PDE WITH RANDOM COEFFICIENTS AND EUCLIDEAN FIELD THEORY

JOSEPH G. CONLON

ABSTRACT. In this paper an identity is proved relating the 2 point correlation function of a Euclidean field theory to the expectation of the Green's function for a pde with random coefficients. The Euclidean field theory is assumed to have convex potential. An inequality of Brascamp and Lieb therefore implies Gaussian bounds on the Fourier transform of the 2 point correlation function. By an application of results from random pde, the previously mentioned identity implies pointwise Gaussian bounds on the 2 point correlation function.

1. INTRODUCTION.

The joint papers of Elliott Lieb with Brascamp in the 1970's are among his most influential. In this paper we shall be concerned with an inequality in one of the Brascamp-Lieb papers [1]. This inequality implies a Gaussian bound on the 2 point correlation function for a Euclidean field theory with convex potential. In 1994 Helffer and Sjöstand [9] gave a remarkable new proof of the Brascamp-Lieb inequality. The key point in the proof is a new representation for the correlation function. Later Naddaf and Spencer [12] realised that the Helffer-Sjöstand representation allowed one to make a connection between the 2 point correlation function for a Euclidean field theory and pde with random coefficients. Using this connection and the theory of homogenization for pde with random coefficients [10, 14] they were able to prove that the scaling limit of the Euclidean field theory is Gaussian.

In this paper we make precise the connection between the 2 point correlation function for the Euclidean field theory and pde with random coefficients. This is stated in Theorem 1.2. It follows then from results of the author [4] on Green's functions for pde with random coefficients (see also [5]), that derivatives of the 2 point correlation function up to second order have Gaussian bounds. The Gaussian bounds on second derivatives were conjectured by Spencer [15]. One should also note that the Gaussian scaling limit for the field theory follows directly from Theorem 1.2 and the homogenization result of Landim et al [11]. The statement of Theorem 1.2 was formulated in discussions between the author and Ali Naddaf in 1999.

The Euclidean field theories we shall be interested in are determined by a potential $V : \mathbf{R}^d \to \mathbf{R}$ which is a C^2 uniformly convex function. Thus there are positive constants λ, Λ such that

$$\lambda I_d \le V''(z) \le \Lambda I_d, \qquad z \in \mathbf{R}^d, \tag{1.1}$$

1991 *Mathematics Subject Classification.* 81T08, 82B20, 35R60, 60J75.

Key words and phrases. Euclidean field theory, pde with random coefficients, homogenization.

where I_d is the identity $d \times d$ matrix and the inequality (1.1) is in the sense of quadratic forms. We assume also that V is an even function, whence $V(z) = V(-z), \quad z \in \mathbf{R}^d$. Next consider functions $\omega : \mathbf{Z}^d \to \mathbf{R}$ on the integer lattice in $\mathbf{R}^d$. Let Ω be the space of all such functions and $\mathcal{F}$ be the Borel algebra generated by finite dimensional rectangles $\{\omega \in \Omega : \ |\omega(x_i) - a_i| < r_i, \ i = 1, ..., N\}$, $x_i \in \mathbf{Z}^d, \ a_i \in \mathbf{R}, \ r_i > 0, \ i = 1, ..., N, \ N \geq 1$. If $d \geq 3$ then one can define [7] a unique translation invariant probability measure P on $(\Omega, \mathcal{F})$ which depends on the function V. The measure is formally given as

$$\exp\left[-\sum_{x \in \mathbf{Z}^d} V\Big(\nabla\omega(x)\Big)\right] \prod_{x \in \mathbf{Z}^d} d\omega(x)/\text{normalization}. \tag{1.2}$$

Here $\nabla\omega$ is the gradient of the function ω. Thus $\nabla\omega(x) = \big(\nabla_1\omega(x), ... \ \nabla_d\omega(x)\big)$ where

$$\nabla_i\omega(x) = \omega(x + \mathbf{e}_i) - \omega(x), \quad 1 \leq i \leq d, \tag{1.3}$$

and $\mathbf{e}_i \in \mathbf{Z}^d$ is the vector with 1 as the ith coordinate and 0 for the other coordinates. Let $< \cdot >_\Omega$ denote expectation w.r. to the measure (1.2). The Brascamp-Lieb inequality implies a Gaussian bound on the Fourier transform of the 2 point correlation function $\langle\omega(x)\omega(0)\rangle_\Omega$, $x \in \mathbf{Z}^d$,

$$\left|\sum_{x \in \mathbf{Z}^d} e^{ix\cdot\xi}\Big\langle\omega(x)\omega(0)\Big\rangle_\Omega\right| \leq 1/2 \ \lambda \sum_{i=1}^d [1 - \cos(\xi \cdot \mathbf{e}_i)] , \quad \xi \in [-\pi, \pi]^d.$$

From the representation for $\langle\omega(x)\omega(0)\rangle_\Omega$ in Theorem 1.2 and the bounds on Green's functions in Theorem 1.2 of [4], we can obtain pointwise bounds on $\langle\omega(x)\omega(0)\rangle_\Omega$.

Theorem 1.1. *For $d \geq 3$ there is a constant $C_d > 0$ depending only on d such that*

(a) $| \langle\omega(x)\omega(0)\rangle_\Omega | \leq \frac{C_d}{\Lambda} \ \left(\frac{\Lambda}{\lambda}\right)^{3d+6} \ \frac{1}{1+|x|^{d-2}}$,

(b) $| \langle\nabla\omega(x)\omega(0)\rangle_\Omega | \leq \frac{C_d}{\Lambda} \ \left(\frac{\Lambda}{\lambda}\right)^{3d+6} \ \frac{1}{1+|x|^{d-1}}$.

Let δ satisfy $0 \leq \delta < 1$. Then there is a constant $C_{d,\delta}$ depending only on d and δ such that

(c) $| \langle\nabla\omega(x)\nabla\omega(0)\rangle_\Omega | \leq \frac{C_{d,\delta}}{\Lambda} \ \left(\frac{\Lambda}{\lambda}\right)^{3d+6} \ \frac{1}{1+|x|^{d-2+2\delta}}$.

Theorem 1.1 (a) was proved by Naddaf and Spencer [12] by using a discrete version of the Aronson inequality [8, 16]. Theorem 1.1(c) was conjectured by Spencer [15] based on corresponding estimates for sine-Gordon field theories obtained by Brydges and Keller [2]. One should note that in [2] there is a requirement analogous to $\delta < 1$. Thus for second derivatives of 2 point correlation functions, we have almost but not exact Gaussian bounds.

The bounds obtained in [2] are on expectations for trigonometric polynomials of the gradient of the field. It seems possible that one could extend the methodology of the present paper to obtain these bounds. This would however be a complicated task. A main point of the work here is to show that one can obtain by conventional pde methods, estimates on correlation functions which are as sharp as those obtained using multi-scale perturbation theory. Our method has the added advantage that there is no small parameter as in the multi-scale perturbation theory. Nevertheless, the method is still based on perturbation theory. We should also note that multi-scale perturbation theory has yielded results which do not appear to be provable by the methods in this paper. In particular, the analyticity of the pressure for the sine-Gordon field theory [3] is among these.

In order to state Theorem 1.2 we introduce a stochastic differential equation with the probability measure (1.2) as its invariant measure. Thus consider the infinite dimensional stochastic equation,

$$d\omega(x,t) = -\frac{\partial}{\partial\omega(x)}\sum_{x'\in\mathbf{Z}^d} V\Big(\nabla\omega(x',t)\Big)dt + \sqrt{2}dB(x,t),\ x\in\mathbf{Z}^d,\ t>0, \tag{1.4}$$

where $B(x,t),\ x\in\mathbf{Z}^d,\ t>0$, are independent copies of Brownian motion. Let $\hat\Omega$ be the space of functions $\omega:\mathbf{Z}^d\times\mathbf{R}\to\mathbf{R}$ which are continuous. Thus for each $x\in\mathbf{Z}^d$ the function $\omega(x,t),\ t\in\mathbf{R}$, is a continuous function of t. Let $\mathcal{F}$ be the Borel algebra generated by all finite dimensional rectangles $\{\omega\in\hat\Omega : |\omega(x_i,t_i)-a_i| < r_i,\ i=1,...,N\},\ x_i\in\mathbf{Z}^d,\ t_i\in\mathbf{R},\ a_i\in\mathbf{R},\ r_i>0,\ i=1,...,N,\ N\ge 1$. If $d\ge 3$ one can define a unique probability measure $\hat P$ on $(\hat\Omega,\hat{\mathcal{F}})$ corresponding to the stationary process associated with the stochastic equation (1.4). Thus for any fixed $t\in\mathbf{R}$ the variables $\omega(x,t),\ x\in\mathbf{Z}^d$, have distribution given by the probability measure $(\Omega,\mathcal{F},P)$ corresponding to (1.2). We may define translation operators $\tau_{x,t},\ x\in\mathbf{Z}^d,\ t\in\mathbf{R}$, on $\hat\Omega$ by $\tau_{x,t}\omega(z,s)=\omega(x+z,t+s),\ z\in\mathbf{Z}^d,\ s\in\mathbf{R}$. It is clear that $\tau_{x,t}\ \tau_{x',t'}=\tau_{x+x',t+t'}$ and $\tau_{0,0}=$ identity. One can also see that the mapping $\tau_{x,t}:\hat\Omega\to\hat\Omega$ is Borel measurable and measure preserving. Finally, it follows from the fact that the space $\hat\Omega$ consists of continuous functions, that the mapping $(t,\omega)\to\tau_{0,t}\omega,\ t\in\mathbf{R},\ \omega\in\hat\Omega$, from $\mathbf{R}\times\hat\Omega$ to $\hat\Omega$ is measurable. Hence the probability space $(\hat\Omega,\hat{\mathcal{F}},\hat P)$ together with the translation operators $\tau_{x,t},\ x\in\mathbf{Z}^d,\ t\in\mathbf{R}$, satisfy the conditions of Theorem 1.2 of [4]. We denote by $<\cdot>_{\hat\Omega}$ expectation w.r. to the space $(\hat\Omega,\hat{\mathcal{F}},\hat P)$.

