A signal-recovery system: asymptotic properties, and construction of an infinite-volume limit

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Abstract

We consider a linear sequence of 'nodes', each of which can be in state 0 ('off') or 1 ('on'). Signals from outside are sent to the rightmost node and travel instantaneously as far as possible to the left along nodes which are 'on'. These nodes are immediately switched off, and become on again after a recovery time. The recovery times are independent exponentially distributed random variables.

We present properties for finite systems and use some of these properties to construct an infinite-volume extension, with signals 'coming from infinity'. This construction is related to a question by D. Aldous and we expect that it sheds some light on, and stimulates further investigation of, that question.

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1 Introduction

Let $X_1(t), \ldots X_n(t)$ be 0–1-valued random processes described as follows: When X_i equals 0 it becomes 1 at rate ρ_i, independently of the other X_j-s. If each of $X_i, X_{i+1}, \ldots, X_n$ equals 1, then at rate ρ they all become simultaneously 0. We start at time $t = 0$ with all $X_i's$ equal to 0. The ρ_i-s and ρ are parameters of the model, called *recovery rates* and the *input rate*, respectively.

This system can be interpreted as a simple model of a communication line, and we will frequently use terminology motivated by this interpretation: the indices $1, 2, \ldots, n$ correspond to nodes which can be 'on' (have value 1) or 'off' (have value 0). Signals from outside are sent at rate ρ to the rightmost node n and are transmitted instantaneously as far as possible to the left until they are blocked by an off-node. The nodes passed by the signal are switched off immediately. When a node i is 'off', it becomes 'on' after an exponentially distributed (with mean $1/\rho_i$) recovery time. Recovery times are completely independent.

Another interpretation is in terms of forest fires (or infections): the numbers $1, \ldots, n$ are possible locations of trees. At the rightmost location ignition attempts are made at rate ρ, and an attempt succeeds if that position is occupied. When a tree is on fire, it immediately sets the tree on the next location to its left (if that location is occupied) on fire and disappears (i.e. its position becomes empty). If position i is empty, a new tree appears there at rate ρ_i. Since in a real forest the growth of new trees is much slower than the propagation of fire, the instantaneous spread of fire (or infection) in our model is not as unrealistic as it may seem at first sight. (The one-dimensionality *is* of course a serious simplification in this context).

In the above description the incoming signals correspond to a Poisson process. More generally we will consider signals corresponding to a renewal process. The distribution function of the intervals between consecutive incoming signals will be denoted by $F^{(n+1)}$. (The reason for this notation, with the superscript $n + 1$, will become clear later).

So, more precisely, this more general model is as follows: The parameters of the model are $F^{(n+1)}$ (a distribution function with $F^{(n+1)}(0) = 0$), and the recovery rates ρ_i, $i = 1, \ldots, n$. Introduce i.i.d. random variables $\tau_1, \tau_2, \ldots$ with distribution function $F^{(n+1)}$, and call the values $T_i := \sum_{j=1}^{i} \tau_j$, $i = 0, 1, 2, \ldots$ *input times.* At the zeroth input time $T_0 = 0$ we set each X_i, $i = 1, \ldots, n$ equal to 0. When $X_i = 0$ it becomes 1 at rate ρ_i, independent of the other X_j-s and of the τ_j-s. If, at time t, $X_i = X_{i+1} = \ldots, = X_n = 1$, then each X_j, $j \geq i$ becomes 0 at the smallest input time larger than t. We call this model a *(size n) on-off system with recovery rates $\rho_1, \ldots, \rho_n$ and input interval distribution function $F^{(n+1)}$*. The case mentioned in the first paragraph, when the input signals arrive according to a Poisson process of intensity ρ, corresponds to $F^{(n+1)} = E^{\rho}$, where $E^{\rho}(t) = 1 - e^{-\rho t}$ denotes the exponential distribution function with expectation $1/\rho$.

As said before, we will frequently use terminology inspired by the signal interpretation. Although this terminology is practically self-explanatory, we want to define some of these terms more precisely, to avoid confusion: we say that a signal *is sent to node* i at time t, if X_{i+1} switches from on to off at time t (or, in case i is the rightmost node, if t is an input time). And we say that a signal *is received by* i at time t, if X_i itself switches from on to off at time t.

Finally, we will also consider the case that input signals are generated 'permanently'. By this we mean that the rightmost node, n, after each recovery immediately receives a signal (and hence is switched off again). In this case we say (with some abuse of notation, since there are no proper input intervals anymore) that $F^{(n+1)} = [0]$. It is easy to see that this case is (when we only 'observe' the behaviour of the nodes $1, \ldots, n-1$) equivalent to the earlier mentioned case with $n-1$ nodes and with Poisson (intensity ρ_n) input signals, i.e., with input interval distribution function $F^{(n)} = E^{\rho_n}$.

Several interesting questions arise: Suppose the input is Poissonian, and all recovery rates are equal (say 1), and we start with all nodes empty. What is the asymptotic behaviour (as $n \to \infty$) of the expectation of the first time a signal arrives at node 0. This appears to be of order $\log n$. (Note that this means that when the system is in equilibrium, the probability that an arbitrary incoming signal reaches node 0, is of order $1/(\log n)$, which is much larger than one would naively guess). This is done in Section 2. Several arguments in that section are of key importance for Section 3, which deals with the question whether there are non-trivial extensions of this model to *infinite systems, with signals "coming from infinity"*. The answer, as stated in Theorem 1, is positive and is related to a question posed by D. Aldous. We hope the result sheds some more light on that question. The proof of Theorem 1 is presented in Section 4.

