Entropy inequalities for unbounded spin systems

Paolo Dai Pra*
Anna Maria Paganoni†
Gustavo Posta‡

Dipartimento di Matematica
Politecnico di Milano
Piazza L.Da Vinci 32, 20133 Milano, Italy

Abstract

We consider nonconservative, reversible spin systems, with unbounded discrete spins. We show that for a class of these dynamics in a high temperature regime, the relative entropy with respect to the equilibrium distribution decays exponentially in time, although the logarithmic-Sobolev inequality fails. To this end we prove a weaker modification of the logarithmic-Sobolev inequality.

Keywords: spin systems, entropy, logarithmic-Sobolev inequality, spectral gap.
Mathematical Subject Classification (2000): 60K35, 82C22.

*Partially supported by MURST project "Stochastic processes with spatial structure" and GNAFA

†Partially supported by GNAFA

‡Partially supported by MURST project "Stochastic processes with spatial structure" and GNAFA

1 Introduction

In this paper we consider nonconservative spin systems in a finite lattice $\Lambda \subset \mathbf{Z}^d$, whose spins take values in $\mathbf{Z}_+$; in the particle interpretation, this means that every site of the lattice may be occupied by any number of particles. The dynamics are required to satisfy detailed balance with respect to a (finite volume) Gibbs measure given by, for $\sigma = (\sigma_x)_{x\in\Lambda} \in \mathbf{Z}_+^\Lambda$

$$\mu_\Lambda^\tau(\sigma) = \frac{1}{Z_\Lambda^\tau} \mathrm{e}^{-\beta H_\Lambda^\tau(\sigma)} \prod_{x\in\Lambda} \rho(\sigma_x)$$

where $\rho(\cdot)$ is a given (reference) measure on $\mathbf{Z}_+$, H_Λ^τ is the Hamiltonian associated to a bounded, finite range interaction, τ is a fixed configuration in $\mathbf{Z}_+^{\mathbf{Z}^d\setminus\Lambda}$ that contributes, as boundary condition, to the Hamiltonian H_Λ^τ, $\beta > 0$ is the inverse temperature and Z_Λ^τ is the normalization factor. We denote by $\mathcal{L}_\Lambda^\tau$ the infinitesimal generator of one of these dynamics, that will be defined in Section 2.

If the volume Λ and the boundary condition τ are kept fixed, the dynamics we consider have μ_Λ^τ as the unique invariant (reversible) measure, and convergence to equilibrium occurs from any initial distribution. The main objective of this paper is to show that, for certain choices of ρ and for β sufficiently small, the system converges to equilibrium at an exponential rate which is uniform in Λ and τ. In particular, this implies exponential convergence to equilibrium for the limiting infinite volume dynamics on $\mathbf{Z}_+^{\mathbf{Z}^d}$.

For reversible, nonconservative spin systems whose spins take values on either a finite set ([10], [9], [8]) or a compact manifold ([11]), uniform convergence to equilibrium is rather well understood. The rate of convergence to equilibrium is typically given in terms of two constants: the *spectral gap* $\mathrm{gap}(\mathcal{L}_\Lambda^\tau)$ and the logarithmic-Sobolev constant $S(\mathcal{L}_\Lambda^\tau)$, that are defined as the optimal constants of, respectively, the Poincaré inequality

$$\mu_\Lambda^\tau(f,f) \le -l^{-1}\mu_\Lambda^\tau(f\mathcal{L}_\Lambda^\tau f)$$

and the logarithmic-Sobolev inequality

$$\mathrm{Ent}_{\mu_\Lambda^\tau}(f^2) \le -\alpha^{-1}\mu_\Lambda^\tau(f\mathcal{L}_\Lambda^\tau f).$$

Here and in the sequel we let, for a probability measure μ, $\mu(f) = \int f d\mu$, $\mu(f,g) = \mu((f-\mu(f))(g-\mu(g)))$, and $\mathrm{Ent}_\mu(f^2) = \mu(f^2\log f^2) - \mu(f^2)\log\mu(f^2)$.

The spectral gap controls the convergence to equilibrium in $L^2(\mu_\Lambda^\tau)$, while the logarithmic-Sobolev constant is related to hypercontractivity of the semigroup $\exp(t\mathcal{L}_\Lambda^\tau)$ (see [6]), and may be used to give bounds on the decay of the relative entropy of the law of the system at time t with respect to μ_Λ^τ (see e.g. [8]). It is known [8] that $\mathrm{gap}(\mathcal{L}_\Lambda^\tau) \geq 2S(\mathcal{L}_\Lambda^\tau)$, so that a uniform lower bound for $S(\mathcal{L}_\Lambda^\tau)$ implies a uniform lower bound for $\mathrm{gap}(\mathcal{L}_\Lambda^\tau)$. For systems with finite or compact spin space, it is known that, for β small enough, $S(\mathcal{L}_\Lambda^\tau)$ (and, henceforth, $\mathrm{gap}(\mathcal{L}_\Lambda^\tau)$) is uniformly bounded away from zero.

In the case of $\mathbf{Z}_+$-valued spins, and letting ρ be either a Poisson or a geometric measure, it is easy to show that $S(\mathcal{L}_\Lambda^\tau) = 0$ for every fixed β, Λ, τ. As a preliminary result we show that, however, $\inf_{\Lambda,\tau} \mathrm{gap}(\mathcal{L}_\Lambda^\tau) > 0$ for both Poisson and geometric reference measures for β sufficiently small. An analogous result has been recently proved in [1], where birth and death dynamics in the continuum are considered. We actually have some partial evidence that our discrete systems with $\rho =$ Poisson measure should have the same ergodic behavior as the ones in [1].

The main objective of this paper is to go beyond the spectral gap, and show that a suitable variation of the logarithmic-Sobolev inequality holds, uniformly in Λ, τ. To our knowledge, Bobkov and Ledoux in [2] have been the first to propose a "modified" logarithmic-Sobolev inequality in the context of Poisson measures; it is not clear to us, however, which consequences in terms of ergodicity their inequality has. In this work we propose the following inequality, that will be referred to as the *entropy inequality*: for $f > 0$

$$\mathrm{Ent}_{\mu_\Lambda^\tau}(f) \leq -s^{-1}\mu_\Lambda^\tau(f\mathcal{L}_\Lambda^\tau \log f).$$

This inequality arise quite naturally, since it corresponds to exponential decay of the relative entropy with respect to the invariant measure (see Section 2). In general, the entropy inequality is implied by the logarithmic-Sobolev inequality, and implies the Poincaré inequality. Letting $s(\mathcal{L}_\Lambda^\tau)$ to be the optimal constant in the entropy inequality, we show that $\inf_{\Lambda,\tau} s(\mathcal{L}_\Lambda^\tau) > 0$ for $\rho =$ Poisson measure and β sufficiently small, while $s(\mathcal{L}_\Lambda^\tau) = 0$ for every β, Λ, τ, in the case of $\rho =$ geometric measure.

In the proof of the lower bound for $s(\mathcal{L}_\Lambda^\tau)$ we follow, as a guiding strategy, the recursive argument in [9]. The (nontrivial) adaptation of that argument to the entropy inequality had lead us to a different proof, that is actually simpler than the original one in [9]. Our proof is based on a simple inequality that expresses "quasi" additivity of the entropy in disjoint volumes. After our proof was completed, we have been informed that a similar inequality has been independently obtained by F.Cesi [3].