Let $\mathbf{a}:\hat\Omega\to\mathbf{R}^{d(d+1)/2}$ be a bounded Borel measurable function from Ω to the space of symmetric $d\times d$ matrices. We assume that $\mathbf{a}(\omega)$ satisfies an inequality analogous to (1.1). Thus

$$\lambda I_d\le\mathbf{a}(\omega)\le\Lambda I_d,\quad \omega\in\hat\Omega, \tag{1.5}$$

where the inequality is in the sense of quadratic forms. Since the mapping $(t,\omega)\to\tau_{0,t}\ \omega$ is measurable, it follows that for almost every $\omega\in\hat\Omega$ we can consider solutions to the initial value problem,

$$\begin{aligned}\frac{\partial u}{\partial t}(x,t,\omega) &= -\nabla^*\mathbf{a}(\tau_{x,t}\omega)\nabla u(x,t,\omega),\quad x\in\mathbf{Z}^d,\ t>0,\ \omega\in\hat\Omega,\\ u(x,0,\omega) &= f(x,\omega),\quad x\in\mathbf{Z}^d,\ \omega\in\hat\Omega,\end{aligned} \tag{1.6}$$

where $f:\mathbf{Z}^d\times\hat\Omega\to\mathbf{R}$ is a given measurable function. The operator ∇^* in (1.6) is the adjoint of the gradient operator (1.3). The solution of (1.6) may be written in the form,

$$u(x,t,\omega)=\sum_{y\in\mathbf{Z}^d} G_{\mathbf{a}}(x,y,t,\omega)f(y,\omega), \tag{1.7}$$

where $G_{\mathbf{a}}(x,y,t,\omega)$ is the Green's function. By translation invariance there is a function $G_{\mathbf{a}}(x,t),\ x\in\mathbf{Z}^d,\ t\ge 0$, such that

$$\langle G_{\mathbf{a}}(x,y,t,\cdot)\rangle_{\hat\Omega}=G_{\mathbf{a}}(x-y,t). \tag{1.8}$$

We are now able to state our main result.

Theorem 1.2. *Let $d \ge 3$ and $\mathbf{a} : \hat{\Omega} \to \mathbf{R}^{d(d+1)/2}$ be defined by $\mathbf{a}(\omega) = V''(\nabla\omega(0,0))$, $\omega \in \hat{\Omega}$, where V satisfies (1.1). Then $\mathbf{a}(\cdot)$ satisfies (1.5) and the function $G_{\mathbf{a}}(x,t)$ defined by (1.6), (1.7), (1.8) satisfies the identity,*

$$\langle \omega(x)\omega(0)\rangle_\Omega = \int_0^\infty G_{\mathbf{a}}(x,t)dt. \tag{1.9}$$

The proof of Theorem 1.2 is obtained by first establishing a finite dimensional version of (1.9). Then the thermodynamic limit is taken. The main technical issue in the paper is to prove the existence of this limit. In §2 we give a construction of the probability spaces $(\Omega,\mathcal{F},P)$ and $(\hat{\Omega},\hat{\mathcal{F}},\hat{P})$ by means of finite dimensional approximations. To construct the space $(\Omega,\mathcal{F},P)$ by finite dimensional approximation we let L be a positive even integer and $Q = Q_L \subset \mathbf{Z}^d$ be the lattice points contained in the cube centered at the origin with side of length L. Let Ω_Q be the space of all periodic functions $\omega : Q \to \mathbf{R}$. Evidently $\Omega_Q,\ Q = Q_L$ can be identified with $\mathbf{R}^N$ where $N = L^d$. Let $\mathcal{F}_Q$ be the Borel algebra for Ω_Q which is generated by the open sets of $\mathbf{R}^N$. We define a probability measure $P_{Q,m},\ m > 0$, on $(\Omega_Q,\mathcal{F}_Q)$ by

$$\exp\left[-\sum_{x\in Q}\left\{V(\nabla\omega(x)) + \frac{1}{2}\,m^2\omega(x)^2\right\}\right]\prod_{x\in Q} d\omega(x)\ /\ \text{normalization}. \tag{1.10}$$

In the definition (1.10) we are identifying points on the boundary of Q since Ω_Q consists of periodic functions on Q. It is evident that we can define translation operators $\tau_x,\ x \in \mathbf{Z}^d$, on Ω_Q and that the τ_x are measure preserving on the space $(\Omega_Q,\mathcal{F}_Q,P_{Q,m})$. Let $< \cdot >_{Q,m}$ denote expectation for the space $(\Omega_Q,\mathcal{F}_Q,P_{Q,m})$. Suppose for some $N \ge 1$, $f : \mathbf{R}^N \to \mathbf{C}$ is a C^2 function satisfying the inequality,

$$|f''(z)| \le A\exp[B|z|],\quad z \in \mathbf{R}^N, \tag{1.11}$$

where A and B are constants. We define then

$$\left\langle f\Big(\omega(x_1),\omega(x_2),...,\omega(x_N)\Big)\right\rangle_\Omega = \lim_{m\to 0}\lim_{L\to\infty}\left\langle f\Big(\omega(x_1),\omega(x_2),...,\omega(x_N)\Big)\right\rangle_{Q_L,m}. \tag{1.12}$$

In §2 we show that the limit on the RHS of (1.12) exists provided $d \ge 3$. The use of the Brascamp-Lieb inequality [1] is crucial to the proof.

An alternative construction of the space $(\Omega,\mathcal{F},P)$ is given in [7] (see also [13]). In that construction the elements of the probability space are gradients $\nabla\omega$ of the field ω. Hence the construction is valid in all dimensions $d \ge 1$ whereas our construction by means of (1.12) is restricted to $d \ge 3$. One might therefore expect that for $f : \mathbf{R}^{Nd} \to \mathbf{C}$ satisfying (1.11) the limit,

$$\lim_{m\to 0}\lim_{L\to\infty}\left\langle f\Big(\nabla\omega(x_1),\nabla\omega(x_2),...,\nabla\omega(x_N)\Big)\right\rangle_{Q_L,m} \tag{1.13}$$

exists for all $d \ge 1$. Evidently by virtue of (1.12) the limit (1.13) exists for $d \ge 3$. Our proof however breaks down if $d \le 2$.

2. Construction of the probability spaces $(\Omega,\mathcal{F},P)$ and $(\hat{\Omega},\hat{\mathcal{F}},\hat{P})$.

We turn to the construction of the space $(\Omega,\mathcal{F},P)$ as a limit of the spaces $(\Omega_Q,\mathcal{F}_Q,P_{Q,m})$ with probability measure (1.10). Our main task will be to establish the existence of the limit on the R.H.S. of (1.12). Let $L^2(Q)$ be the space of periodic functions $h : Q \to \mathbf{C}$ with the standard Euclidean inner product. For a given

$\omega \in \Omega_Q$ we consider the operator $[\nabla^* V''(\nabla\omega)\nabla + m^2]$ acting on $L^2(Q)$. In view of (1.1) the operator $[\nabla^* V''(\nabla\omega)\nabla + m^2]$ is bounded below by the operator $[-\lambda\Delta + m^2]$ on $L^2(Q)$, where Δ is the standard lattice Laplacian. The eigenfunctions of $-\Delta$ are $\exp[i\xi \cdot x]$, $x \in Q$, where ξ lies in the dual $\hat{Q}$ of Q. In fact if $Q = Q_L$ then $\hat{Q}$ is the set of lattice points of $(2\pi/L)\mathbf{Z}^d$ which lie in the cube $[-\pi,\pi]^d$. Just as with Q, the boundary points of $[-\pi,\pi]^d$ are identified , whence there are exactly L^d distinct points in $\hat{Q}$. Since the eigenvalues of $-\Delta$ are given by

$$-\Delta[e^{i\xi\cdot x}] = 2\sum_{i=1}^{d}[1-\cos(\xi\cdot\mathbf{e}_i)]e^{i\xi\cdot x},$$

the operator $[-\lambda\Delta + m^2]$ is invertible on $L^2(Q)$ and hence $[\nabla^* V''(\nabla\omega)\nabla + m^2]$ is also invertible. Let $J : \Omega_Q \to \mathbf{C}$ be a C^1 function with gradient $\partial J/\partial\omega$, where we are identifying Ω_Q with $\mathbf{R}^N$. Now for any $\omega \in \Omega_Q$ we can think of $\partial J/\partial\omega$ evaluated at ω as a bounded periodic function on Q, whence $\partial J/\partial\omega \in L^2(Q)$. Let $(\cdot,\cdot)$ denote the inner product on $L^2(Q)$ and $|\cdot|$ the corresponding Euclidean norm. Suppose there are constants $A, B > 0$ such that

$$\left|\frac{\partial J(\omega)}{\partial\omega}\right| \le A\exp[B|\omega|], \quad \omega \in \Omega_Q. \tag{2.1}$$

Let $< \cdot >_{Q,m}$ denote expectation on Ω_Q with respect to the measure $P_{Q,m}$. The Brascamp-Lieb inequality states that if J satisfies (2.1) then J and $\frac{\partial J}{\partial\omega}$ are square integrable. Further, there is the inequality,

$$\langle |J(\omega) - < J(\cdot) >_{Q,m}|^2\rangle_{Q,m} \le \left\langle \left(\frac{\partial J}{\partial\omega}, \left[\nabla^* V''(\nabla\omega)\nabla + m^2\right]^{-1}\frac{\partial J}{\partial\omega}\right)\right\rangle_{Q,m}. \tag{2.2}$$

One also has in view of (1.1) the inequality,

$$\langle\exp[(h,\omega)]\rangle_{Q,m} \le \exp\left[\frac{1}{2}\,(h,[-\lambda\Delta+m^2]^{-1}\,h)\right], \tag{2.3}$$

for any $h \in L^2(Q)$.

Lemma 2.1. *Suppose $Q = Q_L$ and m satisfies the inequality $1/L \le m/\sqrt{\lambda} \le 1/2$. Let $f : \mathbf{R} \to \mathbf{C}$ be a C^2 function satisfying the inequality,*

$$|f''(z)| \le A\exp[B|z|], \quad z \in \mathbf{R}, \tag{2.4}$$

for some constants $A, B > 0$. Then for $d \ge 3$ there are constants C_d and functions $g_d : (0,\ 1/2) \to \mathbf{R}$ such that

$$\left|\left\langle\left[\sum_{x\in Q}\omega(x)^2\right]\left[f(\omega(0)) - \langle f(\omega(0))\rangle_{Q,m}\right]\right\rangle_{Q,m}\right| \le \frac{A}{\lambda^2}\,g_d(m/\sqrt{\lambda})\exp\left[B^2\,C_d/\lambda\right]. \tag{2.5}$$

The function g_d can be taken to be $g_d(\rho) = c_d$ if $d \ge 5$, $g_4(\rho) = c_4\ell n[1/\rho]$, $g_3(\rho) = c_3/\rho$, where c_d is a constant depending only on d.

