# The Stability of

# the Non-Equilibrium

# Steady States


Yoshiko Ogata

*Department of Physics, Graduate School of Science*
*The University of Tokyo*
*7-3-1 Hongo, Bunkyo-ku, Tokyo*
*113-0033, Japan*



We show that the non-equilibrium steady state of the free lattice Fermion model far from equilibrium is macroscopically unstable. The problem is translated to that of the spectral analysis of *Liouville Operator*, we use the method of positive commutators to investigate it. We construct a positive commutator on the lattice Fermion system, whose dispersion relation is $\omega(k) = \cos k$.


---


e-mail: ogata@monet.phys.s.u-tokyo.ac.jp

# 1 Introduction

The investigation of equilibrium state in statistical physics has a long history. In mathematical framework, an equilibrium state is defined as a state which satisfies the Kubo-Martin-Schwinger condition. This is a generalization of the Gibbs equilibrium state. Many researches to justify this definition have been made [1]-[5]. One of them is the investigation of "*return to equilibrium*"; an arbitrary initial state that is *normal* with respect to the equilibrium state will converge to the equilibrium state. Recall that the notion of *normality* represents the macroscopic equivalence in quasi-local system.(See Definition 2.1below.) Although it is a physically fundamental phenomenon, to prove it rigorously is not easy. A complete proof of return to equilibrium in the two-sided XY-model was given by H. Araki [6]. The other example is an open quantum system, whichconsists of a finite subsystem and an infinitely extended reservoir in equilibrium [7]-[10]. The open quantum system converges to the asymptotic state, which is the equilibrium state of the coupled systems. The notable fact here is that this state is normal to the initial one, i.e., macroscopically equivalent to the initial state. This shows that the equilibrium state of the reservoir is macroscopically stable when a finite subsystem is connected to.

Recently non-equilibrium steady state (NESS) far from equilibrium has attracted considerable interests. The NESS is introduced as a state asymptotically realized from an inhomogeneous initial state [11]-[12]. A question rises naturally here; is the NESS macroscopically stable? As an analogy of return to equilibrium in open system, we connect a finite small system to the NESS through a bounded interaction. Will the NESS converge to a state that is normal to the initial state or not? We consider this problem about a free Fermion model on one-dimensional lattice. The explicit form of the NESS is known on this model [19]-[20]. We show if the NESS is far from equilibrium, it is macroscopically unstable.

Technically, the investigation corresponds to the study of the spectral property about the Liouville operator, which represents the dynamics on the Gelfand Naimark Segal (GNS) Hilbert space of the initial state. We use the positive commutator method to analyze the spectrum. It is well-known that a radiation field system with energy dispersion relation $\omega(k) = |k|$ has a nice covariance property and the positive commutator is constracted with the aid of it. However, in our system, the dispersion relation is $\omega(k) = \cos k$, a new method for constructing a positive commutator is required. We shall construct such a positive commutator and investigate the macroscopic stability of the NESS in this paper.

The paper is organized as follows. In the next section, we introduce basic definitions and notations, then state the main theorems. In Section 3, we review the standard theory and represent the role of it in the research of the NESS. Section 4 is devoted for the explanation of the modular structure of our model. In Section 5, we introduce the rescaling group which is the key to construct the positive commutator. In order for the positive commutator method to work, we have to introduce the cut off of the interaction. This is done in Section 6. Section 7 is devoted for the construction of positive commutator. Then in

Section 8, we derive the spectral property of the Liouville operator using the method of M.Merkli [10] on Virial Theorem. We complete the proof in Section 9. Below, for self adjoint operator $A$, we denote by $P(A \subset I)$ the spectral projection of $A$ to the subset $I$.

# 2 Main Results

## 2.1 *Macroscopic stability of the NESS in $C^*$-algebraic theory*

A $C^*$-dynamical system is a pair $(\mathcal{O}, \tau)$ where $\mathcal{O}$ is a $C^*$-algebra, and $\tau$ is a strongly continuous one-parameter group of automorphisms of $\mathcal{O}$. The elements of $\mathcal{O}$ describe observables in the physical system and $\tau$ specifies their time evolution. Below we assume that $\mathcal{O}$ has an identity. A physical state is described as a positive linear functional with norm 1. Let $\omega$ be a state on $\mathcal{O}$ with GNS triple $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$. The notion of $\omega$-normal is defined as follows:

**Definition 2.1** *A state $\eta$ is said to be $\omega$-normal if there exists a density matrix $\rho$ on $\mathcal{H}_\omega$ such that $\eta(\cdot) = \mathrm{Tr}\rho\pi_\omega(\cdot)$.*

If a state is not $\omega$-normal, it is called $\omega$-singular. For quasi-local algebra, this is equal to that $\eta$ is approximated in norm topology by local perturbation of $\omega$.

