STRETCHED EXPONENTIAL FIXATION IN STOCHASTIC ISING MODELS AT ZERO TEMPERATURE

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Abstract. We study a class of continuous time Markov processes, which describes ± 1 spin flip dynamics on the hypercubic lattice $\mathbb{Z}^d$, $d \geq 2$, with initial spin configurations chosen according to the Bernoulli product measure with density p of spins $+1$. During the evolution the spin at each site flips at rate $c = 0$, or $0 < \alpha \leq 1$, or 1, depending on whether, respectively, a majority of spins of nearest neighbors to this site exists and agrees with the value of the spin at the given site, or does not exist (there is a tie), or exists and disagrees with the value of the spin at the given site. These dynamics correspond to various stochastic Ising models at 0 temperature, for the Hamiltonian with uniform ferromagnetic interaction between nearest neighbors. In case $\alpha = 1$, the dynamics is also a threshold voter model. We show that if p is sufficiently close to 1, then the system fixates in the sense that for almost every realization of the initial configuration and dynamical evolution, each site flips only finitely many times, reaching eventually the state $+1$. Moreover, we show that in this case the probability $q(t)$ that a given spin is in state -1 at time t satisfies the bound: for arbitrary $\varepsilon > 0$, $q(t) \leq \exp(-t^{(1/d)-\varepsilon})$, for large t. In $d = 2$ we obtain the complementary bound: for arbitrary $\varepsilon > 0$, $q(t) \geq \exp(-t^{(1/2)+\varepsilon})$, for large t.

1. Introduction.

In this work we study the long time behavior of a class of continuous time Markov processes, which describe ± 1 spin flip dynamics on the hypercubic lattice $\mathbb{Z}^d$, $d \geq 2$. To each vertex of $\mathbb{Z}^d$ one attributes a spin which can take values -1 or $+1$, so that the state space of the system is $\Sigma = \{-1, +1\}^{\mathbb{Z}^d}$. We write $\sigma_t(x)$ for the value of the spin at vertex $x \in \mathbb{Z}^d$, at time $t \geq 0$. The system evolves as a Markov process on the state space Σ, with generator L acting on local functions f as

$$(Lf)(\sigma) = \sum_{x \in \mathbb{Z}^d} c^\alpha(x, \sigma)(f(\sigma^x) - f(\sigma)), \tag{1.1}$$

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where σ^x is the configuration obtained from σ by flipping the spin at vertex x, and $c^\alpha(x,\sigma)$ is the flip rate of the spin at vertex x, when the system is in the state σ. Throughout the paper we will consider the following rates

$$c^\alpha(x,\sigma) = \begin{cases} 1 & \text{if more than } d \text{ neighbors of } x \text{ have its opposite spin;} \\ \alpha & \text{if exactly } d \text{ neighbors of } x \text{ are of its opposite spin;} \\ 0 & \text{otherwise,} \end{cases} \tag{1.2}$$

where $0 < \alpha \leq 1$ is a fixed parameter. These dynamics can be regarded as 0-temperature stochastic Ising models corresponding to a uniform ferromagnetic interaction between nearest neighbors (no external magnetic field). The case $\alpha = 1$ corresponds to Metropolis dynamics and the case $\alpha = 1/2$ corresponds to the Gibbs sampler (or heat bath) dynamics. Note that the dynamics is symmetric with respect to the interchange of the roles of -1 and $+1$ and is attractive.

We will consider the long time behavior of these systems when started from Bernoulli product measure with $\mathbf{P}_p(\sigma_0(x) = +1) = p$, for each $x \in \mathbb{Z}^d$. While most of the previous study of these systems has been concentrated on the case $p = 1/2$ (see [NNS], [NS], [CDN] and references therein), we will address the case in which p is close to 1. The evolutions that we consider arise from modeling the behavior of a magnetic system which is initially at high temperature under a strong external magnetic field, and from time 0 on is suddenly subject to a very low temperature and to no external field. For an introduction to the physics of this problem, see, e.g., the review [Bra].

In addition to the motivation from physics discussed above, it is worth pointing out that the models studied here are (generally non-linear) voter models (in the case $\alpha = 1$, the model is a threshold voter model) and the study of such models, started from product measures, is a subject of considerable interest (see [Lig2] and references therein).

For each value of p, we will say that the system fixates if for each $x \in \mathbb{Z}^d$ the spin at x flips only finitely many times (locally the system becomes trapped).

In the $d = 1$, $\alpha = 1/2$ case, our system is the well known linear voter model, which is known not to fixate for any $p \in (0,1)$. In this case the boundaries between blocks of spins -1 and $+1$ behave as annihilating random walks. A similar result should also hold for every $0 < \alpha \leq 1$, since in this more general case the boundaries between blocks of spins -1 and $+1$ behave as the same annihilating random walks which now interact, but only when they are at distance 1 from each other. This minor interaction should not have any major effect on the qualitative behavior of the system.

In $d \geq 2$, the behavior is more challenging. In $d = 2$, [NNS] proved that if $p = 1/2$, the system does not fixate:

$$\mathbf{P}_{1/2}(\sigma_t(0) \text{ flips i.o.}) = 1. \tag{1.3}$$

Recently, stronger results were obtained in this case in [CDN]. It is not known if (1.3) is true in higher dimensions (see [NS] and references therein).

In contrast, here we will prove:

Theorem 1.1. *For $d \geq 2$ and any $0 < \alpha \leq 1$ there exists $p_0 \equiv p_0(d,\alpha) < 1$, such that for any $p > p_0$ and $\varepsilon > 0$*

$$\mathbf{P}_p(\sigma_t(0) = -1) \leq \mathbf{P}_p(\sigma_s(0) = -1 \text{ for some } s \geq t) \leq e^{-t^{(1/d)-\varepsilon}}, \tag{1.4}$$

for t large enough. In particular, for any $p > p_0$

$$\mathbf{P}_p(\sigma_t(0) = +1 \text{ eventually}) = 1. \tag{1.5}$$

Theorem 1.2. *For $d = 2$, any $0 < \alpha \leq 1$, $0 \leq p < 1$, and $\varepsilon > 0$*

$$\mathbf{P}_p(\sigma_t(0) = -1) \;\geq\; \mathbf{P}_p(\sigma_s(0) = -1 \;\forall s \leq t) \;\geq\; e^{-t^{(1/2)+\varepsilon}}, \tag{1.6}$$

for t large enough.

Theorems 1.1 and 1.2 show that in $d = 2$, for large p, fixation occurs as fast as, but not faster than, a stretched exponential. Note that the theorems above also identify in $d = 2$ the correct "stretch" of the exponential as a power $1/2$, i.e.,

$$\lim_{t\to\infty} \frac{\log(-\log(P_p(\sigma_t(0) = -1)))}{\log(t)} \;=\; \frac{1}{2}.$$

In [HF] decay of temporal correlations in equilibrium for stochastic Ising models at low temperatures were analyzed. Non-rigorous arguments in that paper suggested that in 2 dimensions these correlations decay in time as stretched exponentials with exponent $1/2$. As far as we know, the stretched exponential fixation in the context of Theorems 1.1 and 1.2 had not been proposed before in the literature. Some similarities between the arguments in [HF] and those here will be apparent. Specifically, in both cases estimates on the density of droplets of the minority spins and on the time to eliminate them are crucial.

It is natural to ask whether in $d \geq 3$ the fixation also occurs only as fast as a stretched exponential, and if so, with what exponent. The non-rigorous analysis in [HF] of the decay of temporal correlations in equilibrium at low temperatures indicates that dimension 3 plays the role of upper critical dimension for that problem, with the decay being exponential in time in $d \geq 3$. The relations between their analysis and ours suggest that also in our case fixation may actually occur exponentially fast in time (no stretch) in $d \geq 3$.

Comparing (1.3) with Theorem 1.1, raises the question of the values of p for which there is fixation in the state with all spins $+1$. We believe that this happens for every $p > 1/2$. Similarly, it seems reasonable to conjecture that at low positive temperatures the stochastic Ising model started from homogeneous product measure with $\mathbf{P}_p(\sigma_0(0) = +1) = p > 1/2$, converges weakly to the (+)-phase (see Open Problem 7, p.225, of [Lig1]). Our results also suggest the conjecture that in $d = 2$ this convergence occurs as fast as, but not faster than, a stretched exponential.

The proof of Theorem 1.1 will be based on a multiscale analysis. It is worth mentioning that even to just prove fixation for large p, i.e., (1.5), we felt the need to use this technique.

A basic ingredient in the proofs of the theorems above are estimates on the time needed for the dynamics to "eliminate droplets", by which we mean the following. Suppose that the process is started from the configuration in which the cube $R = \{1, 2, ..., L\}^d$ is occupied by spins -1 and all other spins are $+1$. It is clear that the spins outside R will never flip and that eventually the configuration with all spins $+1$ will be reached, with the system becoming trapped there. Let T be the random time when the system reaches this trapping configuration. We need to estimate how T grows with L. Heuristically, one expects the droplet to shrink with its boundary being well approximated, when L is large, by a surface moving by mean-curvature (see, e.g., [Lif], [HF], [Spo], [CCS] and references therein). This leads to T being of the order of L^2. The following theorem contains rigorous estimates compatible with this picture in any dimension and supporting it in case $d = 2$. For later use, we also consider the random time $\widehat{T}$ when the spin at the site $(\lfloor L/2 \rfloor, \lfloor L/2 \rfloor)$ is for the first time in state $+1$. Clearly $\widehat{T} \leq T$.

Theorem 1.3.

(a) *There exist* $C, \gamma \in (0, \infty)$ *such that for any* $d \geq 2$ *and any* $0 < \alpha \leq 1$

$$\mathbf{P}(T > (C/\alpha)L^d) \;\leq\; e^{-\gamma L},$$

for large L.

(b) *If* $d = 2$, *for any* $0 < \alpha \leq 1$ *and any* $\delta > 0$,

$$\lim_{L\to\infty} \mathbf{P}(T < L^2/(\log L)^{1+\delta}) \;=\; \lim_{L\to\infty} \mathbf{P}(\widehat{T} < L^2/(\log L)^{1+\delta}) \;=\; 0.$$

The paper is organized as follows. In Section 2 we introduce some terminology, notation and basic tools, including the bootstrap percolation process. In Section 3 we prove Theorem 1.3. In Section 4 we prove Theorem 1.1, after explaining the heuristics behind this proof. In Section 5 we prove Theorem 1.2.

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2. Terminology, notation and basic tools.