Proof. Observe that the LHS of (2.5) is zero if f is an odd function. This follows from our assumption that V is an even function. We may assume then wlog that f is an even function. We use translation invariance and the Schwarz inequality to estimate the LHS of (2.5). Thus we have that

$$\left| \left\langle \left[\sum_{x \in Q} \omega(x)^2 \right] \left[f(\omega(0)) - \langle f(\omega(0)) \rangle_{Q,m} \right] \right\rangle_{Q,m} \right|^2 \le \tag{2.6}$$

$$\frac{1}{L^{2d}} \left\langle \left\{ \sum_{x \in Q} \left[\omega(x)^2 - \left\langle \omega(x)^2 \right\rangle_{Q,m} \right] \right\}^2 \right\rangle_{Q,m} \left\langle \left| \sum_{x \in Q} \left[f(\omega(x)) - \langle f(\omega(x)) \rangle_{Q,m} \right] \right|^2 \right\rangle_{Q,m} .$$

Let $\hat{\omega}(\xi), \ \xi \in \hat{Q}$, be the Fourier transform of $\omega(x)$,

$$\hat{\omega}(\xi) = \sum_{x \in Q} \omega(x) e^{ix \cdot \xi}, \ \ \xi \in \hat{Q}.$$

From (2.2) it follows that

$$\left\langle \left\{ \sum_{x \in Q} \left[\omega(x)^2 - < \omega(x)^2 >_{Q,m} \right] \right\}^2 \right\rangle_{Q,m} \le$$

$$\frac{4}{(2\pi)^d \lambda} \int_{\hat{Q}} d\xi \left\langle |\hat{\omega}(\xi)|^2 \right\rangle_{Q,m} \Big/ \left[m^2/\lambda + 2 \sum_{i=1}^d \{1 - \cos(\xi \cdot \mathbf{e}_i)\} \right],$$

where

$$\int_{\hat{Q}} d\xi = \left(\frac{2\pi}{L} \right)^d \sum_{\xi \in \hat{Q}} \quad .$$

We also have from (2.2) that

$$\left\langle |\hat{\omega}(\xi)|^2 \right\rangle_{Q,m} \le \frac{L^d}{\lambda} \Big/ \left[m^2/\lambda + 2 \sum_{i=1}^d \{1 - \cos(\xi \cdot \mathbf{e}_i)\} \right].$$

We conclude then that if $g_d(\rho)$ is defined by

$$g_d(\rho) = \frac{1}{(2\pi)^d} \int_{\hat{Q}} d\xi \Big/ \left[\rho^2 + 2 \sum_{i=1}^d \{1 - \cos(\xi \cdot \mathbf{e}_i)\} \right]^2, \tag{2.7}$$

there is the inequality,

$$\left\langle \left\{ \sum_{x \in Q} \left[\omega(x)^2 - \left\langle \omega(x)^2 \right\rangle_{Q,m} \right] \right\}^2 \right\rangle_{Q,m} \le \frac{4L^d}{\lambda^2} \, g_d(m/\sqrt{\lambda}). \tag{2.8}$$

The function g_d of (2.7) is bounded by the function g_d in the statement of the Lemma for ρ in the region $\{1/L \le \rho \le 1/2\}$. Hence (2.5) follows from (2.6), (2.8) if f is the function $f(z) = Az^2$ for some constant A.

We can prove (2.5) for general f satisfying (2.4) in a similar fashion. By (2.2) we have that

$$\left\langle \left| \sum_{x \in Q} [f(\omega(x)) - < f(\omega(x)) >_{Q,m}] \right|^2 \right\rangle_{Q,m} \le \tag{2.9}$$

$$\frac{1}{(2\pi)^d \lambda} \int_{\hat{Q}} d\xi \left\langle |\hat{f}_1(\omega,\xi)|^2 \right\rangle_{Q,m} \Big/ \left[m^2/\lambda + 2 \sum_{i=1}^d \{1 - \cos(\xi \cdot \mathbf{e}_i)\} \right],$$

where

$$\hat{f}_1(\omega,\xi)=\sum_{x\in Q} f'(\omega(x))e^{ix\cdot\xi}.$$

Since we are assuming f is an even function it follows again from (2.2) that

$$\left\langle|\hat{f}_1(\omega,\xi)|^2\right\rangle_{Q,m}\le\frac{L^d}{\lambda}\sum_{x\in Q} H(x)G(x)e^{ix\cdot\xi},\tag{2.10}$$

where

$$H(x)=\left\langle f''(\omega(x))\overline{f''(\omega(0))}\right\rangle_{Q,m}\quad,$$

and $G(x)$ is the Green's function satisfying

$$[-\Delta+m^2/\lambda]\ G(x)=\delta(x),\qquad x\in Q,$$

with δ being the Kronecker δ function. By the Plancherel theorem it follows from (2.9), (2.10) that

$$\left\langle\left|\sum_{x\in Q}[f(\omega(x))-<f(\omega(x))>_{Q,m}]\right|^2\right\rangle_{Q,m}\le\frac{L^d}{\lambda^2}\sum_{x\in Q} H(x)G(x)^2.$$

Since $m/\sqrt{\lambda}\ge 1/L$ it follows from (2.3), (2.4) that

$$|H(x)|\le H(0)\le A^2\exp[2B^2C_d/\lambda],$$

for some constant C_d depending only on $d\ge 3$. The inequality (2.5) follows now on using the identity,

$$\sum_{x\in Q} G(x)^2=g_d(m/\sqrt{\lambda}),$$

where g_d is given by (2.7). □

Lemma 2.2. *Let $m>0$. Then for any C^2 function $f:\mathbf{R}\to\mathbf{C}$ satisfying (2.4) the limit,* $\lim_{L\to\infty}\ <f(\omega(0))>_{Q_{L,m}}$ *exists and is finite.*

Proof. Suppose L,L' are positive even integers with $L'>2L,\ L>3+\sqrt{\lambda}/m$. Let ∂Q_L be the lattice points of $\mathbf{Z}^d$ which form the boundary of the cube $Q_L\subset\mathbf{Z}^d$ and $\mathrm{Int}(Q_L)=Q_L\backslash\partial Q_L$. We define the set U_L by

$$U_L=\{x\in\partial Q_L: x+\mathbf{e}_i\notin Q_L\ \text{ for some }\ i,\ 1\le i\le d,\text{and }\ x-\mathbf{e}_j\in Q_L\ \text{ for all }\ j,\ 1\le j\le d\}.$$

Observe that $\mathrm{Int}(Q_L)\cup U_L$ is a cube containing L^d lattice points. For each $x\in\partial Q_L,\ 0\le t\le 1$, we define a function $V_{x,t}:\Omega_{L'}\to\mathbf{R}$ as follows:

(a): If $x\in U_L$ then

$$V_{x,t}(\omega)=V\Big(\omega(x+\mathbf{e}_1)-\omega(x),.....,t[\omega(x+\mathbf{e}_j)-\omega(x)]$$
$$+(1-t)[\omega(x-(L-1)\mathbf{e}_j)-\omega(x)],.....,\omega(x+\mathbf{e}_d)-\omega(x)\Big),$$

where we introduce the interpolation parameter t for any j with $x+\mathbf{e}_j\notin Q_L$.

(b): If $x\in\partial Q_L\backslash U_L$ then

$$V_{x,t}(\omega)=V\Big(\omega(x+\mathbf{e}_1)-\omega(x),.....,t[\omega(x+\mathbf{e}_j)-\omega(x)],....,\omega(x+\mathbf{e}_d)-\omega(x)\Big)\ ,$$

where we introduce the parameter t for any j with $x+\mathbf{e}_j\in Int(Q_L)\cup U_L$.

For $0 \le t \le 1$ we introduce a corresponding Lagrangian $\mathcal{L}_t$ on $\Omega_{L'}$ by

$$\mathcal{L}_t(\omega) = \sum_{x \in Q_{L'} - \partial Q_L} V(\nabla \omega(x)) + \sum_{x \in \partial Q_L} V_{x,t}(\omega) + \sum_{x \in Q_{L'}} \frac{1}{2} m^2 \omega(x)^2,$$

where as in the definition of the measure (1.10) we are identifying points on the boundary of $Q_{L'}$. We associate with $\mathcal{L}_t$ a measure

$$\exp[-\mathcal{L}_t(\omega)] \prod_{x \in Q_{L'}} d\omega(x)/\text{normalization}. \tag{2.11}$$

Evidently if $t = 1$ the measure (2.11) is identical to the measure (1.10) with $Q = Q_{L'}$. If $t = 0$ then in the measure (2.11) the variables $\omega(x),\ x \in Int(Q_L) \cup U_L$, are independent from the variables $\omega(x),\ x \in Q_{L'} \backslash [Int(Q_L) \cup U_L]$. Hence if we denote expectation w.r. to the measure (2.11) by $< \cdot >_{Q_{L',m,t}}$ we have that

$$\begin{aligned} \Big\langle f(\omega(0)) \Big\rangle_{Q_{L',m,1}} &= \Big\langle f(\omega(0)) \Big\rangle_{Q_{L',m}} , \\ \Big\langle f(\omega(0)) \Big\rangle_{Q_{L',m,0}} &= \Big\langle f(\omega(0)) \Big\rangle_{Q_{L,m}} . \end{aligned}$$