The NESS of dynamics $\tau_t$ associated with the initial state $\omega_0$ are the weak$-*$ accumulation points of the set of states

$$\frac{1}{T}\int_0^T \omega_0 \circ \tau_t dt$$

as $T \to \infty$. We denote the set of the NESS by $\Sigma_\tau(\omega_0)$. As the set of states on $\mathcal{O}$ is weak $*$ -compact by Alaoglu's Theorem, $\Sigma_\tau(\omega_0)$ is a non-empty set whose elements are $\tau$-invariant.

Now we are interested in the stability of elements in $\Sigma_\tau(\omega_0)$. Let us perturb the dynamics $\tau$ by a bounded operator $V \in \mathcal{O}$, and denote by $\tau_V$ the perturbed dynamics. We shall say that a NESS $\omega \in \Sigma_\tau(\omega_0)$ is *macroscopically unstable under a perturbation $V$*, if no element of $\Sigma_{\tau_V}(\omega)$ is $\omega$-normal. Let us explain it in detail for our system. First we divide the one-dimensional lattice to the left and the right, and consider a state that each side is in equilibrium at different temperature. This is the initial state $\omega_0$. The corresponding NESS $\Sigma_\tau(\omega_0)$ under the time evolution of free lattice Fermion $\tau$ consists of only one point: $\Sigma_\tau(\omega_0) = \{\omega\}$. To investigate the stability of $\omega$, we prepare an external finite system described in a finite dimensional Hilbert space $\mathbb{C}^d$. We connect it to $\omega$ through a bounded interaction $V$ and investigate if $\Sigma_{\tau_V}(\omega)$ has an $\omega$-normal state or not. If there is no $\omega$-normal state in $\Sigma_{\tau_V}(\omega)$, $\omega$ is macroscopically unstable.

## 2.2 *The model*

In this paper we consider the free lattice Fermion system in one dimension. The explicit form of the NESS is known for this system [19].

Let $\mathfrak{h} \equiv l^2(\mathbb{Z})$ be a Hilbert space of a single Fermion. By Fourier transformation, it is unitary equivalent to $L^2([-\pi,\pi))$. The Hamiltonian $h$ of a single Fermion is given by

$$(hf)(n) = \frac{1}{2}(f(n-1) + f(n+1)) - \gamma f(n), \tag{1}$$

which is described in Fourier representation as

$$\widehat{hf}(k) = \omega(k)\hat{f}(k),$$
$$\omega(k) = \cos(k) - \gamma.$$

The $\gamma$-term represents the interaction with the external field. The free Fermi gas is described as the CAR-algebra $\mathcal{O}_f$ over $\mathfrak{h}$. And its dynamics is given by

$$\alpha_t^f(a(f)) = a\left(\mathrm{e}^{ith}f\right). \tag{2}$$

In the initial state $\omega_0$, the lattice is separated into the left and the right. And they are kept at different inverse temperature $\beta_-, \beta_+$, respectively. The NESS $\omega_\rho$ associated with $\omega_0$ is realized as the asymptotic state under the dynamics (2). The explicit form of $\omega_\rho$ was obtained by Ho and Araki: $\omega_\rho$ is a state whose $n$-point functions have a structure

$$\omega_\rho(a(f_n)^* \cdots a(f_1)^* a(g_1) \cdots a(g_m)) = \delta_{nm} \det(\langle f_i, \rho g_j \rangle), \tag{3}$$

where $\rho$ is represented as a multiplication operator,

$$\rho(k) = \begin{cases} \left(1 + e^{-\beta_+ \epsilon(k)}\right)^{-1} & k \in [0,\pi) \\ \left(1 + e^{-\beta_- \epsilon(k)}\right)^{-1} & k \in [-\pi, 0) \end{cases}, \tag{4}$$
$$\epsilon(k) = \cos(k) - \gamma,$$

in the Fourier representation. In this paper, the stability of this state is considered when we connect a small finite system.

The observables of the small system are described as $C^*$-algebra $\mathcal{O}_S \equiv B(\mathfrak{H}_S)$ on a finite $d$-dimensional Hilbert space $\mathfrak{H}_S = \mathbb{C}^d$. We denote by $H_S$ the free Hamiltonian of the system. The free dynamics $\alpha_t^S$ is given by

$$\alpha_t^S(A) = \mathrm{e}^{itH_S} A \mathrm{e}^{-itH_S}.$$

The combined system $\mathcal{S} + \mathcal{R}$ is described as the $C^*$-algebra

$$\mathcal{O} \equiv \mathcal{O}_S \otimes \mathcal{O}_f.$$

The free dynamics of the combined system is given by $\alpha_t^0 = \alpha_t^S \otimes \alpha_t^f$. We denote by $\delta_0$ the derivation of $\alpha_t^0$.