2 Properties of the finite system

Consider a size n on-off system (as defined in Section 1) with input interval distribution function $F^{(n+1)}$ and recovery rates $\rho_1, \ldots, \rho_n$. As stated before, the input signals correspond to a renewal process. It is easy to see that the times at which signals are *received* by n (i.e. the times at which X_n switches from "on" to "off") also form a renewal process (because, whenever X_n switches from "on" to "off", the process, as far as node n is concerned, starts afresh). Since each signal received by n is sent instantaneously to $n-1$, we can repeat the above argument and conclude that the reception times of signals at $n-1$ also form a renewal process etc. We call the distribution function of the difference between consecutive times at which node i receives a signal, its *interreception time distribution function*.

The following lemma relates the interreception time distributions of two consecutive nodes:

Lemma 1. *Let, for* $1 \le i \le n$, $F^{(i)}$ *be the interreception time distribution function of node* i *and* $F^{(n+1)}$ *the input interval distribution. Define:*

$$\phi^{(i)}(s) = 1 - \int_0^\infty e^{-sx}\, dF^{(i)}(x), \quad i = 1, 2, \dots, n+1$$

Then:

$$\phi^{(i)}(s) = \frac{\phi^{(i+1)}(s)}{\phi^{(i+1)}(s+\rho_i)}, \quad i = 1, \dots, n.$$

Proof. Let τ be the first time node i switches from 'off' to 'on', and let Y be the first time it receives a signal. Further, let Z_k be the k-th time node $i+1$ receives a signal, and let $\xi_k = Z_k - Z_{k-1}$, $k = 1, 2, \dots$. The random variable τ is exponentially distributed with parameter ρ_i. Furthermore, the random variables ξ_k, $k = 1, 2, \dots$ are i.i.d. and also independent of τ. So we have:

$$\begin{aligned}
\phi^{(i)}(s) = 1 - \mathbf{E}\big(e^{-sY}\big) &= 1 - \sum_{k=0}^{\infty} \mathbf{E}\big(e^{-sZ_{k+1}} \mathbb{1}_{\{\tau \in [Z_k, Z_{k+1})\}}\big) \\
&= 1 - \sum_{k=0}^{\infty} \mathbf{E}\big(e^{-sZ_{k+1}} \big(e^{-\rho_i Z_k} - e^{-\rho_i Z_{k+1}}\big)\big) \\
&= 1 - \sum_{k=0}^{\infty} \mathbf{E}\big(e^{-(s+\rho_i)Z_k} \big(e^{-s\xi_{k+1}} - e^{-(s+\rho_i)\xi_{k+1}}\big)\big) \\
&= 1 - \frac{\mathbf{E}\big(e^{-s\xi_1}\big) - \mathbf{E}\big(e^{-(s+\rho_i)\xi_1}\big)}{1 - \mathbf{E}\big(e^{-(s+\rho_i)\xi_1}\big)} \\
&= \frac{\phi^{(i+1)}(s)}{\phi^{(i+1)}(\rho_i + s)}. \qquad (1)
\end{aligned}$$

□

By repeated application of the above Lemma 1, and using induction, we get

Lemma 2. *For* $1 \le i \le j \le n+1$*:*

$$\phi^{(i)}(s) = \frac{\prod_{A \subset \{i, \dots, j-1\} : |A| \text{ even}} \phi^{(j)}(s + \sum_{k \in A} \rho_k)}{\prod_{A \subset \{i, \dots, j-1\} : |A| \text{ odd}} \phi^{(j)}(s + \sum_{k \in A} \rho_k)}, \qquad (2)$$

where $|A|$ *denotes the number of elements of* A.

This immediately gives the following result:

Lemma 3. *The interreception time distribution of node i, $F^{(i)}$, is invariant under permutations of the sequence of recovery rates $\rho_i, \rho_{i+1}, \dots, \rho_n$.*

Remark: In spite of its apparent simplicity, this observation is rather surprising: it easily follows from identity (2) but we do not see any simple direct 'pathwise' argument for its proof.

Lemma 3 is important in the construction of an infinite-volume system in the next section. We will illustrate its strength in the remainder of the present section. We consider the special case when all ρ_i-s are equal, say 1, and the inputs come permanently (that is, $F^{(n+1)} = [0]$). As already mentioned, this is equivalent to a system of $n-1$ nodes with recovery rates 1 and Poissonian input with rate 1, so that $\phi^{(n)}(s) = s/(1+s)$. Using the identity $\binom{n-1}{l-1} + \binom{n-1}{l} = \binom{n}{l}$, from (2) we get

$$\phi^{(1)}(s) = \frac{\prod_{0 \le k \le n:k \text{ even}} (s+k)^{\binom{n}{k}}}{\prod_{0 \le k \le n:k \text{ odd}} (s+k)^{\binom{n}{k}}}.$$

We denote here by T_n the first time a signal is received by node 1. (As we are interested in the asymptotics for long chains of nodes, we denote explicitly by the subscript n the length of the string of identical nodes considered). Thus $\phi^{(1)}(s) = 1 - \mathbf{E}\big(\exp(-sT_n)\big)$. By evaluating the derivative of the above expression at $s = 0$, we get

$$\mathbf{E}\big(T_n\big) = \frac{\prod_{1 \le k \le n:k \text{ even}} k^{\binom{n}{k}}}{\prod_{1 \le k \le n:k \text{ odd}} k^{\binom{n}{k}}}. \tag{3}$$

