On the Singularities of the Magnetic Spectral Shift Function at the Landau Levels

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Abstract. We consider the three-dimensional Schrödinger operators H_0 and $H_\pm$ where $H_0 = (i\nabla + A)^2 - b$, A is a magnetic potential generating a constant magnetic field of strength $b > 0$, and $H_\pm = H_0 \pm V$ where $V \geq 0$ decays fast enough at infinity. Then, A. Pushnitski's representation of the spectral shift function (SSF) for the pair of operators $H_\pm$, H_0 is well-defined for energies $E \neq 2qb$, $q \in \mathbb{Z}_+$. We study the behaviour of the associated representative of the equivalence class determined by the SSF, in a neighbourhood of the Landau levels $2qb$, $q \in \mathbb{Z}_+$. Reducing our analysis to the study of the eigenvalue asymptotics for a family of compact operators of Toeplitz type, we establish a relation between the type of the singularities of the SSF at the Landau levels and the decay rate of V at infinity.

1 Introduction

In this paper we analyze the singularities of the spectral shift function (SSF) for the three-dimensional Schrödinger operator with constant magnetic field, perturbed by an electric potential which decays fast enough at infinity. Let us recall the definition of the abstract SSF for a pair of self-adjoint operators. First, let us consider two self-adjoint operators $\mathcal{T}_0$ and $\mathcal{T}$ acting in the same Hilbert space, such that $\mathcal{T} - \mathcal{T}_0 \in S_1$ where S_1 denotes the space of trace class operators. Then, there exists a unique function $\xi(.;\mathcal{T},\mathcal{T}_0) \in L^1(\mathbb{R})$ such that *the Lifshits-Krein trace formula*

$$\mathrm{Tr}(\phi(\mathcal{T}) - \phi(\mathcal{T}_0)) = \int_{\mathbb{R}} \xi(E;\mathcal{T},\mathcal{T}_0)\phi'(E)dE, \quad \phi \in C_0^\infty(\mathbb{R}), \tag{1.1}$$

holds (see e.g. [16, Theorem 8.3.3]). Let now $\mathcal{H}_0$ and $\mathcal{H}$ be two lower-bounded self-adjoint operators acting in the same Hilbert space. Assume that for some $\gamma > 0$, and $\lambda_0 \in \mathbb{R}$ lying strictly below the infima of the spectra of $\mathcal{H}_0$ and $\mathcal{H}$, we have that

$$(\mathcal{H} - \lambda_0)^{-\gamma} - (\mathcal{H}_0 - \lambda_0)^{-\gamma} \in S_1. \tag{1.2}$$

Set

$$\xi(E;\mathcal{H},\mathcal{H}_0) := \begin{cases} -\xi((E-\lambda_0)^{-\gamma}; (\mathcal{H}-\lambda_0)^{-\gamma}, (\mathcal{H}_0-\lambda_0)^{-\gamma}) & \text{if} \quad E > \lambda_0, \\ 0 \quad \text{if} \quad E \leq \lambda_0. \end{cases}$$

Then, similarly to (1.1),

$$\mathrm{Tr}(\phi(\mathcal{H}) - \phi(\mathcal{H}_0)) = \int_{\mathbb{R}} \xi(E;\mathcal{H},\mathcal{H}_0)\phi'(E)dE, \quad \phi \in C_0^\infty(\mathbb{R}),$$

(see [16, Theorem 8.9.1]). The function $\xi(.;\mathcal{H},\mathcal{H}_0)$ is called the SSF for the pair of the operators $\mathcal{H}$ and $\mathcal{H}_0$. Evidently, it does not depend on the particular choice of γ and λ_0 in (1.2). If E lies below the infimum of the spectrum of $\mathcal{H}_0$, then the spectrum of $\mathcal{H}$ below E could be at most discrete, and we have

$$\xi(E;\mathcal{H},\mathcal{H}_0) = -N(E;\mathcal{H}) \tag{1.3}$$

where $N(E; \mathcal{H})$ denotes the number of eigenvalues of $\mathcal{H}$ in the interval $(-\infty, E)$, and counted with the multiplicities. On the other hand, for almost every E in the absolutely continuous spectrum of $\mathcal{H}_0$, the SSF $\xi(E; \mathcal{H}, \mathcal{H}_0)$ is related to the scattering determinant det $S(E; \mathcal{H}, \mathcal{H}_0)$ for the pair $(\mathcal{H}, \mathcal{H}_0)$ by *the Birman-Krein* formula

$$\det S(E; \mathcal{H}, \mathcal{H}_0) = e^{-2\pi i \xi(E; \mathcal{H}, \mathcal{H}_0)}$$

(see [2] or [16, Section 8.4]). A survey of various asymptotic results concerning the SSF for numerous quantum Hamiltonians is contained in [14].

In the present paper the role of $\mathcal{H}_0$ is played by the operator $H_0 := (i\nabla + \mathbf{A})^2 - b$, which is essentially self-adjoint on $C_0^\infty(\mathbb{R}^3)$. Here the magnetic potential $\mathbf{A} = \left(-\frac{bx_2}{2}, \frac{bx_1}{2}, 0\right)$ generates the constant magnetic field $\mathbf{B} = \text{curl } \mathbf{A} = (0, 0, b)$, $b > 0$. It is well-known that $\sigma(H_0) = \sigma_{\rm ac}(H_0) = [0, \infty)$ (see [1]), where $\sigma(H_0)$ denotes the spectrum of H_0, and $\sigma_{\rm ac}(H_0)$ its absolutely continuous spectrum. Moreover, the so-called Landau levels $2bq$, $q \in \mathbb{Z}_+ := \{0, 1, \ldots\}$, play the role of thresholds in $\sigma(H_0)$.

For $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ we denote by $X_\perp = (x_1, x_2)$ the variables on the plane perpendicular to the magnetic field. We assume that V satisfies

$$V \not\equiv 0, \quad V \in C(\mathbb{R}^3), \quad 0 \leq V(\mathbf{x}) \leq C_0 \langle X_\perp \rangle^{-m_\perp} \langle x_3 \rangle^{-m_3}, \quad \mathbf{x} = (X_\perp, x_3) \in \mathbb{R}^3, \tag{1.4}$$

with $C_0 > 0$, $m_\perp > 2$, $m_3 > 1$, and $\langle x \rangle := (1 + |x|^2)^{1/2}$, $x \in \mathbb{R}^d$, $d \geq 1$. Most of our results will hold under a more restrictive assumption than (1.4), namely

$$V \not\equiv 0, \quad V \in C(\mathbb{R}^3), \qquad 0 \leq V(\mathbf{x}) \leq C_0 \langle \mathbf{x} \rangle^{-m}, \quad m > 3, \quad \mathbf{x} \in \mathbb{R}^3. \tag{1.5}$$

Note that (1.5) implies (1.4) with any $m_3 \in (0, m)$ and $m_\perp = m - m_3$. In particular, we can choose $m_3 \in (1, m-2)$ so that $m_\perp > 2$.

Set $H_\pm := H_0 \pm V$ so that the electric potential $\pm V$ has a definite sign. Obviously, $\inf \sigma(H_+) = 0$, $\inf \sigma(H_-) \geq -C_0$. The role of the perturbed operator $\mathcal{H}$ is played in this paper by $H_\pm$. By (1.4) and the diamagnetic inequality (see e.g. [1]), $V^{1/2}(H_0 - \lambda_0)^{-1}$ with $\lambda_0 < 0$ is a Hilbert-Schmidt operator. Therefore, the resolvent identity implies $(H_\pm - \lambda_0)^{-1} - (H_0 - \lambda_0)^{-1} \in S_1$ for $\lambda_0 < \inf \sigma(H_\pm) \leq \inf \sigma(H_0)$, i.e. (1.2) holds with $\mathcal{H} = H_\pm$, $\mathcal{H}_0 = H_0$, and $\gamma = 1$, and, hence, the SSF $\xi(.; H_\pm, H_0)$ exists.

A priori the SSF $\xi(E; H_\pm, H_0)$ is defined only for almost every $E \in \mathbb{R}$. In Section 2 below we introduce a representative $\tilde{\xi}(.; H_\pm, H_0)$ of the class of equivalence determined by $\xi(.; H_\pm, H_0)$, which is well-defined and uniformly bounded on each compact subset of the complement of the Landau levels. Moreover, $\tilde{\xi}$ is continuous on $\mathbb{R} \setminus 2b\mathbb{Z}_+$ everywhere except at the eigenvalues, isolated or embedded in the continuous spectrum, of the operator $H_\pm$.

The main goal of the paper is the study the asymptotic behaviour as $\lambda \to 0$ of $\tilde{\xi}(2bq + \lambda; H_\pm, H_0)$ with fixed $q \in \mathbb{Z}_+$. Our main results establish the asymptotic coincidence of $\tilde{\xi}(2bq + \lambda; H_\pm, H_0)$ with the traces of certain functions of compact Toeplitz operators. Many of the spectral properties of those Toeplitz operators are well-known, which allows us to describe explicitly the asymptotics as $\lambda \to 0$ of $\tilde{\xi}(2bq + \lambda; H_\pm, H_0)$ in several generic cases. These asymptotic results allow an interpretation directly in the terms of the SSF, which is independent of the choice of the representative of the equivalence class. In particular, these results reveal the link between the type of the singularities of the SSF at the Landau levels, and the decay rate of V at infinity.

The paper is organized as follows. In Section 2 we introduce the representative $\tilde{\xi}$ of the SSF. In Section 3 we formulate our main results, summarize some known spectral properties of compact Toeplitz operators, and obtain as corollaries explicit asymptotic formulas describing the singularities of the SSF at the Landau levels. Section 4 contains some preliminary estimates. The proofs of our main results can be found in Section 5. Finally, in Section 6 we prove some of the corollaries of the main results.