2 Notation and results

The Gibbs measures. For $\Lambda \subset \mathbf{Z}^d$ we denote by $|\Lambda|$ the cardinality of Λ, and if $|\Lambda| < +\infty$ we write $\Lambda \subset\subset \mathbf{Z}^d$. We consider the *configuration space* $\Omega = \mathbf{Z}_+^{\mathbf{Z}^d}$. An element $\eta \in \Omega$ is a function from $\mathbf{Z}^d$ in $\mathbf{Z}_+$; we denote by η_x its value at $x \in \mathbf{Z}^d$. We endow Ω with the Borel σ–field $\mathcal{F}$ generated by the open sets of the product topology. If $\Lambda \subset \mathbf{Z}^d$, we denote by $\Omega_\Lambda := \mathbf{Z}_+^\Lambda$ the reduced configuration space and by $\mathcal{F}_\Lambda$ the σ–field generated by $\{\eta_x : x \in \Lambda\}$. If f is a function defined on Ω we write $f \in \mathcal{F}_\Lambda$ to indicate that f is $\mathcal{F}_\Lambda$–measurable. Given $\eta \in \Omega$ and $\Lambda \subset \mathbf{Z}^d$ we denote with η_Λ the restriction of η on Ω_Λ. For $A, B \subset \mathbf{Z}^d$, $\eta \in \Omega_A$ and $\omega \in \Omega_B$ we define $\eta\omega \in \Omega_{A\Delta B}$, where $A\Delta B := (A \setminus B) \cup (B \setminus A)$ denotes the *symmetric difference* between A and B, as

$$(\eta\omega)_x := \begin{cases} \eta_x & x \in A \setminus B \\ \omega_x & x \in B \setminus A. \end{cases}$$

For $x \in \mathbf{Z}^d$ we denote by $|x| := \max_{k=1,\ldots,d} |x_k|$, and, for $A, B \subset \mathbf{Z}^d$, the *distance beetwen A and B* is denoted by

$$\mathrm{dist}(A, B) := \min\{|x - y| : x \in A,\ y \in B\}.$$

Let $\Phi := \{\Phi_R : R \subset \mathbf{Z}^d\}$ be a *bounded finite range potential*, with range $r > 0$, *i.e.* for every $R \subset \mathbf{Z}^d$, Φ_R is a function $\Phi_R : \Omega_R \to \mathbf{R}$ such that

$$\|\Phi\| := \sup_{x \in \mathbf{Z}^d} \sum_{R \ni x} \|\Phi_R\|_{+\infty} < +\infty$$

and $\Phi_R \equiv 0$ if $\mathrm{diam}(R) > r$. Given the potential $\{\Phi_R : R \subset \mathbf{Z}^d\}$ and $\Lambda \subset\subset \Omega$, the Hamiltonian $H_\Lambda \equiv H_{\Lambda,\Phi}$ is defined by

$$H_\Lambda(\eta) := \sum_{R \cap \Lambda \neq \emptyset} \Phi_R(\eta_R).$$

For $\eta, \tau \in \Omega$ we also set $H_\Lambda^\tau(\eta) := H_\Lambda(\eta_\Lambda \tau_{\Lambda^c})$; τ in this case is called the *boundary condition.*

The reference probability for a single spin $n \in \mathbf{Z}_+$ will be denoted by $\rho(\cdot)$ and it will be taken to be either a Poisson measure

$$\rho(n) = \mathrm{e}^{-\lambda} \frac{\lambda^n}{n!}$$

for some $\lambda > 0$, or a geometric measure

$$\rho(n) = (1-c)c^n$$

for some $0 < c < 1$. From now on the constants λ and c will be kept fixed, and dependence of other constants on λ and c will be omitted.

For $\Lambda \subset\subset \mathbf{Z}^d$, $\tau \in \Omega$ and $\beta > 0$ the finite volume *Gibbs measure* in Λ at inverse temperature β and boundary condition τ is the probability measure on $(\Omega_\Lambda, \mathcal{F}_\Lambda)$ given by

$$\mu_\Lambda^\tau(\eta) := (Z_\Lambda^\tau(\beta))^{-1} \exp\left[-\beta H_\Lambda^\tau(\eta)\right] \prod_{x\in\Lambda} \rho(\eta_x), \tag{2.1}$$

where $Z_\Lambda^\tau(\beta)$, the *partition function*, is the appropriate normalization factor. For $f, g \in \mathcal{F}$ we denote with $\mu_\Lambda^\tau(f)$ the expectation of f w.r.t. μ_Λ^τ, and with $\mu_\Lambda^\tau(f,g) = \mu_\Lambda^\tau(fg) - \mu_\Lambda^\tau(f)\mu_\Lambda^\tau(g)$ the covariance w.r.t. μ_Λ^τ. For $A \subset \Lambda$, we denote with $\mu_{\Lambda,A}$ the *marginal* of μ_Λ in Ω_A, *i.e.* $\mu_\Lambda(f) = \mu_{\Lambda,A}(f)$ for any f $\mathcal{F}_A$–measurable. Given two probability measures μ and ν on $(\Omega_\Lambda, \mathcal{F}_\Lambda)$, we denote the *relative entropy* of ν w.r.t. μ by

$$\mathbf{H}(\nu|\mu) := \begin{cases} \mu(f\log f) & \text{if } d\nu = f d\mu,\ f\log f \in L^1(\mu) \\ +\infty & \text{if } \nu \not\ll \mu \end{cases}$$

and the *total variation distance* between μ and ν by

$$\|\mu - \nu\|_{\mathrm{TV}} = \frac{1}{2} \sum_{\eta\in\Omega_\Lambda} |\mu(\eta) - \nu(\eta)| = \max_{X\subset\Omega_\Lambda} |\mu(X) - \nu(X)|.$$

Finally, for $f > 0$, we let

$$\mathrm{Ent}_\mu(f) = \mu(f\log f) - \mu(f)\log\mu(f)$$

for $f\log f \in L^1(\mu)$, and $\mathrm{Ent}_\mu(f) = +\infty$ otherwise. It is known that $\mathrm{Ent}_\mu(f) = 0$ if and only if $f = 1$ μ-a.s. . We shall write $f \not\equiv 1$ for $\mathrm{Ent}_\mu(f) \neq 0$. Note finally that $\mathbf{H}(\nu|\mu) = \mathrm{Ent}_\mu\left(\frac{d\nu}{d\mu}\right)$.

The dynamics. For a given function f on Ω we let

$$(\nabla_x^- f)(\eta) := \mathbf{1}_{\{\eta_x>0\}}[f(\eta-\delta^x)-f(\eta)] \qquad (\nabla_x^+ f)(\eta) := f(\eta+\delta^x)-f(\eta) \qquad x\in \mathbf{Z}^d,$$

where $\delta^x \in \Omega$, $(\delta^x)_x = 1$, $(\delta^x)_y = 0$ for any $y \neq x$. The stochastic dynamics we want to study are determined by the Markov generator $\mathcal{L}_\Lambda^\tau$, $\Lambda \subset\subset \mathbf{Z}^d$, $\tau\in\Omega$, defined by

$$(\mathcal{L}_\Lambda^\tau f)(\eta) := \sum_{\substack{x\in\Lambda\\ *\in\{-,+\}}} c_\Lambda^\tau(x,\eta,*)(\nabla_x^* f)(\eta).$$