Let us consider the dynamics including the interaction between $\mathcal{S}$ and $\mathcal{R}$. In this paper, we define the interaction term $V$ by

$$V = \lambda \cdot Y \otimes \left(a\left(f\right) + a^{*}\left(f\right)\right).$$

Here $\lambda$ is a coupling constant and $Y$ is a self-adjoint operator on $\mathcal{H}_S$. Note that $V$ is an element of $\mathcal{O}$. The perturbed dynamics $\alpha_t$ is generated by $\delta = \delta_0 + i[V, \cdot]$ with $D(\delta) = D(\delta_0)$. $\alpha_t$ is expanded as follows;

$$\alpha_t\left(A\right) \equiv \alpha_t^0\left(A\right) + \sum_{n\geq 1} i^n \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \left[\alpha_{t_n}^0\left(V\right), \left[\cdots, \left[\alpha_{t_1}^0\left(V\right), \alpha_t^0\left(A\right)\right]\right]\right]. \tag{5}$$

The right hand side converges in norm topology in $\mathcal{O}$. $\alpha_t$ is strongly continuous one parameter group of automorphisms.

### 2.3 *Positive commutator method*

By the standard theory (see Section 3), the problem of the existence of a $\omega$-normal state in $\Sigma_\tau(\omega)$ is translated to the problem of the nontriviality of the Liouville operator's kernel. To investigate the kernel, we use the positive commutator method. In order to construct the positive commutator for the field with dispersion relation $\omega(k) = \cos k - \gamma$, we carry out the variable transformation from $k \in [-\pi, \pi)$ to $t \in \mathbb{R}$, by $t = \tan\frac{k}{2}$. Under this ariable translation, we identify $\mathfrak{h} = L^2([-\pi, \pi), dk)$ with $L^2(\mathbb{R}, \frac{2}{t^2+1}dt)$. Then we define the rescaling group with respect to $t$ (Section 5). The positive commutator is constructed with the aid of the generator of the unitary representation of this group. Due to the property of the dispersion relation $\omega(k) = \cos k - \gamma$, we need a cut off of the interaction term. (See Section 6.)

### 2.4 *Main Theorem*

For the analysis, we need several assumptions on the small system and the interaction $V$.

**Assumption 2.1** *Under the identification $\mathfrak{h} = L^2([-\pi, \pi), dk) = L^2(\mathbb{R}, \frac{2}{t^2+1}dt)$, let $u(\theta)$ be a strongly continuous one parameter unitary gruop on $\mathfrak{h}$ defined by*

$$\left(u(\theta)f\right)(t) \equiv \mathrm{e}^{\frac{\theta}{2}} \sqrt{\frac{t^2+1}{\mathrm{e}^{2\theta}t^2+1}} f\left(\mathrm{e}^{\theta}t\right), \quad \theta \in \mathbb{R}, \quad f \in L^2([-\infty, \infty), \frac{2}{t^2+1}dt),$$

*and let $p$ be the generator of $u(\theta)$. For a constant $0 < v < 1$, let $\Lambda_v$ be the interval of $\mathbb{R}$ defined by*

$$\Lambda_v = \{t \in \mathbb{R}\ ;\ \ s_1(t) = \frac{4t^2}{(1+t^2)^2} \geq v\}.$$

*Define* $g_i^j (i,j=1,2)$ *by*

$$g_1^1 = (1-\rho)^{\frac{1}{2}} f, \quad g_1^2 = \rho^{\frac{1}{2}} \bar{f}$$
$$g_2^1 = \rho^{\frac{1}{2}} f, \quad g_2^2 = (1-\rho)^{\frac{1}{2}} \bar{f}.$$

*We assume that* $g_i^j$ $(i,j=1,2)$ *are in the domain* $D(p^3)$. *Furthermore, we assume that there exists* $0<v<1$ *such that the support of* $g_i^j$ *satisfy*

$$\mathrm{supp} g_i^j \subset \Lambda_v.$$

**Assumption 2.2** *Let* $e_i, e_j$ *be an arbitrary pair of eigenvalues of* $L_S = H_S \otimes 1 - 1 \otimes H_S$ *such that* $e_i \neq e_j$*.There exists* $k_{ij}$ *such that*

$$\omega(\pm k_{ij}) = \cos k_{ij} - \gamma = e_i - e_j,$$
$$0 \leq k_{ij}, \quad k_{ij} \in \Lambda_v.$$

**Assumption 2.3** *1. The Hamiltonian* $H_S$ *has no degenerated eigenvalue.*

*2. There exists* $a>0$ *such that*

$$\min_{e_i \neq e_j} \{|g_1^1(k_{ij})|, |g_1^2(k_{ij})|, |g_2^1(k_{ij})||g_2^2(k_{ij})|\} \geq a.$$