2 A. Pushnitski's representation of the SSF

2.1. In this subsection we introduce some basic notations used throughout the paper.

We denote by S_∞ the class of linear compact operators acting in a fixed Hilbert space.

Let $T = T^* \in S_\infty$. Denote by $\mathbb{P}_I(T)$ the spectral projection of T associated with the interval $I \subset \mathbb{R}$. For $s > 0$ set

$$n_\pm(s;T) := \text{rank}\, \mathbb{P}_{(s,\infty)}(\pm T).$$

For an arbitrary (not necessarily self-adjoint) operator $T \in S_\infty$ put

$$n_*(s;T) := n_+(s^2; T^*T), \quad s > 0. \tag{2.1}$$

If $T = T^*$, then evidently

$$n_*(s;T) = n_+(s,T) + n_-(s;T), \quad s > 0. \tag{2.2}$$

Moreover, if $T_j = T_j^* \in S_\infty$, $j = 1, 2$, then the Weyl's inequalities

$$n_\pm(s_1 + s_2, T_1 + T_2) \le n_\pm(s_1, T_1) + n_\pm(s_2, T_2) \tag{2.3}$$

hold for each s_1, $s_2 > 0$.
Further, we denote by S_p, $p \in [1, \infty)$, the Schatten-von Neumann class of compact operators for which the norm $\|T\|_p := \left(p \int_0^\infty s^{p-1} n_*(s;T)\, ds\right)^{1/p}$ is finite. If $T \in S_p$, $p \in [1,\infty)$, then the following elementary inequality

$$n_*(s;T) \le s^{-p} \|T\|_p^p \tag{2.4}$$

holds for every $s > 0$. If $T = T^* \in S_p$, $p \in [1, \infty)$, then (2.1) and (2.2) imply

$$n_\pm(s;T) \le s^{-p} \|T\|_p^p, \quad s > 0. \tag{2.5}$$

Finally, we define the self-adjoint operators $\text{Re}\, T := \frac{1}{2}(T + T^*)$ and $\text{Im}\, T := \frac{1}{2i}(T - T^*)$. Evidently,

$$n_\pm(s; \text{Re}\, T) \le 2n_*(s;T), \quad n_\pm(s; \text{Im}\, T) \le 2n_*(s;T). \tag{2.6}$$

2.2. In this subsection we summarize several results due to A. Pushnitski on the representation of the SSF for a pair of lower-bounded self-adjoint operators (see [8] – [10]).
Let $I \in \mathbb{R}$ be a Lebesgue measurable set. Set $\mu(I) := \frac{1}{\pi} \int_I \frac{dt}{1+t^2}$. Note that $\mu(\mathbb{R}) = 1$.

Lemma 2.1. [8, Lemma 2.1] *Let $T_1 = T_1^* \in S_\infty$ and $T_2 = T_2^* \in S_1$. Then*

$$\int_{\mathbb{R}} n_\pm(s_1 + s_2; T_1 + t\, T_2)\, d\mu(t) \le n_\pm(s_1; T_1) + \frac{1}{\pi s_2} \|T_2\|_1, \quad s_1, s_2 > 0. \tag{2.7}$$

Let $\mathcal{H}_\pm$ and $\mathcal{H}_0$ be two lower-bounded self-adjoint operators acting in the same Hilbert space. Let $\lambda_0 < \inf \sigma(\mathcal{H}_\pm) \cup \sigma(\mathcal{H}_0)$. First of all, assume that (1.2) holds with $\mathcal{H} = \mathcal{H}_\pm$ for some $\gamma > 0$. Further, let

$$\mathcal{V} := \pm(\mathcal{H}_\pm - \mathcal{H}_0) \ge 0, \tag{2.8}$$

$$\mathcal{V}^{1/2} (\mathcal{H}_0 - \lambda_0)^{-1/2} \in S_\infty. \tag{2.9}$$

Finally, suppose that

$$\mathcal{V}^{1/2} (\mathcal{H}_0 - \lambda_0)^{-\gamma'} \in S_2 \tag{2.10}$$

holds for some $\gamma' > 0$. For $z \in \mathbb{C}$ with $\text{Im}\, z > 0$ set $\mathcal{T}(z) := \mathcal{V}^{1/2} (\mathcal{H}_0 - z)^{-1} \mathcal{V}^{1/2}$.

Lemma 2.2. [8, Lemma 4.1] *Let (1.2) with $\mathcal{H} = \mathcal{H}_\pm$, and (2.8) – (2.10) hold. Then for almost every $E \in \mathbb{R}$ the operator-norm limit $\mathcal{T}(E + i0) := \text{n} - \lim_{\delta \downarrow 0} \mathcal{T}(E + i\delta)$ exists, and by (2.9) we have $\mathcal{T}(E + i0) \in S_\infty$. Moreover, $\text{Im}\, \mathcal{T}(E + i0) \in S_1$.*

Theorem 2.1. [8, Theorem 1.2] *Let (1.2) with $\mathcal{H} = \mathcal{H}_\pm$, and (2.8) – (2.10) hold. Then for almost every $E \in \mathbb{R}$ we have*

$$\xi(E; \mathcal{H}_\pm, \mathcal{H}_0) = \pm \int_{\mathbb{R}} n_\mp(1; \text{Re}\, \mathcal{T}(E + i0) + t\, \text{Im}\, \mathcal{T}(E + i0))\, d\mu(t).$$

Remark: The representation of the SSF described in the above theorem has been generalized to non-sign-definite perturbations in [6] in the case of trace-class perturbations, and in [10] in the case of relatively trace-class perturbations. These generalizations are based on the concept of the index of orthogonal projections. We will not use them in the present paper.

Suppose now that V satisfies (1.4). Then relations (1.2) and (2.8) – (2.10) hold with $\mathcal{V} = V$, $\mathcal{H}_0 = H_0$, and $\gamma = \gamma' = 1$. For $z \in \mathbb{C}$, $\operatorname{Im} z > 0$, set $T(z) := V^{1/2}(H_0 - z)^{-1}V^{1/2}$. By Lemma 2.2, for almost every $E \in \mathbb{R}$ the operator-norm limit

$$T(E + i0) := \mathrm{n} - \lim_{\delta \downarrow 0} T(E + i\delta) \tag{2.11}$$

exists, and

$$\operatorname{Im} T(E + i0) \in S_1. \tag{2.12}$$

For trivial reasons the limit in (2.11) exists and (2.12) holds for *each* $E < 0$. In Corollary 4.3 below we show that this is also true for each $E \in [0, \infty) \setminus 2b\mathbb{Z}_+$. Hence, by Lemma 2.1, the quantity $\int_{\mathbb{R}} n_{\mp}(1; \operatorname{Re} T(E + i0) + t \operatorname{Im} T(E + i0))\, d\mu(t)$ is well-defined for every $E \in \mathbb{R} \setminus 2b\mathbb{Z}_+$. Set

$$\tilde{\xi}(E; H_\pm, H_0) = \pm \int_{\mathbb{R}} n_{\mp}(1; \operatorname{Re} T(E + i0) + t \operatorname{Im} T(E + i0))\, d\mu(t), \quad E \in \mathbb{R} \setminus 2b\mathbb{Z}_+. \tag{2.13}$$

By Theorem 2.1 we have

$$\xi(E; H_\pm, H_0) = \tilde{\xi}(E; H_\pm, H_0) \tag{2.14}$$

for almost every $E \in \mathbb{R}$.
Remark: In [4] it is shown that the function $\tilde{\xi}$ defined on $\mathbb{R} \setminus 2b\mathbb{Z}_+$ is continuous away from the eigenvalues of the operator $H_\pm$. Note that, in contrast to the case $b = 0$, we cannot rule out the possibility of existence of embedded eigenvalues, by imposing short-range assumptions of the type of (1.4) or (1.5): Theorem 5.1 of [1] shows that there are axisymmetric potentials V of compact support such that below each Landau level $2bq$, $q \in \mathbb{Z}_+$, there exists an infinite sequence of eigenvalues of $H_\pm$ which converges to $2bq$. On the other hand, generically, the only possible accumulation points of the eigenvalues of the operators $H_\pm$ are the Landau levels (see [1, Theorem 4.7], [5, Theorem 3.5.3 (iii)]). Further information of the location of these eigenvalues can be found in [4].