We require that the rates $c_\Lambda^\tau(\cdot,\cdot,\cdot)$ satisfy the *detailed balance condition* w.r.t. μ_Λ^τ:

$$c_\Lambda^\tau(x,\eta,\pm)\mu_\Lambda^\tau(\eta) = c_\Lambda^\tau(x,\eta\pm\delta_x,\mp)\mu_\Lambda^\tau(\eta\pm\delta_x) \qquad \forall \eta\in\Omega_\Lambda, \forall x\in\Lambda,$$

i.e. $\mathcal{L}_\Lambda^\tau$ is self adjoint in $L^2(\mu_\Lambda^\tau)$ (note: the identities involving $\eta-\delta_x$ hold whenever $\eta_x>0$) . Moreover we assume that there exists a constant $C>0$ dependent only on β and $\|\Phi\|$, such that

$$C^{-1}\bar{c}(x,\eta,\pm) \le c_\Lambda^\tau(x,\eta,\pm) \le C\bar{c}(x,\eta,\pm) \qquad \forall \eta,\tau\in\Omega_\Lambda,\ x\in\mathbf{Z}^d \tag{2.2}$$

where $\bar{c}(x,\eta,\pm)$ are the rates for a systems with no interaction, that will be chosen as follows:

$$\bar{c}(x,\eta,+) = 1, \quad \bar{c}(x,\eta,-) = \frac{\eta_x}{\lambda}$$

when ρ is a Poisson measure of parameter λ, and

$$\bar{c}(x,\eta,+) = 1, \quad \bar{c}(x,\eta,-) = \frac{1}{c}$$

when ρ is a geometric measure of parameter c. An example of rates satisfying these conditions is:

$$c_\Lambda^\tau(x,\eta,\pm) := \bar{c}(x,\eta,\pm)\exp\left[-\frac{\beta}{2}\nabla_x^\pm H_\Lambda^\tau(\eta)\right].$$

The *Dirichlet form* associated with $\mathcal{L}_\Lambda^\tau$ is given by

$$\mathcal{E}_\Lambda^\tau(f,g) := -\mu_\Lambda^\tau\left(f\mathcal{L}_\Lambda^\tau g\right) = \sum_{x\in\Lambda}\mu_\Lambda^\tau\Big(c_\Lambda^\tau(x,\cdot,+)(\nabla_x^+ f)^2(\cdot)\Big),$$

and we denote by $\{\mathrm{e}^{t\mathcal{L}_\Lambda^\tau}\}_{t\geq 0}$ the Markov semigroup generated by $\mathcal{L}_\Lambda^\tau$ in $L^2(\mu_\Lambda^\tau)$.

The spectral gap, $\mathrm{gap}(\mathcal{L}_\Lambda^\tau)$, is defined by

$$\mathrm{gap}(\mathcal{L}_\Lambda^\tau) := \inf\left\{\frac{\mathcal{E}_\Lambda^\tau(f,f)}{\mu_\Lambda^\tau(f,f)},\ f \in L^2(\mu_\Lambda^\tau),\ \mu_\Lambda^\tau(f,f) \neq 0\right\}. \tag{2.3}$$

It is well known that the following inequality, concerning convergence to equilibrium for the semigroup $\{\mathrm{e}^{t\mathcal{L}_\Lambda^\tau}\}_{t\geq 0}$, holds:

$$\|\mathrm{e}^{t\mathcal{L}_\Lambda^\tau} f - \mu_\Lambda^\tau(f)\|_{L^2(\mu_\Lambda^\tau)} \leq \|f\|_{L^2(\mu_\Lambda^\tau)} \mathrm{e}^{-\frac{gt}{2}}$$

where $g = \mathrm{gap}(\mathcal{L}_\Lambda^\tau)$.

The *entropy constant* $s(\mathcal{L}_\Lambda^\tau)$ is defined as

$$s(\mathcal{L}_\Lambda^\tau) := \inf\left\{\frac{\mathcal{E}_\Lambda^\tau(f,\log f)}{\mathrm{Ent}_{\mu_\Lambda^\tau}(f)}:\ f \geq 0,\ f\log f \in L^1(\mu_\Lambda^\tau),\ f \not\equiv 1\right\} \tag{2.4}$$

or, equivalently, as the optimal constant for the *entropy inequality*

$$\mathrm{Ent}_{\mu_\Lambda^\tau}(f) \leq s^{-1}\mathcal{E}_\Lambda^\tau(f,\log f).$$

The entropy inequality forces the relative entropy with respect to the invariant measure to decay exponentially (see *e.g.* [10]):

$$\mathbf{H}(\nu_t|\mu_\Lambda^\tau) \leq e^{-st}\,\mathbf{H}(\nu|\mu_\Lambda^\tau)$$

where $\nu_t := \nu\mathrm{e}^{t\mathcal{L}_\Lambda^\tau}$ is the distribution at time t of the system having ν as initial distribution. In particular, by using the Csizar's inequality

$$\|\mu - \nu\|_{\mathrm{TV}} \leq \sqrt{\frac{1}{2}\,\mathbf{H}(\nu|\mu)}$$

we get

$$\|\nu_t - \mu_\Lambda^\tau\|_{\mathrm{TV}} \leq \sqrt{\frac{1}{2}\,\mathbf{H}(\nu|\mu_\Lambda^\tau)}\mathrm{e}^{-st/2}.$$

Finally, the logarithmic-Sobolev constant $S(\mathcal{L}_\Lambda^\tau$ is defines as

$$S(\mathcal{L}_\Lambda^\tau) := \inf\left\{\frac{\mathcal{E}_\Lambda^\tau(\sqrt{f},\sqrt{f})}{\mathrm{Ent}_{\mu_\Lambda^\tau}(f)}:\ f \geq 0,\ f\log f \in L^1(\mu_\Lambda^\tau),\ f \not\equiv 1\right\}.$$

In order to state and prove our main result we need to assume a mixing property of the Gibbs measure. We introduce the family of *fat rectangles of size smaller than* L (see [8]) in the following way. Let $R(x; l_1, \ldots, l_d)$ denote the rectangle

$$R(x; l_1, \ldots, l_d) = x + ([1, l_1] \times \cdots \times [1, l_d]) \cap \mathbf{Z}^d, \quad x \in \mathbf{Z}^d,\ (l_1, \ldots, l_d) \in \mathbf{Z}_+^d.$$

The *size* of $R(x; l_1, \ldots, l_d)$ is the number $\max\{l_k : k = 1, \ldots, d\}$, and we say that $R(x; l_1, \ldots, l_d)$ is *fat* if $\min\{l_k : k = 1, \ldots, d\} \geq \frac{1}{10} \max\{l_k : k = 1, \ldots, d\}$. Let $\mathcal{R}_L$ be the class of all fat rectangles in $\mathbf{Z}^d$ of size $L \in \mathbf{Z}_+$, and $\mathcal{R} = \bigcup_{L \geq 1} \mathcal{R}_L$.

Condition 2.1 (Mixing condition) *Let $\Lambda \in \mathcal{R}$, for example $\Lambda \in \mathcal{R}_L$, and $A, B \subset \Lambda$ such that $A, B \in \mathcal{R}_L$ and $A \cap B = \emptyset$. Then there exist constants $C_1(\beta, d, \|\Phi\|), C_2(\beta, d, \|\Phi\|) > 0$ such that*

$$\sup_{\tau, \sigma \in \Omega} \left| \frac{\mu_\Lambda^\tau(\eta : \eta_A = \sigma_A)\, \mu_\Lambda^\tau(\eta : \eta_B = \sigma_B)}{\mu_\Lambda^\tau(\eta : \eta_{A \cup B} = \sigma_{A \cup B})} - 1 \right| \leq C_1 \mathrm{e}^{-C_2 \operatorname{dist}(A,B)}.$$

It is standard to derive this condition from a condition on the exponential decay of covariances, see *e.g.* [8] Proposition 2.12, which can be derived by the “Dobrushin Condition” [5], which, in turn, holds true for β sufficiently small.