András Lukács, [5], drew our attention to the survey-article [3] by Flajolet and Sedgewick, about the use of contour integrals (and Melling transforms) to study the asymptotic behaviour as $n \to \infty$ of expressions of the form $\sum_{k=1}^{n} (-1)^k \binom{n}{k} f(k)$ for a wide range of functions f. The case $f(k) = \log k$ is one of the examples they handle (see their Theorem 4), and according to their paper the expression in the r.h.s. of (3) is asymptotic to $e^{\gamma} \log n$. So

$$\lim_{n \to \infty} \frac{\mathbf{E}\big(T_n\big)}{\log n} = e^{\gamma}, \tag{4}$$

where $\gamma = 0.577\dots$ is Euler's constant. Although the following quite elementary probabilistic argument, based on Lemma 3 above, does not give the precise value of the limit in (4), it does give the correct order of magnitude of $\mathbf{E}\big(T_n\big)$ One of our reasons for working this out here is that a similar argument is used in the construction of the infinite-volume system in Section 4. Another reason is that from the paper by Flajolet and Sedgewick, [3], one gets the impression that no elementary way is known to obtain the order of magnitude of the r.h.s. of (3).

Proposition 1. *Consider, for each n, a finite on-off system with nodes $\{1, \dots, n\}$, where all recovery rates are 1, and with permanent input signals. Let T_n denote the first time node 1 receives a signal. Then there exist constants $C_1, C_2 > 0$ such that for all n*

$$C_1 < \frac{\mathbf{E}(T_n)}{\log n} < C_2 \tag{5}$$

Proof. We use stochastic domination in proving both bounds.

The *lower bound* is easy: Note that before the first receival time at node 1 all nodes $1, 2, \dots n$ must recover at least once. So T_n stochastically dominates $\max\{\tau_i : 1 \le i \le n\}$, where $\tau_1, \tau_2, \dots, \tau_n$ are i.i.d. exponentially distributed random variables with mean 1. It follows that

$$\mathbf{E}\big(T_n\big) \ge \mathbf{E}\big(\max\{\tau_i : 1 \le i \le n\}\big) = \sum_{i=1}^{n} \frac{1}{i} = \log n + \mathcal{O}(1),$$

which proves the lower bound.

The *upper bound* uses a little trick: Suppose we add an extra node 0 at the left of node 1, with recovery rate $1/\log n$. Denote this new system by II and the old system by I. Let $\tilde{T}$ be the first time in system II that node 0 receives a signal. It is clear that system II is an extension of the old one, in the sense that the nodes $1, \dots, n$ 'do not feel the change', so that obviously $T_n \le \tilde{T}$. Finally consider the system, denoted III, obtained from system I by putting an extra node $n+1$ at the right of n, with recovery rate $\rho_{n+1} = 1/(\log n)$. (So, in system III the input signals are sent to $n+1$ which, if it is 'on', sends them to n, etc). Let $\hat{T}$ denote the first time node 1 receives a signal in system III. By Lemma 2, $\hat{T}$ has the same distribution as $\tilde{T}$. So we have

$$\mathbf{E}(T_n) \le \mathbf{E}(\tilde{T}) = \mathbf{E}(\hat{T}).$$

The following computation is for system III. Let k be a non-negative integer. Let A be the event that an input signal is sent in the time interval $(4k \log n, (4k+1)\log n)$, B the event that node $n+1$ has no recovery in the interval $(4k \log n, (4k+2)\log n)$, but does have a recovery in $((4k+2)\log n, (4k+3)\log n))$, C the event that each of the nodes $1, \dots, n$ which is off at time $(4k+1)\log n$ has a recovery before time $(4k+2)\log n$, and D the event that an input signal is sent to $n+1$ in the interval $((4k+3)\log n, (4k+4)\log n)$. It is easy to see that the conditional probability of $A \cap B \cap C \cap D$ given all information up to time $4k \log n$ is at least

$$(1 - e^{-\log n})\, e^{-2}\, (1 - e^{-1})\, (1 - e^{-\log n})^n\, (1 - e^{-\log n}),$$

which is larger than $\alpha := e^{-3}(1 - e^{-1})/2 > 0$, uniformly in k, for sufficiently large n. Moreover, if all the events A-D happen, node 1 will receive a signal in the interval $((4k+3)\log n, (4k+4)\log n)$ (and hence in $(4k \log n, 4(k+$

$1)\log n)$). So, for each integer $k \geq 1$, we have $P(\hat{T} > 4k \log n) \leq \alpha^k$, from which the required result follows.

$\square$

3 Infinite-volume models

Note that a finite on-off system, as introduced in Section 1, could be described as a collection $X_1(t), \ldots, X_n(t)$ of 0–1 valued processes with the property that the time intervals during which a process has value 0 are independent, exponentially distributed (those for X_i with mean $1/\rho_i$), and that, after independent time intervals with distribution $F^{(n+1)}$, the string of 1-s connected to node n is turned into 0-s. In this section we investigate the question whether there are suitable infinite-volume systems with such properties. There are several cases to distinguish, depending on the asymptotic behaviour of the ρ_i-s and the nature of the input signal 'at infinity' (which will be made precise later). The most interesting appears to be the case where

$$\sum_{i=1}^{\infty} e^{-t\rho_i} < \infty, \qquad \forall t > 0, \tag{6}$$

and with 'permanent input signals at infinity'. In the present paper we consider only this case in detail. However, see Remark (iii) after Theorem 1 stated below for a concise description of the other possibilities.