We begin this section with several standard definitions. As already mentioned we shall consider models on the lattice $\mathbb{Z}^d$, where $d \geq 2$ is the space dimensionality. The cardinality of a set $\Lambda \subset \mathbb{Z}^d$ will be denoted by $|\Lambda|$. The distance between vertices of $\mathbb{Z}^d$ will be measured by the L^1 norm $||\cdot||$, and denoted $\mathrm{dist}(x, y) = ||x - y||$. A (selfavoiding) path is a sequence of distinct vertices $x_1, x_2, \ldots, x_n$, such that $||x_i - x_{i+1}|| = 1$, and the vertices x_1 and x_n are called the start- and end-points of the path.

At each vertex in $\mathbb{Z}^d$ there is a spin which can take values -1 or $+1$. The configurations will therefore be elements of the set $\Sigma = \{-1, +1\}^{\mathbb{Z}^d}$. Given $\sigma \in \Sigma$, we write $\sigma(x)$ for the value of the spin at vertex $x \in \mathbb{Z}^d$. The single spin space $\{-1, +1\}$ is endowed with the discrete topology and Σ is endowed with the corresponding product topology.

The time evolution (a spin flip system) is a Markov process on the state space Σ, whose generator L and flip rates $c^\alpha(x, \sigma)$ are given by the formulae (1.1) and (1.2).

Graphical construction. In most of our proofs it will be convenient to use the following graphical construction of the process, called *Harris' graphical representation*, and which provides a specific version of what is known as basic coupling (it provides versions of the whole family of processes either on the infinite lattice $\mathbb{Z}^d$ or on any of its finite subsets with arbitrary boundary conditions and starting from any initial configuration, all on the same probability space). Namely, to each vertex $x \in \mathbb{Z}^d$ we associate two independent Poisson processes, each one of rate $c_{\max}$, the maximal among all possible rates (in our case it equals 1). We will denote the successive arrival times (after time 0) of these Poisson processes by $\{\tau^+_{x,n}\}_{n=1,2...}$ and $\{\tau^-_{x,n}\}_{n=1,2...}$, assuming that the Poisson processes associated to different vertices are also mutually independent. We say that at each space-time point of the form $(x, \tau^+_{x,n})$ there is an upward mark and at each point of the form $(x, \tau^-_{x,n})$

there is a downward mark. We associate to each arrival time $\tau^*_{x,n}$ (where $*$ stands for $+$ or $-$) a random variable $U^*_{x,n}$ with uniform distribution between 0 and 1. All these random variables are supposed independent among themselves and independent from the previously introduced Poisson processes. This finishes the construction of the probability space. The corresponding probability and expectation will be denoted, respectively by $\mathbf{P}$ and $\mathbf{E}$. The process then is defined as follows: we update the state of the process at each space-time location where there is a mark according to the following rules. If the mark that we are considering is at the point $(x, \tau^*_{x,n})$, and the configuration immediately before time $\tau^*_{x,n}$ restricted to x and its neighboring sites was identical to a configuration σ, then

i) if $\sigma(x) = -1$ (resp. $\sigma(x) = +1$), then the spin at x can flip only if the mark is an upward (resp. downward) one;

ii) if the mark is upward and $\sigma(x) = -1$, or if the mark is downward and $\sigma(x) = +1$, then the spin at x flips if and only if $U^*_{x,n} < c^\alpha(x,\sigma)/c_{\max}$.

This construction gives the proper rates and defines a process according to eqs. (1.1-2) (see for instance [Dur] for details). The use of upward and downward marks, in connection to the fact that the rates are attractive, assures that if we compare the evolution started from two configurations which are comparable in the sense that wherever the former has a spin $+1$, so does the latter, then this property is preserved by the evolution.

We will refer to the Poisson marks and the associated uniformly distributed random variables, as the "graphical marks".

The probability measure corresponding to chosing initially a random configuration according to homogeneous product measure with density p of spins $+1$, and letting the system evolve using the graphical marks will be denoted by $\mathbf{P}_p$.

Bootstrap percolation. In the definition below we consider configurations in $\{u,s\}^{\mathbb{Z}^d}$, where u (for "unstable") and s (for "stable") are arbitrary symbols.

Definiton 2.1. *The d-dimensional $(u \to s)$ bootstrap percolation process with threshold θ, defined in a finite or infinite volume $\Lambda \subseteq \mathbb{Z}^d$, starting from the initial configuration $\eta_0 \in \{u,s\}^\Lambda$ is a cellular automaton which evolves in discrete time $t = 0,\ 1,\ 2, \dots,$ and such that at each time unit $t \geq 1$ the current configuration is updated according to the following rules. For each $x \in \Lambda$,*

1. If $\eta_{t-1}(x) = s$, then $\eta_t(x) = s$.

2. If $\eta_{t-1}(x) = u$, and at time $t-1$ the vertex x has at least θ nearest neighbors in Λ in state s, then $\eta_t(x) = s$; otherwise the spin at vertex x remains unchanged, i.e., $\eta_t(x) = u$.

If $|\Lambda| < \infty$, it is immediate from the Definition 2.1, that the procedure described above converges in a finite time to a fixed configuration. In this context, we say that we *bootstrap* the initial configuration η_0, the result of which is the final configuration of the bootstrap percolation dynamics.

Of special importance to us will be the cases in which the threshold θ is d or 2. The former is important for its close relation to the spin flip systems that we are considering, as explained in the heuristics background to the proof of Theorem 1.1 (see the beginning of Section 4). The latter is of technical importance for us, and will be used in the proof of Theorem 1.1.

In the case of threshold $\theta = 2$, when Λ is a rectangle, the bootstrapped configuration can be described as follows (see [Ent], [AL]). Say that a family of disjoint subsets of $\mathbb{Z}^d$ is well separated if there is no vertex in $\mathbb{Z}^d$ at distance less than or equal to 1 from two sets in the family. For any initial η_0, the bootstrapped configuration takes the value s on the smallest collection of well separated rectangles contained in Λ which contains all the sites x at which $\eta_0(x) = s$. We say that the rectangle $R \subset \Lambda$ is *internally spanned* by the configuration $\eta \in \{u, s\}^\Lambda$, if bootstrap percolation restricted to the volume R, started from $\eta_0 = \eta\big|_R$, ends up with all sites in R in state s.

The following lemma provides us with a crucial estimate.

Lemma 2.1. *(Aizenman-Lebowitz [1988], [AL]). Suppose that $n \in \{1, 2, ...\}$ is fixed. Suppose that Λ is a finite rectangle and $\eta \in \{u, s\}^\Lambda$ is a configuration with the property that when we perform $(u \to s)$ $\theta = 2$ bootstrap percolation in Λ started from η, we obtain a final configuration which includes a rectangle with one of its sides larger than n fully occupied by s. Then there must exist some rectangle $R \subset \Lambda$ with larger side in $\{\lfloor n/2 \rfloor - 1, \ldots, n\}$ which is internally spanned by η.*

This lemma may be seen as providing the existence of a "bottleneck event": if a large rectangle full of s is going to occur in the bootstrapped configuration, then, for arbitrary n between 1 and the size of the larger side of this rectangle, also rectangles of "size of order n" are going to be internally spanned. One can then optimize on the choice of n to obtain an upper bound on the probability of large rectangles occurring in the bootstrapped configuration. (The "bottleneck event" is the occurrence of an internally spanned rectangle of "size of the order of the optimal n", i.e., the n which minimizes the probability of this event.)

3. Proof of Theorem 1.3.

Proof of Claim (a). Step 1: case $d = 2$. We will compare the evolution of our system to that in which the initial configuration has spins -1 in the quadrant $\{1, 2, ...\}^2$, and spins $+1$ at the other sites. We will refer to this system as the quadrant evolution. Note that in this evolution, when the site (x, y), $x, y \geq 1$ has a spin $+1$, the sites (x', y') with $x' \leq x$ and $y' \leq y$ also have spins $+1$. Let T' be the first time that in the quadrant evolution the site (L, L) has a spin $+1$. Note that, in the quadrant evolution, at time T' the square R is fully occupied by spins $+1$. Therefore, by attractiveness, T' is stochastically larger than T, i.e., for any $t \geq 0$,

$$\mathbf{P}(T > t) \ \leq \ \mathbf{P}(T' > t). \tag{3.1}$$

In order to estimate T', we use a well known relation between the quadrant evolution and the exclusion process. In the relevant version of the exclusion process, particles are initially located at each negative site in $\mathbb{Z}$, with the positive sites and the origin being empty. Particles try to jump independently of everything else one step to the right at rate $1/2$ and one step to the left at rate $1/2$. The only interaction among the particles being that a jump is suppressed if a particle is attempting to jump to an occupied site. After accelerating the quadrant evolution by a factor of $1/(2\alpha)$, it can be mapped into the evolution of this exclusion process in the following way, as explained in [Lig1], pp. 411, 412. Let $X_i(t)$ be the position at time t of the exclusion process particle initially at the site $-i$, $i = 1, 2, \ldots$. In the quadrant evolution, the site (x, y), $x, y \geq 1$ is occupied by a spin $+1$ at time t if and only if, in the exclusion process at time t the displacement $X_y(t) + y$ of the particle initially at $-y$ is at least x. In particular,

$$\mathbf{P}(T' > (C/\alpha)L^2) \ \leq \ \mathbf{P}(X_L(2CL^2) < 0) \ = \ \mathbf{P}\left(\sum_{i \geq 1} \mathbb{1}_{\{X_i(2CL^2) \geq 0\}} < L\right), \tag{3.2}$$

where $\mathbb{1}_A$ is the indicator of the set A.