We can now state our main results.

Theorem 2.1 *If condition* (2.1) *holds true and the reference measure ρ is either Poisson or geometric, then there exists a constant $\alpha > 0$ independent of Λ and τ, such that*

$$\mu_\Lambda^\tau(f, f) \leq \alpha \mathcal{E}_\Lambda^\tau(f, f),$$

for all $f \in L^2(\mu_\Lambda^\tau)$. In particular $\operatorname{gap}(\mathcal{L}_\Lambda^\tau) \geq \alpha^{-1}$.

Theorem 2.2 *If condition* (2.1)*, holds true and the reference measure is a Poisson measure, then there exists a constant $\alpha > 0$ independent of Λ and τ, such that*

$$\operatorname{Ent}_{\mu_\Lambda^\tau}(f) \leq \alpha \mathcal{E}_\Lambda^\tau(f, \log f),$$

for all $f \geq 0$ such that $f \log f \in L^1(\mu_\Lambda^\tau)$. In particular $s(\mathcal{L}_\Lambda^\tau) \geq \alpha^{-1}$.

3 Single Spin Dynamics

In this section we analyze spectral gap and entropy constant for the non-interacting system with rates $\bar{c}(x, \eta, \pm)$. For such system, spins at different sites evolve through independent dynamics, whose infinitesimal generators are given by

$$\mathcal{L}_0 f(n) = [f(n+1) - f(n)] + \frac{n}{\lambda}[f(n-1) - f(n)] \tag{3.1}$$

for $\rho =$ Poisson measure, and

$$\mathcal{L}_0 f(n) = [f(n+1) - f(n)] + \frac{\mathbf{1}_{n>0}}{c}[f(n-1) - f(n)] \tag{3.2}$$

for $\rho =$ geometric measure. Note that in both (3.1) and (3.2) ρ is a reversible measure for $\mathcal{L}_0$ (i.e. $\mathcal{L}_0$ is self adjoint in $L^2(\rho)$), and the associated Dirichet form is given by

$$\mathcal{E}_0(f, g) = -\sum_n \rho(n) f(n) \mathcal{L}_0 f(n) = \sum_n \rho(n) [f(n+1) - f(n)]^2.$$

First of all, for $\rho(n) = \mathrm{e}^{-\lambda}\lambda^n/n!$, we note that the Logarithmic-Sobolev constant

$$\inf\left\{\frac{\mathcal{E}_0(\sqrt{f}, \sqrt{f})}{\mathrm{Ent}_\rho(f)} :\ f \geq 0,\ f\log f \in L^1(\rho),\ f \not\equiv 1\right\}$$

vanishes, as shown by using the test functions $f_N = \mathbf{1}_{[N+1,+\infty)}$, and letting $N \uparrow +\infty$.

In the case of $\rho(n) = (1-c)c^n$, the entropy constant

$$\inf\left\{\frac{\mathcal{E}_0(f, \log f)}{\mathrm{Ent}_\rho(f)} :\ f \geq 0,\ f\log f \in L^1(\rho),\ f \not\equiv 1\right\}$$

is shown to be zero by using the test functions $f_a(n) = a^n$, and letting $a \uparrow \frac{1}{c}$. Because of the general inequality $\mathcal{E}_0(f, \log f) \geq 4\mathcal{E}_0(\sqrt{f}, \sqrt{f})$ (see *e.g.* [4]), also the Logarithmic-Sobolev constant is zero for geometric measures.

We now turn to positive results.

Proposition 3.1 *The entropy constant for a Poisson measure of mean λ is greater or equal to λ^{-1}, i.e. the inequality*

$$\mathrm{Ent}_\rho(f) \leq \lambda \mathcal{E}_0(f, f)$$

holds for every $f > 0$.

Proof. We follow the scheme of the proof given in [2] for the inequality $\mathrm{Ent}_\rho(f) \le \lambda \sum_n \rho(n) \frac{[f(n+1)-f(n)]^2}{f(n)}$. It is not clear to us whether there is any direct implication between this inequality and the entropy inequality.

The proof is based on classical binomial approximation for Poisson measures. Let μ_p be the Bernoulli measure on $\mathbf{Z}_2 = \mathbf{Z}/2\mathbf{Z}$, with $\mu_p(0) = p \in (0,1)$. For $f : \mathbf{Z}_2 \to \mathbf{R}$, we let $Df(x) = f(x+1) - f(x)$. We first show that, for all $f > 0$ and all $p \in (0,1)$:

$$\mathrm{Ent}_{\mu_p}(f) \le p(1-p)\mu_p[(Df)(D\log f)]. \tag{3.3}$$

To show (3.3), it is enough to observe the identity

$$\mathrm{Ent}_{\mu_p}(f) = p(1-p)\mu_p[(Df)(D\log f)] + \mu_p(f)\mu_p\left[\log\left(\frac{f}{\mu_p(f)}\right)\right], \tag{3.4}$$

and that, by Jensen's inequality, $\mu_p\left[\log\left(\frac{f}{\mu_p(f)}\right)\right] \le 0$.

The rest of the proof is based on tensorization on (3.3), mimicking the argument in [2]. For $n > 1$, let $\mu_p^n = \mu_p \otimes \cdots \otimes \mu_p$. For $f : \mathbf{Z}_2^n \to \mathbf{R}^+$ we let $D_i f$ denote its discrete derivative w.r.t. the i-th variable, and $\mathrm{Ent}_{\mu_p}(f_i)$ denote the entropy of f seen as a function of the i-th variable only, all others being frozen. The following inequality is a general property of entropy for product measures

$$\mathrm{Ent}_{\mu_p^n}(f) \le \sum_{i=1}^n \int \mathrm{Ent}_{\mu_p}(f_i)d\mu_p^n. \tag{3.5}$$

Therefore one gets from (3.3)

$$\mathrm{Ent}_{\mu_p^n}(f) \le p(1-p)\sum_{i=1}^n E_{\mu_p^n}[(D_i f)(D_i \log f)] \tag{3.6}$$

for all $f : \mathbf{Z}_2^n \to \mathbf{R}^+$. Now, take f of the form $f(x_1, \dots, x_n) = \varphi(x_1 + x_2 + \cdots + x_n) \equiv \varphi(S_n)$, where $\varphi : \mathbf{Z}_+ \to \mathbf{R}^+$ is such that $\log\varphi$ is bounded. Note that

$$\sum_{i=1}^n (D_i f)(D_i \log f)(x_1, \dots, x_n)$$

$$= (n - S_n)[\varphi(S_{n+1}) - \varphi(S_n)][\log\varphi(S_{n+1}) - \log\varphi(S_n)] +$$

$$+ S_n[\varphi(S_{n-1}) - \varphi(S_n)][\log\varphi(S_{n-1}) - \log\varphi(S_n)].$$

Thus, letting $p = \lambda/n$ in (3.6), we get

$$\begin{aligned} &\mathrm{Ent}_{\mu^n_{\lambda/n}}(\varphi(S_n)) \\ &\le \frac{\lambda}{n}\left(1-\frac{\lambda}{n}\right)\mu^n_{\lambda/n}\Big[(n-S_n)[\varphi(S_{n+1}) - \varphi(S_n)][\log\varphi(S_{n+1}) - \log\varphi(S_n)] + \\ &\qquad + S_n[\varphi(S_{n-1}) - \varphi(S_n)][\log\varphi(S_{n-1}) - \log\varphi(S_n)]\Big]. \end{aligned} \tag{3.7}$$