The above condition (6) on the ρ_i-s means, by Borel-Cantelli, that if we start with all nodes in state 0, and there would be NO input signals, there is an infinite connected string of 1-s at any positive time $t > 0$. So, when we do take into account permanent input signals at infinity we expect, intuitively, that in every time interval, no matter how small, infinite connected strings of 1's are formed and immediately destroyed (i.e. turned into 0-s). It is not at all clear at this stage that a dynamics with such kind of behaviour exists; see Remark (i) below about existence problems for so-called frozen-percolation models, and Remark (ii).

The main result of the present paper is a proof that such a system does indeed exist. More precisely, we prove the following theorem.

Theorem 1. *Let $\rho_i, i = 1, 2, \ldots$ be positive numbers satisfying (6) There exist 0–1-valued processes $X_i : \mathbb{R}_+ \mapsto \{0, 1\}$, $i \in \mathbb{N}$, defined jointly on the same probability space, with the following properties:*

(a) Almost surely, for all $i \in \mathbb{N}$, $X_i(0) = 0$.

(b) Almost surely, for all $i \in \mathbb{N}$, $t \mapsto X_i(t)$ is continuous from the right having left limits (c.a.d.l.a.g.).

(c) *Let T_k^i denote the length of the kth interval during which $X_i(\cdot)$ equals 0. Then each T_k^i is exponentially distributed with mean $1/\rho_i$, and the random variables $\left(T_k^i\right)_{i,k\in\mathbb{N}}$ are independent.*

(d) *Almost surely, for all $t \in \mathbb{R}_+$ and $k \in \mathbb{N}$ with $X_k(t^-) = 1$: if for all $l \geq k$ $X_l(t^-) = 1$ then $X_k(t) = 0$, else $X_k(t) = 1$.*

Moreover, the collection of processes $t \mapsto X_i(t)$, $i = 1, 2, \dots$ has the following additional properties:

(e) *Almost surely, there are no t and k such that $X_l(t) = 1$ for all $l \geq k$.*

(f) *Almost surely, the reception times of signals are dense. That is, for all $t \geq 0$ and $\varepsilon > 0$ there exist $i \in \mathbb{N}$ and $s \in (t, t+\varepsilon)$, such that for all $j \geq i$ $X_j(s^-) = 1$ and $X_j(s) = 0$.*

Remarks:

(i) The following remark illustrates why the existence of such a process is not obvious: D. Aldous , [1], has introduced a percolation model in which infinite clusters are 'frozen' (we will refer to this model as 'frozen-percolation'). Informally, that model is as follows. Each vertex (or, for *bond percolation*, each edge) of a countably infinite, locally finite connected graph G can have state 0 or 1. At time 0 they are all in state 0. Now, assign to each vertex i a time τ_i. The (τ_i) are iid random variables with a continuous distribution. Each vertex i remains 0 until time τ_i. Then it switches to 1 (and stays 1 forever), unless some neighbour of i already belongs to an infinite cluster of 1-s, in which case i remains 0 forever. Aldous constructed such a process for the case where G is the regular binary tree, and posed the question whether it exists for $\mathbb{Z}^d$. Benjamini and Schramm, [2], have pointed out that it does not exist for $\mathbb{Z}^2$. The following simple, deterministic, 1-dimensional example, due to Antal Járai, [4], shows very clearly the essence of the difficulty:

Claim 1 (A. Járai, [4]). *Let $t_1, t_2, \dots$ be a sequence of distinct, strictly positive numbers which tends to 0. There is* no *sequence of functions $\omega_i : \mathbb{R}_+ \mapsto \{0,1\}$, $i = 1, 2, \dots$ with the following properties:*

$$\omega_i(t) := \begin{cases} 0 & \text{if } t < t_i \text{ or } \omega_j(t_i^-) = 1 \text{ for all } j > i \\ 1 & \text{otherwise} \end{cases}$$

Proof. Suppose such a sequence does exist. There are two possibilities: either there exist t and i with $\omega_j(t) = 1$ for all $j \geq i$ or there exist no such t and i. In the latter case we have (by the rules above) that $\omega_j(t) = 1$ for all j and all $t \geq t_j$. Since all t_j are smaller than some number $t_{\max}$, every ω_j equals 1 at time $t_{\max}$, a contradiction. As to the former case, let t and i be as stated there. Let j be the smallest number larger than i with $t_k < t_i$ for all $k \geq j$. From the rules given above (and the assumption for this case) it follows that $\omega_k(t_{j-1}^-) = 1$ for all $k \geq j$ and so $\omega_{j-1} = 0$ at *every* time, in particular at time t: again a contradiction. Since both cases lead to a contradiction, the proposition has been proved.

$\square$

Note that, when the t_i are not deterministic but independent, exponentially distributed random variables with mean $1/\rho_i$, $i = 1, 2, \dots$, with the (ρ_i) as in Theorem 1 (i.e., in our terminology, when they are the first recovery times of the nodes in the system Theorem 1 deals with) the condition in Járai's example is satisfied with probability 1. This shows that the frozen percolation model does not exist for the half-line with ρ_i-s as in Theorem 1. Although the situation for $\mathbb{Z}^2$ looks more complicated than for the half-line, the reason why frozen percolation does not exist is essentially the same: Consider frozen percolation on $\mathbb{Z}^2$, with identically (say: exponentially) distributed holding times τ_i, $i \in \mathbb{Z}^2$. At the critical time (just before an infinite cluster is formed), there are infinitely many separate (not connected with each other) open circuits around the origin. If we then consider the sequence $(t_i), i = 1, 2, \dots$ of (random) times needed to connect consecutive circuits, we are exactly in the situation of Járai's example. This illustrates how study of the half-line can give more insight on what happens on $\mathbb{Z}^2$.