We can therefore compare $\Big\langle f(\omega(0)) \Big\rangle_{Q_{L',m}}$ and $\Big\langle f(\omega(0)) \Big\rangle_{Q_{L,m}}$ by differentiating (2.11) w.r. to t and using the fundamental theorem of calculus. We have then,

$$\Big\langle f(\omega(0)) \Big\rangle_{Q_{L',m}} - \Big\langle f(\omega(0)) \Big\rangle_{Q_{L,m}} = \frac{1}{L^d} \sum_{x \in Int(Q_L) \cup U_L} \Big\{ \langle f(\omega(x)) \rangle_{Q_{L',m,1}} - \langle f(\omega(x)) \rangle_{Q_{L',m,0}} \Big\} =$$

$$\frac{-1}{L^d} \int_0^1 dt \Big\langle \left[\frac{d\mathcal{L}_t(\omega)}{dt} - \Big\langle \frac{d\mathcal{L}_t(\omega)}{dt} \Big\rangle_{Q_{L',m,t}} \right] \left[\sum_{x \in Int(Q_L) \cup U_L} \Big\{ f(\omega(x)) - \langle f(\omega(x)) \rangle_{Q_{L',m,t}} \Big\} \right] \Big\rangle_{Q_{L',m,t}} .$$

Hence if we use the Schwarz inequality we have that

$$\begin{aligned} \Big| \Big\langle f(\omega(0)) \Big\rangle_{Q_{L',m}} - \Big\langle f(\omega(0)) \Big\rangle_{Q_{L,m}} \Big| \le \\ \frac{1}{L^d} \sup_{0 \le t \le 1} \Big\langle \left[\frac{d\mathcal{L}_t(\omega)}{dt} - \Big\langle \frac{d\mathcal{L}_t(\omega)}{dt} \Big\rangle_{Q_{L',m,t}} \right]^2 \Big\rangle_{Q_{L',m,t}}^{1/2} \\ \sup_{0 \le t \le 1} \Big\langle \Big| \sum_{x \in Int(Q_L) \cup U_L} f(\omega(x)) - \langle f(\omega(x)) \rangle_{Q_{L',m,t}} \Big|^2 \Big\rangle_{Q_{L',m,t}}^{1/2} . \end{aligned} \tag{2.12}$$

We can estimate the RHS of (2.12) by using the Brascamp-Lieb inequality provided we can obtain a suitable lower bound on the Hessian $\mathcal{L}_t''(\omega)$ of the Lagrangian $\mathcal{L}_t(\omega)$. We have already observed that

$$\mathcal{L}_1''(\omega) \ge -\lambda \Delta_{L'} + m^2 ,$$

where $\Delta_{L'}$ is the lattice Laplacian on $Q_{L'}$ with periodic boundary conditions. It is easy to see also that

$$\mathcal{L}_0''(\omega) \ge -\lambda [\Delta_L \otimes I + I \otimes \Delta_{L',L}] + m^2 ,$$

where Δ_L is the periodic Laplacian acting on $Int(Q_L) \cup U_L$ and $\Delta_{L',L}$ is the Laplacian on $Q_{L'} \backslash [Int(Q_L) \cup U_L]$ with Neumann boundary conditions on the boundary of $Int(Q_L) \cup U_L$. More generally, for $0 \le t \le 1$ there is the inequality,

$$\mathcal{L}_t''(\omega) \ge -\lambda \left[t^2 \Delta_{L'} + (1-t)^2 \left\{ \Delta_L \otimes I + I \otimes \Delta_{L',L} \right\} \right] + m^2. \tag{2.13}$$

It follows from (2.13) and Brascamp-Lieb that if $0 \le t \le 1/2$, then

$$\left\langle \Big| \sum_{x \in Int(Q_L) \cup U_L} f(\omega(x)) - \langle f(\omega(x)) \rangle_{Q_{L',m,t}} \Big|^2 \right\rangle_{Q_{L',m,t}}^{1/2} \le \frac{4L^d}{\lambda} \sum_{x \in Int(Q_L) \cup U_L} H(x) G(x), \tag{2.14}$$

where

$$\begin{aligned} H(x) &= \left\langle f'(\omega(x)) \overline{f'(\omega(0))} \right\rangle_{Q_{L',m,t}} , \\ \left[-\Delta_L + \frac{4m^2}{\lambda} \right] G(x) &= \delta(x), \quad x \in Int(Q_L) \cup U_L \ . \end{aligned}$$

We conclude that the LHS of (2.14) is bounded by

$$\frac{4L^d}{\lambda} H(0) \sum_{x \in Int(Q_L) \cup U_L} G(x) = \frac{L^d}{m^2} \ H(0).$$

In view of (2.13) one has an inequality like (2.3) for the measure $< \cdot >_{Q_{L',m,t}}$. Since f satisfies (2.4) it follows that there is a constant C_d depending only on $d \ge 3$ such that

$$H(0) \le A^2 \exp[2B^2 C_d / \lambda].$$

We conclude then that if $0 \le t \le 1/2$, there is the inequality,

$$\left\langle \Big| \sum_{x \in Int(Q_L) \cup U_L} f(\omega(x)) - \langle f(\omega(x)) \rangle_{Q_{L',m,t}} \Big|^2 \right\rangle_{Q_{L',m,t}} \le \frac{L^d}{m^2} \ A^2 \exp[2B^2 C_d / \lambda]. \tag{2.15}$$

We can make a similar argument for $1/2 \le t \le 1$ by replacing Δ_L by $\Delta_{L'}$ in the proof of (2.15). We can estimate the expectation of $d\mathcal{L}_t(\omega)/dt$ on the RHS of (2.12) similarly to the way we obtained (2.15). Thus by Brascamp-Lieb there is the inequality,

$$\left\langle \left[\frac{d\mathcal{L}_t(\omega)}{dt} - \left\langle \frac{d\mathcal{L}_t(\omega)}{dt} \right\rangle_{Q_{L',m,t}} \right]^2 \right\rangle_{Q_{L',m,t}} \le \sum_{x,y \in Q_{L'}} \langle g_t(\omega, x) \ G_t(x,y) \ g_t(\omega, y) \rangle_{Q_{L',m,t}} \ , \tag{2.16}$$

where $G_t(x,y)$ is the kernel of the inverse of the operator on the RHS of (2.13). The function g_t has the property that

$$g_t(\omega, x) \quad = \quad 0, \quad x \notin N(\partial Q_L),$$

$$|g_t(\omega,x)| \quad\le\quad C\Lambda \sum_{y\in N(x)\cap N(\partial Q_L)} |\omega(y)|, \quad x\in N(\partial Q_L),$$

for some universal constant C. Here $N(\partial Q_L)$ is the neighborhood of ∂Q_L with radius 1. By $N(x)$ we mean the union of the neighborhoods of x and $x-(L-1)\mathbf{e}_j,\ \ 1\le j\le d$, with radius 1. Observe that G_t satisfies

$$G_t(x,y)\ge 0, \quad x,y\in Q_{L'}, \quad \sum_{y\in Q_{L'}} G_t(x,y)=\frac{1}{m^2}.$$

We conclude that the LHS of (2.16) is bounded above by

$$\frac{C_d\Lambda^2}{m^2}\sum_{x\in N(\partial Q_L)} \left\langle \omega(x)^2\right\rangle_{Q_{L'},m,t} \le \frac{C_d\Lambda^2}{m^2}\sum_{x\in N(\partial Q_L)} G_t(x,x)\le \frac{C_d\Lambda^2}{m^4}|N(\partial Q_L)|,$$

for some constant C_d depending only on d. Evidently the number of lattice points $|N(\partial Q_L)|$ in $N(\partial Q_L)$ is bounded by C_dL^{d-1} for some constant C_d depending only on d. Hence there is a constant C_d depending only on d such that

$$\left\langle \left[\frac{d\mathcal{L}_t(\omega)}{dt}-\left\langle\frac{d\mathcal{L}_t(\omega)}{dt}\right\rangle_{Q_{L'},m,t}\right]^2\right\rangle_{Q_{L'},m,t} \le \frac{C_d\Lambda^2L^{d-1}}{m^4}. \tag{2.17}$$

The result follows now from (2.12), (2.15), (2.17). $\square$

Proposition 2.1. *Let Ω be the space of functions $\omega:\mathbf{Z}^d\to\mathbf{R}$ and $\mathcal{F}$ be the corresponding Borel algebra generated by finite dimensional rectangles. Then if $d\ge 3$ there is a unique probability measure P on $(\Omega,\mathcal{F})$ with the properties:*

(a): *If $N\ge 1$ and $f:\mathbf{R}^N\to\mathbf{C}$ is a C^2 function satisfying the inequality*

$$|f''(z)|\le A\exp[B|z|], \quad z\in\mathbf{R}^N, \tag{2.18}$$

for some constants A,B, then for any $x_i\in\mathbf{Z}^d,\ 1\le i\le N$, the function $f(\omega(x_1),\omega(x_2),...,\omega(x_N))$ is integrable w.r. to $(\Omega,\mathcal{F},P)$.

(b): *Denoting expectation w.r. to $(\Omega,\mathcal{F},P)$ by $<\cdot>_\Omega$, there is the identity,*

$$\langle f(\omega(x_1),\omega(x_2),...,\omega(x_N))\rangle_\Omega =$$

$$\lim_{m\to0}\lim_{L\to\infty}\ \langle f(\omega(x_1),\omega(x_2),...,\omega(x_N))\rangle_{Q_L,m}.$$

(c): *The translation operators $\tau_x,\ x\in\mathbf{Z}^d$, are measure preserving and ergodic w.r. to $(\Omega,\mathcal{F},P)$.*

Proof. We have already seen in Lemma 2.2 that if $f:\mathbf{R}\to\mathbf{C}$ is C^2 and satisfies (2.4) then $\lim_{L\to\infty}\ \langle f(\omega(0))\rangle_{Q_L,m}$ exists. Suppose now that m,m' satisfy the inequality $\sqrt{\lambda}/L\le m'<m$. Define t_0 by $\sqrt{t_0}m=m'$. Then there is the identity

$$\langle f(\omega(0))\rangle_{Q,m}-\langle f(\omega(0))\rangle_{Q,m'} =$$

$$-\int_{t_0}^1 dt\left\langle\left[\frac{m^2}{2}\sum_{x\in Q}\omega(x)^2\right]\left[f(\omega(0))-\langle f(\omega(0))\rangle_{Q,\sqrt{t}m}\right]\right\rangle_{Q,\sqrt{t}m}.$$