By binomial approximation for Poisson distribution, the law of S_n under $\mu^n_{\lambda/n}$ converges to ρ, and $\frac{1}{n}\mu^n_{\lambda/n}(S_n) \to 0$. Therefore, by letting $n \to \infty$ in (3.7), one obtains the desired entropy inequality for $\varphi > 0$ with $\log\varphi$ bounded. An elementary approximation procedure extends it to all $\varphi > 0$ with $\varphi \log\varphi \in L^1(\rho)$. ■

Proposition 3.2 *For $\rho(n) = (1-c)c^n$, the generator $\mathcal{L}_0$ has a positive spectral gap.*

Proof. The result follows from [7], Corollary 5.5. ■

After having shown positivity of the entropy constant and of the spectral gap for the reference dynamics of a single spin, we now show the basic fact that, for a fixed Λ, the corresponding constants for $\mathcal{L}^\tau_\Lambda$ are strictly positive, although possibly depending on Λ, if and only if they are strictly positive for the reference dynamics.

Proposition 3.3 *Let $k(\mathcal{L})$ be either one of $\mathrm{gap}(\mathcal{L})$, $s(\mathcal{L})$ or $S(\mathcal{L})$ associated to a generator $\mathcal{L}$. For every $\Lambda \subset\subset \mathbf{Z}^d$, there is a constant $A > 0$, possibly depending on $|\Lambda|$ but not on $\tau \in \Omega$, such that*

$$A^{-1}k(\mathcal{L}_0) \le k(\mathcal{L}^\tau_\Lambda) \le Ak(\mathcal{L}_0). \tag{3.8}$$

Proof. For the given Λ, we consider a system in Ω_Λ whose spins evolve independently with dynamics generated by $\mathcal{L}_0$. The infinitesimal generator of this process is

$$\mathcal{L}_{0,\Lambda} f(\eta) := \sum_{x\in\Lambda} \mathcal{L}_x f(\eta)$$

where $\mathcal{L}_x$ denotes the operator $\mathcal{L}_0$ acting on the spin η_x. The homogeneous product measure $\rho_\Lambda := \rho^{\otimes\Lambda}$ is reversible for $\mathcal{L}_{0,\Lambda}$. We denote by $\mathcal{E}_{0,\Lambda}$ the corresponding Dirichlet form. As in the proof of Proposition 3.1 (see (3.5)), the behavior of the entropy under tensorization, guarantees that $s(\mathcal{L}_{0,\Lambda}) = s(\mathcal{L}_0)$. Similarly (see *e.g.* [4]), one shows that $\operatorname{gap}(\mathcal{L}_{0,\Lambda}) = \operatorname{gap}(\mathcal{L}_0)$. Moreover, trivially, $S(\mathcal{L}_{0,\Lambda}) = 0$. Thus, to get (3.8), it is enough to show that

$$B^{-1}\mathcal{E}_{0,\Lambda}(f,f) \le \mathcal{E}^\tau_\Lambda(f,f) \le B\mathcal{E}_{0,\Lambda}(f,f), \tag{3.9}$$

$$B^{-1}\mathcal{E}_{0,\Lambda}(f,\log f) \le \mathcal{E}^\tau_\Lambda(f,\log f) \le B\mathcal{E}_{0,\Lambda}(f,\log f), \tag{3.10}$$

$$B^{-1}\rho_\Lambda(f,f) \le \mu^\tau_\Lambda(f,f) \le B\rho_\Lambda(f,f), \tag{3.11}$$

$$B^{-1}\operatorname{Ent}_{\rho_\Lambda}(f) \le \operatorname{Ent}_{\mu^\tau_\Lambda}(f) \le B\operatorname{Ent}_{\rho_\Lambda}(f), \tag{3.12}$$

for every f (positive in (3.10) and (3.12)), with the constant B depending on $|\Lambda|$ but not on τ. Inequalities (3.9) and (3.10) are simple consequences of (2.2), after having noticed that $\bar{c}(x,\eta,\pm)$ are just the rates corresponding to $\mathcal{L}_{0,\Lambda}$. Inequality (3.11) is easily derived from the uniform estimate

$$\mathrm{e}^{-2|\Lambda|\beta\|\Phi\|} \le \frac{d\mu^\tau_\Lambda}{d\rho_\Lambda} \le \mathrm{e}^{2|\Lambda|\beta\|\Phi\|}. \tag{3.13}$$

Finally, inequality (3.12) comes from (3.13) as in Lemma 3.5 in [11]. ■

4 Spectral Gap

This Section is devoted to the proof of Theorem 2.1. We begin by stating two lemmas concerning conditional variance.These lemmas would simplify the existing proofs of uniform positivity of the spectral gap even in the compact spin case.

In what follows, given a Probability space $(\Omega,\mathcal{F},\mu)$ and a σ-field $\mathcal{G}\subset\mathcal{F}$, we let $\mu(f,g|\mathcal{G})$ be the conditional covariance of the random variables f and g.

Lemma 4.1 *Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, $\mathcal{F}_1, \mathcal{F}_2$ be sub-σ-fields of $\mathcal{F}$. Then the following inequality holds for every $f \in L^2(\mu)$:*

$$\mu(f,f) \leq \mu\Big(\mu(f,f|\mathcal{F}_1) + \mu(f,f|\mathcal{F}_2)\Big) + 2\mu\Big(\mu(f|\mathcal{F}_1), \mu(f|\mathcal{F}_2)\Big). \tag{4.1}$$

Proof. It is enough to prove (4.1) for f such that $\mu(f) = 0$. In this case (4.1) is equivalent to

$$\mu(f^2) - \mu\Big(\mu(f|\mathcal{F}_1)^2\Big) - \mu\Big(\mu(f|\mathcal{F}_2)^2\Big) + 2\mu\Big(\mu(f|\mathcal{F}_1)\mu(f|\mathcal{F}_2)\Big) \geq 0. \tag{4.2}$$

Letting $P_i f = \mu(f|\mathcal{F}_i)$, $i = 1, 2$, we have that P_i is an orthogonal projection in the Hilbert space $L^2(\mu)$. If we denote by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ the norm and the scalar product in $L^2(\mu)$ respectively, then (4.2) follows from the general inequality for orthogonal projection in Hilbert spaces

$$\|f\|^2 - \|P_1 f\|^2 - \|P_2 f\|^2 + 2\langle P_1 f, P_2 f\rangle \geq 0$$

that is proved as follows:

$$\begin{aligned}
\|P_1 f\|^2 + \|P_2 f\|^2 - 2\langle P_1 f, P_2 f\rangle &= \|(P_1 - P_2)f\|^2 \\
&= \|[P_1(\mathbf{1} - P_2) - (\mathbf{1} - P_1)P_2]f\|^2 \\
&= \|P_1(\mathbf{1} - P_2)f\|^2 + \|(\mathbf{1} - P_1)P_2 f\|^2 \\
&\leq \|(\mathbf{1} - P_2)f\|^2 + \|P_2 f\|^2 = \|f\|^2.
\end{aligned}$$

■

Next elementary Lemma, provides an estimate for the last summand of the right hand side of (4.1).