We compare now the exclusion process described above to the one that follows the same rules but starts from the configuration with particles only at the sites $-4L, -4L+1, ..., -2, -1$. For this process let $\bar{X}_i(t)$ be the position at time t of the particle initially at $-i$, $i = 1, ..., 4L$. By monotonicity of the exclusion process (Prop. VIII.2.12 of [Lig1]),

$$\mathbf{P}\left(\sum_{i\geq 1} \mathbb{1}_{\{X_i(2CL^2)\geq 0\}} < L\right) \;\leq\; \mathbf{P}\left(\sum_{i=1}^{4L} \mathbb{1}_{\{\bar{X}_i(2CL^2)\geq 0\}} < L\right). \tag{3.3}$$

Set $\bar{I}_i = \mathbb{1}_{\{\bar{X}_i(2CL^2)\geq 0\}}$. Then, for arbitrary $\theta > 0$,

$$\begin{aligned}\mathbf{P}\left(\sum_{i=1}^{4L} \mathbb{1}_{\{\bar{X}_i(2CL^2)\geq 0\}} < L\right) &= \mathbf{P}\left(\sum_{i=1}^{4L} \bar{I}_i < L\right) \\ &= \mathbf{P}\left(\exp\left(-\theta \sum_{i=1}^{4L} \bar{I}_i\right) > \exp(-\theta L)\right) \;\leq\; \frac{\mathbf{E}\left(\exp\left(-\theta \sum_{i=1}^{4L} \bar{I}_i\right)\right)}{\exp(-\theta L)}.\end{aligned} \tag{3.4}$$

Let $\widetilde{X}_1(t), \widetilde{X}_2(t), ..., \widetilde{X}_{4L}(t)$ be independent continuous time simple symmetric rate 1 random walks on Z, started from $\widetilde{X}_i(0) = -i$, $i = 1, ..., 4L$. (These random walks behave like the exclusion process particles, but without the exclusion rule.) We will exploit now a relation between $(X_1(t), ..., X_{4L}(t))$ and $(\widetilde{X}_1(t), ..., \widetilde{X}_{4L}(t))$ which will allow us to estimate the r.h.s. of (3.4). [Thanks to T. Liggett for suggesting this approach.] Set $\widetilde{I}_i = \mathbb{1}_{\{\widetilde{X}_i(2CL^2)\geq 0\}}$. Note that the function $f(x_1, ..., x_{4L}) = \exp(-\theta \sum_{i=1}^{4L} \mathbb{1}_{\{x_i \geq 0\}})$ satisfies the condition of Proposition VIII.1.7 of [Lig1], so that this proposition yields

$$\mathbf{E}\left(\exp\left(-\theta \sum_{i=1}^{4L} \bar{I}_i\right)\right) \;\leq\; \mathbf{E}\left(\exp\left(-\theta \sum_{i=1}^{4L} \widetilde{I}_i\right)\right) \;=\; \prod_{i=1}^{4L} \mathbf{E}(\exp(-\theta \widetilde{I}_i)) \;\leq\; \left(\mathbf{E}(\exp(-\theta \widetilde{I}_{4L}))\right)^{4L}. \tag{3.5}$$

By the Central Limit Theorem,

$$\lim_{L\to\infty} \mathbf{P}(\widetilde{I}_{4L} = 1) \;=\; \lim_{L\to\infty} \mathbf{P}(\widetilde{X}_{4L}(2CL^2) \geq 0) \;=\; \frac{1}{\sqrt{2\pi}} \int_{\frac{4}{\sqrt{2C}}}^{\infty} e^{-z^2/2} dz.$$

Choosing C large enough, the right-hand-side of the last display can be made as close to $1/2$ as desired. Hence, once an appropriate C is chosen, there exists L_0 such that $\mathbf{P}(\widetilde{I}_{4L} = 1) > 1/3$ for $L \geq L_0$. Consequently,

$$\mathbf{E}(\exp(-\theta \widetilde{I}_{4L})) \;\leq\; \frac{e^{-\theta} + 2}{3}, \tag{3.6}$$

for $L \geq L_0$.

From (3.1) - (3.6), we obtain,

$$\mathbf{P}(T > (C/\alpha)L^2) \;\leq\; \left(e^{\theta}\left(\frac{e^{-\theta}+2}{3}\right)^4\right)^L \;=\; e^{-\gamma(\theta)L}, \tag{3.7}$$

for $L \geq L_0$. Elementary calculus shows that

$$\lim_{\theta \searrow 0} \gamma(\theta) \;=\; 0 \qquad \text{and} \qquad \lim_{\theta \searrow 0} \frac{d\gamma}{d\theta}(\theta) \;=\; \frac{1}{3}.$$

Therefore $\gamma(\theta) > 0$, for small $\theta > 0$ and Claim (a) follows from (3.7), in the case $d = 2$.

Proof of Claim (a). Step 2: case $d > 2$. For $i = (i_3, i_4, ..., i_d) \in \{1, ..., L\}^{d-2}$, set

$$R_i = \{(x_1, x_2, i_3, ..., i_d) : x_1, x_2 \in \{1, ..., L\}\}, \qquad S_i = \{(x_1, x_2, i_3, ..., i_d) : x_1, x_2 \in \mathbb{Z}\}.$$

Then for each $i \in \{1, ..., L\}^{d-2}$, $R_i \subset S_i$, S_i is isomorphic to $\mathbb{Z}^2$, and R_i is isomorphic to $\{1, ..., L\}^2$. Note also that $\{R_i : i \in \{1, ..., L\}^{d-2}\}$ is a partition of R.

Let $i(1) < i(2) < ... < i(N)$, $N = L^{d-2}$, be the lexicographic order in $\{1, ..., L\}^{d-2}$. Observe that for each $x \in \mathbb{Z}^d$, in the lexicographic order, half of the neighbors of x are larger than x and half of the neighbors of x are smaller than x. It follows that for each $j \in \{1, ..., L\}^{d-2}$ and each $x \in R_j$, among the $2d$ neighbors of x, 4 are in S_j, $d-2$ are in $(\cup_{k<j} R_k) \cup R^C$, and $d-2$ are in $(\cup_{k>j} R_k) \cup R^C$. Therefore, for each $i \in \{1, ..., L\}^{d-2}$, once all spins in the sets R_j, $j < i$, are $+1$, they will stay in this state forever. Moreover, the sites in R_i will have then at least $d-2$ neighbors which are not in S_i and which are frozen in the state $+1$. Therefore, the spins in R_i will behave then in a way that favors spins $+1$ as compared to the 2 dimensional version of the dynamics restricted to S_i and started from the configuration which has spins -1 in R_i and spins $+1$ in $S_i \backslash R_i$ (meaning that one can construct a coupling of these evolutions in which the dynamics which is said to favor spins $+1$ has a spin $+1$ wherever the other one has one).

Motivated by the observations above, modify now the dynamics, by only allowing spins -1 in each set R_i to flip after all the spins in the sets R_j, $j < i$, are in state $+1$. Let T_k be the time in this modified dynamics when all spins in $R_{i(k)}$, $k = 1, ..., N$, are first in state $+1$. Then $T_1, T_2 - T_1, ..., T_N - T_{N-1}$ are mutually independent, and each one is stochastically smaller than a random variable $T^{(2)}$ which corresponds to the time to erode an $L \times L$ square in the 2 dimension version of the dynamics. From Step 1, we know that there exists $C, \gamma \in (0, \infty)$ such that

$$\mathbf{P}\left(T^{(2)} > \frac{C}{2\alpha} L^2\right) \;\leq\; e^{-2\gamma L}, \tag{3.8}$$

for large L.

Attractiveness allows the following comparison between our original dynamics and the modified one:

$$\mathbf{P}(T > (C/\alpha) L^d) \;\leq\; \mathbf{P}(T_N > (C/\alpha) L^d). \tag{3.9}$$

Now we modify further the modified dynamics, by observing it at times $(C/(2\alpha)) L^2 m$, $m = 1, 2, ...$, and resetting to -1 at those times all the spins in each of the $R_{i(k)}$, $k = 1, ..., N$, which still has some spin -1 at this time. (This will affect at most one of the $R_{i(k)}$ at each of these times, since the others will either be full of spins $+1$ or full of spins -1.) Using attractiveness again, we have the following comparison, where $(S_m)_{m=0,1,...}$ is a random walk which starts from 0 and at each unit of time moves 1 unit up, with probability $\mathbf{P}(T^{(2)} \leq (C/(2\alpha)) L^2)$, and otherwise remains still:

$$\mathbf{P}(T_N > (C/\alpha) L^d) \;\leq\; \mathbf{P}\left(S_{\frac{(C/\alpha) L^d}{(C/(2\alpha)) L^2}} < N\right) \;=\; \mathbf{P}(S_{2L^{d-2}} < L^{d-2}). \tag{3.10}$$

Using (3.8), we have

$$\mathbf{P}(S_{2L^{d-2}} < L^{d-2}) \;\le\; \binom{2L^{d-2}}{L^{d-2}} \left(\mathbf{P}\left(T^{(2)} > \frac{C}{2\alpha}L^2\right)\right)^{L^{d-2}} \;\le\; 2^{2L^{d-2}} \left(e^{-2\gamma L}\right)^{L^{d-2}} \;\le\; e^{-\gamma L^{d-1}}, \tag{3.11}$$

for large L. Claim (a) follows for $d > 2$, from (3.9), (3.10), and (3.11).

Proof of Claim (b). We will compare the evolution of our system to 4 quadrant evolutions, as defined in Step 1 of the proof of Claim (a). Each one of these 4 quadrant evolutions will be related to the evolution of our system close to one of the corners of the square R. Consider first a single quadrant evolutions, and define T_l as the first time when in this evolution the vertex $(l, 1)$ has spin $+1$. We will compare the distribution of $\widehat{T}$ to that of T_l, with l close to $L/2$. For this purpose, let $\widetilde{T}$ be the first time in our original evolution, when one of the following 8 sites is occupied by a spin $+1$: $(\lfloor L/2\rfloor, 1), (\lfloor L/2\rfloor + 1, 1), (1, \lfloor L/2\rfloor), (1, \lfloor L/2\rfloor + 1), (L, \lfloor L/2\rfloor), (L, \lfloor L/2\rfloor + 1)(\lfloor L/2\rfloor, L), (\lfloor L/2\rfloor + 1, L)$. (These are pairs of sites, roughly at the center of each side of the square R.) Due to the geometric properties of our original evolution, before time $\widetilde{T}$, we can see the $+1$ spins inside R in this evolution as a superposition of 4 independent quadrant evolutions. (with the axes properly reoriented and shifted), one for each corner of the square. Exploiting the symmetries of the model, we have then, for $t \ge 0$,

$$\mathbf{P}(T < t) \;\le\; \mathbf{P}(\widehat{T} < t) \;\le\; \mathbf{P}(\widetilde{T} < t) \;\le\; 8\mathbf{P}(T_{\lfloor L/2\rfloor - 1} < t). \tag{3.12}$$

In order to estimate T_l, we use again the relation between the quadrant evolution and the exclusion process, used in the proof of Claim (a) in the $d = 2$ case. With the notation introduced in Step 1, above, let $M_s = X_1(s)$ be the position at time $s \ge 0$ of the rightmost particle in the exclusion process. Then

$$\mathbf{P}\left(T_l < \frac{t}{2\alpha}\right) \;\le\; \mathbf{P}\left(\sup_{0\le s\le t} M_s \ge l - 1\right).$$

It is known from [Arr] that

$$\frac{M_s}{\sqrt{s}} - \sqrt{\log s} \to 0 \quad \text{a.s.} \quad \text{as} \quad s \to \infty.$$

This implies that, for $\delta > 0$

$$\mathbf{P}\left(T_l < l^2/(\log l)^{1+\delta/2}\right) \to 0 \quad \text{as} \quad l \to \infty.$$

Hence

$$\mathbf{P}\left(T_{\lfloor L/2\rfloor - 1} < L^2/(\log L)^{1+\delta}\right) \to 0 \quad \text{as} \quad L \to \infty. \tag{3.13}$$

Claim (b) follows from (3.12) and (3.13).