3 Main Results

3.1. In this subsection we formulate our main results. To this end we need some more notations. Introduce the Landau Hamiltonian

$$h(b) := \left(i\frac{\partial}{\partial x_1} - \frac{bx_2}{2}\right)^2 + \left(i\frac{\partial}{\partial x_2} + \frac{bx_1}{2}\right)^2 - b, \tag{3.1}$$

i.e. the two-dimensional Schrödinger operator with constant scalar magnetic field $b > 0$, essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$. It is well-known that $\sigma(h(b)) = \cup_{q=0}^\infty \{2bq\}$, and each eigenvalue $2bq$, $q \in \mathbb{Z}_+$, has infinite multiplicity (see e.g. [1]).
For $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^2$ denote by $\mathcal{P}_{q,b}(\mathbf{x}, \mathbf{x}')$ the integral kernel of the orthogonal projection $p_q(b)$ onto the subspace $\operatorname{Ker}(h(b) - 2bq)$, $q \in \mathbb{Z}_+$. It is well-known that

$$\mathcal{P}_{q,b}(\mathbf{x}, \mathbf{x}') = \frac{b}{2\pi} L_q\left(\frac{b|\mathbf{x} - \mathbf{x}'|^2}{2}\right) \exp\left(-\frac{b}{4}(|\mathbf{x} - \mathbf{x}'|^2 + 2i(x_1'x_2 - x_1x_2'))\right) \tag{3.2}$$

(see [7] or [12, Subsection 2.3.2]) where

$$L_q(t) := \frac{1}{q!}e^t\frac{d^q(t^q e^{-t})}{dt^q} = \sum_{k=0}^q \binom{q}{k}\frac{(-t)^k}{k!}, \quad t \in \mathbb{R}, \quad q \in \mathbb{Z}_+,$$

are the Laguerre polynomials. Note that $\mathcal{P}_{q,b}(\mathbf{x},\mathbf{x}) = \frac{b}{2\pi}$ for each $q \in \mathbb{Z}_+$ and $\mathbf{x} \in \mathbb{R}^2$. Introduce the orthogonal projections $P_q : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$, $q \in \mathbb{Z}_+$, by

$$(P_q u)(X_\perp, x_3) = \int_{\mathbb{R}^2} \mathcal{P}_{q,b}(X_\perp, X'_\perp)\, u(X'_\perp, x_3)\, dX'_\perp, \quad u \in L^2(\mathbb{R}^3). \tag{3.3}$$

Assume that (1.4) holds. Set

$$W(X_\perp) := \int_{\mathbb{R}} V(X_\perp, x_3) dx_3, \quad X_\perp \in \mathbb{R}^2. \tag{3.4}$$

If, moreover, V satisfies (1.5), then

$$0 \le W(X_\perp) \le C'_0 \langle X_\perp \rangle^{-m+1}, \quad X_\perp \in \mathbb{R}^2, \tag{3.5}$$

where $C'_0 = C_0 \int_{\mathbb{R}} \langle x \rangle^{-m} dx$. For $q \in \mathbb{Z}_+$ and $\lambda > 0$ introduce the operator

$$\omega_q(\lambda) := \frac{1}{2\sqrt{\lambda}} p_q W p_q. \tag{3.6}$$

Evidently, $\omega_q(\lambda)$ is self-adjoint and non-negative in $L^2(\mathbb{R}^2)$.

Lemma 3.1. *Let $U \in L^r(\mathbb{R}^2)$, $r \ge 1$, and $q \in \mathbb{Z}_+$. Then $p_q U p_q \in S_r$.*

Proof. If $U \in L^\infty(\mathbb{R}^2)$, then evidently $\|p_q U p_q\| \le \|U\|_{L^\infty}$. If $U \in L^1(\mathbb{R}^2)$, we write $p_q U p_q = p_q |U|^{1/2} \mathrm{sign}\, U\; |U|^{1/2} p_q$, check that

$$\|p_q |U|^{1/2}\|_2^2 = \frac{b}{2\pi} \|U\|_{L^1}, \quad \|\mathrm{sign}\, U\; |U|^{1/2} p_q\|_2^2 = \frac{b}{2\pi} \|U\|_{L^1},$$

and conclude that $\|p_q U p_q\|_1 \le \frac{b}{2\pi} \|U\|_{L^1}$. Interpolating, we get $\|p_q U p_q\|_r^r \le \frac{b}{2\pi} \|U\|_{L^r}^r$ which implies the desired result. $\square$

Remark: The proof of Lemma 3.1 follows the idea of the proof of [11, Lemma 5.1]. We include it here in order to make the exposition self-contained.

If $\lambda > 0$, and V satisfies (1.4) with $m_\perp > 2$ and $m_3 > 1$, then Lemma 3.1 with $U = W$ implies $\omega_q(\lambda) \in S_1$.

Theorem 3.1. *Assume that* (1.5) *is valid. Let $q \in \mathbb{Z}_+$, $b > 0$. Then the asymptotic estimates*

$$\tilde{\xi}(2bq - \lambda; H_+, H_0) = O(1), \tag{3.7}$$

and

$$-n_+((1-\varepsilon); \omega_q(\lambda)) + O(1) \le \tilde{\xi}(2bq - \lambda; H_-, H_0) \le -n_+((1+\varepsilon); \omega_q(\lambda)) + O(1), \tag{3.8}$$

hold as $\lambda \downarrow 0$ for each $\varepsilon \in (0,1)$.

Suppose that the potential V satisfies (1.4). For $\lambda > 0$ define the matrix-valued function

$$\mathcal{W}_\lambda = \mathcal{W}_\lambda(X_\perp) := \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}, \quad X_\perp \in \mathbb{R}^2, \tag{3.9}$$

where

$$w_{11} := \int_{\mathbb{R}} V(X_\perp, x_3) \cos^2(\sqrt{\lambda} x_3) dx_3, \quad w_{22} := \int_{\mathbb{R}} V(X_\perp, x_3) \sin^2(\sqrt{\lambda} x_3) dx_3,$$

$$w_{12} = w_{21} := \int_{\mathbb{R}} V(X_\perp, x_3) \cos(\sqrt{\lambda} x_3) \sin(\sqrt{\lambda} x_3) dx_3.$$

Introduce the operator

$$\Omega_q := \frac{1}{2\sqrt{\lambda}} p_q \mathcal{W}_\lambda p_q. \tag{3.10}$$

Evidently, $\Omega_q(\lambda)$ is self-adjoint and non-negative in $L^2(\mathbb{R}^2)^2$. Moreover, using the fact that $\omega_q(\lambda) \in S_1$, it is easy to check that $\Omega_q(\lambda) \in S_1$ as well.

Theorem 3.2. *Assume that* (1.5) *is valid. Let* $q \in \mathbb{Z}_+$, $b > 0$. *Then the asymptotic estimates*

$$\pm\frac{1}{\pi}\mathrm{Tr}\ \arctan\left((1 \pm \varepsilon)^{-1}\Omega_q(\lambda)\right) + O(1) \leq \tilde{\xi}(2bq + \lambda; H_\pm, H_0) \leq \pm\frac{1}{\pi}\mathrm{Tr}\ \arctan\left((1 \mp \varepsilon)^{-1}\Omega_q(\lambda)\right) + O(1) \tag{3.11}$$

hold as $\lambda \downarrow 0$ *for each* $\varepsilon \in (0, 1)$.

The proofs of Theorems 3.1 and 3.2 can be found in Section 5.

3.2. Relations (3.8) and (3.11) allow us to reduce the analysis of the behaviour as $\lambda \to 0$ of $\tilde{\xi}(2bq + \lambda; H_\pm, H_0)$, to the study of the asymptotic distribution of the eigenvalues of Toeplitz-type operators $p_q U p_q$. The following three lemmas concern the spectral asymptotics of such operators.

Lemma 3.2. [11, Theorem 2.6] *Let the function* $U \in C^1(\mathbb{R}^2)$ *satisfy the estimates*

$$0 \leq U(X_\perp) \leq C_1 \langle X_\perp \rangle^{-\alpha}, \quad |\nabla U(X_\perp)| \leq C_1 \langle X_\perp \rangle^{-\alpha-1}, \quad X_\perp \in \mathbb{R}^2,$$

for some $\alpha > 0$ *and* $C_1 > 0$. *Assume, moreover, that*

$$U(X_\perp) = u_0(X_\perp/|X_\perp|)|X_\perp|^{-\alpha}(1 + o(1)), \quad |X_\perp| \to \infty,$$

where u_0 *is a continuous function on* $\mathbb{S}^1$ *which is non-negative and does not vanish identically. Then for each* $q \in \mathbb{Z}_+$ *we have*

$$n_+(s; p_q U p_q) = \frac{b}{2\pi}\left|\left\{X_\perp \in \mathbb{R}^2 | U(X_\perp) > s\right\}\right| (1 + o(1)) = s^{-2/\alpha}\frac{b}{4\pi}\int_{\mathbb{S}^1} u_0(t)^{2/\alpha} dt\ (1 + o(1)), \quad s \downarrow 0,$$

where $|.|$ *denotes the Lebesgue measure.*

Remark: Theorem 2.6 of [11] contains a considerably more general result than Lemma 3.2. For the sake of the simplicity of exposition, here we reproduce only the special case of asymptotically homogeneous U.

Lemma 3.3. [13, Theorem 2.1, Proposition 4.1] *Let* $0 \leq U \in L^\infty(\mathbb{R}^2)$. *Assume that*

$$\ln U(X_\perp) = -\mu |X_\perp|^{2\beta}(1 + o(1)), \quad |X_\perp| \to \infty,$$

for some $\beta \in (0, \infty)$, $\mu \in (0, \infty)$. *Then for each* $q \in \mathbb{Z}_+$ *we have*

$$n_+(s; p_q U p_q) = \varphi_\beta(s)(1 + o(1)), \quad s \downarrow 0,$$

where

$$\varphi_\beta(s) = \varphi_\beta(s; \mu, b) := \begin{cases} \frac{b}{2\mu^{1/\beta}} |\ln s|^{1/\beta} & \text{if} \quad 0 < \beta < 1, \\ \frac{1}{\ln(1 + 2\mu/b)} |\ln s| & \text{if} \quad \beta = 1, \\ \frac{\beta}{\beta - 1} (\ln|\ln s|)^{-1} |\ln s| & \text{if} \quad 1 < \beta < \infty. \end{cases} \qquad s \in (e, \infty).$$

Lemma 3.4. [13, Theorem 2.2, Proposition 4.1] *Let* $0 \leq U \in L^\infty(\mathbb{R}^2)$. *Assume that the support of* U *is compact, and that there exists a constant* $C > 0$ *such that* $U \geq C$ *on an open non-empty subset of* $\mathbb{R}^2$. *Then for each* $q \in \mathbb{Z}_+$ *we have*

$$n_+(s; p_q U p_q) = \varphi_\infty(s)\ (1 + o(1)), \quad s \downarrow 0,$$

where

$$\varphi_\infty(s) := (\ln|\ln s|)^{-1}|\ln s|, \quad s \in (e, \infty).$$

Remark: For each $\beta \in (0, \infty]$ and $c > 0$ we have $\varphi_\beta(cs) = \varphi_\beta(s)(1 + o(1))$ as $s \downarrow 0$.