(ii) In Section 5 of his paper, D. Aldous poses some open questions related to the frozen-percolation model. One of them is whether a dynamics exists where vertices (or, for bond percolation, edges) become open (in state 1) at rate 1 and where infinite clusters of 1-s are destroyed (i.e. turned into 0's) immediately. Although this question was formulated for graphs which have critical percolation probability less than 1, like the binary tree or $\mathbb{Z}^d$, $d \geq 2$, we think, based on the previous Remark, that results for analogous problems on the half-line, like our Theorem 1, can help to better understand these problems.

(iii) Now, returning to the setup of the present paper: given the recovery rates ρ_i, $i = 1, 2, \dots$ define

$$\theta := \sup\{t \in \mathbb{R}_+ : \sum_{i=1}^{\infty} e^{-\rho_i t} = \infty\} = \inf\{t \in \mathbb{R}_+ : \sum_{i=1}^{\infty} e^{-\rho_i t} < \infty\}.$$

There are four essentially different cases with essentially different behaviour of the infinitely extended system. Theorem 1 refers to Case 4, the only really interesting one. The claims below for Cases 1,2 and 3, which are formulated in a quite informal way, can be stated more precisely, and proved by straightforward applications of Borel-Cantelli lemmas.

Case 1: If $\theta = \infty$, then by a simple Borel-Cantelli argument one can see, that in the infinitely extended system *no signals coming from infinity will penetrate the system.* This is the case when $\rho_k \ll \log k$, as $k \to \infty$. The system with constant recovery rates, $\rho_i = 1$, belongs to this case.

Case 2: If $\theta < \infty$ and $\sum_{i=1}^{\infty} e^{-\rho_i \theta} = \infty$ then one can construct an infinite dynamics which satisfies properties (a-c) stated in Theorem 1, but not properties (d-f) (inclusion of property d in this case leads to the same kind of problems as in Jarai's example (see Remark (i) above)). In particular, there will be nonempty time intervals during which infinite connected strings of 1s are present in the system. This makes the dynamics uninteresting for us. Typical example is $\rho_k = \theta^{-1} \log k$.
Case 3: If $\theta < \infty$ and $\sum_{i=1}^{\infty} e^{-\rho_i \theta} < \infty$ then one can construct an infinitely extended dynamics with moderately interesting behaviour. Namely: in this case, if at some time t_0 all but finitely many nodes are in state 0, then exactly at time $t_0 + \theta$ an infinite connected string of 1-s emerges (Borel-Cantelli), which is instantaneously erased by a signal penetrating from infinity and sweeping through the system, down to the rightmost node in state 0. So, one can construct with 'bare hands' a dynamics where periodically, with period θ, signals penetrate from infinity and erase an infinite connected string of 1s, just emerging. Typical example is $\rho_k = \theta^{-1} \log k + \alpha \log \log k$, with $\alpha > 1$.
Case 4: The only really interesting case is $\theta = 0$. In this case infinite connected strings of 1-s try to emerge 'in no time' and are immediately swept away by signals penetrating from infinity. So the constructed system is in a permanent state of excitation. This behaviour is intuitively somewhat related to the so-called *self-organized criticality* phenomenon which receives a lot of attention in the physics literature. This case is the subject of Theorem 1.

(iv) A very natural question to ask is whether properties (a)-(d) listed in Theorem 1 determine *uniquely* the process. Under the extra condition that the signal reception times at each node form a *renewal process*, we can prove uniqueness. This uses very similar ideas to the ones presented in the proof of Theorem 1 in the next section. We do not include this proof in the present note. We can not anwer this question in full generality, without the extra assumption mentioned above.

4 Proof of Theorem 1

To prove the main theorem we will first revisit the finite case studied in Sections 1 and 2, and introduce some more terminology and notation. So consider a finite on-off system with nodes $\{1, \cdots, n\}$, recovery rates $\rho_1, \cdots, \rho_n$, and input interval distribution function $F^{(n+1)}$. Suppose at time 0 all nodes are off. Let, for $1 \le i \le n$ and $k = 1, 2, \dots$, R_k^i denote the kth recovery time at node i, i.e., the kth time it switches from 'off' to 'on'. Also, let S_k^i be the kth time a signal is received by node i. For convenience, we will define $S_0^i = 0$. Let $\mu(F^{(n+1)}; \rho_n, \rho_{n-1}, \cdots, \rho_1)$ denote the joint distribution of the collection $\big((R_k^i, S_k^i),\ 1 \le i \le n,\ k = 1, 2, \dots\big)$.