□

4. Proof of Theorem 1.1.

The proof of this theorem will be substantially involved, so that to make it more readable, we first explain some ideas behind it.

At first sight, one could think that if p is close to 1, so that the spins -1 form initially only finite clusters dispersed in a sea of spins $+1$, the spins $+1$ would easily take over and eliminate the spins -1. The actual subtlety in the behavior of the system, even in this case, may be seen as resulting from the subtlety of the behavior of bootstrap percolation. One can roughly see the system as a competition between a $(-1 \to +1)$ threshold $\theta = d$ bootstrap percolation process and a $(+1 \to -1)$ threshold $\theta = d$ bootstrap percolation process. (We say "roughly" because both bootstrap percolation processes are running in continuous time, and α may be less than 1.) In particular, if the spins -1 did not flip at all, and the spins $+1$ flipped with the rules of our dynamics, then the set of sites which would eventually be in state -1 would be precisely the set resulting from the $(+1 \to -1)$ threshold $\theta = d$ bootstrap percolation dynamics applied to the initial configuration. And it is known (see [Ent] and [Sch]) that, contrary to what one could first guess, even when $p < 1$ is very close to 1, this dynamics leads to each site being eventually in state -1. This means that to show that our dynamics in reality leads to each site being eventually in state $+1$, one must show that the significantly larger initial density of spins $+1$ allows them, as they expand via their own bootstrap percolation mechanism, to prevent the -1 spins from expanding.

Motivated by the picture discussed above, it is natural to try to use results and techniques which allow one to control from above the speed with which -1 spins can expand via their bootstrap percolation mechanism. Results of this kind are available from [AL], specifically, in the form of Lemma 2.1 (here) and its uses. Once control is gained on how fast -1 spins can expand, one can hope to use the estimate in Theorem 1.3(a) to show that before the -1's can expand, the spins $+1$ will typically eliminate them.

The need to consider a hierarchy of space and time scales in the analysis (multiscale analysis) arises then as follows. The approach mentioned in the previous paragraph can be used to show that there are space and time scales, l_1 and t_1, so that for regions of size l_1, the spins $+1$ will typically eradicate the spins -1 in a time t_1. But on the infinite lattice $\mathbb{Z}^d$, there will exist rare regions of size l_1 which at time t_1 still contain spins -1. Our approach relies then on considering a coarser (rescaled) lattice, with rescaled sites corresponding to original blocks of size l_1. One starts then at time t_1 with a very low density of rescaled sites containing spins -1. The arguments from the previous paragraph can be applied at this rescaled level, and produce space and time scales, l_2 and t_2, so that for regions of size l_2 in the rescaled lattice, typically only rescaled sites with no spins -1 are left in a time t_2. The procedure can be iterated, producing rapidly growing sequences of space and time scales. To control the eradication of spins -1 in the k-th scale, one uses induction in k, taking as input the results of the analysis of the $(k-1)$-th scale.

One further idea is needed in the implementation of the scheme sketched above. In order to analyze the behavior of the system at each scale, one needs to know not only that the starting configuration in this step has a low density of rescaled sites which contain spins -1, but one needs also to know that the joint distribution of these rare "bad" rescaled sites is well behaved. To recursively obtain this sort of control, we will be considering a modified dynamics in which interaction is restricted in range. The comparison between the original dynamics and this modified one will be obtained from attractiveness and from estimates on the speed with which effects can propagate.

We should mention also that an approach in spirit similar to ours, by using a combination of multiscale arguments with estimates involving bootstrap percolation, was taken by H.-N. Chen (see [Che1], [Che2]), studying the stability of "growth models with sexual reproduction".

We turn now to the specification of the space and time scales that will be used in the analysis, and various related objects.

We will use the notation $q = 1 - p$. For $\varepsilon > 0$, and small positive $\varkappa > 0$, to be chosen later, we set

$$q_0 = q, \qquad \ell_0 = 1, \qquad t_0 = 0, \tag{4.1}$$

and inductively define

$$q_k = e^{-\varkappa/(q_{k-1})^{\frac{1}{d-1}}}, \qquad \ell_k = \left\lfloor \left(\frac{1}{q_{k-1}}\right)^{\frac{d+2\varepsilon}{d-1}} \right\rfloor, \qquad t_k = \left(\frac{1}{q_{k-1}}\right)^{\frac{d+\varepsilon}{d-1}}. \tag{4.2}$$

Set also

$$L_k = \ell_0 \cdot \ell_1 \cdot \ldots \cdot \ell_k, \qquad T_k = t_0 + t_1 + \cdots + t_k. \tag{4.3}$$

Now we define the hypercubes of scale k, $k = 0, 1, \ldots$ as

$$B_k^i = \{0, ..., L_k - 1\}^d + L_k\, i, \quad i \in \mathbb{Z}^d.$$

Note that the hypercubes of scale 0 are of size 1, that for each k, $\{B_k^i,\, i \in \mathbb{Z}^d\}$ is a partition of $\mathbb{Z}^d$, and that for $k \geq 1$, each hypercube of scale k is the disjoint union of $(l_k)^d$ hypercubes of scale $k-1$.

For $k \geq 1$, with each B_k^i, $i \in \mathbb{Z}^d$, we also associate a larger hypercube, denoted by $\widetilde{B}_k^i$, which roughly speaking is concentric with B_k^i and has sidelength $(1 + \frac{2}{3})L_k$ (see Fig. 1). We will need $\widetilde{B}_k^i$ to be a union of hypercubes of scale $k-1$, and with this in mind we define

$$\widetilde{B}_k^i = \left(\cup_{j \in \bar{B}_k} B_{k-1}^j\right) + L_k i, \qquad \text{where} \qquad \bar{B}_k = \left\{- \left\lfloor \frac{1}{3} l_k \right\rfloor, ..., l_k + \left\lfloor \frac{1}{3} l_k \right\rfloor \right\}^d. \tag{4.4}$$

For simplicity we abbreviate $B_k = B_k^0$ and $\widetilde{B}_k = \widetilde{B}_k^0$.

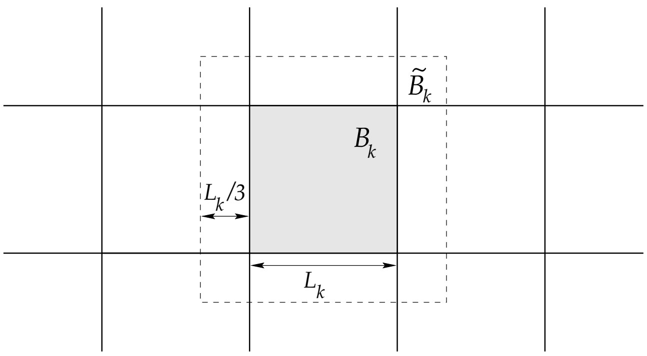

FIGURE 1. Solid lines separate blocks B_k^i, $i \in \mathbb{Z}^d$; dashed lines indicate the block $\widetilde{B}_k$, associated with the block B_k, which is shaded. The outer corridors between the boundaries of $\widetilde{B}_k$ and B_k are of width $L_k/3$.

Theorem 1.1 will be obtained through a comparison of the original system (1.1-2) with a modified dynamics, further referred to as block-dynamics, coupled to the original dynamics in that it will be constructed on the same probability space, using the same graphical marks, and defined by Rules 1-3 below. The block-dynamics will be so constructed as to have the following properties.

(A) The block-dynamics favors -1 spins, in the sense that at any site and time where the original dynamics has a -1 spin, also the block-dynamics has a -1 spin.

(B) In the block-dynamics at the time T_k all hypercubes of the k^{th} scale will be 'monochromatic', i.e., they will be entirely filled in either by $+1$ or -1 spins. In the former case we will say that the block is in state $+1$, and in the latter case that it is in state -1.

(C) For each $k \geq 1$, the random field η_k that associates to each $i \in \mathbb{Z}^d$ a random variable $\eta_k(i)$ which takes the value $+1$ (resp. -1) if at time T_k the block B_k^i is in state $+1$ (resp. -1) is a 1-dependent random field. This means that for each n, if $i_1, ..., i_n \in \mathbb{Z}^d$ are such that $\text{dist}(x_i, x_j) > 1$ for $i \neq j$, then the random variables $\eta_k(i_1), ..., \eta_k(i_n)$ are independent.

To assure that (C) above is satisfied, in the block-dynamics, the evolution of the spins in each box B_k^i, $i \in \mathbb{Z}^d$, during the interval of time $[T_{k-1}, T_k]$, will depend only on the configuration at time T_{k-1} and the graphical marks inside the corresponding box $\widetilde{B}_k^i$.

To define the block-dynamics so that (A) above is also satisfied, we first introduce a notion of *'influence time'*, associated with the box $\widetilde{B}_k^i$. Let $(\sigma^{\xi, T_{k-1}}_{\widetilde{B}_k^i, \zeta; s})_{s \geq T_{k-1}}$ be the evolution in the box $\widetilde{B}_k^i$, with boundary condition ζ outside this box, started at time T_{k-1} from the configuration ξ inside the box and obtained using the graphical marks. Set, now,

$$\tau_k^i = \inf \left\{ s \geq T_{k-1} : \sigma^{\xi, T_{k-1}}_{\widetilde{B}_k^i, +; s}(x) \neq \sigma^{\xi, T_{k-1}}_{\widetilde{B}_k^i, -; s}(x) \text{ for some } x \in B_k^i \text{ and some } \xi \in \{-1, +1\}^{\widetilde{B}_k^i} \right\}. \tag{4.5}$$

Note that by attractiveness, evolutions with other boundary conditions will be bounded from above and below by the two evolutions which appear in the definition of τ_k^i. Therefore we can think of τ_k^i as the first (random) time after T_{k-1}, when spins in B_k^i can suffer any influence from the spins outside of $\widetilde{B}_k^i$ at time T_{k-1}.