Employing Lemma 3.2, 3.3, 3.4, and the above remark, we find that (3.8) immediately entails the following corollary.

Corollary 3.1. *Let* (1.5) *hold with* $m > 3$.
i) Assume that the hypotheses of Lemma 3.2 hold with $U = W$ *and* $\alpha = m - 1$. *Then we have*

$$\tilde{\xi}(2bq - \lambda; H_-, H_0) = -\frac{b}{2\pi}\left|\left\{X_\perp \in \mathbb{R}^2 | W(X_\perp) > 2\sqrt{\lambda}\right\}\right|\ (1 + o(1)) =$$

$$-\lambda^{-1/(m-1)}\frac{b}{4\pi}\int_{\mathbb{S}^1}(u_0(t)/2)^{2/(m-1)}dt\ (1 + o(1)), \quad \lambda \downarrow 0. \tag{3.12}$$

ii) Assume that the hypotheses of Lemma 3.3 hold with $U = W$. *Then we have*

$$\tilde{\xi}(2bq - \lambda; H_-, H_0) = -\varphi_\beta(\sqrt{\lambda}; \mu, b)\ (1 + o(1)), \quad \lambda \downarrow 0, \quad \beta \in (0, \infty).$$

iii) Assume that the hypotheses of Lemma 3.4 hold with $U = W$. *Then we have*

$$\tilde{\xi}(2bq - \lambda; H_-, H_0) = -\varphi_\infty(\sqrt{\lambda})\ (1 + o(1)), \quad \lambda \downarrow 0.$$

Remark: In the special case $q = 0$ when $-\tilde{\xi}(-\lambda; H_-, H_0)$ coincides with the eigenvalue counting function for the operator H_- (see (1.3)), relation (3.12) was established for the first time in [15]. Here we use a different approach related to the one developed in [11].

Similarly, the combination of Theorem 3.2 with Lemmas 3.2 – 3.4 yields the following corollary.

Corollary 3.2. *i) Let* (1.5) *hold with* $m > 3$. *Assume that the hypotheses of Lemma 3.2 are fulfilled for* $U = W$ *and* $\alpha = m - 1$. *Then we have*

$$\tilde{\xi}(2bq + \lambda; H_\pm, H_0) = \pm\frac{b}{2\pi^2}\int_{\mathbb{R}^2}\arctan\left((2\sqrt{\lambda})^{-1}W(X_\perp)\right)dX_\perp\ (1 + o(1)) =$$

$$\pm\lambda^{-1/(m-1)}\frac{b}{8\pi\cos\left(\pi/(m-1)\right)}\int_{\mathbb{S}^1}(u_0(t)/2)^{2/(m-1)}dt\ (1 + o(1)), \quad \lambda \downarrow 0.$$

ii) Let (1.5) *hold with* $m > 3$. *Suppose in addition that* V *satisfies* (1.4) *for some* $m_\perp > 2$ *and* $m_3 > 2$. *Finally, assume that the hypotheses of Lemma 3.3 are fulfilled for* $U = W$. *Then we have*

$$\tilde{\xi}(2bq + \lambda; H_\pm, H_0) = \pm\varphi_\beta(\sqrt{\lambda}; \mu, b)\ (1 + o(1)), \quad \lambda \downarrow 0, \quad \beta \in (0, \infty).$$

iii) Let the assumptions of the previous part be fulfilled, except that the hypotheses of Lemma 3.3 are replaced by those of Lemma 3.4. Then we have

$$\tilde{\xi}(2bq + \lambda; H_\pm, H_0) = \pm\varphi_\infty(\sqrt{\lambda})\ (1 + o(1)), \quad \lambda \downarrow 0.$$

The proof of Corollary 3.2 can be found in Section 6.

3.3. In this subsection we present a possible interpretation of our results directly in the terms of the SSF $\xi(.; H_\pm, H_0)$ which is invariant of the choice of the representative of the class of equivalence determined by the SSF. For $\lambda > 0$, and $q \in \mathbb{Z}_+$, introduce the averaged values of the SSF

$$\Xi^{\pm}_{q,<}(\lambda) := \frac{1}{\lambda}\int_{2bq-\lambda}^{2bq} \xi(s; H_\pm, H_0)ds = \frac{1}{\lambda}\int_0^\lambda \xi(2bq - t; H_\pm, H_0)dt,$$

$$\Xi^{\pm}_{q,>}(\lambda) := \frac{1}{\lambda}\int_{2bq}^{2bq+\lambda} \xi(s; H_\pm, H_0)ds = \frac{1}{\lambda}\int_0^\lambda \xi(2bq + t; H_\pm, H_0)dt.$$

Since $\xi(.; H_\pm, H_0) \in L^1_{\rm loc}(\mathbb{R})$, the quantities $\Xi^{\pm}_{q,<}(\lambda)$ and $\Xi^{\pm}_{q,>}(\lambda)$ are well-defined for every $\lambda > 0$. Applying (2.14), we find that the asymptotic bound $\Xi^{+}_{q,<}(\lambda) = O(1)$ as $\lambda \downarrow 0$ follows from (3.7). Further, Corollary 3.1 i) implies

$$\Xi^{-}_{q,<}(\lambda) = -\lambda^{-1/(m-1)}\frac{b(m-1)}{4(m-2)\pi}\int_{\mathbb{S}^1}(u_0(t)/2)^{2/(m-1)}dt\,(1+o(1)), \quad \lambda \downarrow 0, \quad m > 3,$$

while Corollary 3.1 ii) – iii) entails

$$\Xi^{-}_{q,<}(\lambda) = -\varphi_\beta(\sqrt{\lambda}; \mu, b)\,(1+o(1)), \quad \lambda \downarrow 0, \quad \beta \in (0, \infty].$$

Finally, it follows from Corollary 3.2 i) that

$$\Xi^{\pm}_{q,>}(\lambda) = \pm\lambda^{-1/(m-1)}\frac{b(m-1)}{8(m-2)\pi\cos{(\pi/(m-1))}}\int_{\mathbb{S}^1}(u_0(t)/2)^{2/(m-1)}dt\,(1+o(1)), \quad \lambda \downarrow 0, \quad m > 3,$$

while Corollary 3.2 ii) – iii) implies

$$\Xi^{\pm}_{q,>}(\lambda) = \pm\varphi_\beta(\sqrt{\lambda}; \mu, b)\,(1+o(1)), \quad \lambda \downarrow 0, \quad \beta \in (0, \infty].$$

4 Preliminary estimates

For $z \in \mathbb{C}$ with Im $z > 0$, define the operator $R(z) := \left(-\frac{d^2}{dx_3^2} - z\right)^{-1}$ bounded in $L^2(\mathbb{R})$, as well as the operators

$$T_q(z) := V^{1/2}P_q(H_0 - z)^{-1}V^{1/2}, \quad q \in \mathbb{Z}_+,$$

bounded in $L^2(\mathbb{R}^3)$ (see (3.3) for the definition of the orthogonal projection P_q). The operator $R(z)$ admits the integral kernel $\mathcal{R}_z(x_3 - x_3')$ where $\mathcal{R}_z(x) = ie^{i\sqrt{z}|x|}/(2\sqrt{z})$, $x \in \mathbb{R}$, the branch of $\sqrt{z}$ being chosen so that Im $\sqrt{z} > 0$. Moreover, $T_q(z) = V^{1/2}\left(p_q(b) \otimes R(z - 2bq)\right)V^{1/2}$.
For $\lambda \in \mathbb{R}$, $\lambda \neq 0$, define $R(\lambda)$ as the operator with integral kernel $\mathcal{R}_\lambda(x_3 - x_3')$ where

$$\mathcal{R}_\lambda(x) := \lim_{\delta \downarrow 0}\mathcal{R}_{\lambda+i\delta}(x) = \begin{cases} \frac{e^{-\sqrt{-\lambda}|x|}}{2\sqrt{-\lambda}} & \text{if} \quad \lambda < 0, \\ \frac{ie^{i\sqrt{\lambda}|x|}}{2\sqrt{\lambda}} & \text{if} \quad \lambda > 0, \end{cases} \qquad x \in \mathbb{R}. \tag{4.1}$$

Evidently, if $w \in L^2(\mathbb{R})$ and $\lambda \neq 0$, then $wR(\lambda)\bar{w} \in S_2$. For $E \in \mathbb{R}$, $E \neq 2bq$, $q \in \mathbb{Z}_+$, set

$$T_q(E) := V^{1/2}\left(p_q(b) \otimes R(E - 2bq)\right)V^{1/2}.$$

Proposition 4.1. *Let $E \in \mathbb{R}$, $q \in \mathbb{Z}_+$, $E \neq 2bq$. Let* (1.4) *hold. Then $T_q(E) \in S_2$, and*

$$\|T_q(E)\|_2^2 \leq C_1 b/|E - 2bq| \tag{4.2}$$

with C_1 independent of E, b, and q. Moreover,

$$\lim_{\delta \downarrow 0} \|T_q(E + i\delta) - T_q(E)\|_2 = 0. \tag{4.3}$$

Proof. The operator $T_q(E)$ admits the representation

$$T_q(E) = M\ (G_q \otimes J_{E-2bq})\ M \tag{4.4}$$

where $M : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ is the multiplier by $V(X_\perp, x_3)^{1/2}\langle X_\perp\rangle^{m_\perp/2}\langle x_3\rangle^{m_3/2}$, $G_q : L^2(\mathbb{R}^2_{X'_\perp}) \to L^2(\mathbb{R}^2_{X_\perp})$ is the operator with integral kernel

$$\langle X_\perp\rangle^{-m_\perp/2}\mathcal{P}_{q,b}(X_\perp, X'_\perp)\langle X'_\perp\rangle^{-m_\perp/2}, \quad X_\perp, X'_\perp \in \mathbb{R}^2,$$