Lemma 4. *Let F and F_m, $m = 1, 2, \dots$ be probability distribution functions with $F(0) = 0$ and $F_m(0) = 0$ for all $m = 1, 2, \dots$. If F_m converges weakly to F then $\mu(F_m; \rho_n, \dots, \rho_1)$ also converges weakly to $\mu(F; \rho_n, \dots, \rho_1)$, as $m \to \infty$,*

Sketch of proof. The most natural (and rather standard) way to see this is by use of a space-time diagram. This enables us to couple two on-off systems with the same recovery rates but different input interval distributions, say F and F'. We give a short outline of the argument: Let $0 < I_1 < I_2 < I_3, \dots$ denote the points of a renewal process with interval distribution F. (That is, $(I_{k+1} - I_k)_{k=1,2,\dots}$ are i.i.d. random variables with distribution function F). Now assign to each node i, independently of the other nodes and of the above renewal process, a Poisson point process with intensity ρ_i. These Poisson points are interpreted as *potential* recovery points. This means that if t is such a point for node i, and node i is in state 0 just before time t, it switches to state 1 at time t (otherwise the point is ignored). The R_k^i and S_k^i can be defined in a natural way in terms of the above Poisson processes and the renewal process. If we now replace F by F', we can compare the new situation with the old one with the help of a suitable natural coupling: use the same realization of the above mentioned Poisson point processes and take an obvious coupling of F and F'. Details are left to the reader.

□

We will need the following notation: If F is the input interval distribution function at node n, then let, for $i \le n$, $F_{(\rho_n, \cdots, \rho_i)}$ denote the probability distribution function of the intervals between successive signals received at node i, i.e. the distribution of S_1^i. It is clear from the description of the system that for $i \le k \le n$

$$\left(F_{(\rho_n, \dots, \rho_k)}\right)_{(\rho_{k-1}, \cdots, \rho_i)} = F_{(\rho_n, \cdots, \rho_i)}.$$

If F and G are two probability distribution functions, we write $F \preceq G$ (or $G \succeq F$) if for any x we have $F(x) \ge G(x)$, i.e., if the distribution G stochastically dominates the distribution F. We have the following lemma:

Lemma 5. *For any $\rho_k, \dots, \rho_n, \rho_{n+1} > 0$ and any probability distribution function F,*

$$F_{(\rho_{n+1}, \rho_n, \dots, \rho_k)} \succeq F_{(\rho_n, \dots, \rho_k)}.$$

Proof. Using Lemma 3 we have $F_{(\rho_{n+1}, \rho_n, \dots, \rho_k)} = F_{(\rho_n, \dots, \rho_k, \rho_{n+1})} = \left(F_{(\rho_n, \dots, \rho_k)}\right)_{(\rho_{n+1})}$, which obviously stochastically dominates $F_{(\rho_n, \dots, \rho_k)}$. □

Remarks: (i) This lemma is not as obvious as it looks. For instance, it is *not* true in general that if $F \succeq G$, then $F_{(\rho)} \succeq G_{(\rho)}$. The above argument *essentially relies* on Lemma 3.
(ii) Although, strictly speaking, Lemma 3 has not been proved for the case

with permanent input (i.e. the case where the input interval distribution function is [0]), its analog for that case follows easily from the fact that, as remarked earlier, for such input signals the sequence of signals received at node n (and sent to node $n-1$) corresponds to a Poisson process with rate ρ_n, so that formally

$$[0]_{(\rho_n,\dots,\rho_k)} = E^{\rho_n}_{(\rho_{n-1},\dots,\rho_k)}, \tag{7}$$

with E^{ρ_n} the exponential distribution with mean $1/\rho_n$. In the sequel we shall use this notation for the exponential distribution. Using (7), and the (easy to check) fact that, if E^{ρ} and $E^{\rho'}$ are exponential distributions with parameter ρ and ρ', respectively, then

$$\left(E^{\rho}\right)_{(\rho')} = E^{\rho} * E^{\rho'} = \left(E^{\rho'}\right)_{(\rho)},$$

one can easily extend Lemma 3 to the case $F = [0]$.

The following lemma is a deterministic statement. First we give some more definitions and terminology.

A collection of non-negative numbers s^i_{k-1}, r^i_k, $1 \le i \le n$, $k = 1, 2, \dots$ is called a (volume-n) *signal/recovery sequence* if the following hold:

(i) For each i, $0 = s^i_0 < r^i_1 < s^i_1 < r^i_2 < s^i_2 < \dots$.

(ii) For each i the set $\{s^i_{k-1}, r^i_k : k = 1, 2, \dots\}$ is discrete.

(iii) For each $i < n$, and $k \ge 1$

$$s^i_k = \min\{s^{i+1}_l : s^{i+1}_l > r^i_k\}.$$

The motivation for this definition is that the r^i_k-s and s^i_k-s can be interpreted as a realization of the recovery and reception times in an on-off system.

We denote $\mathcal{S}^i := \{s^i_k : k \ge 0\}$. Property (iii) above is clearly equivalent to saying that (iiia) *and* (iiib) below hold for all $i < n$.

(iiia) $\mathcal{S}^i \subset \mathcal{S}^{i+1}$

(iiib) $\mathcal{S}^{i+1} \setminus \mathcal{S}^i \subset \cup_{k=1}^{\infty} (s^i_{k-1}, r^i_k]$.