In order to define the evolution in the block-dynamics of the spins inside each box B_k^i, $i \in \mathbb{Z}^d$, from T_{k-1} to T_k, we use now the following rules.

Rule 1. During the interval of time $[T_{k-1}, T_k)$ we observe the evolution inside the box $\widetilde{B}_k^i$ with $+1$ boundary conditions. We assign to the spins in the box B_k^i up to time $\min\{\tau_k^i, T_k\}$ the values that we see in that evolution.

Rule 2. If it is the case that $\tau_k^i < T_k$, then at the time τ_k^i all spins in B_k^i will be declared to be -1, and persist at this state without change up to time T_k.

Rule 3. If, following the two rules above, there is any spin in state -1 in B_k^i at times which are arbitrarily close to T_k, then at time T_k all the spins in B_k^i are declared to be in state -1. Otherwise, at time T_k all the spins in B_k^i are declared to be in state $+1$.

It is immediate from *Rules 1-3* that the block-dynamics satisfies properties (A), (B) and (C) above. (Note that for this we need the initial distribution to be 1-dependent, which is the case, since it is a product measure. The properties can then be verified by induction in k.)

Let $\widetilde{q}_k$, $k \geq 0$, denote the probability that at the time T_k the block B_k is in the state -1. Note that $\widetilde{q}_0 = q_0$. The following is our main technical estimate in order to prove Theorem 1.1.

Lemma 4.1. *If q is small enough, then $\widetilde{q}_m \le q_m$ for all $m \ge 0$.*

The following estimates will be needed in the proof of Lemma 4.1 and Theorem 1.1. Let

$$\hat{q} = \sup\{x > 0 : \text{if } q \in (0,x), \text{ then } q_k \le q,\ k = 0,1,...\}.$$

Note that if $q > 0$ is small enough, then $q_1 \le q_0 = q$ and then, by induction, q_k is decreasing in k. Therefore $\hat{q} > 0$. Several times we will need to take $q \in (0,\hat{q})$, to assure that certain inequalities hold regardless of the value of k. In other words, we will need this assumption in order to assure uniformity over scales in the choice of constants in the multiscale scheme.

We claim that for arbitrary $\delta > 0$, if $q \in (0,\hat{q})$ is small enough, then

$$q_{k-1}q_{k-2}\cdots q_1 q_0 \ \ge\ (q_k)^{\delta}, \tag{4.6}$$

for $k \ge 1$. Indeed, for $k = 1$ we have, when q is small, $q_0 \ge \exp(-\delta\varkappa/(q_0)^{\frac{1}{d-1}}) = (q_1)^{\delta}$. Suppose now that (4.6) is true for $k-1$ in place of k, i.e., $q_{k-2}q_{k-3}\cdots q_1 q_0 \ge (q_{k-1})^{\delta}$. Then, for $q \in (0,\hat{q})$ small, $q_{k-1}q_{k-2}\cdots q_0 \ge (q_{k-1})^{\delta+1} \ge \exp(-\delta\varkappa/(q_{k-1})^{\frac{1}{d-1}}) = (q_k)^{\delta}$, which is (4.6). This proves, by induction, that (4.6) holds for all $k \ge 1$.

From the definitions of L_k and l_k, it is clear that for small $q \in (0,\hat{q})$,

$$L_k \ \ge\ l_k \ \ge\ \frac{1}{2}\left(\frac{1}{q_{k-1}}\right)^{\frac{d+2\varepsilon}{d-1}}, \tag{4.7}$$

for $k \ge 1$. For a bound in the opposite direction, we use (4.6) to obtain the following. For arbitrary $\delta > 0$, if $q \in (0,\hat{q})$ is small enough,

$$L_k \ \le\ \left(\frac{1}{q_{k-1}q_{k-2}\cdots q_1 q_0}\right)^{\frac{d+2\varepsilon}{d-1}} \ \le\ \left(\left(\frac{1}{q_k}\right)^{\frac{(d-1)\delta}{d+2\varepsilon}}\right)^{\frac{d+2\varepsilon}{d-1}} \ =\ \left(\frac{1}{q_k}\right)^{\delta}, \tag{4.8}$$

for $k \ge 1$.

Proof of Lemma 4.1. We use induction on m. The statement is obviously true for $m = 0$. Assume now that it is true for $m = k$, and we will show that $\widetilde{q}_{k+1} \le q_{k+1}$.

Following Rule 1, we observe the evolution inside the box $\widetilde{B}_{k+1}$ with $+1$ boundary conditions, during the interval of time $[T_k, T_{k+1})$. Let F_{k+1} be the event that in this evolution -1 spins are present in the box B_{k+1} at times which are arbitrarily close to T_{k+1}. We will show that

$$\mathbf{P}_p(F_{k+1}) \ \le\ \frac{q_{k+1}}{2}. \tag{4.9}$$

We will also show that

$$\mathbf{P}_p(\tau_{k+1} < T_{k+1}) \ \le\ \frac{q_{k+1}}{2}, \tag{4.10}$$

where $\tau_{k+1} = \tau^0_{k+1}$. Combining (4.9) with (4.10) yields the desired inequality $\widetilde{q}_{k+1} \le q_{k+1}$, since $\widetilde{q}_{k+1} \le \mathbf{P}_p(F_{k+1}) + \mathbf{P}_p(\tau_{k+1} < T_{k+1})$.

The proof of (4.9) will be divided into two steps. In the first step, we will analyze the random configuration inside $\widetilde{B}_{k+1}$ at the time T_k, and use methods from the study of bootstrap percolation to show that all the -1 spins in this configuration are likely to be contained in a collection of well separated rectangles which are not too large. In the second step, we will analyze the evolution inside the box $\widetilde{B}_{k+1}$ with $+1$ boundary conditions, during the interval of time $[T_k, T_{k+1})$, conditioned on the configuration at time T_k having this property.

The third step in the proof of Lemma 4.1 will be the proof of (4.10).

Step 1. Control of bootstrapping at time T_k. We will cover all the sites in $\widetilde{B}_{k+1}$ which have a spin -1 at time T_k with a collection R_1, R_2, ..., R_N of well separated rectangles. (Recall that a family of disjoint subsets of $\mathbb{Z}^d$ is well separated if there is no vertex in $\mathbb{Z}^d$ at distance less than or equal to 1 from two sets in the family.) At time T_k all blocks B_k^i of scale k are monochromatic, i.e., entirely occupied by spins -1 or $+1$, and now they will play the role of "renormalized sites" of $\widetilde{B}_{k+1}$. For the sake of notation, we identify these "renormalized sites" with elements of $\bar{B}_{k+1}$ (see (4.4)). For $i \in \bar{B}_{k+1}$, let $\eta_k(i)$ be $+1$ (resp. -1) if the block B_k^i is in state $+1$ (resp. -1) at time T_k. We obtain first a the collection of rectangles $\bar{R}_1$, $\bar{R}_2$, ..., $\bar{R}_N$, by applying the $(+1 \to -1)$ $\theta = 2$ bootstrap percolation rule to the random field η_k in $\bar{B}_{k+1}$. Recall that this means that $\bar{R}_1$, $\bar{R}_2$, ..., $\bar{R}_N$ is the smallest collection of well separated rectangles in $\bar{B}_{k+1}$ which contain all the renormalized sites $i \in \bar{B}_{k+1}$ which have $\eta_k(i) = -1$. Let now $R_n = \cup_{j \in \bar{R}_n} B_k^j$, $n = 1, ..., N$, and note that $R_1, R_2, ..., R_N$ have the desired properties.

By translation invariance and from the induction hypothesis,

$$\mathbf{P}_p(\eta_k(i) = -1) \;=\; \widetilde{q}_k \leq q_k, \tag{4.11}$$

for each i.

For $b > 0$ to be chosen later (small enough), we define the following event.

$$E_{k+1} \;=\; \left\{\text{the rectangles } \bar{R}_1, \bar{R}_2, ..., \bar{R}_N \text{ have all sides of length at most } \lfloor b/(q_k)^{\frac{1}{d-1}} \rfloor\right\}.$$

Our goal in this step is to show that if b and $\varkappa$ are chosen properly (independently of k) and q is small enough, then (4.11) implies

$$\mathbf{P}_p((E_{k+1})^C) \;\leq\; \frac{e^{-\varkappa/(q_k)^{\frac{1}{d-1}}}}{4} \;=\; \frac{q_{k+1}}{4}. \tag{4.12}$$

In order to show (4.12), we will use Lemma 2.1, with the choice

$$n \;=\; \left\lfloor \frac{b}{(q_k)^{\frac{1}{d-1}}} \right\rfloor. \tag{4.13}$$

We have

$$\begin{array}{l}\text{\# of rectangles inside } \bar{B}_{k+1} \text{ with the length} \\ \text{of the larger side being in } \{\lfloor n/2 \rfloor - 1, ..., n\}\end{array} \;\leq\; \left(\frac{5}{3}\ell_{k+1}\right)^d \left(\left\lfloor \frac{b}{(q_k)^{\frac{1}{d-1}}} \right\rfloor\right)^d, \tag{4.14}$$

where the first term on the right hand side of (4.14) is an upper bound on the number of choices of positions for the corner of the rectangle with all maximal coordinates, and the second term is an upper bound on the number of choices of sidelengths of the rectangle.

We claim that if $R \subset \bar{B}_k$ is a rectangle of size $n_1 \times n_2 \times \cdots \times n_d$, with $n_1 \le n_2 \le ... \le n_d$, then for the bootstrap percolation process that we are considering,

$$\mathbf{P}_p(R \text{ is internally spanned}) \ \le \ (2n_1 n_2 \cdots n_{d-1} q_k)^{\lfloor n_d/3 \rfloor}. \tag{4.15}$$

To see this, let $R'_1, R'_2, ..., R'_{n_d}$ be the rectangles of size $n_1 \times n_2 \times ... \times n_{d-1} \times 1$ which partition R according to the value of the d-th coordinate, and ordered according to increasing values of this coordinate. If R is internally spanned, then for each $1 \le j \le n_d - 1$, $R'_j \cup R'_{j+1}$ must contain some vertex i at which $\eta_k(i) = -1$ (otherwise each of these vertices will have at most 1 neighboring vertex $i' \in R$ with $\eta_k(i') = -1$ and therefore will remain in state $+1$ in the bootstrap percolation process in R that we are considering). We can now use the fact that the random field $(\eta_k(i))_{i\in\mathbb{Z}^d}$ is 1-dependent, to conclude that for $j = 1, 4, 7, 10, ..., (\lfloor n_d/3 \rfloor - 1)\cdot 3 + 1$, the corresponding events are independent. In combination with (4.11), this leads to (4.15), since for each j, $|R'_j \cup R'_{j+1}| = 2n_1 n_2 \cdots n_{d-1}$.