(see (3.2) for the definition of $\mathcal{P}_{q,b}$), while $J_\lambda : L^2(\mathbb{R}_{x'_3}) \to L^2(\mathbb{R}_{x_3})$ is the operator with integral kernel

$$\langle x_3\rangle^{-m_3/2}\mathcal{R}_\lambda(x_3 - x'_3)\langle x'_3\rangle^{-m_3/2}, \quad x_3, x'_3 \in \mathbb{R}, \quad \lambda \in \mathbb{R} \setminus \{0\}. \tag{4.5}$$

Evidently, $\|T_q(E)\|_2^2 \leq \|M\|^4\|G_q\|_2^2\|J_{E-2bq}\|_2^2$. By (1.4) we have $\|M\|^4 \leq C_0^2$. Further,

$$\|J_{E-2bq}\|_2^2 = \int_{\mathbb{R}} |\mathcal{R}_{E-2bq}(x_3 - x'_3)|^2\langle x_3\rangle^{-m_3}\langle x'_3\rangle^{-m_3}\,dx_3\,dx'_3 \leq \frac{1}{4|E-2bq|}\left(\int_{\mathbb{R}}\langle x_3\rangle^{-m_3}dx_3\right)^2.$$

Finally, since $\|G_q\| \leq 1$, we have $\|G_q\|_2^2 \leq \|G_q\|_1 = \frac{b}{2\pi}\int_{\mathbb{R}^2}\langle X_\perp\rangle^{-m_\perp}dX_\perp$. Hence, (4.2) holds with $C_1 = \frac{C_0^2}{8\pi}\left(\int_{\mathbb{R}}\langle x_3\rangle^{-m_3}dx_3\right)^2\int_{\mathbb{R}^2}\langle X_\perp\rangle^{-m_\perp}dX_\perp$. To prove (4.3), we write

$$\|T_q(E+i\delta) - T_q(E)\|_2^2 = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}} V(X_\perp, x_3)V(X'_\perp, x'_3)|\mathcal{P}_{q,b}(X_\perp, X'_\perp)|^2$$

$$|\mathcal{R}_{E-2bq+i\delta}(x_3 - x'_3) - \mathcal{R}_{E-2bq}(x_3 - x'_3)|^2 dx_3 dx'_3 dX_\perp dX'_\perp,$$

note that $\lim_{\delta\downarrow 0}\mathcal{R}_{E-2bq+i\delta}(x) = \mathcal{R}_{E-2bq}(x)$ for each $x \in \mathbb{R}$, and that the integrand in the above integral is upper-bounded for each $\delta > 0$ by the $L^1(\mathbb{R}^6)$-function

$$\frac{1}{|E-2bq|}V(X_\perp, x_3)V(X'_\perp, x'_3)|\mathcal{P}_{q,b}(X_\perp, X'_\perp)|^2, \quad (X_\perp, x_3, X'_\perp, x'_3) \in \mathbb{R}^6.$$

Therefore, the dominated convergence theorem implies (4.3). □

Remark: Using more sophisticated tools than those of the proof of Proposition 4.1, it is shown in [4] that for $E \neq 2bq$ we have not only $T_q(E) \in S_2$, but also $T_q(E) \in S_1$. We will not use this fact here.

Corollary 4.1. *Assume that* (1.4) *holds. Let $E \in \mathbb{R}$, $q \in \mathbb{Z}_+$, $E \neq 2bq$. Then* Im $T_q(E) \geq 0$. *Moreover, if $E < 2bq$, then* Im $T_q(E) = 0$.

Proof. The non-negativity of Im $T_q(E)$ follows from the representation

$$\text{Im}\ T_q(E + i\delta) = \delta V^{1/2}P_q((H_0 - E)^2 + \delta^2)^{-1}P_qV^{1/2}, \quad \delta > 0,$$

and the limiting relation Im $T_q(E) = \text{n} - \lim_{\delta\downarrow 0}\text{Im}\ T_q(E + i\delta)$, $E \neq 2bq$, which on its turn is implied by (4.3). Moreover, if $E < 2bq$, then (4.1) entails $T_q(E) = T_q(E)^*$ so that Im $T_q(E) = 0$. □

Corollary 4.2. *Under the assumptions of Corollary 4.1 we have* $\operatorname{Im} T_q(E) \in S_1$. *Furthermore, if* $E > 2bq$, *then*

$$\|\operatorname{Im} T_q(E)\|_1 = \operatorname{Tr} \operatorname{Im} T_q(E) = \frac{b}{4\pi}(E-2bq)^{-1/2}\int_{\mathbb{R}^3} V(\mathbf{x})d\mathbf{x}. \tag{4.6}$$

Proof. Bearing in mind the representation (4.4), we find that the inclusion $\operatorname{Im} T_q(E) \in S_1$ would be implied by the inclusion $G_q \in S_1$ and $\operatorname{Im} J_{E-2bq} \in S_1$. The first inclusion follows from Lemma 3.1, and the second one from the obvious fact that $\operatorname{rank} \operatorname{Im} J_{E-2bq} \leq 2$.
Further, the first equality in (4.6) follows from the non-negativity of the operator $\operatorname{Im} T_q(E+i0)$ which is guaranteed by Corollary 4.1. Since the operator $\operatorname{Im} T_q(E+i0)$ with $E > 2bq$ admits the kernel

$$\frac{1}{2\sqrt{E-2bq}}\sqrt{V(X_\perp,x_3)}\cos\left(\sqrt{E-2bq}(x_3-x_3')\right)\mathcal{P}_{q,b}(X_\perp,X_\perp')\sqrt{V(X_\perp',x_3')},\ (X_\perp,x_3),(X_\perp',x_3')\in\mathbb{R}^3,$$

the Mercer theorem implies the second equality in (4.6). □

Proposition 4.2. *Let* $q \in \mathbb{Z}_+$, $\lambda \in \mathbb{R}$, $|\lambda| \in (0,b)$, *and* $\delta > 0$. *Assume that* V *satisfies* (1.4). *Then the operator series*

$$T_q^+(2bq+\lambda+i\delta) := \sum_{l=q+1}^{\infty} T_l(2bq+\lambda+i\delta), \tag{4.7}$$

$$T_q^+(2bq+\lambda) := \sum_{l=q+1}^{\infty} T_l(2bq+\lambda) \tag{4.8}$$

are convergent in S_2. *Moreover,*

$$\|T_q^+(2bq+\lambda)\|_2^2 \leq \frac{C_0 b}{8\pi}\sum_{l=q+1}^{\infty}(2b(l-q)-\lambda)^{-3/2}\int_{\mathbb{R}^3}V(\mathbf{x})d\mathbf{x}. \tag{4.9}$$

Finally,

$$\lim_{\delta\downarrow 0}\|T_q^+(2bq+\lambda+i\delta) - T_q^+(2bq+\lambda)\|_2 = 0. \tag{4.10}$$

Proof. For each $q', q'' \in \mathbb{Z}_+$ such that $q+1 \leq q' < q'' < \infty$ we have

$$\|\sum_{l=q'}^{q''} T_l^+(2bq+\lambda)\|_2^2 = \|\sum_{l=q'}^{q''}\sqrt{V}p_l \otimes \left(-\frac{d^2}{dx_3^2}+2b(l-q)-\lambda\right)^{-1}\sqrt{V}\|_2^2 \leq$$

$$C_0\sum_{l=q'}^{q''}\operatorname{Tr}\left(p_l\otimes\left(-\frac{d^2}{dx_3^2}+2b(l-q)-\lambda\right)^{-2}V\right) =$$

$$\frac{C_0 b}{(2\pi)^2}\sum_{l=q'}^{q''}\int_{\mathbb{R}}\frac{d\eta}{(\eta^2+2b(l-q)-\lambda)^2}\int_{\mathbb{R}^3}V(\mathbf{x})d\mathbf{x} = \frac{C_0 b}{8\pi}\sum_{l=q'}^{q''}(2b(l-q)-\lambda)^{-3/2}\int_{\mathbb{R}^3}V(\mathbf{x})d\mathbf{x}. \tag{4.11}$$

Since the numerical series $\sum_{l=q+1}^{\infty}(2b(l-q)-\lambda)^{-3/2}$ is convergent, and S_2 is a Hilbert (hence, complete) space, we conclude that (4.11) entails the convergence in S_2 of the operator series in (4.8), as well as the validity of (4.9). The convergence of the series in (4.7) is proved in exactly the same manner. Finally, (4.10) follows from the estimate

$$\|T_q^+(2bq+\lambda+i\delta) - T_q^+(2bq+\lambda)\|_2^2 \leq$$

$$\delta^2 \frac{C_0 b}{(2\pi)^2} \sum_{l=q+1}^{\infty} \int_{\mathbb{R}} \frac{d\eta}{(\eta^2+2b(l-q)-\lambda)^2((\eta^2+2b(l-q)-\lambda)^2+\delta^2)} \int_{\mathbb{R}^3} V(\mathbf{x})d\mathbf{x} \leq$$

$$\delta^2 \frac{C_0 b}{2\pi^2} \sum_{l=q+1}^{\infty} (2b(l-q)-\lambda)^{-7/2} \int_0^{\infty} \frac{d\eta}{(\eta^2+1)^4} \int_{\mathbb{R}^3} V(\mathbf{x})d\mathbf{x}.$$

□

For $E = 2bq + \lambda$ with $q \in \mathbb{Z}_+$, and $\lambda \in \mathbb{R}$, $|\lambda| \in (0, b]$, set $T_q^-(E) := T(E) - T_q(E) - T_q^+(E)$ (see (4.8)). Note that if $q = 0$, then $T_q^-(E) = 0$, and if $q \geq 1$, then $T_q^-(E) = \sum_{l=0}^{q-1} T_l(E)$.

