\mathbb{R}^n$, let $dx^2$ be the usual n-dimensional Euclidean metric and let g, with corresponding distance function $d(x,y)$, be an arbitrary Riemannian metric satisfying*

$$dx^2/(1+\delta) \le g \le (1+\delta)dx^2$$

*for some* $\delta > 0$. *Then*

$$1/(1+\delta)^{1/2} \le \frac{|a-b|}{d(a,b)} \le (1+\delta)^{1/2}.$$

*Furthermore, let $\gamma$ be a minimal geodesic in the metric g between two points a and c and let b be a point on $\gamma$ such that $d(a,b)/d(b,c)$ and $d(b,c)/d(a,b)$ are bounded. Then, in the Euclidean metric, the angle between the lines through a and b and through b and c respectively is $\pi + O(\delta^{1/2})$.*

*Proof.* First we estimate the ratio between the Euclidean distance function $|a-b|$ and the Riemannian distance $d(a,b)$. If $\eta$ is a piecewise smooth curve with $\eta(0) = a$ and $\eta(1) = b$ we have

$$|a-b| \le \int_0^1 |\eta'|\, dt \le (1+\delta)^{1/2} \int_0^1 g(\eta', \eta')^{1/2}\, dt.$$

Taking infimum over all such curves $\eta$ gives us $|a-b| \le (1+\delta)^{1/2} d(a,b)$. The role of the metrics may be reversed giving us the opposite inequality $d(a,b) \le (1+\delta)^{1/2}|a-b|$.

If $\beta$ denotes the angle we want to estimate, the cosine theorem gives for the Euclidean triangle with corners $a$, $b$ and $c$

$$1+\cos\beta = \frac{2\,|a-b|\,|b-c| + |a-b|^2 + |b-c|^2 - |a-c|^2}{2\,|a-b|\,|b-c|}$$
$$\leq \frac{(1+\delta)\,(d(a,b)+d(b,c))^2 - d(a,c)^2/(1+\delta)}{2d(a,b)d(b,c)/(1+\delta)} = \left(\delta+\delta^2/2\right)\frac{(d(a,b)+d(b,c))^2}{d(a,b)d(b,c)},$$

since $d(a,c) = d(a,b)+d(b,c)$. Then using that $(d(a,b)+d(b,c))^2/d(a,b)d(b,c)$ is bounded by hypothesis we get $\cos\beta = -1+O(\delta)$ and the lemma follows. □

Now we are ready to estimate the regularity of a minimal geodesic.

PROPOSITION 2.2. *Let the metric $g$ be Hölder continuous of order $\alpha \in (0,1)$. Let $\gamma(t)$, $0 \leq t \leq L$ be a minimal geodesic parametrized by arc-length. Then $d\gamma/dt$ exists and is Hölder continuous of order $\alpha/2$.*

That the condition that the metric is Hölder continuous of positive order is important shows the following modification of Example 1.2.

EXAMPLE 2.3. As in Example 1.2, let $M = \mathbb{R}^2$ with Euclidean metric $g$, which in polar coordinates centred at the origin can be written $dr^2 + r^2 d\varphi^2$, and let $\gamma : [-1/2, 1/2] \ni t \to (t,0)$ be a minimal geodesic through the origin. With $a(r) = \log(-\log|r|)$ the mapping $(r,\varphi) \mapsto (s,\theta)$ given by $s = r$ and $\theta = \varphi - a(r)$ for $r < 1/2$ is continuous and gives us a metric

$$\left(1+\frac{1}{(\log s)^2}\right)ds^2 - \frac{2s}{\log s}ds\,d\theta + s^2 d\theta^2$$

which is continuous at the origin but not Hölder continuous of any positive order due to the term $1/(\log s)^2$. Going back to rectangular coordinates the metric can be written

$$\hat{g} = dx^2 + dy^2 + \frac{4}{(\log(x^2+y^2))^2}\frac{(x\,dx+y\,dy)^2}{x^2+y^2} - \frac{4}{\log(x^2+y^2)}\frac{(x\,dx+y\,dy)(-y\,dx+x\,dy)}{x^2+y^2}.$$

The transformation maps the geodesic $\gamma$ to $t(\cos(a(t)), \sin(a(t)))$ that winds spirally towards the origin. It is continuous at the origin but not differentiable. The length of the tangent vector tends to 1 when $\gamma$ approaches the origin but the angle between the tangent and the $x$-axis has no limit.