Lemma 4.2 *With the notations of Lemma 4.1, suppose there exists a further probability measure $\bar{\mu}$ on $(\Omega, \mathcal{F})$ such that $\mathcal{F}_1$ and $\mathcal{F}_2$ are independent under $\bar{\mu}$, $\mu \ll \bar{\mu}$, and $\mu|_{\mathcal{F}_i} = \bar{\mu}|_{\mathcal{F}_i}$, $i = 1, 2$, where $\mu|_{\mathcal{F}_i}$ denotes the restriction of μ to $\mathcal{F}_i$. Then, for every $f \in L^2(\mu)$*

$$\mu\Big(\mu(f|\mathcal{F}_1), \mu(f|\mathcal{F}_2)\Big) \leq \|h - 1\|_\infty \mu(f,f) \tag{4.3}$$

where $h = \frac{d\mu}{d\bar{\mu}}$.

Proof. We may assume $\mu(f)=0$. We obtain:

$$\begin{aligned}
\mu\Big(\mu(f|\mathcal{F}_1),\mu(f|\mathcal{F}_2)\Big) &= \bar\mu\Big((h-1)\mu(f|\mathcal{F}_1)\mu(f|\mathcal{F}_2)\Big)+\bar\mu\Big(\mu(f|\mathcal{F}_1)\mu(f|\mathcal{F}_2)\Big)\\
&\le \|h-1\|_\infty \mu(|\mu(f|\mathcal{F}_1)|)\mu(|\mu(f|\mathcal{F}_2)|)\\
&\le \|h-1\|_\infty \mu(f^2)
\end{aligned}$$

where we have used the facts that $\bar\mu\Big(\mu(f|\mathcal{F}_1)\mu(f|\mathcal{F}_2)\Big)=0$ and $\mu(|\mu(f|\mathcal{F}_i)|)\le \sqrt{\mu(\mu(f|\mathcal{F}_i)^2)}\le\sqrt{\mu(f^2)}$. ∎

Now we want to use inequality (4.1) to prove the existence of a spectral gap independent of the volume Λ and of the boundary condition τ. This can be done by a recursive analysis similar to the one in [8]. Let

$$g(L):=\inf_{R\in\mathcal{R}_L}\inf_{\tau\in\Omega}\mathrm{gap}(\mathcal{L}^\tau_{\mathrm{R}}).$$

By Proposition 3.3, $g(L)>0$ for every $L>0$, for both geometric and Poisson reference measure. Theorem 2.1 is an immediate consequence of the following result.

Proposition 4.1 *Assume the Mixing Condition 2.1. Then there exists a positive constant $k=k(\beta,\|\Phi\|)$ such that*

$$g(2L)\ge\left(1-\frac{k}{\sqrt{L}}\right)g(L) \tag{4.4}$$

for L large enough. In particular this imply $\inf_L g(L)>0$

Proof. Let us consider a rectangle $R:=R(x;l_1,\dots,l_d)\in\mathcal{R}_{2L}$. Inequality (4.4) follows if we show that

$$\mathrm{gap}(\mathcal{L}^\tau_R)\ge\left(1-\frac{k}{\sqrt{L}}\right)g(L)$$

for a constant k independent of R,L and τ. Without loss of generality we can take $x=0$, $\underline{l}:=l_1$, $\bar{l}:=l_d$ and $L<l_d\le 2L$. We set $a=[\sqrt{L}]$ and for every integer $1\le n\le\frac{L}{10a}$ we cover R with the pair of rectangles

$$A_n=\left\{x\in R:\frac{l_d}{2}+(n-1)a<x_d\le l_d\right\},$$

$$B_n = \left\{ x \in R : 1 \le x_d \le \frac{l_d}{2} + na \right\}.$$

Note that, for n fixed, A_n and B_n are two overlapping rectangles, and the shortest side of the overlap has length proportional to $\sqrt{L}$. In what follows we apply Lemmas 4.1 and 4.2 with

$$\mu = \mu_R^\tau, \quad \mathcal{F}_1 = \sigma\{\eta_i : i \in R \setminus A_n\}, \quad \mathcal{F}_2 = \sigma\{\eta_i : i \in R \setminus B_n\}$$

and

$$\bar{\mu}(\eta) = \mu_{A_n \cap B_n}^{\eta\tau}(\eta_{A_n \cap B_n}) \mu_{R, R\setminus A_n}^\tau (\eta_{R \setminus A_n}) \mu_{R, R\setminus B_n}^\tau (\eta_{R\setminus B_n}).$$

Note that the assumptions of Lemma 4.2 are satisfied. Moreover, by Condition 2.1, for $h = \frac{d\mu}{d\bar{\mu}}$, we have

$$\|h - 1\|_\infty \le e^{-C\sqrt{L}} \tag{4.5}$$

for a suitable constant C and L large enough.

Now let us take a function f with $\mu_R^\tau(f) = 0$. By Lemma 4.1 we have

$$\mu_R^\tau(f,f) \le \mu_R^\tau\Big(\mu_{A_n}^\cdot(f,f) + \mu_{B_n}^\cdot(f,f)\Big) + 2\mu_R^\tau\Big(\mu_{A_n}^\cdot(f)\mu_{B_n}^\cdot(f)\Big), \tag{4.6}$$

while by Lemma 4.2 and (4.5),

$$\mu_R^\tau(\mu_{A_n}^\cdot(f)\mu_{B_n}^\cdot(f)) \le \mathrm{e}^{-C\sqrt{L}} \mu_R^\tau(f,f).$$

Moreover, by definition

$$\mu_{A_n}^\sigma(f,f) \le \big(\operatorname{gap}(\mathcal{L}_{A_n}^\sigma)\big)^{-1} \mathcal{E}_{A_n}^\sigma(f,f)$$

and the same for the set B_n. So from (4.6) we obtain

$$\begin{aligned}\mu_R^\tau(f,f) \le &\left[\inf_{\sigma\in\Omega}\big(\operatorname{gap}(\mathcal{L}_{A_n}^\sigma) \wedge \operatorname{gap}(\mathcal{L}_{B_n}^\sigma)\big)\right]^{-1} \mu_R^\tau\big[\mathcal{E}_{A_n}^\tau(f,f) + \mathcal{E}_{B_n}^\tau(f,f)\big] + \\ &+ 2\mathrm{e}^{-C\sqrt{L}} \mu_R^\tau(f,f).\end{aligned} \tag{4.7}$$

and so

$$\mu_R^\tau(f,f) \le \gamma(L, A_n, B_n)\left\{ \mathcal{E}_R^\tau(f,f) + \mu_R^\tau\Big(\sum_{x\in A_n\cap B_n} c_\Lambda^\tau(x,\cdot,+)|\nabla_x^+ f|^2\Big)\right\}; \tag{4.8}$$

where

$$\gamma(L,A_n,B_n)=\frac{1}{1-2\mathrm{e}^{-C\sqrt{L}}}\left(\inf_{\sigma\in\Omega}\mathrm{gap}(\mathcal{L}^{\sigma}_{A_n})\wedge\mathrm{gap}(\mathcal{L}^{\sigma}_{B_n})\right)^{-1}. \tag{4.9}$$

Now, let us observe that, as n varies, the sets $A_n\cap B_n$ are disjoints, so that if we average over n the equation (4.8), we obtain:

$$\mu^{\tau}_R(f,f)\le\left(1+\left[\frac{L}{10a}\right]^{-1}\right)\sup_n\gamma(L,A_n,B_n)\mathcal{E}^{\tau}_R(f,f).$$