Corollary 4.3. *For $E = 2bq + \lambda$ with $q \in \mathbb{Z}_+$ and $\lambda \in \mathbb{R}$, $|\lambda| \in (0, b]$ the operator-norm limit* (2.11) *exists, and*

$$T(E + i0) = T_q^-(E) + T_q(E) + T_q^+(E). \tag{4.12}$$

Moreover,

$$\mathrm{Re}\, T(E + i0) = \mathrm{Re}\, T_q^-(E) + \mathrm{Re}\, T_q(E) + T_q^+(E), \tag{4.13}$$

$$\mathrm{Im}\, T(E + i0) = \mathrm{Im}\, T_q^-(E) + \mathrm{Im}\, T_q(E) \in S_1. \tag{4.14}$$

Proof. Let $\delta > 0$. Evidently, $T(E + i\delta) = \sum_{l=0}^{q} T_l(E + i\delta) + T_q^+(E + i\delta)$ (see (4.7)). Proposition 4.1 implies that $\mathrm{n} - \lim_{\delta\downarrow 0} \sum_{l=0}^{q} T_l(E + i\delta) = T_q^-(E) + T_q(E)$, while Proposition 4.2 implies that $\mathrm{n} - \lim_{\delta\downarrow 0} T_q^+(E) = T_q^+(E)$. Combining the above two relations, we get (4.12). Relation (4.13) follows immediately from (4.12) and $T_q^+(E) = T_q^+(E)^*$, while (4.14) is implied by (4.12) and Corollary 4.2. □

5 Proof of the Main Results

5.1. This subsection contains a general estimate which will be used in the proofs of all our main results. Informally speaking, we show that we can replace the operator $T(E+i0)$ by $T_q(E)$ in the r.h.s of (2.14) when we deal with the first asymptotic term of $\tilde{\xi}(E; H_\pm, H_0)$ (see (2.13)) as the energy E approaches a given Landau level $2bq$, $q \in \mathbb{Z}_+$.

Proposition 5.1. *Assume that* (1.4) *holds. Let $E = 2bq + \lambda$ with $q \in \mathbb{Z}_+$, and $\lambda \in \mathbb{R}$, $|\lambda| \in (0, b)$. Then the asymptotic estimates*

$$\int_{\mathbb{R}} n_\pm(1+\varepsilon; \mathrm{Re}\, T_q(E) + t\, \mathrm{Im}\, T_q(E))\, d\mu(t) + O(1) \leq$$

$$\int_{\mathbb{R}} n_\pm(1; \mathrm{Re}\, T(E+i0) + t\, \mathrm{Im}\, T(E+i0))\, d\mu(t) \leq$$

$$\int_{\mathbb{R}} n_\pm(1-\varepsilon; \mathrm{Re}\, T_q(E) + t\, \mathrm{Im}\, T_q(E))\, d\mu(t) + O(1) \tag{5.1}$$

hold as $\lambda \to 0$ for each $\varepsilon \in (0, 1)$.

Proof. Using (4.13) and (4.14), and applying the Weyl's inequalities (2.3), we get

$$\int_{\mathbb{R}} n_\pm(1+\varepsilon; \mathrm{Re}\, T_q(E) + t\, \mathrm{Im}\, T_q(E))\, d\mu(t) -$$

$$\int_{\mathbb{R}} n_\mp(\varepsilon; \mathrm{Re}\, T_q^-(E) + T_q^+(E+i0) + t\, \mathrm{Im}\, T_q^-(E))\, d\mu(t) \leq$$

$$\int_{\mathbb{R}} n_\pm(1; \operatorname{Re} T(E+i0) + t \operatorname{Im} T(E+i0))\, d\mu(t) \leq$$
$$\int_{\mathbb{R}} n_\pm(1-\varepsilon; \operatorname{Re} T_q(E) + t \operatorname{Im} T_q(E))\, d\mu(t) +$$
$$\int_{\mathbb{R}} n_\pm(\varepsilon; \operatorname{Re} T_q^-(E) + T_q^+(E) + t \operatorname{Im} T_q^-(E))\, d\mu(t). \tag{5.2}$$

In order to conclude that (5.2) implies (5.1), it remains to show that

$$\int_{\mathbb{R}} n_\pm(\varepsilon; \operatorname{Re} T_q^-(E) + T_q^+(E) + t \operatorname{Im} T_q^-(E))\, d\mu(t) = O(1), \quad \lambda \to 0, \tag{5.3}$$

for each $\varepsilon > 0$. Employing (2.7) and (2.3), we find that

$$\int_{\mathbb{R}} n_\pm(\varepsilon; \operatorname{Re} T_q^-(E) + T_q^+(E) + t \operatorname{Im} T_q^-(E))\, d\mu(t) \leq$$
$$n_\pm(\varepsilon/2; \operatorname{Re} T_q^-(E) + T_q^+(E)) + \frac{2}{\varepsilon\pi} \|\operatorname{Im} T_q^-(E)\|_1 \leq$$
$$n_\pm(\varepsilon/4; \operatorname{Re} T_q^-(E)) + n_\pm(\varepsilon/4; T_q^+(E)) + \frac{2}{\varepsilon\pi} \|\operatorname{Im} T_q^-(E)\|_1, \quad \varepsilon > 0. \tag{5.4}$$

If $q = 0$, and, hence, $T_q^-(E) = 0$, we need only to apply (2.5) with $p = 2$, and (4.9), in order to get

$$n_\pm(\varepsilon/4; T_q^+(E)) \leq 16\varepsilon^{-2} \|T_q^+(E)\|_2^2 \leq \frac{2C_0 b}{\varepsilon^2 \pi} \sum_{l=q+1}^{\infty} (2b(l-q) - \lambda)^{-3/2} \int_{\mathbb{R}^3} V(\mathbf{x}) d\mathbf{x}, \tag{5.5}$$

which combined with (5.4) yields (5.3). If $q \geq 1$ we should also utilize the estimate

$$n_\pm(\varepsilon/4; \operatorname{Re} T_q^-(E)) \leq 32\varepsilon^{-2} \|T_q^-(E)\|_2^2 \leq 32\varepsilon^{-2} q C_1 b \sum_{l=0}^{q-1} (2b(q-l) + \lambda)^{-1} \tag{5.6}$$

which follows from (2.6), (2.4) with $p = 2$, (4.2), and

$$\|\operatorname{Im} T_q^-(E)\|_1 \leq \frac{b}{2\pi} \sum_{l=0}^{q-1} (2b(q-l) + \lambda)^{-1/2} \int_{\mathbb{R}^3} V(\mathbf{x}) d\mathbf{x}, \tag{5.7}$$

which follows from (4.6). Thus, in the case $q \geq 1$, estimate (5.3) is implied by from the combination of (5.4) – (5.7). $\square$

5.2. In this subsection we complete the proof of the first part of Theorem 3.1.
Since Im $T_q(2bq - \lambda) = 0$ and Re $T_q(2bq - \lambda) = T_q(2bq - \lambda) \geq 0$ if $\lambda > 0$, Proposition 5.1 implies immediately the following corollary.

Corollary 5.1. *Under the hypotheses of Proposition 5.1 the asymptotic estimates*

$$\int_{\mathbb{R}} n_+(1; \operatorname{Re} T(2bq - \lambda + i0) + t \operatorname{Im} T(2bq - \lambda + i0))\, d\mu(t) = O(1), \tag{5.8}$$

and

$$n_+(1+\varepsilon; T_q(2bq - \lambda)) + O(1) \leq$$
$$\int_{\mathbb{R}} n_+(1; \operatorname{Re} T(2bq - \lambda + i0) + t \operatorname{Im} T(2bq - \lambda + i0))\, d\mu(t) \leq$$
$$n_+(1-\varepsilon; T_q(2bq - \lambda)) + O(1) \tag{5.9}$$

hold as $\lambda \downarrow 0$ *for each* $\varepsilon \in (0, 1)$.

Now the combination of (2.13) and (5.8) yields (3.7).

5.3. In this section we complete the proof of the second part of Theorem 3.1. For $q \in \mathbb{Z}_+$ and $\lambda > 0$ define $\mathcal{O}_q(\lambda) : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ as the operator with integral kernel

$$\frac{1}{2\sqrt{\lambda}}\sqrt{V(X_\perp, x_3)}\, \mathcal{P}_{q,b}(X_\perp, X'_\perp)\sqrt{V(X'_\perp, x'_3)}, \quad (X_\perp, x_3), (X'_\perp, x'_3) \in \mathbb{R}^3.$$

Proposition 5.2. *Under the hypotheses of Theorem 3.1 the asymptotic estimates*

$$n_+((1+\varepsilon)s; \mathcal{O}_q(\lambda)) + O(1) \leq n_+(s; T_q(2bq - \lambda)) \leq n_+((1-\varepsilon)s; \mathcal{O}_q(\lambda)) + O(1) \tag{5.10}$$

hold as $\lambda \downarrow 0$ for each $\varepsilon \in (0,1)$ and $s > 0$.