If $d=3$ it follows from (2.5) that

$$\Big|\langle f(\omega(0))\rangle_{Q,m}-\langle f(\omega(0))\rangle_{Q,m'}\Big| \le \frac{c_3A}{\lambda^{3/2}}\exp[B^2C_3/\lambda]\frac{m}{2}\int_{t_0}^1\frac{dt}{\sqrt{t}}.$$

We conclude from this last inequality that

$$\lim_{m\to 0}\lim_{L\to\infty}\langle f(\omega(0))\rangle_{Q_{L,m}} \quad \text{exists if} \quad d=3.$$

We can similarly conclude from Lemma 2.1 that the limit exists for all $d\ge 3$. Since the measures $<\cdot>_{Q_{L,m}}$ form a tight sequence we have that there exists a unique Borel probability measure μ_0 on $\mathbf{R}$ such that

$$\langle f(\omega(0))\rangle_{\mu_0} = \lim_{m\to 0}\lim_{L\to\infty}\langle f(\omega(0))\rangle_{Q_{L,m}}$$

for any C^2 function $f:\mathbf{R}\to\mathbf{C}$ satisfying (2.4). Generalizing Lemmas 2.1, 2.2, one sees that for any $N\ge 1$ and $x_i\in\mathbf{Z}^d$, $1\le i\le N$, there exists a unique Borel probability measure $\mu_{x_1,\dots,x_N}$ on $\mathbf{R}^N$ such that

$$\langle f\big(\omega(x_1),\omega(x_2),\dots,\omega(x_N)\big)\rangle_{\mu_{x_1,\dots,x_N}} = \lim_{m\to 0}\lim_{L\to\infty}\langle f\big(\omega(x_1),\omega(x_2),\dots,\omega(x_N)\big)\rangle_{Q_{L,m}}\ , \tag{2.19}$$

for any C^2 function $f:\mathbf{R}^N\to\mathbf{C}$ satisfying (2.18). It is easy to see that the set of measures $\mu_{x_1,\dots,x_N}$, $x_i\in\mathbf{Z}^d$, $1\le i\le N$, $N\ge 1$, form a consistent set. Hence the probability measure P on $(\Omega,\mathcal{F})$ exists by the Kolmogorov extension theorem [6]. Evidently (b) follows from (2.19). The fact that the translation operators τ_x, $x\in\mathbf{Z}^d$, are measure preserving w.r. to $(\Omega,\mathcal{F},P)$ is clear from (b). The ergodicity is a consequence of the Brascamp-Lieb inequality. □

We turn now to the construction of the probability space $(\hat\Omega,\hat{\mathcal{F}},\hat P)$ which gives the stationary process for the stochastic equation (1.4). We shall first do this for a finite cube Q. We consider as before the space Ω_Q of periodic functions $\omega:Q\to\mathbf{R}$ and let $\mathcal{L}_{Q,m}:\Omega_Q\to\mathbf{R}$ be defined by

$$\mathcal{L}_{Q,m}(\omega)=\sum_{x\in Q}\left\{V(\nabla\omega(x))+\frac{1}{2}m^2\omega(x)^2\right\},$$

whence the measure (1.10) is the probability measure corresponding to Lebesgue measure on Ω_Q weighted by $\exp[-\mathcal{L}_{Q,m}(\omega)]$. Let $f:\Omega_Q\to\mathbf{C}$ be a C^2 function satisfying (2.18), where $Q=Q_L$, $N=L^d$. Then one can solve the initial value problem

$$\begin{aligned}\frac{\partial u}{\partial t}(\omega,t) &= \sum_{x\in Q}\left\{-\left[\frac{\partial\mathcal{L}_{Q,m}(\omega)}{\partial\omega(x)}\right]\frac{\partial}{\partial\omega(x)}+\left[\frac{\partial}{\partial\omega(x)}\right]^2\right\}u(\omega,t), \quad \omega\in\Omega_Q,\ t>0,\\ u(\omega,0) &= f(\omega), \quad \omega\in\Omega_Q.\end{aligned}$$

The solution can be written as

$$u(\omega,t)=\int_{\mathbf{R}^N}G_{Q,m}(\omega,\omega',t)f(\omega')d\omega'\ ,$$

where $G_{Q,m}(\omega,\omega',t)>0$ is the Green's function and satisfies the identity,

$$\int_{\mathbf{R}^N}G_{Q,m}(\omega,\omega',t)d\omega'=1,\quad \omega\in\Omega_Q,\ t>0.$$

Let $\hat\Omega_Q$ be the space of functions $\omega:Q\to\mathbf{R}$, periodic on Q and continuous in the $\mathbf{R}$ variable. We denote by $\hat{\mathcal{F}}_Q$ the Borel algebra of subsets of $\hat\Omega_Q$ generated by all finite dimensional rectangles $\{\omega\in\hat\Omega_Q : |\omega(x_i,t_i)-a_i|<r_i, i=1,...,n\}$,

$x_i \in Q,\ t_i \in \mathbf{R},\ a_i \in \mathbf{R},\ r_i > 0,\ i = 1,...,n,\ n \ge 1$. Then there is a probability measure $\hat{P}_{Q,m}$ on $(\hat{\Omega}_Q, \hat{\mathcal{F}}_Q)$ such that for any C^2 function $f : \mathbf{R}^n \to \mathbf{C}$ satisfying (2.18) the function $f(\omega(x_1,t_1),....,\omega(x_n,t_n))$ is integrable on $(\hat{\Omega}_Q, \hat{\mathcal{F}}_Q, \hat{P}_{Q,m})$ for any $x_i \in Q,\ 1 \le i \le n,\ t_1 < t_2 < ... < t_n$, and there is the identity,

(2.20)
$$\Big\langle f\Big(\omega(x_1,t_1),...,\omega(x_n,t_n)\Big)\Big\rangle_{Q,m} = \Big\langle \int_{\mathbf{R}^{N(n-1)}} d\omega_2...d\omega_n\ f(\omega_1(x_1),....,\omega_n(x_n))$$
$$G_{Q,m}(\omega_1,\omega_2,t_2-t_1)G_{Q,m}(\omega_2,\omega_3,t_3-t_2)\cdots G_{Q,m}(\omega_{n-1},\omega_n,t_n-t_{n-1})\Big\rangle_{Q,m}.$$

The expectation $< \cdot >_{Q,m}$ on the LHS of (2.20) refers to the expectation w.r. to the probability space $(\hat{\Omega}_Q, \hat{\mathcal{F}}_Q, \hat{P}_{Q,m})$. The expectation $< \cdot >_{Q,m}$ on the RHS of (2.20) refers to the expectation w.r. to the measure (1.10) on functions $\omega_1 \in \Omega_Q$. It is evident that the translation operators $\tau_{x,t},\ x \in \mathbf{Z}^d,\ t \in \mathbf{R}$, defined by $\tau_{x,t}\omega(z,s) = \omega(x+z,t+s)$, $z \in Q,\ s \in \mathbf{R}$, are measure preserving on the space $(\hat{\Omega}_Q, \hat{\mathcal{F}}_Q, \hat{P}_{Q,m})$. One can also further see that the time translation operators are ergodic.

We need now to show that the probability space $(\hat{\Omega}_Q, \hat{\mathcal{F}}_Q, \hat{P}_{Q,m})$ defined by (2.20) has a limit as $|Q| \to \infty$. To do this we shall construct $(\hat{\Omega}_Q, \hat{\mathcal{F}}_Q, \hat{P}_{Q,m})$ by using stochastic differential equations instead of with Green's functions as in (2.20). For $x \in Q$ let $B(x,t),\ t \ge 0$, be independent copies of Brownian motion, where we identity boundary points of Q. Thus each realization of the Brownian motion yields a continuous function $B_t,\ t \ge 0$, with values in Ω_Q and B_0 is the zero function in Ω_Q. Since the vector field $\partial\mathcal{L}_{Q,m}(\omega)/\partial\omega$ on Ω_Q is Lipschitz there is for any $\omega_0 \in \Omega_Q$ a unique solution $\omega_t,\ t \ge 0$, of the integral equation,

$$\omega_t = \omega_0 + \sqrt{2}B_t - \int_0^t \frac{\partial\mathcal{L}_{Q,m}(\omega_s)}{\partial\omega} ds. \tag{2.21}$$

The equation (2.20) is then equivalent to

$$\langle f(\omega(x_1,t_1),.....,\omega(x_n,t_n))\rangle_{Q,m} = \langle f\left(\omega_0(x_1),\omega_{t_2-t_1}(x_2),\omega_{t_3-t_1}(x_3),.....,\omega_{t_n-t_1}(x_n)\right)\rangle_{Q,m,W}, \tag{2.22}$$

where $< \cdot >_{Q,m,W}$ means that integration over ω_0 is with respect to the measure (1.10), and integration over ω_t with given ω_0 is with respect to the Wiener measure. Now the solution of (2.21) can be generated by an iteration process at least for small t. Thus for $N = 0,1,2,...$ we define $\omega_{t,N}$ inductively by

$$\omega_{t,N} = \omega_0 + \sqrt{2}B_t - \int_0^t \frac{\partial\mathcal{L}_{Q,m}(\omega_{s,N-1})}{\partial\omega}\ ds, \tag{2.23}$$

with $\omega_{t,0} = \omega_0 + \sqrt{2}B_t$. We can obtain a rate of convergence of the Nth iterate of (2.23) to the solution of (2.21).

Lemma 2.3. *There is a constant $c_d > 0$, depending only on d, such that if $c_d(\Lambda + m^2)t < 1$ then the Nth iterate $\omega_{t,N}$ of (2.23) and the solutions of (2.21) satisfy the inequality,*

$$\left\langle \left[\sup_{0\le s\le t} |\omega_s(0) - \omega_{s,N}(0)|\right]^2 \right\rangle_{Q,m,W} \le$$

$$\frac{[c_d(\Lambda+m^2)t]^{2N}}{[1-c_d(\Lambda+m^2)t]^2}\ C_d(\Lambda+m^2)^2t\left[t+\langle\omega_0^2(0)\rangle_{Q,m}\right],$$

for some constant C_d *depending only on* d.