**Assumption 2.4** *Let* $\varphi_n$ *be the* $n$*-th eigenvector of* $H_S$ *with eigenvalue* $E_n$*, and let* $p_n$ *be the spectral projection onto* $\varphi_n$*. We denote the eigenvalue of* $L_S = H_S \otimes 1 - 1 \otimes H_S$ *as* $E_{n,m} \equiv E_n - E_m$*, and introduce the following subsets:*

$$N_l^{(j)} \equiv \{i; E_{i,j} = e\}, \quad N_r^{(i)} \equiv \{j; E_{i,j} = e\}$$
$$N_l \equiv \bigcup_j N_l^{(j)}, \quad N_r \equiv \bigcup_i N_r^{(i)}.$$

*1. For* $e \neq 0$*, we assume*

$$2\delta_0 \equiv \min_{n \in N_l} \inf \sigma \left( p_{N_r^{(n)}} \bar{Y} p_{N_r^c} \bar{Y} p_{N_r^{(n)}} \Big|_{p_{N_r^{(n)}} \mathcal{H}_S} \right) + \min_{m \in N_r} \inf \sigma \left( p_{N_l^{(m)}} Y p_{N_l^c} Y p_{N_l^{(m)}} \Big|_{p_{N_l^{(m)}} \mathcal{H}_S} \right) > 0.$$

*2. Let* $Y_{mn} \equiv \langle \varphi_m | Y | \varphi_n \rangle$*. We assume* $|Y_{mn}| > 0$ *for all* $m \neq n$

We obtain the following results.

**Theorem 2.2** *The NESS which is far from equilibrium is macroscopically unstable.*

**Theorem 2.3** *Suppose that Assumption 2.1 to 2.4 are satisfied. Then for sufficiently small* $\lambda$*, the system shows return to equilibrium,i.e.,any state which is normal to the equilibrium state converges to the equilibrium state.*

# 3 The Standard Theory and NESS

In this section, we recapitulate the standard theory (see [14]) and explain its role in the investigation of the NESS. Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathfrak{H}$ with a cyclic and separating vector $\Omega$. A positive linear functional $\omega$ on $\mathcal{M}$ is said to be *normal* if there exists a density matrix $\rho$ such that

$$\omega(\cdot) = \mathrm{Tr}\left(\rho(\cdot)\right).$$

We denote by $\mathcal{M}_{*+}$ the set of all normal positive linear functional. We can define an operator $S_0$ on a dense set $\mathcal{M}\Omega$ by

$$S_0 x\Omega = x^*\Omega.$$

$S_0$ is closable and we represent the polar decomposition of the closure $\bar{S}_0$ as

$$\bar{S}_0 = J\Delta^{\frac{1}{2}}.$$

$J$ is called the *modular conjugation* and $\Delta$ is called the *modular operator*. By the Tomita-Takesaki Theory we have $J\mathcal{M}J = \mathcal{M}'$, where $\mathcal{M}'$ is the commutant of $\mathcal{M}$. Let $j : \mathcal{M} \to \mathcal{M}'$ be the antilinear $*$-isomorphism defined by $j(x) = JxJ$. We define the *natural positive cone* $\mathcal{P}$ by the closure of the set $\{xj(x)\Omega; x \in \mathcal{M}\}$. We have $J\xi = \xi$ for all $\xi$ in $\mathcal{P}$. For $\xi \in \mathcal{P}$, define the normal positive functional $\omega_\xi \in \mathcal{M}_{*+}$ by

$$\omega_\xi(x) = \langle \xi, x\xi \rangle \quad \forall x \in \mathcal{M}.$$

We have the following Theorems;

**Theorem 3.1** *For any $\omega \in \mathcal{M}_{*+}$, there exists a unique $\xi(\omega) \in \mathcal{P}$ such that $\omega = \omega_{\xi(\omega)}$. Moreover, the following estimates are valid for any $\xi, \eta \in \mathcal{P}$;*

$$\|\xi - \eta\|^2 \leq \|\omega_\xi - \omega_\eta\| \leq \|\xi - \eta\| \, \|\xi + \eta\|.$$

**Theorem 3.2** *For any $*$-automorphism $\alpha$ of $\mathcal{M}$, there exists a unique unitary operator $U(\alpha)$ satisfying the following property:*

1. $U(\alpha)\, xU(\alpha)^* = \alpha(x), \quad x \in \mathcal{M};$
2. $U(\alpha)\,\mathcal{P} \subset \mathcal{P}$ *and* $U(\alpha)\,\xi(\omega) = \xi\left(\alpha^{-1*}(\omega)\right)\ \omega \in \mathcal{M}_{*+}$, *where* $(\alpha^*\omega)(A) \equiv \omega(\alpha(A))$,
3. $[U(\alpha), J] = 0$

The following corollary can be easily checked.