Proof. Fix $s > 0$ and $\varepsilon \in (0,1)$. Then the Weyl's inequalities entail

$$n_+((1+\varepsilon)s; \mathcal{O}_q(\lambda)) - n_-(\varepsilon s; T_q(2bq - \lambda) - \mathcal{O}_q(\lambda)) \leq$$

$$n_+(s; T_q(2bq - \lambda)) \leq$$

$$n_+((1-\varepsilon)s; \mathcal{O}_q(\lambda)) + n_+(\varepsilon s; T_q(2bq - \lambda) - \mathcal{O}_q(\lambda)).$$

In order to get (5.10), it suffices to show that

$$n_\pm(t; T_q(2bq - \lambda) - \mathcal{O}_q(\lambda)) = O(1), \quad \lambda \downarrow 0, \tag{5.11}$$

for every fixed $t > 0$. Denote by $\tilde{T}_q$ the operator with integral kernel

$$-\frac{1}{2}\sqrt{V(X_\perp, x_3)}\, |x_3 - x'_3|\, \mathcal{P}_{q,b}(X_\perp, X'_\perp)\sqrt{V(X'_\perp, x'_3)}, \quad (X_\perp, x_3), (X'_\perp, x'_3) \in \mathbb{R}^3. \tag{5.12}$$

Pick $m' \in (3, m)$, and write

$$\tilde{T}_q = \tilde{M}_{m,m'} \tilde{G}_{q,m-m'} \otimes \tilde{J}^{(0)}_{m'} \tilde{M}_{m,m'}$$

where $\tilde{M}_{m,m'}$ is the multiplier by the bounded function $\sqrt{V(X_\perp, x_3)}\langle X_\perp\rangle^{(m-m')/2}\langle x_3\rangle^{m'/2}$, $(X_\perp, x_3) \in \mathbb{R}^3$, $\tilde{G}_{q,m-m'} : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is the operator with integral kernel

$$\langle X_\perp\rangle^{-(m-m')/2}\mathcal{P}_{q,b}(X_\perp, X'_\perp)\langle X'_\perp\rangle^{-(m-m')/2}, \quad X_\perp, X'_\perp \in \mathbb{R}^2,$$

and $\tilde{J}^{(0)}_{m'} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the operator with integral kernel

$$-\frac{1}{2}\langle x_3\rangle^{-m'/2}|x_3 - x'_3|\langle x'_3\rangle^{-m'/2}, \quad x_3, x'_3 \in \mathbb{R}.$$

Since $m - m' > 0$, Lemma 3.1 implies that the operator $\tilde{G}_{q,m-m'}$ is compact, and since $m' > 3$ we have $\tilde{J}^{(0)}_{m'} \in S_2$. Finally, since $\tilde{M}_{m,m'}$ is bounded, we find that the operator $\tilde{T}_q$ is compact. Further,

$$T_q(2bq - \lambda) - \mathcal{O}_q(\lambda) = \tilde{M}_{m,m'} \tilde{G}_{q,m-m'} \otimes \tilde{J}^{(\lambda)}_{m'} \tilde{M}_{m,m'}$$

where $\tilde{J}^{(\lambda)}_{m'}$, $\lambda > 0$, is the operator with integral kernel

$$-\frac{1}{\sqrt{\lambda}}\langle x_3\rangle^{-m'/2} e^{-\frac{1}{2}\sqrt{\lambda}|x_3 - x'_3|} \sinh\left(\frac{\sqrt{\lambda}|x_3 - x'_3|}{2}\right)\langle x'_3\rangle^{-m'/2}, \quad x_3, x'_3 \in \mathbb{R}.$$

Applying the dominated convergence theorem, we easily find that $\lim_{\lambda\downarrow 0}\|\tilde J_{m'}^{(\lambda)}-\tilde J_{m'}^{(0)}\|_2=0$. Therefore, $\tilde T_q = \mathrm{n}-\lim_{\lambda\downarrow 0}(T_q(2bq-\lambda+i0)-\mathcal{O}_q(\lambda))$. Fix $t>0$. Choosing $\lambda>0$ so small that $\|T_q(2bq-\lambda+i0)-\mathcal{O}_q(\lambda)-\tilde T_q\|<t/2$, and applying the Weyl's inequalities, we get

$$n_\pm(t;T_q(2bq-\lambda)-\mathcal{O}_q(\lambda))\leq$$
$$n_\pm(t/2;T_q(2bq-\lambda)-\mathcal{O}_q(\lambda)-\tilde T_q)+n_\pm(t/2;\tilde T_q)=n_\pm(t/2;\tilde T_q). \tag{5.13}$$

Since the r.h.s. of (5.13) is finite and independent of λ, we conclude that (5.13) entails (5.11). □

Proposition 5.3. *Assume that* (1.4) *holds. Then for each $q\in\mathbb{Z}_+$, $\lambda>0$, and $s>0$ we have*

$$n_+(s;\mathcal{O}_q(\lambda))=n_+(s;\omega_q(\lambda)) \tag{5.14}$$

(see (3.6) *for the definition of the operator $\omega_q(\lambda)$).*

Proof. Define the operator $K:L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^2)$ by

$$(Ku)(X_\perp):=\int_{\mathbb{R}^2}\int_{\mathbb{R}}\mathcal{P}_{q,b}(X_\perp,X_\perp')\sqrt{V(X_\perp',x_3')}u(X_\perp',x_3')\,dx_3'\,dX_\perp',\quad X_\perp\in\mathbb{R}^2,$$

where $u\in L^2(\mathbb{R}^3)$. The adjoint operator $K^*:L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^3)$ is given by

$$(K^*v)(X_\perp):=\sqrt{V(X_\perp,x_3)}\int_{\mathbb{R}^2}\mathcal{P}_{q,b}(X_\perp,X_\perp')v(X_\perp')\,dX_\perp',\quad (X_\perp,x_3)\in\mathbb{R}^3,$$

where $v\in L^2(\mathbb{R}^2)$. Obviously,

$$\mathcal{O}_q(\lambda)=\frac{1}{2\sqrt{\lambda}}K^*K,\quad \omega_q(\lambda)=\frac{1}{2\sqrt{\lambda}}K\,K^*.$$

Since $n_+(s;K^*K)=n_+(s;K\,K^*)$ for each $s>0$, we get (5.14). □

Putting together (2.13), (5.9), (5.10), and (5.14), we get (3.8). Thus, we are done with the proof of Theorem 3.1.

5.4. In this subsection we complete the proof of Theorem 3.2.

Proposition 5.4. *Let $q\in\mathbb{Z}_+$ and $b>0$. Assume that* (1.5) *holds. Then the asymptotic estimates*

$$n_\pm(s;\mathrm{Re}\,T_q(2bq+\lambda))=O(1) \tag{5.15}$$

are valid as $\lambda\downarrow 0$ for each $s>0$.

Proof. The operator $\mathrm{Re}\,T_q(2bq+\lambda+i0)$ admits the integral kernel

$$-\frac{1}{2\sqrt{\lambda}}\sqrt{V(X_\perp,x_3)}\sin\left(\sqrt{\lambda}|x_3-x_3'|\right)\mathcal{P}_{q,b}(X_\perp,X_\perp')\sqrt{V(X_\perp',x_3')},\quad (X_\perp,x_3),(X_\perp',x_3')\in\mathbb{R}^3.$$

Arguing as in the proof of Proposition 5.2, we find that $\mathrm{n}-\lim_{\lambda\downarrow 0}\mathrm{Re}\,T_q(2bq+\lambda+i0)=\tilde T_q$ (see (5.12)). Fix $s>0$. Choosing $\lambda>0$ so small that $\|\mathrm{Re}\,T_q(2bq+\lambda+i0)-\tilde T_q\|<s/2$, and applying the Weyl's inequalities, we get $n_\pm(s;\mathrm{Re}\,T_q(2bq+\lambda+i0))\leq n_\pm(s/2;\tilde T_q)$ which implies (5.15). □

Taking into account Propositions 5.1 and 5.4, and applying the Weyl's inequalities and the evident identities

$$\int_{\mathbb{R}}n_\pm(s;tT)d\mu(t)=\frac{1}{\pi}\mathrm{Tr}\ \arctan\,(s^{-1}T),\quad s>0,$$

where $T=T^*\geq 0$, $T\in S_1$, we obtain the following

Corollary 5.2. *Let $q \in \mathbb{Z}_+$, $b > 0$. Assume that V satisfies* (1.5)*. Then the asymptotic estimates*

$$\frac{1}{\pi}\mathrm{Tr}\ \arctan\left((1+\varepsilon)^{-1}\mathrm{Im}\ T_q(2bq+\lambda)\right) + O(1) \leq$$

$$\int_{\mathbb{R}} n_{\pm}(1; \mathrm{Re}\ T_q(2bq+\lambda) + t\ \mathrm{Im}\ T_q(2bq+\lambda))d\mu(t) \leq$$

$$\frac{1}{\pi}\mathrm{Tr}\ \arctan\left((1-\varepsilon)^{-1}\mathrm{Im}\ T_q(2bq+\lambda)\right) + O(1) \tag{5.16}$$

are valid as $\lambda \downarrow 0$ for each $\varepsilon \in (0,1)$.

Proposition 5.5. *Assume that* (1.4) *holds. Then for each $q \in \mathbb{Z}_+$, $\lambda > 0$, and $s > 0$, we have*

$$n_+(s; \mathrm{Im}\ T_q(2bq+\lambda)) = n_+(s; \Omega_q(\lambda)) \tag{5.17}$$

(see (3.10) *for the definition of the operator $\Omega_q(\lambda)$). Consequently,*

$$\mathrm{Tr}\ \arctan\left(s^{-1}\mathrm{Im}\ T_q(2bq+\lambda)\right) = \mathrm{Tr}\ \arctan\left(s^{-1}\Omega_q(\lambda)\right) \tag{5.18}$$

for each $q \in \mathbb{Z}_+$, $\lambda > 0$.