It follows that

$$\mathrm{gap}(\mathcal{L}^{\tau}_R)\ge\left(1-\frac{C}{\sqrt{L}}\right)\min_n\left(\mathrm{gap}(\mathcal{L}^{\tau}_{A_n})\wedge\mathrm{gap}(\mathcal{L}^{\tau}_{B_n})\right). \tag{4.10}$$

In order to derive (4.4) we have to analyze the spectral gap of the generator in the rectangles A_n and B_n. Let us consider first the rectangle B_n (the analysis is similar for A_n) and let us distinguish two different cases.

a) If $\max\{l_k : k=1,\dots,d-1\}\le\frac{3}{2}L$ then the longest side of B_n is less or equal to $\frac{3}{2}L$, because also the side corresponding to the coordinate x_d, by the definition of B_n is less or equal to $(1.2)L$. In this case $B_n\in\mathcal{R}_{\frac{3}{2}L}$ and so $\mathrm{gap}(\mathcal{L}^{\tau}_{B_n})\ge g(\frac{3}{2}L)$.

b) If there exists a index i such that $l_i>\frac{3}{2}L$, then $B_n\in\mathcal{R}_{2L}$, with the property that $\bar{l}:=l_i>\frac{3}{2}L$ and $\underline{l}\le l_d\le(1.2)L$. In this case

$$\mathrm{gap}(\mathcal{L}^{\tau}_{B_n})\ge\min_{\substack{R\in\mathcal{R}_{2L}\\ \underline{l}\le(1.2)L,\bar{l}\ge3/2L}}\mathrm{gap}(\mathcal{L}^{\tau}_R).$$

In conclusion we obtain that the right hand side of (4.10) is larger than

$$\left(1-\frac{C}{\sqrt{L}}\right)\left(g\left(\frac{3}{2}L\right)\wedge\min_{\substack{R\in\mathcal{R}_{2L}\\ \underline{l}\le1.2L,\bar{l}\ge3/2L}}\mathrm{gap}(\mathcal{L}^{\tau}_R)\right) \tag{4.11}$$

Now we consider a rectangle in the set $\mathcal{R}_{2L}$ such that the shortest side is less or equal to $(1.2)L$ and the longest one is greater or equal to $3/2L$. Iterating the argument leading to (4.11) at most $d-1$ times on such rectangle, we end up with a rectangle whose longest side has length less or equal to $\frac{3}{2}L$. Thus

$$\min_{\substack{R\in\mathcal{R}_{2L}\\ l\le(1.2)L,\bar{l}\ge 3/2L}} \mathrm{gap}(\mathcal{L}_R^\tau) \ge \left(1-\frac{C}{\sqrt{L}}\right)^{d-1} g\left(\frac{3}{2}L\right). \tag{4.12}$$

By combining (4.11) and (4.12) we get

$$g(2L) \ge \left(1-\frac{C}{\sqrt{L}}\right)^{d} g\left(\frac{3}{2}L\right). \tag{4.13}$$

If we iterate two more times (4.13) we obtain

$$g(2L) \ge \left(1-\frac{C}{\sqrt{L}}\right)^{d}\left(1-\frac{C}{\sqrt{\frac{3}{4}L}}\right)^{d}\left(1-\frac{C}{\sqrt{\frac{9}{16}L}}\right)^{d} g(L), \tag{4.14}$$

from which the conclusion follows. ■

5 Entropy Decay

This Section is devoted to the proof of Theorem 2.2. The proof follows closely the proof of Theorem 2.1. In what follows, given a probability space $(\Omega,\mathcal{F},\mu)$ and a σ-field $\mathcal{G}\subset\mathcal{F}$, we define, for $f>0$, the *conditional entropy* $\mathrm{Ent}_\mu(f|\mathcal{G})$ as

$$\mathrm{Ent}_\mu(f|\mathcal{G}) = \mu(f\log f|\mathcal{G}) - \mu(f|\mathcal{G})\log\mu(f|\mathcal{G}).$$

Similarly to Section 4, we state two lemmas concerning conditional entropy.

Lemma 5.1 *Let $(\Omega,\mathcal{F},\mu)$ be a probability space, $\mathcal{F}_1,\mathcal{F}_2$ be sub-σ-fields of $\mathcal{F}$. Then the following inequality holds for every $f\ge 0$ such that $f\log f\in L^1(\mu)$ and $\mu(f)=1$:*

$$\mathrm{Ent}_\mu(f) \le \mu\left[\mathrm{Ent}_\mu(f|\mathcal{F}_1)+\mathrm{Ent}_\mu(f|\mathcal{F}_2)\right] + \log\mu\left[\mu(f|\mathcal{F}_1)\mu(f|\mathcal{F}_2)\right]. \tag{5.1}$$

Proof. The right end side of (5.1) is equal to

$$2\mu(f\log f)-\mu\left[f\log\mu(f|\mathcal{F}_1)\right]-\mu\left[f\log\mu(f|\mathcal{F}_2)\right]+\log\mu\left[\mu(f|\mathcal{F}_1)\mu(f|\mathcal{F}_2)\right].$$

So (5.1) is equivalent to

$$\mu(f\log f)-\mu\left[f\log\mu(f|\mathcal{F}_1)\mu(f|\mathcal{F}_2)\right]+\log\mu\left[\mu(f|\mathcal{F}_1)\mu(f|\mathcal{F}_2)\right]\geq 0. \quad (5.2)$$

This is trivially true because the left hand side of (5.2) equals the the relative entropy of $fd\mu$ with respect to $\frac{\mu(f|\mathcal{F}_1)\mu(f|\mathcal{F}_2)}{\mu[\mu(f|\mathcal{F}_1)\mu(f|\mathcal{F}_2)]}d\mu$. ■

Lemma 5.2 *With the same notations of Lemma 5.1, suppose there exists a further probability measure $\bar{\mu}$ on $(\Omega,\mathcal{F})$ such that $\mathcal{F}_1$ and $\mathcal{F}_2$ are independent under $\bar{\mu}$, $\mu\ll\bar{\mu}$, and $\mu|_{\mathcal{F}_i}=\bar{\mu}|_{\mathcal{F}_i}$, $i=1,2$. Then, for every $f\geq 0$ such that $f\log f\in L^1(\mu)$ and $\mu(f)=1$*

$$\left|\log\mu\left[\mu(f|\mathcal{F}_1)\mu(f|\mathcal{F}_2)\right]\right|\leq 4\|h-1\|_\infty \operatorname{Ent}_\mu(f), \quad (5.3)$$

where $h=\frac{d\mu}{d\bar{\mu}}$.