Maximizing the r.h.s. of (4.15) over choices of $n_1 \le n_2 \le \cdots \le n_d$, with $n_d \in \{\lfloor n/2 \rfloor - 1, \dots, n\}$, with n given by (4.13), we obtain

$$\mathbf{P}_p(R \text{ is internally spanned}) \ \le \ (2(n_d)^{d-1} q_k)^{\lfloor n_d/3 \rfloor} \ \le \ \left(2b^{d-1}\right)^{b/(7(q_k)^{\frac{1}{d-1}})} \ = \ \exp\left\{\frac{b\,\log(2b^{d-1})}{7(q_k)^{\frac{1}{d-1}}}\right\}, \tag{4.16}$$

provided $2b^{d-1} < 1$ and $q \in (0, \hat{q})$ is small enough. Using Lemma 2.1 with the choice made above of n, (4.14) and (4.16) we have

$$\mathbf{P}_p((E_{k+1})^C) \ \le \ \left(\frac{5b\ell_{k+1}}{3(q_k)^{\frac{1}{d-1}}}\right)^d \exp\left\{\frac{b\,\log(2b^{d-1})}{7(q_k)^{\frac{1}{d-1}}}\right\} \ \le \ \left(\frac{5b}{3(q_k)^{\frac{d+2+2\varepsilon}{d-1}}}\right)^d \exp\left\{\frac{b\,\log(2b^{d-1})}{7(q_k)^{\frac{1}{d-1}}}\right\}.$$

For small $b > 0$, $b\log(2b^{d-1})$ is negative. So, choosing $b > 0$ which maximizes $-b\log(2b^{d-1})$ and $\varkappa = -\frac{1}{8}b\log(2b^{d-1}) > 0$, yields (4.12).

Step 2. Erosion of (-1)*-rectangles.* We will show in this step that, for q small enough,

$$\mathbf{P}_p(F_{k+1}|E_{k+1}) \ \le \ \frac{q_{k+1}}{4}. \tag{4.17}$$

In combination with (4.12), this implies (4.9), since $\mathbf{P}_p(F_{k+1}) \le \mathbf{P}_p(F_{k+1}|E_{k+1}) + \mathbf{P}_p((E_{k+1})^C)$.

By the Markov property, we need to estimate the probability that starting at time T_k from a configuration in $\widetilde{B}_{k+1}$ compatible with the event E_{k+1} and letting the system evolve with $+1$ boundary conditions, some spin -1 will be present at time T_{k+1}. By attractiveness, an upper bound can be obtained on this probability by starting the evolution inside $\widetilde{B}_{k+1}$ at time T_k with -1 spins at all sites of the rectangles $R_1, ..., R_N$ described in Step 1. Clearly, no spin -1 can appear in this evolution outside of the rectangles $R_1, ..., R_N$. Also, once a rectangle is "destroyed", meaning that it contains no -1 spins, no -1 spins will ever be created inside of it again, in the evolution that we are considering here.

If E_{k+1} occurs, then each rectangle R_n, $n = 1, ..., N$, is contained in a cube of sidelength bounded above by

$$\left\lfloor \frac{b}{(q_k)^{\frac{1}{d-1}}} \right\rfloor L_k \ \le \ \left(\frac{1}{q_k}\right)^{\frac{1+\varepsilon/(2d)}{d-1}},$$

for small $q \in (0, \hat{q})$, where we used (4.8). By attractiveness, the time needed to erode R_n is therefore stochastically bounded above by the time needed to erode a cube with sidelength equal to this upper bound. From Theorem 1.3(a) we conclude then that, for small $q \in (0, \hat{q})$, for each $n = 1, ..., N$, the probability that at time $T_{k+1} = T_k + t_{k+1} = T_k + (1/q_k)^{(d+\varepsilon)/(d-1)}$ there is any spin -1 inside R_n is bounded above by

$$\exp\left(-\gamma\left(\frac{1}{q_k}\right)^{\frac{1+\varepsilon/(2d)}{d-1}}\right),$$

where γ is a positive constant. But, clearly, for small $q \in (0, \hat{q})$,

$$N \;\le\; \left|\widetilde{B}_{k+1}\right| \;\le\; \left(\frac{5}{3}L_{k+1}\right)^d \;\le\; \frac{1}{q_{k+1}},$$

by (4.8). From the bounds in the last two displays, we obtain

$$\mathbf{P}_p(F_{k+1}|E_{k+1}) \;\le\; \frac{1}{q_{k+1}}\;\exp\left(-\gamma\left(\frac{1}{q_k}\right)^{\frac{1+\varepsilon/(2d)}{d-1}}\right) \;\le\; \frac{q_{k+1}}{4},$$

for small $q \in (0, \hat{q})$. This completes the proof of (4.17), and hence of (4.9).

Step 3. Control of the outer influence. In this step we will prove (4.10).

We begin with the notion of a *discrepancy process*. Consider two evolutions $\sigma^{\xi,T_k}_{\widetilde{B}_{k+1},+;t}$ and $\sigma^{\xi,T_k}_{\widetilde{B}_{k+1},-;t}$, $t \ge T_k$, starting at time T_k from the same configuration ξ in $\widetilde{B}_{k+1}$, with $(+)$ and $(-)$ boundary condition outside $\widetilde{B}_{k+1}$, respectively, and using the same graphical marks. We say that at time $t \ge T_k$ there is a *discrepancy* at a vertex $x \in \widetilde{B}_{k+1}$ if $\sigma^{\xi,T_k}_{\widetilde{B}_{k+1},+;t}(x) \ne \sigma^{\xi,T_k}_{\widetilde{B}_{k+1},-;t}(x)$, for some ξ. Otherwise, the vertex x is called an *agreement* vertex.

According to this terminology and our choice of the conditions for the dynamics at T_k, all vertices in $\widetilde{B}_{k+1}$ at time T_k are agreement vertices and all vertices outside $\widetilde{B}_{k+1}$ have discrepancies. Moreover, the influence time τ_{k+1} is the time of occurrence of the first discrepancy in B_{k+1}.

We observe that the occurrence of a discrepancy at time s at vertex $z \in \widetilde{B}_{k+1}$ implies, that at time $s-$ there must be at least one neighboring vertex of z occupied by a discrepancy. This follows straightforwardly from the fact that the two evolutions that we are considering use the same graphical marks. For $x \in \widetilde{B}_{k+1}$, we define

$$T(x) \;=\; \inf\left\{s \ge T_k :\, \sigma^{\xi,T_k}_{\widetilde{B}_{k+1},+;t}(x) \ne \sigma^{\xi,T_k}_{\widetilde{B}_{k+1},-;t}(x) \text{ for some } \xi\right\}.$$

We recall that a (self-avoiding) path π on $\mathbb{Z}^d$ is a sequence of distinct vertices $\pi = \{x_1, x_2, ..., x_n\}$, such that $||x_i - x_{i+1}|| = 1,\ 1 \le i < n$. A *chronological path* in the epoch (T_k, T_{k+1}) is a path $\pi = \{x_1, x_2, ..., x_n\}$ such that

$$T_k \;<\; T(x_1) \;<\; T(x_2) \;<\cdots<\; T(x_n) \;<\; T_{k+1}.$$

In this case we say that the chronological path starts at x_1 and ends at x_n.

In what follows, for any volume Λ

$$\partial\Lambda = \{x \in \Lambda : \text{ there is a nearest neighbor of } x \text{ not in } \Lambda\};$$
$$\bar{\partial}\Lambda = \{x \notin \Lambda : \text{ there is a nearest neighbor of } x \text{ in } \Lambda\}.$$

Proceeding with the argument, we claim that if some vertex $x \in \partial B^i_{k+1}$ at time $t > T_k$ is occupied by a discrepancy, then there exists a chronological path in the epoch (T_k, t) which starts at some vertex of $\bar{\partial}\widetilde{B}^i_{k+1}$ and ends at $x \in \partial B^i_{k+1}$. Note that the above claim is a deterministic statement and does not depend on the randomness of the evolution. Once this is realized, the proof of (4.10) is completed by estimating the probability of such an event.

To prove the above claim we will construct a chronological path ending at $x \in \partial B^i_{k+1}$ by moving backwards in time. Assume that $x = \tilde{x}_1 \in \partial B^i_{k+1}$ and $T(\tilde{x}_1) < t$. So at time $T(\tilde{x}_1)$ there must exist at least one neighboring vertex $\tilde{x}_2 \in \widetilde{B}^i_{k+1}$ of x_1 occupied by a discrepancy with $T(\tilde{x}_2) < T(\tilde{x}_1)$. At time $T(\tilde{x}_2)$ there must exist at least one neighboring vertex $\tilde{x}_3 \in \widetilde{B}^i_{k+1}$ of $\tilde{x}_2$, occupied by a discrepancy with $T(\tilde{x}_3) < T(\tilde{x}_2)$. Note that $\tilde{x}_3 \neq \tilde{x}_1$. It also follows that $T(\tilde{x}_3) < T(\tilde{x}_2)$. Now we iterate the procedure, thus producing a self-avoiding path moving backwards in time, such that $\tilde{x}_j \neq \tilde{x}_i$ for $j \neq i$ and $T(\tilde{x}_j) < T(\tilde{x}_{j-1}) < \cdots < T(\tilde{x}_2) < T(\tilde{x}_1) < t$.

This time-reversed path can be made to end in a site x_r in $\bar{\partial}\widetilde{B}^i_{k+1}$ after finitely many steps. For the finiteness of the number of steps, notice that during the epoch (T_k, t), with $t < T_{k+1}$, only finitely many Poisson marks occur in the interior of $\widetilde{B}^i_{k+1}$ $\mathbf{P}$-a.s. Due to the coupling we are using, a discrepancy at some vertex cannot be created without the presence of at least one other discrepancy at some neighboring vertex and at time T_k, the discrepancies are located only outside $\widetilde{B}^i_{k+1}$. So our path must be traced back to some $x_r \in \bar{\partial}\widetilde{B}^i_{k+1}$.

To get a forwards in time path, we invert the order and set $x_j = \tilde{x}_{r+1-j}$, $1 \le j \le r$, and obtain a chronological path starting at $x_1 \in \bar{\partial}\widetilde{B}^i_{k+1}$ and ending at $x_r \in \partial B^i_{k+1}$.