If $\chi \in C_0^\infty(\mathbb{R}^2)$, $0 \leq \chi \leq 1$, is supported in the circle of radius $1/2$ centred at the origin and equal to 1 in a neighbourhood of the origin, then $g_0 = (1-\chi)g + \chi\hat{g}$ is a continuous metric in $\mathbb{R}^2$ where minimal geodesics passing the origin are not differentiable there. Since $g_0$ is a continuous metric which is equal to the Euclidean metric outside a compact set, the metric has bounded maximum norm. Let $g_a$ denote the metric translated so that the origin is translated to the point $a$. If $a_j$ is a suitable sequence of positions then

$$\sum_{j=0}^{\infty} 2^{-j} g_{a_j}$$

is a continuous metric where a given minimal geodesic is not even piecewise differentiable.

The example is easily generalized to $\mathbb{R}^n$ for $n > 2$ and by working in a local coordinate system to any finite dimensional manifold with dimension greater than one.

*Proof of Proposition 2.2.* Let $\gamma(t_0)$ be an interior point on $\gamma$. We may choose local coordinates $x = (x_1, \ldots, x_n)$ centred at $\gamma(t_0)$ such that $g$ is Euclidean at $T_{\gamma(t_0)}M$. First we study the right derivative of $\gamma$ at $t_0$. Let $t > 0$ be so small that $\gamma(t_0 + t)$ is still an interior point on $\gamma$, and so small that we do not leave the local coordinate patch in the following. Since $g$ is Hölder continuous of order $\alpha$ we get the estimate

$$dx^2/(1+Ct^\alpha) < g < (1+Ct^\alpha)\,dx^2 \tag{2.1}$$

with some constant $C > 0$ in a neighbourhood of radius $2t$ around $\gamma(t_0)$. If $y$ lies in such a neighbourhood, then the inequalities follow from

$$\begin{aligned} |g(y)(\xi) - dx^2(\xi)| &= |g(y)(\xi) - g(\gamma(t_0))(\xi)| \leq \sum_{i,j=1}^{n} \left|g_{ij}(y) - g_{ij}(\gamma(t_0))\right| |\xi^i \xi^j| \\ &\leq \sum_{i,j=1}^{n} ct^\alpha |\xi^i\xi^j| \leq ct^\alpha \frac{1}{2} \sum_{i,j=1}^{n} ((\xi^i)^2 + (\xi^j)^2) = cnt^\alpha \sum_{i=1}^{n} (\xi^i)^2 = \frac{Ct^\alpha}{2} dx^2(\xi) \end{aligned}$$

and since $1/(1+Ct^\alpha) \leq 1 - Ct^\alpha/2$ this gives (2.1). The choice of parametrization implies that $d(\gamma(t+t_0), \gamma(t_0)) = t$ and according to Lemma 2.1 we get the estimate

$$1/\left(1+Ct^\alpha\right)^{1/2} < \frac{|\gamma(t+t_0) - \gamma(t_0)|}{t} < \left(1+Ct^\alpha\right)^{1/2}.$$

Letting $t \to 0$ shows that $|\gamma(t+t_0) - \gamma(t_0)|/t \to 1$.