**Corollary 3.3** *If unitary operator $U$ satisfies*

$$UxU^* = \alpha(x), \quad Uj(x)U^* = j(\alpha(x)) \quad x \in \mathcal{M},$$

$$U\Omega \in \mathcal{P},$$

*then $U$ is the unique unitary operator which satisfies the conditions of Theorem 3.2.*

Let $\alpha_t$ be a one parameter group of automorphisms and let $U(t)$ be a one parameter group of unitaries associated with $\alpha_t$. If $U(t)$ is strongly continuous, it is written as $U(t) = \mathrm{e}^{itL}$ with self adjoint operator $L$. We call $L$ the Liouville operator of $\alpha_t$.

Now we move from $W^*$-dynamical systems to $C^*$-dynamical systems, and explain the role of the standard theory in the investigation of the NESS. Let $\alpha_t$ be a one parameter group of automorphisms which describes the dynamics of a unital $C^*$-algebra $\mathcal{O}$, and let $\omega$ be a state of $\mathcal{O}$. What we are interested in is roughly speaking, the asymptotic state $w - \lim_{T\to\infty} \omega \circ \alpha_T$. In practice, the existence of the limit point is not trivial. However, as the space of states is weak-$*$ compact, the set $\{\omega \circ \alpha_t \ \ t \in \mathbb{R}\}$ has accumulation points. But they may not be invariant under $\alpha_t$. To make invariant states, we integrate the states;

$$\omega^T = \frac{1}{T}\int_0^T \omega \circ \alpha_t dt.$$

Again $\omega^T$ has weak-$*$ accumulation points, which are invariant under $\alpha_t$ this time. We call them the non-equilibrium steady states (NESS) and denote the set of them by $\Sigma^+(\omega)$.

Let $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ be the GNS triple of $\omega$. Suppose that there exists an extension $\tilde{\alpha}_t$ of $\alpha_t$ to the von Neumann algebra $\pi_\omega(\mathcal{O})''$. Then by Theorem 3.2 there is one parameter group of unitary operators $U_t$ such that

$$\begin{aligned} &\tilde{\alpha}_t(x) = U_t x U_t^* \quad x \in \pi_\omega(\mathcal{O})'' \\ &U_t^* \mathcal{P} \subset \mathcal{P}. \end{aligned} \tag{6}$$

In addition, suppose that $U_t$ is strongly continuous and let $L$ be the Liouville operator. If $\omega_\infty \in \Sigma^+(\omega)$ is $\omega$-normal, there exists a vector $\xi_\infty$ in $\mathcal{P}$ s.t.

$$\omega_\infty(A) = \langle \xi_\infty, \pi_\omega(A)\xi_\infty \rangle, \quad A \in \mathcal{O},$$

by Theorem 3.1. As $\omega_\infty \in \Sigma^+(\omega)$, $\omega_\infty$ is invariant under $\alpha_t$: $\omega_\infty \circ \alpha_t = \omega_\infty$. Hence we have

$$\langle U_t^*\xi_\infty, \pi_\omega(A) U_t^*\xi_\infty \rangle . = \omega_\infty \circ \alpha_t(A) = \omega_\infty(A) = \langle \xi_\infty, \pi_\omega(A)\xi_\infty \rangle, \quad A \in \mathcal{O}.$$

As $\xi_\infty, U_t\xi_\infty \in \mathcal{P}$, we get

$$U_t^*\xi_\infty = \xi_\infty,$$

by Theorem 3.1. This means

$$\xi_\infty \in \mathrm{Ker} L.$$

In other words, we have the following proposition:

**Proposition 3.4** *If* $\mathrm{Ker} L = \{0\}$*, any state in* $\Sigma^+(\omega)$ *is* $\omega$*-singular.*

# 4 The Modular Structure of the Model

First we consider the modular structure of a finite system. Let $\mathcal{O}_S = B(\mathfrak{H}_S)$ be the $C^*$-algebra of bounded operators on a finite dimensional Hilbert space $\mathfrak{H}_S$. Let $\psi \to \bar{\psi}$ be a complex conjugation with respect to the basis which orthogonarizes $H_S$. If $A$ is a linear operator on $\mathfrak{H}_S$, we denote by $\bar{A}$ the linear operator

$$\bar{A}\psi = \overline{A\bar{\psi}}.$$

Any state $\omega_S$ on $\mathcal{O}_S$ is described by a density matrix $\rho_S$ as $\omega_S(A) = \mathrm{Tr}(\rho_S A)$.