Proof. For $\omega\in\Omega_Q$ we define $\|\omega\|$ to be

$$\|\omega\|^2=\sum_{x\in Q}|\omega(x)|^2\ e^{-2|x|}\ .$$

it is easy to see from (2.23) that

$$\|\omega_{t,N+1}-\omega_{t,N}\|\le c_d(\Lambda+m^2)t\sup_{0\le s\le t}\|\omega_{s,N}-\omega_{s,N-1}\|,$$

for some constant $c_d>0$ depending only on d. Hence we have by induction that

$$\sup_{0\le s\le t}\|\omega_{s,N+1}-\omega_{s,N}\|\le\left[c_d(\Lambda+m^2)t\right]^N\sup_{0\le s\le t}\|\omega_{s,1}-\omega_{s,0}\|.$$

Using the fact that the iterates of (2.23) converge to the solution of (2.21) we have from the previous inequality that

$$\sup_{0\le s\le t}\|\omega_s-\omega_{s,N}\|\le\frac{[c_d(\Lambda+m^2)t]^N}{[1-c_d(\Lambda+m^2)t]}\sup_{0\le s\le t}\|\omega_{s,1}-\omega_{s,0}\|. \tag{2.24}$$

Since $|\omega_s(0)-\omega_{s,N}(0)|\le\|\omega_s-\omega_{s,N}\|$ the result will follow if we can show that

$$\left\langle\left[\sup_{0\le s\le t}\|\omega_{s,1}-\omega_{s,0}\|\right]^2\right\rangle_{Q,m,W}\le C_d(\Lambda+m^2)^2\left[t+\left\langle\omega_0^2(0)\right\rangle_{Q,m}\right], \tag{2.25}$$

for some constant C_d depending only on d. It is clear there is a constant C'_d such that

$$\sup_{0\le s\le t}\|\omega_{s,1}-\omega_{s,0}\|^2\le C'_d(\Lambda+m^2)^2t\Big[\|\omega_0\|^2+\sup_{0\le s\le t}\|B_s\|^2\Big].$$

The inequality (2.25) now follows from the inequalities,

$$\begin{aligned}\left\langle\|\omega_0\|^2\right\rangle_{Q,m,W}&=\left[\sum_{x\in Q}e^{-2|x|}\right]\left\langle\omega_0^2(0)\right\rangle_{Q,m},\\ \left\langle\sup_{0\le s\le t}\|B_s\|^2\right\rangle_{Q,m,W}&\le\left[\sum_{x\in Q}e^{-2|x|}\right]4t.\end{aligned}$$

□

Next we use the method of proving Proposition 2.1(b) to show that one can let $|Q|\to\infty,\ m\to 0$ on the RHS of (2.22).

Lemma 2.4. *Let* $n\ge 1$ *and* $f:\mathbf{R}^n\to\mathbf{C}$ *be a* C^2 *function satisfying (2.18). Then the limit*

$$\lim_{m\to 0}\lim_{L\to\infty}\langle f(\omega(x_1,t_1),...,\omega(x_n,t_n))\rangle_{Q_L,m} \tag{2.26}$$

exists for any $x_1,...,x_n\in\mathbf{Z}^d,\ t_1\le t_2\le\cdots\le t_n$, *provided* $d\ge 3$.

Proof. We shall first prove the existence of the limit (2.26) when ω_t is replaced by $\omega_{t,N}$ for any N, on the RHS of (2.22). To see this first note that the conditional expectation,

$$\left\langle f\left(\omega_0(x_1), \omega_{t_2-t_{1,N}}(x_2), \ldots.., \omega_{t_n-t_{1,N}}(x_n)\right) |\omega_0\right\rangle_{Q,m,W} = g\left(\omega_0(y_1), \omega_0(y_2), \ldots, \omega_0(y_p)\right), \tag{2.27}$$

where the variables $y_1, y_2, \ldots, y_p$ include the variables $x_1, x_2, \ldots, x_n$ and neighbors of them in $\mathbf{Z}^d$ within a distance N. Since the function V of (1.1) is C^2 and f satisfies (2.18) it follows that g is a C^1 function which satisfies the inequality,

$$|g'(z)| \le A \exp[B|z|], \quad z \in \mathbf{R}^p \ ,$$

for some constants A, B. We conclude then from Proposition 2.1(b) by approximating g by a C^2 function that

$$\lim_{m\to 0}\lim_{L\to\infty} \langle g\left(\omega_0(y_1), \ldots, \omega_0(y_p)\right)\rangle_{Q_L,m} \qquad \text{exists.}$$

Next we use Lemma 2.3 to show that the limit (2.26) exists provided $c_d\Lambda(t_n - t_1) < 1$. In fact on restricting m to satisfy $c_d(\Lambda + m^2)(t_n - t_1) < 1$ we have that

$$\begin{aligned} &\Big| \langle f\left(\omega_0(x_1), \omega_{t_2-t_1}(x_2), \ldots.., \omega_{t_n-t_1}(x_n)\right)\rangle_{Q,m,W} \\ &\quad - \left\langle f\left(\omega_0(x_1), \omega_{t_2-t_{1,N}}(x_2), \ldots.., \omega_{t_n-t_{1,N}}(x_n)\right)\right\rangle_{Q,m,W} \Big| \\ &\le \frac{[c_d(\Lambda+m^2)(t_n-t_1)]^N}{[1-c_d(\Lambda+m^2)(t_n-t_1)]} \ \frac{C_d^{1/2}}{c_d^{1/2}} \left[t_n - t_1 + \left\langle \omega_0^2(0)\right\rangle_{Q,m}\right]^{1/2} \\ &\quad \int_0^1 d\gamma \sum_{j=2}^n \Big\langle \Big| f_j'\Big(\omega_0(x_1), \omega_{t_2-t_1}(x_2), \ldots, \\ &\gamma\omega_{t_j-t_1}(x_j) + (1-\gamma)\omega_{t_j-t_1,N}(x_j), \omega_{t_{j+1}-t_1,N}(x_{j+1}), \ldots, \omega_{t_n-t_1,N}(x_n)\Big)\Big|^2\Big\rangle_{Q,m,W}^{1/2}, \end{aligned} \tag{2.28}$$

where f_j' denotes the derivative of f w.r. to the jth variable. Observe now from (2.3) that

$$\begin{aligned} \langle\exp\left[A|\omega_t(x)|\right]\rangle_{Q,m,W} &= \langle\exp\left[A|\omega_t(0)|\right]\rangle_{Q,m,W} \\ &= \ \langle\exp\left[A|\omega_0(0)|\right]\rangle_{Q,m} \ \le 2\ \exp[C_dA^2/\lambda], \end{aligned}$$

provided $d \ge 3$. We also have from the argument of Lemma 2.3 that

$$\begin{aligned} \langle\exp\left[A|\omega_t(x) - \omega_{t,N}(x)|\right]\rangle_{Q,m,W} &= \langle\exp\left[A|\omega_t(0) - \omega_{t,N}(0)|\right]\rangle_{Q,m,W} \\ &\le \left\langle \exp\left[A \sup_{0\le s\le t} \|\omega_{s,1} - \omega_{s,0}\| \Big/ \left\{1 - c_d(\Lambda + m^2)t\right\}\right]\right\rangle_{Q,m,W} \end{aligned}$$

provided $c_d(\Lambda + m^2)t < 1$. We also have the inequality,

$$\sup_{0\le s\le t} \|\omega_{s,1} - \omega_{s,0}\| \le C_d(\Lambda + m^2)\sum_{x\in Q} e^{-|x|}\left[|\omega_0(x)| + \sup_{0\le s\le t}|B_s(x)|\right].$$

If we use the inequality,

$$e^{\varepsilon|z|} \le 4^{\sqrt{\varepsilon}} \cosh \sqrt{\varepsilon}\, z, \quad 0 < \varepsilon \le \frac{1}{4}, \quad z \in \mathbf{R},$$

then it is clear from (2.3) that

$$\left\langle \exp\left[A \sum_{x \in Q} e^{-|x|} |\omega_0(x)| \right] \right\rangle_{Q,m} \le C_d \exp[C_d A^2/\lambda], \tag{2.29}$$

for some constant C_d depending only on $d \ge 3$. There is also the inequality,

$$\left\langle \exp\left[A \sup_{0 \le s \le t} |B_s(x)| \right] \right\rangle_W \le \exp[CA^2 t], \tag{2.30}$$

for some universal constant C. We can conclude now from the last two inequalities and (2.28) that the limit (2.26) holds provided $c_d\Lambda(t_n - t_1) < 1$. It is not difficult to generalize the previous argument to remove the restriction $c_d\Lambda(t_n - t_1) < 1$. We can assume wlog that $c_d\Lambda(t_k - t_{k-1}) < 1$, $k = 2, ..., n$. We then define $\omega_{t_2,t_1,N} = \omega_{t_2-t_1,N}$ as before but now $\omega_{t_3,t_2,N}$ as the Nth iterate of (2.23) over a time interval of length $t_3 - t_2$ with initial condition $\omega_{t_2,t_1,N}$. We similarly define $\omega_{t_k,t_{k-1},N}$, $3 \le k \le n$. Then we obtain analogues of the identity (2.27) and the inequality (2.28). □

Lemma 2.4 enables us to prove the analogue of Proposition 2.1.

Proposition 2.2. *Let $\hat{\Omega}$ be the space of functions $\omega : \mathbf{Z}^d \times \mathbf{R} \to \mathbf{R}$ continuous in the $\mathbf{R}$ variable. Let $\hat{\mathcal{F}}$ be the corresponding Borel algebra generated by finite dimensional rectangles. Then if $d \ge 3$ there is a unique probability measure $\hat{P}$ on $(\hat{\Omega}, \hat{\mathcal{F}})$ with the properties:*

(a) *If $f : \mathbf{R}^n \to \mathbf{C}$ is a continuous function satisfying the inequality*

$$|f(z)| \le A \exp[B|z|], \qquad z \in \mathbf{R}^n\ , \tag{2.31}$$

for some constants A, B, then for any $x_i \in \mathbf{Z}^d$, $t_i \in \mathbf{R}$, $1 \le i \le n$, the function $f(\omega(x_1, t_1), ..., \omega(x_n, t_n))$ is integrable w.r. to $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$.