Next we consider the sequence $\gamma(t_0 + 2^{-k}t)$ for $k \in \mathbb{N}$. Applying Lemma 2.1, with $\delta = Ct^\alpha$, to the Euclidean triangle with corners $\gamma(t_0)$, $\gamma(t_0+t)$ and $\gamma(t_0 + 2^{-1}t)$ shows that the angle at $\gamma(t_0 + 2^{-1}t)$ is $\pi + O(t^{\alpha/2})$. Hence the angle between the two chords through $\gamma(t_0)$ is $O(t^{\alpha/2})$. Repeating this argument with the triangle with corners $\gamma(t_0)$, $\gamma(t_0 + 2^{-k}t)$ and $\gamma(t_0 + 2^{-k-1}t)$ shows that the angle between the two chords through $\gamma(t_0)$ is $O(2^{-k\alpha/2}t^{\alpha/2})$. Holding $t$ fixed and letting $k \to \infty$ we see that $(\gamma(t_0 + 2^{-k}t) - \gamma(t_0))/(2^{-k}t)$ has a limit vector of length one. This limit is independent of the choice of $t$ which is easily seen by mixing two different sequences. Hence $\gamma'(t_0+0)$ exists with length one. A similar argument shows that $\gamma'(t_0-0)$ exists and is equal to $\gamma'(t_0+0)$. By repeating the arguments above we also find that $\gamma'(t_0+t)$ exists and has an angle with the chord through $\gamma(t_0)$ and $\gamma(t_0+t)$ of order $O(t^{\alpha/2})$. Since the arguments are uniform in $t_0$ it follows that $\gamma'(t_0+t)$ exists for all $t$.

Next we study the regularity of $\gamma'$. To estimate how the length of $\gamma'$ varies we get in the coordinates centred at $\gamma(t_0)$ and small $s$

$$\begin{aligned} \frac{|s|}{1+Ct^\alpha} = \frac{d(\gamma(t_0+t+s), \gamma(t_0+t))}{1+Ct^\alpha} &\leq |\gamma(t_0+t+s) - \gamma(t_0+t)| \\ &\leq (1+Ct^\alpha) d(\gamma(t_0+t+s), \gamma(t_0+t)) = (1+Ct^\alpha)\,|s| \end{aligned}$$

that is

$$\frac{1}{1+Ct^{\alpha}} \leq \left| \frac{\gamma(t_0+t+s)-\gamma(t_0+t)}{s} \right| \leq 1+Ct^{\alpha}$$

and hence

$$1/(1+Ct^{\alpha}) \leq |\gamma'(t_0+t)| \leq 1+Ct^{\alpha}.$$

Since $|\gamma'(t_0)| = 1$ it follows that

$$-Ct^{\alpha} \leq -\frac{Ct^{\alpha}}{1+Ct^{\alpha}} \leq |\gamma'(t_0+t)| - |\gamma'(t_0)| \leq Ct^{\alpha}$$

that is

$$\big||\gamma'(t_0+t)| - |\gamma'(t_0)|\big| \leq Ct^{\alpha}$$

and hence the length of $\gamma'$ is Hölder continuous of order $\alpha$. But the angle is Hölder continuous of order $\alpha/2$. Hence, by letting $t \to 0$, we conclude that $\gamma'$ is Hölder continuous of order $\alpha/2$ for the argument is uniform in $t_0$. □

When $\alpha = 1$, that is, when the metric is Lipschitz continuous, then the metric is regular enough so that regularization will yield a stronger result.

PROPOSITION 2.4. *Let $g$ be a Lipschitz continuous, positive Riemannian metric on $M$. If $[0,1] \ni t \mapsto \gamma(t)$ is a minimal geodesic with the parameter proportional to the arc-length, then $\gamma$ is in $C^1$ with Lipschitz continuous first derivative.*

This generalizes a corresponding result for geodesics on a 2-dimensional surface in $\mathbb{R}^3$ in [4].