**Theorem 4.1** *Suppose that* $\rho_S > 0$ *and*

$$\rho_S = \sum_i p_i \left|\psi_i\right\rangle\left\langle\psi_i\right|.$$

*Then the GNS triple* $(\mathcal{H}_S, \pi_S, \Omega_S)$ *associated to* $\omega_S$ *is*

$$\mathcal{H}_S = \mathfrak{H}_S \otimes \mathfrak{H}_S, \quad \pi_S(A) = A \otimes 1, \quad \Omega_S = \sum p_i^{\frac{1}{2}} \psi_i \otimes \bar{\psi}_i.$$

*Furthermore, the modular conjugation* $J_S$ *is*

$$J_S(\phi \otimes \psi) = \bar{\psi} \otimes \bar{\phi},$$

*the dual representation* $\pi_S^\sharp = J_S \pi_S J_S$ *is*

$$\pi_S^\sharp(A) = 1 \otimes \bar{A},$$

*and the natural positive cone is*

$$\mathcal{P}_S = \left\{ \sum_i \varphi_i \otimes \bar{\varphi}_i ; \varphi_i \in \mathcal{H}_S \right\}.$$

As the dimension of $\mathfrak{H}_S$ is finite, $\pi_S(\mathcal{O}_S)$ is itself a von Neumann algebra. Note that $\mathcal{O}_S$ has no non-trivial two-sided ideal and $\pi_S$ is faithful. Hence it is possible to define the free dynamics on the von Neumann algebra $\pi_S(\mathcal{O}_S)$ by

$$\tilde{\alpha}_S \circ \pi_S(A) \equiv \pi_S \circ \alpha_S(A), \quad A \in \mathcal{O}_S.$$

Let an operator $L_S$ on $\mathcal{H}_S$ be

$$L_S = H_S \otimes 1 - 1 \otimes H_S.$$

Then, we have

$$\begin{aligned}
&\mathrm{e}^{itL_S}\pi_S(A)\,\mathrm{e}^{-itL_S} = \pi_S\left(\mathrm{e}^{itH_S}A\mathrm{e}^{-itH_S}\right),\\
&\mathrm{e}^{itL_S}\pi_S^{\sharp}(A)\,\mathrm{e}^{-itL_S} = \pi_S^{\sharp}\left(\mathrm{e}^{itH_S}A\mathrm{e}^{-itH_S}\right),\\
&\mathrm{e}^{itL_S}\Omega \in \mathcal{P}_S.
\end{aligned} \tag{7}$$

Hence by Corollary 3.3, $L_S$ is the Liouville operator associated to $\tilde{\alpha}_S$.

Next we discuss the GNS construction of quasi-free states of the CAR-algebra over $\mathfrak{h}$. A quasi-free state is a state $\omega_\rho$ whose $n$-point functions have the struture

$$\omega_\rho(a^*(f_n)\cdots a^*(f_1)a(g_1)\cdots a(g_m)) = \delta_{n,m}\det\left((g_i,\rho f_j)\right), \tag{8}$$

where $\rho$ is a linear operator on $\mathfrak{h}$ such that $0 \le \rho \le 1$.

Let $\mathcal{F}(\mathfrak{h})$ be a Fermi Fock space over $\mathfrak{h}$, and let $\Omega_0$ be a Fermi vacuum. Let $\mathfrak{j}$ be a conjugate operator on $\mathfrak{h}$, defined by

$$\widehat{(\mathfrak{j}f)}(k) = \overline{\hat{f}(k)},$$

in the Fourier representation, and let $T$ be the antiunitaly operator on $\mathcal{F}(\mathfrak{h})$ defined as

$$T\Omega_0 = \Omega_0 \quad Ta^*(f)T = a^*(\mathfrak{j}f) \quad Ta(f)T = a(\mathfrak{j}f).$$

**Theorem 4.2** *Suppose that $0 < \rho < 1$. Let $(\mathcal{H}_\rho, \pi_\rho, \Omega_\rho)$ be the Araki-Wyss representation defined by*

$$\begin{aligned}
\mathcal{H}_\rho &= \mathcal{F}(\mathfrak{h}) \otimes \mathcal{F}(\mathfrak{h}),\\
\Omega_\rho &= \Omega_0 \otimes \Omega_0,\\
\pi_\rho(a(f)) &= a\left((1-\rho)^{\frac{1}{2}} f\right) \otimes 1 + \vartheta \otimes a^*\left(\rho^{\frac{1}{2}} \bar{f}\right),\\
\pi_\rho(a^*(f)) &= a^*\left((1-\rho)^{\frac{1}{2}} f\right) \otimes 1 + \vartheta \otimes a\left(\rho^{\frac{1}{2}} \bar{f}\right),
\end{aligned}$$