(b) *Denoting expectation w.r. to $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$ by $< \cdot >_{\hat{\Omega}}$, there is the identity,*

$$\langle f(\omega(x_1, t_1), ..., \omega(x_n, t_n)) \rangle_{\hat{\Omega}} = \lim_{m \to 0} \lim_{L \to \infty} \langle f(\omega(x_1, t_1), ..., \omega(x_n, t_n)) \rangle_{Q_L,m}\ , \tag{2.32}$$

where the RHS of (2.32) is defined by (2.22).

(c) *The translation operators $\tau_{x,t}$, $x \in \mathbf{Z}^d$, $t \in \mathbf{R}$, are measure preserving.*

Proof. Arguing as in Proposition 2.1, one sees that for any $n \ge 1$ and $x_i \in \mathbf{Z}^d$, $t_i \in \mathbf{R}$, $1 \le i \le n$, there exists a unique Borel probability measure $\mu_{(x_1,t_1),...,(x_n,t_n)}$ on $\mathbf{R}^n$ such that

$$\langle f(\omega(x_1, t_1), ..., \omega(x_n, t_n)) \rangle_{\mu_{(x_1,t_1),...,(x_n,t_n)}} = \lim_{m \to 0} \lim_{L \to \infty} \langle f(\omega(x_1, t_1), ..., \omega(x_n, t_n)) \rangle_{Q_L,m}\ ,$$

for any continuous function $f : \mathbf{R}^n \to \mathbf{C}$ satisfying (2.31). Evidently the set of measures $\mu_{(x_1,t_1),...,(x_n,t_n)}$, $x_i \in \mathbf{Z}^d$, $t_i \in \mathbf{R}$, $1 \le i \le n$, form a consistent set. Hence by the Kolmogorov extension theorem the probability measure $\hat{P}$ on $(\hat{\Omega}, \hat{\mathcal{F}})$

exists provided we can show that $\omega(x,t)$ is continuous w.r. to t with probability 1 on $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$. To see this observe from the argument of Lemma 2.3 that

$$\sup_{0\le s\le t} \|\omega_s\| \le \left[\|\omega_0\| + \sqrt{2} \sup_{0\le s\le t} \|B_s\| \right] \Big/ [1 - c_d(\Lambda+m^2)t], \tag{2.33}$$

provided $c_d(\Lambda+m^2)t < 1$. We also have from (2.21) that for $h>0$ small there is the inequality,

$$\sup_{\substack{0\le s,s'\le t,\\ |s-s'|\le h}} |\omega_s(0)-\omega_{s'}(0)| \le \sqrt{2} \sup_{\substack{0\le s,s'\le t,\\ |s-s'|\le h}} |B_s(0)-B_{s'}(0)| + c_d(\Lambda+m^2)h \sup_{0\le s\le t} \|\omega_s\|. \tag{2.34}$$

Observe from (2.33), (2.29), (2.30) that $\sup_{0\le s\le t}\|\omega_s\|$ is exponentially integrable for $c_d(\Lambda+m^2)t<1$. Hence it follows from (2.34) that $\omega_s(0)$ is uniformly continuous in the interval $0\le s\le t$ with probability 1. □

3. Proof of Theorem 1.2

As a first step in proving Theorem 1.2 we obtain a finite dimensional version of the theorem. Corresponding to (1.6) we consider the initial value problem,

$$\begin{aligned} \frac{\partial u}{\partial t}(x,t,\omega) &= -\nabla^* \mathbf{a}(\tau_{x,t}\,\omega)\nabla u(x,t,\omega), \ x\in Q,\ t>0,\ \omega\in\hat{\Omega}_Q, \\ u(x,0,\omega) &= f(x,\omega), \quad x\in Q, \quad \omega\in\hat{\Omega}_Q\ , \end{aligned} \tag{3.1}$$

where $f: Q\times\hat{\Omega}_Q \to \mathbf{R}$ is a given measurable function, periodic on Q. The solution of (3.1) may be written in the form,

$$u(x,t,\omega) = \sum_{y\in Q} G_{\mathbf{a},Q}(x,y,t,\omega) f(y,\omega), \tag{3.2}$$

where $G_{\mathbf{a},Q}(x,y,t,\omega)$ is the Green's function. It is evident there is a function $G_{\mathbf{a},Q,m}(x,t)$, $x\in Q,\ t\ge 0$, such that

$$\langle G_{\mathbf{a},Q}(x,y,t,\cdot)\rangle_{Q,m} = G_{\mathbf{a},Q,m}(x-y,t). \tag{3.3}$$

We then have the following analogue of Theorem 1.1:

Lemma 3.1. *Let $\mathbf{a}:\hat{\Omega}_Q\to\mathbf{R}^{d(d+1)/2}$ be defined by $\mathbf{a}(\omega)=V''(\nabla\omega(0,0))$, $\omega\in\hat{\Omega}_Q$, where V satisfies (1.1). Then $\mathbf{a}(\cdot)$ satisfies (1.5) and the function $G_{\mathbf{a},Q,m}(x,t)$ defined by (3.1), (3.2), (3.3) satisfies the identity,*

$$\langle \omega(x)\omega(0)\rangle_{Q,m} = \int_0^\infty G_{\mathbf{a},Q,m}(x,t)e^{-m^2 t}\,dt. \tag{3.4}$$

Proof. It follows from the fundamental identity of Helffer-Sjöstrand [9] that

$$\langle \omega(x)\omega(0)\rangle_{Q,m} = \left[\delta_x, \left[\nabla^* V''(\nabla\omega)\nabla + m^2 + A\right]^{-1} \delta_0\right]_{Q,m}, \tag{3.5}$$

where the inner product $[\cdot,\ \cdot]_{Q,m}$ on the RHS of (3.5) refers to an inner product on the space of square integrable functions $L^2(Q\times\Omega_Q)$. Thus for $f,g: Q\times\Omega_Q\to\mathbf{C}$ we define

$$[f,g]_{Q,m} = \sum_{x\in Q} \left\langle f(x,\cdot)\overline{g(x,\cdot)}\right\rangle_{Q,m}.$$

The operator A acts on functions $f : \Omega_Q \to \mathbf{C}$ by

$$Af(\omega) = -\sum_{x\in Q}\left\{-\left[\frac{\partial \mathcal{L}_{Q,m}(\omega)}{\partial\omega(x)}\right]\frac{\partial}{\partial\omega(x)} + \left[\frac{\partial}{\partial\omega(x)}\right]^2\right\} f(\omega).$$

The $\delta_y : Q \times \Omega_Q \to \mathbf{R}$ is defined for any $y \in Q$ by $\delta_y(x,\omega) = \delta(x-y)$ where δ is the Kronecker δ function. We can rewrite the expression on the RHS of (3.5) as

$$\int_0^\infty dt\ e^{-m^2 t}\left[\delta_x,\ \exp\left[-t\left\{\nabla^* V''(\nabla\omega)\nabla + A\right\}\right]\delta_0\right]_{Q,m}.$$

Hence it will be sufficient to show that

$$G_{\mathbf{a},Q,m}(x,t) = \left[\delta_x,\ \exp\left[-t\left\{\nabla^* V''(\nabla\omega)\nabla + A\right\}\right]\delta_0\right]_{Q,m}. \tag{3.6}$$

Let $v(x,t,\omega)$ be the solution to the initial value problem,

$$\begin{aligned} \frac{\partial v}{\partial t}(x,t,\omega) &= -\left\{\nabla^* V''(\nabla\omega)\nabla + A\right\} v(x,t,\omega),\ x\in Q,\ t>0,\ \omega\in\Omega_Q, \\ v(x,0,\omega) &= g(x,\omega),\quad x\in Q,\quad \omega\in\Omega_Q, \end{aligned} \tag{3.7}$$

For $\omega \in \hat\Omega_Q,\ t \in \mathbf{R}$, let $\omega_t \in \Omega_Q$ be the function $\omega_t(x) = \omega(x,t),\ x \in Q$. Suppose now the function $f(x,\omega)$ of (3.1) depends only on x, ω_0. Then the solution $u(x,t,\omega)$ of (3.1) depends only on $\omega_s,\ 0 \le s \le t$. It follows now from (3.1), (3.7) that

$$\frac{d}{ds}\sum_{x\in Q}\left\langle u(x,s,\omega)\overline{v(x,t-s,\omega_s)}\right\rangle_{Q,m} = 0,\quad 0\le s\le t.$$

We conclude that

$$\sum_{x\in Q}\left\langle u(x,t,\omega)\overline{g(x,\omega_t)}\right\rangle_{Q,m} = \sum_{x\in Q}\left\langle f(x,\omega_0)\overline{v(x,t,\omega_0)}\right\rangle_{Q,m}.$$

If we take $f = \delta_0,\ g = \delta_x$ in the previous identity we obtain (3.6). □

Proof of Theorem 1.2: It follows from Proposition 2.1 that the limit of the LHS of (3.4) converges as $|Q| \to \infty,\ m \to 0$ to the LHS of (1.9). Hence we need to show that the limit of the RHS of (3.4) converges to the RHS of (1.9) as $|Q| \to \infty,\ m \to 0$. It follows from a discrete Aronson inequality [8] or by the argument of [4] that if $Q = Q_L$ and $m \ge \sqrt{\lambda}/L$ then

$$e^{-m^2 t}\,|G_{\mathbf{a},Q,m}(x,t)| \le C/[1+t^{d/2}],\quad t\ge 0,$$

for a constant C depending only on d, λ, Λ. Hence for $d \ge 3$ there is the inequality,

$$\int_0^\infty \left[1-e^{-\eta t}\right] e^{-m^2 t}\,|G_{\mathbf{a},Q,m}(x,t)|dt \le C\eta^{d/2\,-\,1},\quad 0\le\eta\le 1,$$

for a constant C depending only on d, λ, Λ. It is therefore sufficient to show that for any $\eta > 0$ there is the limit,

$$\lim_{m\to 0}\lim_{L\to\infty}\int_0^\infty e^{-\eta t}\ G_{\mathbf{a},Q_L,m}(x,t)dt = \int_0^\infty e^{-\eta t}\ G_{\mathbf{a}}(x,t)dt. \tag{3.8}$$