*Proof.* In the metric $\hat{g} = (1+\rho)g$, where $\rho$ is a non-negative, smooth function vanishing on $\gamma([0,1])$ and positive on $M \setminus \gamma([0,1])$, $\gamma$ is the unique, minimal geodesic between $\gamma(0)$ and $\gamma(1)$. Such a function $\rho$ can be constructed, for example, by starting with a partition of unity $\sum \theta_j = 1$, $\theta_j \in C_0^{\infty}(M \setminus \gamma([0,1]))$ where the sum is locally finite and then setting $\rho = \sum \varepsilon_j \theta_j$ with $\varepsilon_j > 0$ vanishing fast enough. The metric $\hat{g}$ is still Lipschitz continuous. If $L$ is the length of $\gamma$ in the metric $\hat{g}$ and $\Omega$ is an open, bounded neighbourhood of $\gamma$, then there exists a constant $\delta > 0$ such that all curves from $\gamma(0)$ to $\gamma(1)$ that leave $\Omega$ have a length at least $(1+\delta)L$. Since $\overline{\Omega}$ is compact and $\hat{g}$ is Lipschitz continuous there is a constant $C_1$ such that

$$|\hat{g}(x+z) - \hat{g}(x)| \leq C_1 |z|, \qquad |z| < \delta.$$

If $\chi \in C_0^{\infty}(M)$, $\chi \geq 0$ with $\int \chi(y)\,dy = 1$, then let $\chi_{\varepsilon}(y) = \varepsilon^{-n}\chi(y/\varepsilon)$ be a standard mollifier and let $\hat{g}^{\varepsilon} = \hat{g} * \chi_{\varepsilon}$ be a standard regularization of $\hat{g}$ such that $\hat{g}^{\varepsilon} \in C^{\infty}$ and has uniformly bounded first derivatives in $\Omega$ and satisfies

$$\hat{g}/(1+C\varepsilon) \leq \hat{g}^{\varepsilon} \leq \hat{g}(1+C\varepsilon)$$

for some $C > 0$. In fact, we have to estimate

$$\partial_{x_j}\hat{g}^\varepsilon = \partial_{x_j}\frac{1}{\varepsilon^n}\int \hat{g}(y)\chi\Big(\frac{x-y}{\varepsilon}\Big)\,dy = \frac{1}{\varepsilon^{n+1}}\int \hat{g}(y)\chi_j'\Big(\frac{x-y}{\varepsilon}\Big)\,dy = \frac{1}{\varepsilon}\int \hat{g}(x-\varepsilon y)\chi_j'(y)\,dy.$$

Observing that $\frac{1}{\varepsilon}\int \hat{g}(x)\chi_j'(y)\,dy = \frac{1}{\varepsilon}\hat{g}(x)\int \chi_j'(y)\,dy = 0$ we get

$$\begin{aligned}\left|\partial_{x_j}\hat{g}^\varepsilon(x)\right| &= \left|\frac{1}{\varepsilon}\int \hat{g}(x-\varepsilon y)\chi_j'(y)\,dy - \frac{1}{\varepsilon}\int \hat{g}(x)\chi_j'(y)\,dy\right| \\ &\le \frac{1}{\varepsilon}\int |\hat{g}(x-\varepsilon y)-\hat{g}(x)|\left|\chi_j'(y)\right|\,dy \\ &\le \frac{1}{\varepsilon}C_1\varepsilon\int |y|\left|\chi_j'(y)\right|\,dy = C_1C_2\end{aligned}$$

for all $x \in \Omega$. Furthermore, since $\hat{g}$ is positive and $\overline{\Omega}$ is compact there exists a positive constant $c$ such that $\hat{g} \ge c > 0$ in $\overline{\Omega}$. Then we have

$$\begin{aligned}|\hat{g}^\varepsilon(x)-\hat{g}(x)| &= \left|\int \hat{g}(x-\varepsilon y)\chi(y)\,dy - \int \hat{g}(x)\chi(y)\,dy\right| \\ &\le \int |\hat{g}(x-\varepsilon y)-\hat{g}(y)|\,|\chi(y)|\,dy \le C_1\int \varepsilon|y|\,|\chi(y)|\,dy = C_1C_2\varepsilon \le \frac{C_1C_2\varepsilon}{c}\hat{g}.\end{aligned}$$