Proof.
Notice that

$$\begin{aligned}\log\mu\left[\mu(f|\mathcal{F}_1)\mu(f|\mathcal{F}_2)\right]&=\log\bar{\mu}\left[h\mu(f|\mathcal{F}_1)\mu(f|\mathcal{F}_2)\right]=\\ &=\log\left\{\left[(h-1)\mu(f|\mathcal{F}_1)\mu(f|\mathcal{F}_2)\right]+\bar{\mu}\left[\mu(f|\mathcal{F}_1)\mu(f|\mathcal{F}_2)\right]\right\}=\\ &=\log\left\{\bar{\mu}\left[(h-1)\mu(f|\mathcal{F}_1)\mu(f|\mathcal{F}_2)\right]+1\right\}\leq\bar{\mu}\left[(h-1)\mu(f|\mathcal{F}_1)\mu(f|\mathcal{F}_2)\right], \quad (5.4)\end{aligned}$$

where we use the elementary inequality $\log(x+1)\leq x$. This implies that

$$\begin{aligned}&\left|\log\mu\left[\mu(f|\mathcal{F}_1)\mu(f|\mathcal{F}_2)\right]\right|\leq\left|\bar{\mu}\left[(h-1)\mu(f|\mathcal{F}_1)\mu(f|\mathcal{F}_2)\right]\right|=\\ &=\left|\bar{\mu}\left\{(h-1)\left[\mu(f|\mathcal{F}_1)-\mu\left(\sqrt{\mu(f|\mathcal{F}_1)}\right)^2\right]\left[\mu(f|\mathcal{F}_2)-\mu\left(\sqrt{\mu(f|\mathcal{F}_2)}\right)^2\right]\right\}\right|\leq\\ &\leq\|h-1\|_\infty\,\bar{\mu}\left[\left|\mu(f|\mathcal{F}_1)-\mu\left(\sqrt{\mu(f|\mathcal{F}_1)}\right)^2\right|\left|\mu(f|\mathcal{F}_2)-\mu\left(\sqrt{\mu(f|\mathcal{F}_2)}\right)^2\right|\right]=\\ &=\|h-1\|_\infty\,\mu\left(\left|\mu(f|\mathcal{F}_1)-\mu\left(\sqrt{\mu(f|\mathcal{F}_1)}\right)^2\right|\right)\mu\left(\left|\mu(f|\mathcal{F}_2)-\mu\left(\sqrt{\mu(f|\mathcal{F}_2)}\right)^2\right|\right).\end{aligned} \quad (5.5)$$

Now notice that

$$\mu\left(\left|\mu(f|\mathcal{F}_k)-\mu\left(\sqrt{\mu(f|\mathcal{F}_k)}\right)^2\right|\right)\le 2\sqrt{\mu\left(\sqrt{f},\sqrt{f}\right)}\qquad k=1,2. \tag{5.6}$$

In fact

$$\begin{aligned}
&\mu\left(\left|\mu(f|\mathcal{F}_k)-\mu\left(\sqrt{\mu(f|\mathcal{F}_k)}\right)^2\right|\right)=\\
&=\mu\left[\left|\sqrt{\mu(f|\mathcal{F}_k)}-\mu\left(\sqrt{\mu(f|\mathcal{F}_k)}\right)\right|\left|\sqrt{\mu(f|\mathcal{F}_k)}+\mu\left(\sqrt{\mu(f|\mathcal{F}_k)}\right)\right|\right]\le\\
&\le\sqrt{\mu\left(\sqrt{\mu(f|\mathcal{F}_k)},\sqrt{\mu(f|\mathcal{F}_k)}\right)}\sqrt{2\left[1+\mu\left(\sqrt{\mu(f|\mathcal{F}_k)}\right)^2\right]}\le\\
&\le 2\sqrt{\mu\left(\sqrt{\mu(f|\mathcal{F}_k)},\sqrt{\mu(f|\mathcal{F}_k)}\right)}\le 2\sqrt{\mu\left(\sqrt{f},\sqrt{f}\right)}.
\end{aligned}$$

By (5.5) and (5.6) we get

$$|\log\mu\left[\mu(f|\mathcal{F}_1)\mu(f|\mathcal{F}_2)\right]|\le 4\,\|h-1\|_\infty\,\mu\left(\sqrt{f},\sqrt{f}\right),$$

and (5.3) follows because

$$\mu\left(\sqrt{f},\sqrt{f}\right)\le\mu\left(f\log f\right) \tag{5.7}$$

for every $f>0$ such that $\mu(f)=1$, see [3].

■

We can now prove Theorem 2.2. Because the proof follows closely the proof of Theorem 2.1 we give only a sketch. Define

$$s(L):=\inf_{R\in\mathcal{R}_L^d}\inf_{\tau\in\Omega}s(\mathcal{L}_R^\tau).$$

By Proposition 3.3, $s(L)>0$ for every L, for the Poisson reference measure. Similarly to Theorem 2.1, Theorem 2.2 follows readily from the following Proposition.

Proposition 5.1 *Assume the Mixing Condition 2.1. Then there exists a positive constant* $k=k(\beta,\|\Phi\|)$ *such that*

$$s(2L)\ge\left(1-\frac{k}{\sqrt{L}}\right)s(L) \tag{5.8}$$

for L large enough. In particular this imply $\inf_L s(L)>0$.

Proof.

With the same notations of Proposition 4.1 let f be a non-negative function such that $\mu^\tau_R(f) = 1$ and $f\log f \in L^1(\mu^\tau_R$. By (5.1): (5.1)

$$\begin{aligned}\mu^\tau_R(f\log f) \le \mu^\tau_R\left[\mu^\cdot_{A_n}\left(f\log\frac{f}{\mu^\cdot_{A_n}(f)}\right) + \mu^\cdot_{B_n}\left(f\log\frac{f}{\mu^\cdot_{B_n}(f)}\right)\right] + \\ + \log\mu^\tau_R\left[\mu^\cdot_{A_n}(f)\mu^\cdot_{B_n}(f)\right],\end{aligned} \tag{5.9}$$

while by Lemma 5.2 and Condition 2.1

$$\left|\log\mu^\tau_R(\mu^\cdot_{A_n}(f)\mu^\cdot_{B_n}(f))\right| \le 2\mathrm{e}^{-C\sqrt{L}}\mu^\tau_R(f\log f).$$

Moreover, by definition,

$$\mu^\sigma_{A_n}\left(f\log\frac{f}{\mu^\cdot_{A_n}(f)}\right) \le s(\mathcal{L}^\sigma_{A_n})^{-1}\mathcal{E}^\sigma_{A_n}(f,\log f)$$

and the same for the set B_n. So from (5.9) we obtain

$$\begin{aligned}\mu^\tau_R(f\log f) \le \left[\inf_{\sigma\in\Omega} s(\mathcal{L}^\sigma_{A_n}) \wedge s(\mathcal{L}^\sigma_{B_n})\right]^{-1} \mu^\tau_R\left[\mathcal{E}^\cdot_{A_n}(f,\log f) + \mathcal{E}^\cdot_{B_n}(f,\log f)\right] + \\ + 2\mathrm{e}^{-C\sqrt{L}}\mu^\tau_R\left(f\log f\right).\end{aligned}$$

Thus

$$\begin{aligned}&\mu^\tau_R(f\log f) \le \\ &\qquad \le \gamma(L,A_n,B_n)\left[\mathcal{E}^\tau_R(f,\log f) + \sum_{x\in A_n\cap B_n}\mu^\tau_R\left(c^\tau_R(x,\cdot,+)|\nabla^+_x f|^2\right)\right].\end{aligned}$$

where

$$\gamma(L,A_n,B_n) = \frac{1}{1-2\mathrm{e}^{-C\sqrt{L}}}\left[\inf_{\sigma\in\Omega} s(\mathcal{L}^\sigma_{A_n}) \wedge s(\mathcal{L}^\sigma_{B_n})\right]^{-1}.$$

After averaging over n we obtain

$$s(\mathcal{L}^\tau_R) \ge \left(1 - \frac{C}{\sqrt{L}}\right)\min_n\inf_{\sigma\in\Omega}\left[s(\mathcal{L}^\sigma_{A_n}) \wedge s(\mathcal{L}^\sigma_{B_n})\right].$$

The rest of the proof is identical to the proof of Proposition 4.1.

∎

Acknowledgments. We thank F.Cesi for teaching us inequality (5.7), that avoids the use of the spectral gap in the proof of the entropy inequality.

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