*where $\vartheta \equiv (-1)^N$. Then $(\mathcal{H}_\rho, \pi_\rho, \Omega_\rho)$ is the GNS triple associated with the state $\omega_\rho$ (8). Moreover, the dual representation $\pi_\rho^\sharp$ is given by*

$$\begin{aligned}
\pi_\rho^\sharp(a^*(f)) &= \vartheta a\left(\rho^{\frac{1}{2}} f\right) \otimes \vartheta + 1 \otimes a^*\left((1-\rho)^{\frac{1}{2}} \bar{f}\right)\vartheta,\\
\pi_\rho^\sharp(a(f)) &= a^*\left(\rho^{\frac{1}{2}} f\right)\vartheta \otimes \vartheta + 1 \otimes \vartheta a\left((1-\rho)^{\frac{1}{2}} \bar{f}\right).
\end{aligned}$$

*The modular conjugation $J$ is given by*

$$J(\Phi \otimes \Psi) \equiv uT\Psi \otimes uT\Phi,$$

*where $u \equiv (-1)^{\frac{N(N-1)}{2}}$. There is a relation that*

$$J\pi_\rho(A)J = \pi_\rho^\sharp(A), \quad A \in \mathcal{O}.$$

Below we denote by $\mathcal{M}_f$ the von Neumann algebra $\pi_\rho(\mathcal{O}_f)''$.

Now let us consider the dynamics of the field $\mathcal{O}_f$. The dynamics of the free Fermi gas is given by

$$\alpha_t^f(a(f)) = a\left(e^{ith}f\right)$$

where $h$ is the one particle Hamiltonian (1). In the following suppose that $[h, \rho] = 0$. The NESS (4) satisfies this property. Under this condition, the quasi-free state $\omega_\rho$ (8) is $\alpha_t^f$-invariant. So there exists a unitary operator $U_t$ such that

$$\begin{aligned} U_t\Omega_\rho &= \Omega_\rho, \\ U_t\pi_\rho(A)\Omega_\rho &= \pi_\rho\left(\alpha_t^f(A)\right)\Omega_\rho, \quad A \in \mathcal{O}_f. \end{aligned}$$

We can extend $\alpha_t^f$ to $\mathcal{M}_f$ by

$$\tilde{\alpha}_t^f(x) = U_t x U_t^* \quad x \in \mathcal{M}_f.$$

Let $L_f$ be a self-adjoint operator defined by $L_f \equiv d\Gamma(h) \otimes 1 - 1 \otimes d\Gamma(h)$ where $d\Gamma(h)$ is the second quantization of $h$ in (1). $L_f$ satisfies

$$\begin{aligned} \mathrm{e}^{itL_f}\Omega &= \Omega, \\ \mathrm{e}^{itL_f}x\mathrm{e}^{-itL_f} &= \tilde{\alpha}_t^f(x) \quad x \in \mathcal{M}_f. \\ \mathrm{e}^{itL_f}j(x)\mathrm{e}^{-itL_f} &= j\left(\tilde{\alpha}_t^f(x)\right) \quad x \in \mathcal{M}, \end{aligned} \tag{9}$$

Hence $L_f$ is the Liouville operator of $\tilde{\alpha}_t^f$ by the Corollary 3.3.

We can translate the tensor product of two Fermi Fock spaces $\mathcal{F}(\mathfrak{h}) \otimes \mathcal{F}(\mathfrak{h})$ into one $\mathcal{F}(\mathfrak{h} \oplus \mathfrak{h})$ by a unitary operator. Define the unitary operator $W$ : $\mathcal{F}(\mathfrak{h}) \otimes \mathcal{F}(\mathfrak{h}) \to \mathcal{F}(\mathfrak{h} \oplus \mathfrak{h})$ by

$$\begin{aligned} &W(a^*(f_1) \otimes 1) \cdots (a^*(f_n) \otimes 1)(\vartheta \otimes a^*(g_1)) \cdots (\vartheta \otimes a^*(g_m))\Omega_0 \otimes \Omega_0 \\ &\qquad \equiv a^*(f_1 \oplus 0) \cdots a^*(f_n \oplus 0)\, a^*(0 \oplus g_1) \cdots a^*(0 \oplus g_m)\Omega, \end{aligned}$$

where $\Omega$ is the vacuum of the Fermi Fock space $\mathcal{F}(\mathfrak{h} \oplus \mathfrak{h})$.