Proof. The proof is quite similar to that of Proposition 5.3. Define the operator $\mathcal{K} : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^2)^2$ by

$$\mathcal{K}u := \mathbf{v} = (v_1, v_2) \in L^2(\mathbb{R}^2)^2, \quad u \in L^2(\mathbb{R}^3),$$

where

$$v_1(X_\perp) := \int_{\mathbb{R}^2}\int_{\mathbb{R}} \mathcal{P}_{q,b}(X_\perp, X_\perp')\cos(\sqrt{\lambda}x_3')\sqrt{V(X_\perp', x_3')}u(X_\perp', x_3')\,dx_3'\,dX_\perp',$$

$$v_2(X_\perp) := \int_{\mathbb{R}^2}\int_{\mathbb{R}} \mathcal{P}_{q,b}(X_\perp, X_\perp')\sin(\sqrt{\lambda}x_3')\sqrt{V(X_\perp', x_3')}u(X_\perp', x_3')\,dx_3'\,dX_\perp', \quad X_\perp \in \mathbb{R}^2.$$

Evidently, the adjoint operator $\mathcal{K}^* : L^2(\mathbb{R}^2)^2 \to L^2(\mathbb{R}^3)$ is given by

$$(\mathcal{K}^*\mathbf{v})(X_\perp, x_3) := \cos(\sqrt{\lambda}x_3)\sqrt{V(X_\perp, x_3)}\int_{\mathbb{R}^2}\mathcal{P}_{q,b}(X_\perp, X_\perp')v_1(X_\perp')\,dX_\perp' +$$

$$\sin(\sqrt{\lambda}x_3)\sqrt{V(X_\perp, x_3)}\int_{\mathbb{R}^2}\mathcal{P}_{q,b}(X_\perp, X_\perp')v_2(X_\perp')\,dX_\perp', \quad (X_\perp, x_3) \in \mathbb{R}^3,$$

where $\mathbf{v} = (v_1, v_2) \in L^2(\mathbb{R}^2)^2$. Obviously,

$$\mathrm{Im}\ T_q(2bq+\lambda) = \frac{1}{2\sqrt{\lambda}}\mathcal{K}^*\mathcal{K}, \quad \Omega_q(\lambda) = \frac{1}{2\sqrt{\lambda}}\mathcal{K}\mathcal{K}^*.$$

Since $n_+(s; \mathcal{K}^*\mathcal{K}) = n_+(s; \mathcal{K}\mathcal{K}^*)$ for each $s > 0$, we get (5.17). $\square$

Now the combination of (2.13), (5.1), (5.16), and (5.18) yields (3.11).

6 Proof of Corollary 2.2

Introduce the matrix-valued functions

$$\mathcal{W}^{(1)}(X_\perp) := \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{W}^{(2)}(X_\perp) = \mathcal{W}^{(2)}(X_\perp;\lambda) := \mathcal{W}_\lambda(X_\perp) - \mathcal{W}^{(1)}(X_\perp) = \begin{pmatrix} -w_{22} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}$$

(see (3.4) and (3.9) for the definitions of W and $\mathcal{W}_\lambda$, respectively), as well as the operators

$$\Omega_q^{(j)}(\lambda) : \frac{1}{2\sqrt{\lambda}} p_q \mathcal{W}^{(j)} p_q, \quad \lambda > 0, \quad q \in \mathbb{Z}_+, \quad j = 1, 2,$$

compact in $L^2(\mathbb{R}^2)^2$. Evidently, $\Omega_q^{(j)}(\lambda) \in S_1$, $j = 1, 2$.

Proposition 6.1. *(i) Let* (1.5) *hold with* $m \in (3, 4]$. *Then for each* $q \in \mathbb{Z}_+$, $s > 0$, *and* $\delta > \frac{4-m}{2}$, *we have*

$$\mathrm{Tr}\,\left(\arctan\,(s^{-1}\Omega_q(\lambda)) - \arctan\,(s^{-1}\Omega_q^{(1)}(\lambda))\right) = O(\lambda^{-\delta}), \quad \lambda \downarrow 0, \tag{6.1}$$

(see (3.10) *for the definition of the operator* $\Omega_q(\lambda)$*).*
(ii) Let (1.4) *hold with* $m_\perp > 2$, $m_3 > 2$. *Then for each* $q \in \mathbb{Z}_+$ *and* $s > 0$ *we have*

$$\mathrm{Tr}\,\left(\arctan\,(s^{-1}\Omega_q(\lambda)) - \arctan\,(s^{-1}\Omega_q^{(1)}(\lambda))\right) = O(1), \quad \lambda \downarrow 0. \tag{6.2}$$

Proof. Applying the Lifshits-Krein trace formula (1.1), we easily get

$$\mathrm{Tr}\,\left(\arctan\,(s^{-1}\Omega_q(\lambda)) - \arctan\,(s^{-1}\Omega_q^{(1)}(\lambda))\right) = \int_{\mathbb{R}} \xi(E; s^{-1}\Omega_q(\lambda); s^{-1}\Omega_q^{(1)}(\lambda))(1+E^2)^{-1}dE, \quad s > 0. \tag{6.3}$$

Therefore,

$$\left|\mathrm{Tr}\,\left(\arctan\,(s^{-1}\Omega_q(\lambda)) - \arctan\,(s^{-1}\Omega_q^{(1)}(\lambda))\right)\right| \leq$$

$$\int_{\mathbb{R}} |\xi(E; s^{-1}\Omega_q(\lambda); s^{-1}\Omega_q^{(1)}(\lambda))|dE \leq \frac{1}{s}\,\|\Omega_q(\lambda) - \Omega_q^{(1)}(\lambda)\|_1 = \frac{1}{s}\,\|\Omega_q^{(2)}(\lambda)\|_1 \tag{6.4}$$

(see [16, Theorem 8.2.1]). Further,

$$\|\Omega_q^{(2)}(\lambda)\|_1 \leq \frac{b}{2\pi\sqrt{\lambda}} \int_{\mathbb{R}^2} \sqrt{w_{22}(X_\perp)^2 + w_{12}(X_\perp)^2}dX_\perp \leq$$

$$\frac{b}{2\pi\sqrt{\lambda}} \sum_{j=1,2} \int_{\mathbb{R}^2} |w_{j2}(X_\perp)|dX_\perp \leq \frac{b}{\pi\sqrt{\lambda}} \int_{\mathbb{R}^2}\int_{\mathbb{R}} V(X_\perp, x_3)|\sin(\sqrt{\lambda}x_3)|dx_3\,dX_\perp. \tag{6.5}$$

Assume now that V satisfies (1.5) with $m \in (3, 4]$. Pick $\delta > \frac{4-m}{2}$, and $m' \in (-2\delta + 2, m-2)$. Then we have

$$\lambda^{-1/2} \int_{\mathbb{R}^2}\int_{\mathbb{R}} V(X_\perp, x_3)|\sin\,(\sqrt{\lambda}x_3)|dx_3\,dX_\perp \leq \lambda^{-\delta} C_0' \int_{\mathbb{R}^2} \langle X_\perp\rangle^{-(m-m')}dX_\perp \int_{\mathbb{R}} \langle x_3\rangle^{-m'}|x_3|^{-2\delta+1}dx_3. \tag{6.6}$$

Since $m - m' > 2$ the integral with respect to $X_\perp \in \mathbb{R}^2$ is convergent, and since $m' + 2\delta - 1 > 1$ the integral with respect to x_3 is convergent as well. Now the combination of (6.3) – (6.6) entails (6.1). Further, suppose that V satisfies (1.4) with $m_\perp > 2$ and $m_3 > 2$. Then

$$\lambda^{-1/2} \int_{\mathbb{R}^2}\int_{\mathbb{R}} V(X_\perp, x_3)|\sin\,(\sqrt{\lambda}x_3)|dx_3\,dX_\perp \leq C_0 \int_{\mathbb{R}^2} \langle X_\perp\rangle^{-m_\perp}dX_\perp \int_{\mathbb{R}} \langle x_3\rangle^{-m_3}|x_3|dx_3. \tag{6.7}$$

Putting together (6.3) – (6.5), and (6.7), we get (6.2). ☐

Now note that if V satisfies (1.5) with $m \in (3, 4]$, we can choose $\frac{4-m}{2} < \delta < \frac{1}{m-1}$ so that in this case Proposition 6.1 entails

$$\mathrm{Tr}\, \left(\arctan\,(s^{-1}\Omega_q(\lambda)) - \arctan\,(s^{-1}\Omega_q^{(1)}(\lambda))\right) = o(\lambda^{-1/(m-1)}), \quad \lambda \downarrow 0. \tag{6.8}$$

Moreover, if V satisfies (1.5) with $m > 4$, then it satisfies (1.4) with $m_\perp > 2$ and $m_3 > 2$, and, hence, (6.2) is valid. Finally,

$$\mathrm{Tr}\, \arctan\,(s^{-1}\Omega_q^{(1)}(\lambda)) = \mathrm{Tr}\, \arctan\,(s^{-1}\omega_q(\lambda)) = \int_0^\infty \frac{n_+(st; \omega_q(\lambda))}{1+t^2} dt, \quad s > 0, \quad \lambda > 0, \tag{6.9}$$

(see (3.6) for the definition of the operator $\omega_q(\lambda)$). Putting together (6.9), (6.8), and (6.2), we conclude that Corollary 3.2 follows easily from Theorem 3.2 and Lemmas 3.2 – 3.4.

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