We now give a natural infinite version of this definition: A collection of non-negative numbers s^i_{k-1}, r^i_k, $i = 1, 2, \dots,$ $k = 1, 2, \dots$ is called an infinite signal/recovery sequence if for each n the collection s^i_{k-1}, r^i_k, $i = 1, 2 \dots, n$, $k = 1, 2, \dots$ is a volume-n signal/recovery sequence. We say that the sequence has *dense signals* if for every interval $V \subset \mathbb{R}_+$ there exist i, k s.t. $s^i_k \in V$. When (s^i_{k-1}, r^i_k) is a, finite or infinite, signal/recovery sequence, we define its corresponding *on-off sequence* as the following sequence of functions $\omega_i : R_+ \mapsto \{0, 1\}$, $i \in \mathbb{N}$.

$$\omega_i(t) := \begin{cases} 0 & \text{if } \ t \in \cup_{k=1}^{\infty} [s^i_{k-1}, r^i_k), \\ 1 & \text{if } \ t \in \cup_{k=1}^{\infty} [r^i_k, s^i_k) \end{cases}$$

Lemma 6. *Let s^i_{k-1}, r^i_k, $i = 1, 2, \dots, k = 1, 2, \dots$ be an infinite signal/-recovery sequence with dense signals. Let $\omega_i(\cdot)$, $i = 1, 2\dots$ be the corresponding on-off sequence. Then:*

(a) For each $i = 1, 2\dots$, the function $t \mapsto \omega_i(t)$ is c.a.d.l.a.g.

(b) There are no t and k for which $\omega_l(t) = 1$, for all $l \geq k$.

(c) For each t and k with $\omega_l(t^-) = 1$ for all $l \geq k$, we have $\omega_k(t) = 0$.

(d) For every k, $l > k$ and $t > 0$ with $\omega_k(t^-) = 1$ and $\omega_l(t^-) = 0$, we have $\omega_k(t) = 1$.

Proof. (a) The c.a.d.l.a.g. property follows immediatley from the definition of the functions ω_i.

(b) Suppose that for some k and $t \geq 0$ $\omega_l(t) = 1$ for all $l \geq k$. Then, because of (a), there is an $\varepsilon > 0$ such that $\omega_k(s) = 1$ for all $s \in (t, t+\varepsilon)$. Hence, by definition of ω_k, $\mathcal{S}^k \cap (t, t+\varepsilon) = \emptyset$. However, because signals are dense, there is a $j > k$ with $\mathcal{S}^j \cap (t, t+\varepsilon) \neq \emptyset$. Let $\tilde{j}$ be the smallest of such $j > k$. So we have $\omega_{\tilde{j}-1}(t) = 1$, $\mathcal{S}^{\tilde{j}-1} \cap (t, t+\varepsilon) = \emptyset$ and $\mathcal{S}^{\tilde{j}} \cap (t, t+\varepsilon) \neq \emptyset$, which contradicts property (iiib) of a signal/recovery sequence.

(c) Suppose that for some $t > 0$ and some k $\omega_l(t^-) = 1$ for all $l \geq k$, and $\omega_k(t) = 1$. By (b) there is an $l > k$ with $\omega_l(t) = 0$. Let m be the smallest. So we have: $\omega_{m-1}(t^-) = \omega_{m-1}(t) = 1$ and $\omega_m(t^-) = 1$, $\omega_m(t) = 0$. This clearly implies that $t \in \mathcal{S}^m$ but at the same time t is in the interior of the set $\cup_k [r^{m-1}_k, s^{m-1}_k)$. This contradicts property (iiib) of signal/recovery systems.

(d) Suppose $\omega_k(t^-) = 1$ and $\omega_l(t^-) = 0$ for some $l > k$ and $\omega_k(t) = 0$. So $t \in \mathcal{S}^k$. But then (by property (iii) of a signal/recovery system) $t \in \mathcal{S}^l$, which is in conflict with the above mentioned fact that $\omega_l(t^-) = 0$.

□

We continue our proof of Theorem 1. Let the ρ_i, $i = 1, 2, \dots$ be as in the statement of the theorem, i.e., for all $t > 0$, $\sum_i e^{-\rho_i t} < \infty$. Let, for $k \leq l$, $F^{(k,l)} = [0]_{(\rho_l, \dots, \rho_k)}$ (see earlier in this section). Using Lemma 5 we have

$$F^{(k,l+1)} = [0]_{(\rho_{l+1}, \rho_l, \dots, \rho_k)} \succeq [0]_{(\rho_l, \dots, \rho_k)} = F^{(k,l)}.$$

Hence, keeping k fixed, the sequence of distributions $F^{(k,l)}$, $l \geq k$, converges weakly, as $l \to \infty$. The following lemma shows that it converges to a probability distribution:

Lemma 7. *For each k, $F^{(k,l)}(t) \to 1$ as $t \to \infty$, uniformly in l.*

Proof. As before, let E^ρ denote the exponential distribution with mean $1/\rho$. For each $\rho > 0$ and $t > 0$ we have (using Lemma 5 again)

$$F^{(k,l)}(t) = [0]_{(\rho_l, \dots, \rho_k)}(t) \preceq [0]_{(\rho, \rho_l, \dots, \rho_k)}(t) = E^\rho_{(\rho_l, \dots, \rho_k)}(t). \qquad (8)$$

Note that this last expression is the probability that in a finite on-off system with $l-k+1$ nodes with recovery rates $\rho_l, \dots, \rho_k$, and where the input signals are generated according to a Poisson process with intensity ρ, the last node receives a signal before time t. This probability is clearly larger than or equal to the probability that each of (a)-(c) below happens:
(a) No input signal is sent in the interval $(0, \sqrt{t})$.
(b) Every node is in state 1 at time $\sqrt{t}$.
(c) An input signal is sent in the interval $(\sqrt{t}, t)$.
This probability is

$$e^{-\rho\sqrt{t}} \prod_{j=k}^{l} (1 - e^{-\rho_j t})(1 - e^{-\rho(t-\sqrt{t})}) \le e^{-\rho\sqrt{t}}(1 - \sum_{j=1}^{\infty} e^{-\rho_j \sqrt{t}})(1 - e^{-\rho(t-\sqrt{t})}).$$

For every ρ this is a lower bound for $F^{(k,l)}(t)$. Now use (6) and take $\rho = t^{-2/3}$ to complete the proof of Lemma 7.