From the claim it follows that, if $\tau_{k+1} \le T_{k+1}$, then there exists a chronological path connecting $\bar{\partial}\widetilde{B}_{k+1}$ to ∂B_{k+1}, thus covering a distance which, for small $q \in (0, \hat{q})$, is at least

$$r_k \;=\; \left\lfloor \frac{1}{4} L_{k+1} \right\rfloor \;\ge\; \frac{1}{9}\left(\frac{1}{q_k}\right)^{\frac{d+2\varepsilon}{d-1}},$$

where in the last inequality we used (4.7).

The number of possible starting points for this chronological path equals

$$\left|\bar{\partial}\widetilde{B}_{k+1}\right| \;\le\; C(L_{k+1})^{d-1} \;\le\; \frac{C}{q_{k+1}},$$

for small $q \in (0, \hat{q})$, where C is a dimension dependent constant, and we used (4.8) in the last inequality.

On the other hand, the probability that any given path of length r is a chronological path during the epoch (T_k, T_{k+1}) is clearly bounded above by $\mathbf{P}(Z \ge r-1)$, where Z is a Poisson random variable with mean $T_{k+1} - T_k = t_{k+1} = (q_k)^{-(d+\varepsilon)/(d-1)}$. Using the standard large deviation estimate for Poisson random variables (see, e.g., (A.2) in [KS], p.467) $\mathbf{P}\{Z \ge r-1\} \le e^{-(\log(r/t_{k+1})-1)r}$, together with the upper bound $(2d)^r$ on the number of self-avoiding paths of length r starting from a given vertex, we obtain the following estimate. If $q \in (0, \hat{q})$ is chosen small enough, then for appropriate constants C' and C'', for any $k \geq 1$,

$$\mathbf{P}_p(\tau_{k+1} \leq T_{k+1}) \;\leq\; \sum_{r \geq r_k} \frac{C}{q_{k+1}} \, (2d)^r \; e^{-2\log(2d) r} \;\leq\; \frac{C'}{q_{k+1}} \, \exp\left(-C'' \left(\frac{1}{q_k}\right)^{\frac{d+2\varepsilon}{d-1}} \right) \;\leq\; \frac{q_{k+1}}{2},$$

where we used the fact that r/t_{k+1} can be made arbitrarily large, uniformly in k and $r \geq r_k$, by taking $q \in (0, \hat{q})$ small enough.

This proves (4.10), and completes the proof of Lemma 4.1.

□

Proof of Theorem 1.1. Set $a = 1/(d+\varepsilon)$ and note for later use that for any $\varepsilon > 0$

$$a > \frac{1}{d} - \varepsilon. \tag{4.18}$$

We will first prove that if q is small enough, then there are $C_1, C_2 \in (0, \infty)$ such that

$$\mathbf{P}_p(\sigma_t(0) = -1) \;\leq\; C_1 e^{-C_2 t^a}, \tag{4.19}$$

for $t > 0$.

We consider first times of the form $t = T_k$ for some $k \geq 1$. From (4.2) it follows that $q_k = e^{-\varkappa(t_k)^a}$. Comparing the original dynamics with the block-dynamics at time T_k (recall Property (A) of the block dynamics, and note that the origin is in the block B_k) and using Lemma 4.1, we get

$$\mathbf{P}_p(\sigma_t(0) = -1) \;=\; \mathbf{P}_p(\sigma_{T_k}(0) = -1) \;\leq\; \tilde{q}_k \;\leq\; q_k \;=\; e^{-\varkappa(t_k)^a}. \tag{4.20}$$

In order to replace t_k with T_k in the exponent, we observe that $T_1 = t_1$, and there exists $0 < c < 1$, such that, for small $q \in (0, \hat{q})$, for all $k \geq 2$,

$$\frac{t_{k-1}}{t_k} \;=\; \left(\frac{q_{k-1}}{q_{k-2}}\right)^{\frac{d+\varepsilon}{d-1}} \;=\; \left(\frac{e^{-\varkappa/(q_{k-2})^{\frac{1}{d-1}}}}{q_{k-2}}\right)^{\frac{d+\varepsilon}{d-1}} \;\leq\; c\,,$$

and thus

$$T_k \;=\; t_1 + \cdots + t_k \;\leq\; t_k\big(1 + c + c^2 + \ldots\big) \;=\; (1-c)^{-1} t_k\,. \tag{4.21}$$

So, for $t = T_k$, from (4.20) and (4.21) we get

$$\mathbf{P}_p(\sigma_t(0) = -1) \;\leq\; e^{-\varkappa(t_k)^a} \;\leq\; e^{-\varkappa((1-c)T_k)^a} = e^{-C(T_k)^a} \;=\; e^{-Ct^a}, \tag{4.22}$$

where we take $C = \varkappa(1-c)^a$.

In order to extend the result to all positive times $t \geq 0$ we will be comparing evolutions started from product measures with different values of q. For this purpose, we write $q_k(q)$, $t_k(q)$ and $T_k(q)$

for the corresponding values of q_k, t_k and T_k defined by (4.2) and (4.3) with $q_0 = q$. Summarizing the conclusion in (4.22), we know that there exists $\widetilde{q} \in (0, \hat{q})$ and $C > 0$, such that for all $q \in (0, \widetilde{q}]$, if for some $k \geq 1$, $t = T_k(q)$, then

$$\mathbf{P}_p(\sigma_t(0) = -1) \;\leq\; e^{-Ct^a}. \tag{4.23}$$

We will write $\widetilde{q}_k = q_k(\widetilde{q})$, $\widetilde{t}_k = t_k(\widetilde{q})$ and $\widetilde{T}_k = T_k(\widetilde{q})$. Since $\widetilde{q} \in (0, \hat{q})$, $\widetilde{q}_k$ decreases with k, and therefore $\widetilde{t}_k$ increases with k.

Note that, for each fixed $k \geq 1$, if we imagine the parameter q decreasing continuously from $\widetilde{q}$ to $\widetilde{q}_1$, we will have the corresponding $T_k(q)$ increasing continuously from $T_k(\widetilde{q}) = \widetilde{T}_k$ to $T_k(\widetilde{q}_1) = t_1(\widetilde{q}_1) + \ldots + t_k(\widetilde{q}_1) = t_2(\widetilde{q}) + \ldots + t_{k+1}(\widetilde{q}) = T_{k+1}(\widetilde{q}) - t_1(\widetilde{q}) = \widetilde{T}_{k+1} - \widetilde{t}_1$. Thus, by continuity, any $t > 0$ which is not in $\cup_{k\geq 1}[\widetilde{T}_k - \widetilde{t}_1, \widetilde{T}_k)$, can be written as $t = T_{k(t)}(q(t))$, for some $k(t) \geq 1$ and some $q = q(t) \in (\widetilde{q}_1, \widetilde{q}]$. Set $p(t) = 1 - q(t)$. Then for any $q \in (0, \widetilde{q}_1)$ we have $p = 1 - q \geq 1 - \widetilde{q}_1 \geq 1 - q(t) = p(t)$. Therefore, by attractiveness and (4.23), we have

$$\mathbf{P}_p(\sigma_t(0) = -1) \;\leq\; \mathbf{P}_{p(t)}(\sigma_t(0) = -1) \;\leq\; e^{-Ct^a}.$$

This establishes the validity of (4.19) for $q < \widetilde{q}_1$ and $t > 0$ which is not in $\cup_{k\geq 1}[\widetilde{T}_k - \widetilde{t}_1, \widetilde{T}_k)$. To extend the result to t in this excluded set, observe that for each k and $t \in [\widetilde{T}_k - \widetilde{t}_1, \widetilde{T}_k]$, if $\sigma_t(0) = -1$ and the spin at the origin does not flip between times t and $\widetilde{T}_k$, then $\sigma_{\widetilde{T}_k}(0) = -1$. Using the Markov property, we obtain then

$$\mathbf{P}_p(\sigma_t(0) = -1) \;\leq\; \frac{1}{e^{-\widetilde{t}_1}}\,\mathbf{P}_p(\sigma_{\widetilde{T}_k}(0) = -1) \;\leq\; C' e^{-C''(\widetilde{T}_k)^a} \;\leq\; C' e^{-C''t^a}, \tag{4.24}$$

where the term $e^{-\widetilde{t}_1}$ is a lower bound on the probability of the event that no flips occur at the origin from t to $\widetilde{T}_k$. This completes the proof of (4.19).

To derive (1.4) from (4.19), we first note that an argument similar to the one used to derive the first inequality in (4.24) (but now using the strong Markov property) gives, for $t \geq 1$,

$$\mathbf{P}_p(\sigma_s(0) = -1 \text{ for some } s \in [t-1, t)) \;\leq\; e\,\mathbf{P}_p(\sigma_t(0) = -1) \;\leq\; e\,C_1 e^{-C_2 t^a}.$$

Consequently

$$\begin{aligned}\mathbf{P}_p(\sigma_s(0) = -1 \text{ for some } s \geq t) \;&\leq\; \sum_{n=0}^{\infty} \mathbf{P}_p(\sigma_s(0) = -1 \text{ for some } s \in [t+n, t+n+1)) \\ &\leq\; \sum_{n=0}^{\infty} e\,C_1 e^{-C_2(t+n+1)^a} \;\leq\; e\,C_1 \int_t^{\infty} e^{-C_2 s^a}\, ds.\end{aligned}$$

Observe now that for arbitrary $\delta > 0$,

$$\int_t^{\infty} e^{-C_2 s^a}\, ds \;\leq\; e^{-C_2 t^{a-\delta}},$$

for large t. To see this, note that $e^{-C_2 s^a} \le C_2\,(a-\delta)s^{a-\delta-1}\,e^{-C_2 s^{a-\delta}}$, for large s, and integrate both sides from t to ∞.

From the last two displays, we obtain

$$\mathbf{P}_p(\sigma_s(0) = -1 \text{ for some } s \ge t) \;\le\; e\,C_1\,e^{-C_2 t^{a-\delta}} \;\le\; e^{-t^{a-2\delta}},$$

for large t.

This completes the proof of (1.4), since from (4.18) we have $a - 2\delta > (1/d) - \varepsilon$, for small $\delta > 0$.

$\square$

5. Proof of Theorem 1.2.

In this section $d = 2$. Set

$$F(t) \;=\; \mathbf{P}_p(\sigma_s(0) = -1 \;\forall s \le t).$$

It is clear that since the flip rates are at most 1,

$$F(t) \;\ge\; qe^{-t}. \tag{5.1}$$

The next lemma is the main step in the proof of Theorem 1.2.