With $C/2 = C_1C_2/c$ we get

$$(1-C\varepsilon/2)\hat{g} \le \hat{g}^\varepsilon \le (1+C\varepsilon/2)\hat{g},$$

and since $1/(1+C\varepsilon) \le 1-C\varepsilon/2$ for $\varepsilon > 0$ small enough, the estimates follow.

In the metric $\hat{g}^\varepsilon$, $\gamma$ has at most the length $L(1+C\varepsilon)^{1/2}$, but all curves from $\gamma(0)$ to $\gamma(1)$ that leave $\Omega$ have length at least $L(1+\delta)(1+C\varepsilon)^{-1/2}$. For $\varepsilon > 0$ so small that $C\varepsilon < \delta$ it follows that any minimal geodesic $\gamma_\varepsilon$ from $\gamma(0)$ to $\gamma(1)$ stays in $\Omega$ since the length is at most $L(1+C\varepsilon)^{1/2}$. Take for all $\varepsilon$ small enough, a minimal geodesic between $\gamma(0)$ and $\gamma(1)$ with a parameter proportional to the arc-length. Then the first derivative is uniformly bounded. Furthermore, by the choice of regularization it follows that the first derivatives of $\hat{g}^\varepsilon$ are uniformly bounded in $\Omega$ by constants independent of $\varepsilon$. Thus the Christoffel symbols are uniformly bounded in $\Omega$ by constants independent of $\varepsilon$ and using the equation of geodesics we conclude that the second derivative of the geodesic is uniformly bounded by a constant independent of $\varepsilon$. Letting $\varepsilon \to 0$, we may choose a subsequence converging to a curve $[0,1] \ni t \mapsto \eta(t)$ that is in $C^1$ and has Lipschitz continuous first derivative. Since $\gamma_\varepsilon''$ is uniformly bounded by a constant $C_3$ it follows from the mean-value theorem that

$$\left|\gamma_\varepsilon'(t)-\gamma_\varepsilon'(s)\right| \le C_3\,|t-s|$$

for all $s$, $t$ and $\varepsilon$. Hence $\gamma_\varepsilon'$ is equicontinuous. Then the Arzela-Ascoli theorem shows that there is a subsequence such that $\gamma_\varepsilon$ converges to a function $[0,1] \ni t \mapsto \eta(t)$ that belongs to $C^1(\Omega)$. Going to the limit in the inequality above shows that $\eta$ has Lipschitz continuous first derivative and since $\hat{g}(\gamma_\varepsilon'(t),\gamma_\varepsilon'(t)) = 1$ for all $t$ and $\varepsilon$ it follows that $\hat{g}(\eta'(t),\eta'(t)) = 1$. Hence $\eta(t) = \gamma(t)$

for $t = 0, 1$, and the length with respect to the metric $\hat{g}$ is at most $L$. Since by the construction of $\hat{g}$, $\gamma$ is the unique minimal geodesic between $\gamma(0)$ and $\gamma(1)$, and it follows that $\gamma(t) = \eta(t)$ for all $0 \leq t \leq 1$. Thus the proposition is proved. $\square$

EXAMPLE 2.5. Repeating the construction of Examples 1.2 and 2.3 with the new function $a(r) = |r|^{\alpha}$, $0 < \alpha \leq 1$ instead, gives us a transformation and a Riemannian metric which are $C^{0,\alpha}$ and a minimal geodesic which is precisely of order $C^{1,\alpha}$ at the origin.

Hence Proposition 2.4 is optimal but there are chances that Proposition 2.2 is not.

## References

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