□

We go on with the proof of Theorem 1. We have seen that $F^{(k,l)}$ converges to a probability distribution function as $l \to \infty$. Denote the limit by $F^{(k)}$, and let

$$\mu_k := \mu(F^{(k)}; \rho_{k-1}, \dots, \rho_1),$$

where we use the notation introduced at the beginning of this section. In this way we get a sequence (μ_k) of probability measures on Σ^{k-1}, where Σ is the set of all sequences $(s_{i-1}, r_i)_{i=1}^{\infty}$ with $0 = s_0 < r_1 < s_1 < r_2 < \dots$. From the definitions it is clear that for each l, the projection of $\mu(F^{(k+1,l)}; \rho_k, \dots, \rho_1)$ on Σ^{k-1} equals $\mu(F^{(k,l)}; \rho_{k-1}, \dots, \rho_1)$. By Lemma 4 it follows that the projection of μ_{k+1} on Σ^{k-1} is μ_k. Hence, by standard extension theorems, there is a measure ν on $\Sigma^{\mathbb{N}}$ whose marginal on Σ^k is μ_{k+1}, $k = 1, 2, \dots$. It is clear that for each k a random element of Σ^k is μ_{k+1}-a.s. a (volume k) signal/recovery sequence. Hence, a random element of $\Sigma^{\mathbb{N}}$ is ν-a.s. an infinite signal/recovery sequence. The theorem now follows from Lemma 6 if we can show that ν-a.s. the system has dense signals. By standard countability arguments this is equivalent to showing that for every open interval $I \subset \mathbb{R}_+$,

$$\nu\{\exists k : \mathcal{S}^k \cap I \neq \emptyset\} = 1. \tag{9}$$

Due to property (iiia) of signal/recovery systems, the l.h.s. of (9) equals $\lim_{k\to\infty} \nu\{\mathcal{S}^k \cap I \neq \emptyset\}$ which, by the construction of ν above, equals

$$\lim_{k\to\infty} \lim_{l\to\infty} \mu([0]; \rho_l, \dots, \rho_k)\{\mathcal{S}^k \cap I \neq \emptyset\}.$$

The required result now follows from the following Lemma:

Lemma 8. *For every open interval $I \subset \mathbb{R}_+$ and for every $\varepsilon > 0$ there exists a finite K such that for all $k \geq K$ and $l \geq k$*

$$\mu([0]; \rho_l, \dots, \rho_k)\{\mathcal{S}^k \cap I \neq \emptyset\} > 1 - \varepsilon.$$

Proof. We have, for any $\rho > 0$,

$$\begin{aligned} \mu([0]; \rho_l, \dots, \rho_k)\{\mathcal{S}^k \cap I \neq \emptyset\} &\geq \mu([0]; \rho_l, \dots, \rho_k, \rho)\{\mathcal{S}^{k-1} \cap I \neq \emptyset\} \\ &= \mu([0]; \rho, \rho_l, \dots, \rho_k)\{\mathcal{S}^k \cap I \neq \emptyset\} \\ &= \mu(E^\rho; \rho_l, \dots, \rho_k)\{\mathcal{S}^k \cap I \neq \emptyset\}, \qquad (10) \end{aligned}$$

where the first two expressions in the r.h.s. refer to a system with leftmost and rightmost nodes $k-1$ and l, and k and $l+1$, respectively. The inequality is obvious from the definition, the first equality follows from Lemma 3. Remind that E^ρ denotes the exponential distribution function with mean $1/\rho$. Note that the last expression in the r.h.s. of (10) is the probability that in a (size $l - k + 1$) on-off system to which input signals are sent according to a Poisson process with intensity ρ, and with recovery rates $\rho_l, \dots, \rho_k$, the last node receives a signal in the time interval I, and the computations below refer to that system. We will choose ρ appropriately, depending on k. First of all, it follows from (6) that there exists a sequence (τ_i) with the properties that $\lim_{i\to\infty} \tau_i = 0$, $\tau_i < |I|/2$ for all i, and $\lim_{i\to\infty} \sum_{j\geq i} e^{-\rho_j \tau_i} = 0$. Now take $\rho = 1/\sqrt{\tau_k}$. Let t and $t + s$ be the infimum and supremum of the interval I. It is clear that the last expression in (10) is larger than or equal to the probability that each of the following events (a)-(c) occur:

(a) No input signal is sent in $(t, t + \tau_k)$.

(b) Each node in the system which had value 0 at time t, has recovered before time $t + \tau_k$.

(c) An input signal is sent in the interval $(t + s/2, t + s)$.

This probability is

$$e^{-\sqrt{\tau_k}} \prod_{j=k}^{l} \left(1 - e^{-\rho_j \tau_k}\right)\left(1 - e^{-\frac{s}{2\sqrt{\tau_k}}}\right) \geq e^{-\sqrt{\tau_k}}\left(1 - \sum_{j\geq k} e^{-\rho_j \tau_k}\right)\left(1 - e^{-\frac{s}{2\sqrt{\tau_k}}}\right).$$

The right hand side in the last inequality does not depend on l and goes to 1 as $k \to \infty$. This completes the proof of Lemma 8 and of Theorem 1.

□

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