Lemma 5.1. *Suppose that for some $a > 0$ there are $C_1, C_2 \in (0,\infty)$ such that*

$$F(t) \;\ge\; C_1 e^{-C_2 t^a}, \tag{5.2}$$

for $t \ge 0$, and that $\delta > 0$. Then there are $C_1', C_2' \in (0,\infty)$ such that

$$F(t) \;\ge\; C_1' e^{-C_2' t^{\frac{2a(1+\delta)}{2a+1}}}, \tag{5.3}$$

for $t \ge 0$.

From (5.1) and Lemma 5.1, we obtain a first stretched exponential lower bound on $F(t)$: For arbitrary $\varepsilon > 0$, there are $C_1, C_2 \in (0,\infty)$ such that

$$F(t) \;\ge\; C_1 e^{-C_2 t^{\frac{2}{3}(1+\varepsilon)}}, \tag{5.4}$$

for $t \ge 0$.

Using now (5.4) as the input in Lemma 5.1, we obtain the better lower bound: For arbitrary $\varepsilon > 0$, there are $C_1, C_2 \in (0,\infty)$ such that

$$F(t) \;\ge\; C_1 e^{-C_2 t^{\frac{4}{7}(1+\varepsilon)}},$$

for $t \ge 0$.

The proof of Theorem 1.2 will be obtained by showing that by iterating this procedure, the exponent of t in the stretched exponential can be chosen as close to $1/2$ as desired.

Proof of Lemma 5.1. If $a \in (0, 1/2]$, then $(2a)/(2a+1) \geq a$, so that the claim is trivially true with $C_1' = C_1$ and $C_2' = C_2$. We suppose therefore that $a > 1/2$.

Set

$$l = \left\lfloor t^{\frac{1+\delta}{2a+1}} \right\rfloor, \qquad K = \left\lfloor \frac{t^{\frac{1+\delta}{2}}}{l} \right\rfloor.$$

Note that, since $a > 1/2$, we have $1 << K << l$, for large t. Define also the following subsets of $\mathbb{Z}$:

$$A = \{-lK, -lK+1, ..., lK, lK+1\}, \qquad B = \bigcup_{i=-K}^{K} \{il, il+1\}$$

and the following subsets of $\mathbb{Z}^2$ (see Fig 2):

$$\bar{Q} = A \times A, \qquad Q = (A \times B) \cup (B \times A).$$

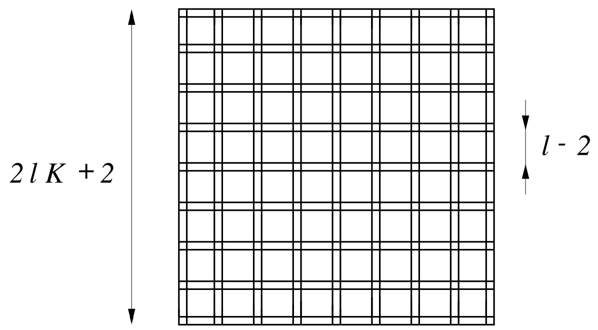

FIGURE 2. The set Q. (The set $\bar{Q}$ is the square obtained by filling all the holes in Q.)

The set $\bar{Q}$ is a square of sidelength $2lK+2$. (Note that $2lK+2 \sim 2t^{(1/2)(1+\delta)}$, as $t \to \infty$.) The set Q is a subset of $\bar{Q}$, which can be seen as a porous structure obtained by removing $(2K)^2$ squares of sidelength $l-2$ from $\bar{Q}$. (Note that $2K \sim 2t^{(2a-1)(1+\delta)/(4a+2)}$, and $l-2 \sim t^{(1+\delta)/(2a+1)}$, as $t \to \infty$.) The horizontal and vertical strips whose union defines Q are of width 2. This assures that the only sites in Q which do not have at least three neighbors also in Q are the four corner sites in Q: $(-lK, -lK)$, $(-lK, lK+1)$, $(lK+1, -lK)$ and $(lK+1, lK+1)$. This implies that if Q is fully occupied by spins -1 at time 0, no one of these spins can flip until one of the spins at a corner site of Q flips.

Define

$$\widetilde{t} \; = \; (C/\alpha)\, t^{2(1+\delta)/(2a+1)},$$

with C as in Theorem 1.3(a). Let E_1 be the event that Q is fully occupied by spins -1 at time 0. Let E_2 be the event that the spins at the four corner sites of Q are -1 from time 0 to time $\widetilde{t}$. Using the fact that E_1 and E_2 are decreasing events, (5.2) and translation invariance, we obtain, for large t,

$$\begin{aligned}\mathbf{P}_p(E_1E_2) &\geq \mathbf{P}_p(E_1)\,\mathbf{P}_p(E_2) \geq q^{|Q|}\left(F\left(\frac{C}{\alpha}\,t^{\frac{2(1+\delta)}{2a+1}}\right)\right)^4 \\ &\geq q^{4(2K+1)(2lK+2)}\left(C_1\,\exp\left(-C_2\left(\frac{C}{\alpha}\right)^a t^{\frac{2a(1+\delta)}{2a+1}}\right)\right)^4 \\ &\geq q^{17\,t^{\frac{2a(1+\delta)}{2a+1}}}\,(C_1)^4\,\exp\left(-4C_2\left(\frac{C}{\alpha}\right)^a t^{\frac{2a(1+\delta)}{2a+1}}\right) \\ &= (C_1)^4\,\exp\left(-C_2'\,t^{\frac{2a(1+\delta)}{2a+1}}\right). \end{aligned} \tag{5.5}$$

If E_1 and E_2 happen, the spins in Q are frozen as -1 from time 0 to time $\widetilde{t}$. We can therefore use Theorem 1.3(a), with the roles of -1 and $+1$ spins switched, to see that for each of the $(2K)^2$ square holes of size $(l-2)\times(l-2)$ in Q, the probability that it is not fully occupied by spins -1 at time $\widetilde{t}$ (which is $\geq (C/\alpha)(l-2)^2$) is bounded above by $\exp(-\gamma(l-2)^2) \leq \exp(-(\gamma/2)t^{2(1+\delta)/(2a+1)})$, for large t. Let E_3 be the event that $\bar{Q}$ is fully occupied by spins -1 at time $\widetilde{t}$. We conclude that for large t,

$$\begin{aligned}\mathbf{P}_p(E_3|E_1E_2) &\geq 1-(2K)^2\,e^{-(\gamma/2)t^{2(1+\delta)/(2a+1)}} \\ &\geq 1-5\,t^{(2a-1)(1+\delta)/(2a+1)}\,e^{-(\gamma/2)t^{2(1+\delta)/(2a+1)}} \geq \frac{1}{2}. \end{aligned} \tag{5.6}$$

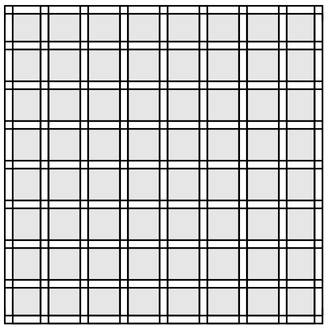

FIGURE 3. Conditioned on E_1, E_2, by the time $\widetilde{t}$ all holes (of size $(l-2)\times(l-2)$) in Q are likely to be entirely filled in by -1 spins (shaded region).

Since the origin is contained in Q, on E_1E_2 the origin has spin -1 from time 0 to time $\widetilde{t}$. Therefore,

$$\mathbf{P}_p\left(\sigma_s(0)=-1\ \forall s\leq t \,|\, E_1E_2E_3\right) = \mathbf{P}_p\left(\sigma_s(0)=-1\ \forall s\in[\widetilde{t},t]\,\middle|\,E_1E_2E_3\right). \tag{5.7}$$

Note that the event $E_1E_2E_3$ only depends on the evolution of the process up to time $\widetilde{t}$, and that on $E_1E_2E_3$, at time $\widetilde{t}$ the origin is at the center of a square of sidelength $2lK+2 \sim 2t^{(1/2)(1+\delta)}$ fully occupied by spins -1. Therefore, from the Markov property, Theorem 1.3(b), and attractiveness, we have

$$\mathbf{P}_p\left(\sigma_s(0) = -1 \ \forall s \in [\widetilde{t}, t] \,\middle|\, E_1 E_2 E_3\right) \ \geq \ \frac{1}{2}, \tag{5.8}$$

for large t.

From (5.5), (5.6), (5.7) and (5.8), we obtain

$$F(t) \ \geq \ \mathbf{P}_p(\sigma_s(0) = -1 \ \forall s \leq t \,|\, E_1 E_2 E_3)\, P_p(E_3|E_1 E_2)\, P_p(E_1 E_2) \ \geq \ \frac{(C_1)^4}{4} \exp\left(-C_2'\, t^{\frac{2a(1+\delta)}{2a+1}}\right),$$

for large t.

By choosing C_1' sufficiently small, we obtain (5.3), for all $t \geq 0$.

□

Proof of Theorem 1.2. We will show that for any $\varepsilon \in (0,2)$ there are $C_1, C_2 \in (0,\infty)$ such that

$$F(t) \ \geq \ C_1 e^{-C_2 t^{\frac{1+\varepsilon}{2}}}, \tag{5.9}$$

for $t \geq 0$. This is clearly equivalent to (1.6) being true for large t.

Set

$$g(a) = \frac{2a(1+\varepsilon/4)}{2a+1}, \quad a \geq 0.$$

We have $g(0) = 0$, $g(1) = (2/3)(1+\varepsilon/4) \in (0,1)$. The equation $g(a) = a$ has solutions $a = 0$ and $a = \bar{a} = 1/2 + \varepsilon/4 \in (0,1)$. Also,

$$\frac{d}{da} g(a) = \frac{2(1+\varepsilon/4)}{(2a+1)^2}$$

is positive and decreasing in $a > 0$. We conclude that $g : [0,1] \to \mathbb{R}$ is a concave increasing function, which has fixed points 0 and $\bar{a}$.

Set

$$a_0 = 1, \qquad a_i = g(a_{i-1}), \quad i \geq 1.$$

It follows from the properties of g that

$$a_i \searrow \bar{a} \quad \text{as} \quad i \to \infty.$$

In particular, there exists j such that $a_j < (1+\varepsilon)/2$.

By (5.1), the hypothesis of Lemma 5.1 is satisfied with $a = a_0 = 1$ and $\delta = \epsilon/4$. (5.9) follows now from iterating this lemma j times, always with $\delta = \epsilon/4$.

□

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