We will prove (3.8) by using Proposition 2.2 and a perturbation expansion. To do this we put $\mathbf{b}(\omega) = [\Lambda I_d - \mathbf{a}(\omega)]/\Lambda$. In view of (1.5) it follows that

$$0 \le \mathbf{b}(\omega) \le [1-\lambda/\Lambda]I_d\ , \tag{3.9}$$

where the inequality in (3.9) is in the sense of quadratic forms. Let $L^2(Q\times\mathbf{R}^+,\ \mathbf{C}^d)$ be the Hilbert space of vector fields $g : Q\times\mathbf{R}^+\to\mathbf{C}^d$ with norm defined by

$$\|g\|^2=\sum_{x\in Q}\int_0^\infty dt|g(x,t)|^2.$$

We define an operator $T_{\Lambda,Q}$ on the vector fields g by $T_{\Lambda,Q}\ g(x,t)=\nabla u(x,t)$ where $u(x,t)$ is the solution to the equation,

$$\begin{aligned}\frac{1}{\Lambda}\ \frac{\partial u}{\partial t}(x,t) &= \Delta u(x,t)+\nabla^* g(x,t),\quad t>0,\\ u(x,0) &= 0,\quad x\in Q.\end{aligned}\tag{3.10}$$

It is easy to see that $T_{\Lambda,Q}$ is a bounded operator on $L^2(Q\times\mathbf{R}^+,\ \mathbf{C}^d)$ with $\|T_{\Lambda,Q}\|\le 1$. For fixed $\omega\in\Omega$ we may also define an operator $\mathbf{b}_\omega$ on $L^2(Q\times\mathbf{R}^+,\ \mathbf{C}^d)$ by

$$\mathbf{b}_\omega\ g(x,t)=\mathbf{b}(\tau_{x,t}\omega)g(x,t),\ x\in Q,\ t>0.$$

Evidently from (3.9) $\mathbf{b}_\omega$ is a bounded operator on $L^2(Q\times\mathbf{R}^+,\ \mathbf{C}^d)$ with $\|\mathbf{b}_\omega\|\le 1-\lambda/\Lambda$. Let $G_Q(x,t),\ x\in Q,\ t>0$, be the Green's function satisfying the equation,

$$\begin{aligned}\frac{\partial G_Q}{\partial t}\ (x,t) &= \Delta\ G_Q(x,t),\ \ x\in Q,\ t>0,\\ G_Q(x,0) &= \delta(x),\quad x\in Q.\end{aligned}\tag{3.11}$$

Observe that $\nabla G_Q\in L^2(Q\times\mathbf{R}^+,\ \mathbf{C})$ with norm satisfying $\|\nabla G_Q\|\le 1/\sqrt{2}$. Now the function $G_{\mathbf{a},Q}(x,0,t,\omega)$ of (3.2) satisfies the equation,

$$\begin{aligned}\frac{1}{\Lambda}\frac{\partial G_{\mathbf{a},Q}}{\partial t} &= \Delta\ G_{\mathbf{a},Q}+\nabla^*(\mathbf{b}(\tau_{x,t}\ \omega))\nabla G_{\mathbf{a},Q}\quad t>0,\\ G_{\mathbf{a},Q}(x,0,0,\omega) &= \delta(x),\quad x\in Q.\end{aligned}\tag{3.12}$$

If we compare (3.12) with (3.10), (3.11) we see that $\nabla G_{\mathbf{a},Q}(x,0,t,\omega)$ satisfies the equation,

$$\nabla G_{\mathbf{a},Q}=\nabla G_{\Lambda,Q}+T_{\Lambda,Q}(\mathbf{b}_\omega\nabla G_{\mathbf{a},Q}),\tag{3.13}$$

where $G_{\Lambda,Q}(x,t)=G_Q(x,\Lambda t)$. Since $\|\nabla G_Q\|\le 1/\sqrt{2}$ it follows that $\|\nabla G_{\Lambda,Q}\|\le 1/\sqrt{2\Lambda}$. Since $\|T_{\Lambda,Q}\mathbf{b}_\omega\|\le 1-\lambda/\Lambda$ the Neumann series for the solution of (3.13) converges. For any $n=1,2,...,$ we write

$$\nabla G_{\mathbf{a},Q}=\sum_{k=0}^n\Big(T_{\Lambda,Q}\mathbf{b}_\omega\Big)^k\nabla G_{\Lambda,Q}+\Big(T_{\Lambda,Q}\mathbf{b}_\omega\Big)^{n+1}\Big[I-T_{\Lambda,Q}\mathbf{b}_\omega\Big]^{-1}\ \nabla G_{\Lambda,Q}.$$

Define $G_{\mathbf{a},Q,n}$ by $G_{\mathbf{a},Q,n}(x,t,\omega)=G_{\Lambda,Q}(x,t)+u(x,t)$ where $u(x,t)$ is the solution to (3.10) with $g(x,t)$ given by

$$g(x,t)=\mathbf{b}(\tau_{x,t}\ \omega)\sum_{k=0}^{n-1}\Big(T_{\Lambda,Q}\mathbf{b}_\omega\Big)^k\nabla G_{\Lambda,Q}(x,t).$$

It follows from Proposition 2.2 that

(3.14)

$$\lim_{m\to 0}\lim_{L\to\infty}\int_0^\infty dt\ e^{-\eta t}\ \langle G_{\mathbf{a},Q_L,n}(x,t,\cdot)\rangle_{Q_{L,m}}=\int_0^\infty dt\ e^{-\eta t}\ \ \langle G_{\mathbf{a},n}(x,t,\cdot)\rangle_{\hat\Omega}\ ,$$

where $G_{\mathbf{a},n}(x,t,\omega) = \lim_{L\to\infty} G_{\mathbf{a},Q_L,n}(x,t,\omega)$. We need to bound the error $G_{\mathbf{a},Q} - G_{\mathbf{a},Q,n}$ as $|Q| \to \infty$. To see this first note from (3.10) that

$$\sum_{x\in Q} |u(x,t)|^2 \le \Lambda \|g\|^2/2,$$

whence the solution $u(x,t)$ of (3.10) satisfies the inequality,

$$\int_0^\infty e^{-\eta t}|u(x,t)|dt \le \sqrt{\Lambda}\|g\|/\eta\,\sqrt{2}. \tag{3.15}$$

Now $G_{\mathbf{a},Q} - G_{\mathbf{a},Q,n}$ is the solution to (3.10) with g given by

$$g = \mathbf{b}(\tau_{x,t}\,\omega)\Big(T_{\Lambda,Q}\mathbf{b}_\omega\Big)^n \Big[I - T_{\Lambda,Q}\mathbf{b}_\omega\Big]^{-1}\,\nabla G_{\Lambda,Q}.$$

It follows therefore from (3.15) that

$$\int_0^\infty e^{-\eta t}|G_{\mathbf{a},Q}(x,0,t,\omega) - G_{\mathbf{a},Q,n}(x,t,\omega)|dt \le \left(1-\frac{\lambda}{\Lambda}\right)^n\,\frac{\Lambda}{2\lambda\eta}. \tag{3.16}$$

Evidently for any $\eta > 0$ we can make the RHS of (3.16) as small as we please by simply taking n large enough. The result follows from this and (3.14). □

Acknowledgement: This research was partially supported by NSF under grant DMS-0138519.

References

[1] H. Brascamp and E. Lieb, *On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation*, J. Functional Analysis **22** (1976), 366-389, MR 56#8774.

[2] D. Brydges and G. Keller, *Correlation functions of general observables in dipole-type systems. I. accurate upper bounds*, Helv. Phys. Acta. **67** (1994), 43-116. MR 95h:82006.

[3] D. Brydges and H. T. Yau, *Grad ϕ perturbations of massless Gaussian fields*, Comm. Math. Phys. **129** (1990), 351-392, MR 91e:81053.

[4] J. Conlon, *Greens Functions for Elliptic and Parabolic Equations with Random Coefficients II*, 2002 preprint, available at www.math.lsa.umich.edu/∼conlon.

[5] J. Conlon and A. Naddaf, *Greens Functions for Elliptic and Parabolic Equations with Random Coefficients*, New York J. Math. **6** (2000), 153-225, MR 2001j:35282.

[6] R. Durrett, *Probability: theory and examples*, second edition, Duxbury Press, Belmont CA. 1996. MR 98m:60001.

[7] T. Funaki and H. Spohn, *Motion by mean curvature from the Ginzburg-Landau $\nabla\phi$ interface model*, Comm. Math. Phys. **185** (1997), 1-36, MR 98f:60206.

[8] G. Giacomin, S. Olla and H. Spohn, *Equilibrium fluctuations for $\nabla\phi$ interface model*, Ann. Probab. **29** (2001), 1138-1172, MR 1872740.

[9] B. Helffer and J. Sjöstrand, *On the correlation for Kac-like models in the convex case*, J. Statist. Phys. **74** (1994), 349-409, MR 95q:82022.

[10] S. Kozlov, *Averaging of random structures*, Dokl. Akad. Nauk. SSSR **241** (1978), 1016-1019, MR 80e:60078.

[11] C. Landim, S. Olla and H. T. Yau, *Convection-diffusion equation with space-time ergodic random flow*, Probab. Theory Relat. Fields **112** (1998), 203-220, MR 99j:35084.

[12] A. Naddaf and T. Spencer, *On homogenization and scaling limit of some gradient perturbations of a massless free field*, Comm. Math. Phys. **183** (1997), 55-84, MR 98m:81089.

[13] H. Osada and H. Spohn, *Gibbs measures relative to Brownian motion*, Ann. Probab. **27** (1999), 1183-1207, MR 2001f:82024.

[14] G. Papanicolaou and S. Varadhan, *Boundary value problems with rapidly oscillating random coefficients*, Volume 2 of *Coll. Math. Soc. Janos Bolya*, **27**, Random fields, Amsterdam, North Holland Publ. Co. 1981, pp. 835-873, MR 84k:58233.

[15] T. Spencer, *Scaling, the free field and statistical mechanics*, in The Legacy of Norbert Wiener: A Centennial Symposium (Cambridge MA, 1994) Proc. Sympos. Pure Math. **60**, Amer. Math. Soc., Providence 1997, MR 98k:82009.
[16] V. Zhikov, S. Kozlov and O. Oleinik, *Homogenization of Differential Operators and Integral Functionals*, Springer Verlag, Berlin, 1994, MR 96h:35003b.

University of Michigan, Department of Mathematics, Ann Arbor, MI 48109-1109
E-mail address: conlon@umich.edu