**Theorem 4.3** *Let $u_1, u_2$ be unitary operators on $\mathfrak{h}$, and $\Gamma(u_1), \Gamma(u_2)$ be the second quantization of them. There is a unitary operator $W : \mathcal{F}(\mathfrak{h}) \otimes \mathcal{F}(\mathfrak{h}) \to \mathcal{F}(\mathfrak{h} \oplus \mathfrak{h})$ such that*

$$\begin{aligned} W(a(f) \otimes 1)W^* = a(f \oplus 0), \quad W(\vartheta \otimes a(f))W^* = a(0 \oplus f), \\ W(\Gamma(u_1) \otimes \Gamma(u_2))W^* = \Gamma(u_1 \oplus u_2). \end{aligned}$$

In particular, $L_f$ is translated as

$$WL_fW^* = d\Gamma(h \oplus -h).$$

Below we denote $d\Gamma(h \oplus -h)$ by the same symbol $L_f$.

Then we have the following proposition.

**Proposition 4.4** *Suppose that $\omega_S$ is faithful, and $0 < \rho < 1$. The GNS triple $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ associated with the state $\omega = \omega_S \otimes \omega_\rho$ is given by*

$$\mathcal{H}_\omega = \mathcal{H}_S \otimes \mathcal{H}_f, \quad \Omega_\omega = \Omega_S \otimes \Omega, \quad \pi_\omega = \pi_S \otimes \pi_f,$$

*where*

$$\begin{aligned}
&\mathcal{H}_S = \mathfrak{H}_S \otimes \mathfrak{H}_S, \quad \Omega_S = \sum p_i^{\frac{1}{2}} \psi_i \otimes \bar{\psi}_i, \quad \pi_S(A) = A \otimes 1 \qquad A \in \mathcal{O}_S, \\
&\mathcal{H}_f = \mathcal{F}(\mathfrak{h} \oplus \mathfrak{h}), \quad \Omega : \text{vacuum vector of } \mathcal{F}(\mathfrak{h} \oplus \mathfrak{h}) \\
&\quad \pi_f(a(f)) = a\left((1-\rho)^{\frac{1}{2}} f \oplus 0\right) + a^*\left(0 \oplus \rho^{\frac{1}{2}} \bar{f}\right), \\
&\quad \pi_f(a^*(f)) = a^*\left((1-\rho)^{\frac{1}{2}} f \oplus 0\right) + a\left(0 \oplus \rho^{\frac{1}{2}} \bar{f}\right).
\end{aligned}$$

*The modular conjugation is*

$$J = J_S \otimes J_f,$$

*and the dual representation is given by*

$$\pi_\omega^\sharp = \pi_S^\sharp \otimes \pi_f^\sharp,$$

*where*

$$\begin{aligned}
&\pi_S^\sharp(A) = 1 \otimes \bar{A}, \\
&\pi_f^\sharp(a^*(f)) = (-1)^N \left(-a^*\left(0 \oplus (1-\rho)^{\frac{1}{2}} \bar{f}\right) + a\left(\rho^{\frac{1}{2}} f \oplus 0\right)\right), \\
&\pi_f^\sharp(a(f)) = (-1)^N \left(-a^*\left(\rho^{\frac{1}{2}} f \oplus 0\right) + a\left(0 \oplus (1-\rho)^{\frac{1}{2}} \bar{f}\right)\right).
\end{aligned}$$

Below we denote by $\mathcal{M}$ the von Neumann algebra $\pi_\omega(\mathcal{O})''$.

Now we consider the total dynamics on $\mathcal{M}$. First we consider the free dynamics $\tilde{\alpha}_t^0$ which is defined by $\tilde{\alpha}_t^0 \equiv \tilde{\alpha}_t^S \otimes \tilde{\alpha}_t^f$ on $\mathcal{M}$. Let $L_0$ be $L_0 = L_S \otimes 1 + 1 \otimes L_f$ defined on $\mathcal{H}_S \otimes D(L_f)$. $L_0$ satisfies

$$\begin{aligned}
&\tilde{\alpha}_t^0(x) = \mathrm{e}^{itL_0} x \mathrm{e}^{-itL_0}, \quad x \in \mathcal{M}, \\
&\mathrm{e}^{itL_0} J x J \mathrm{e}^{-itL_0} = J \tilde{\alpha}_t^0(x) J, \quad x \in \mathcal{M}, \\
&\mathrm{e}^{itL_0} \Omega \in \mathcal{P}.
\end{aligned}$$

By Corollary 3.3, $L_0$ is the Liouville operator corresponding to $\tilde{\alpha}_t^0$. Especially it satisfies $[\mathrm{e}^{itL_0}, J] = 0$.

Next we extend the perturbed dynamics $\alpha_t$ to $\mathcal{M}$. Let $\tilde{\delta}_0$ be the generator of $\tilde{\alpha}_t^0$ with domain $D(\tilde{\delta}_0)$. The perturbed dynamis $\tilde{\alpha}_t$ of $\mathcal{M}$ is defined by the derivation

$$\tilde{\delta} = \tilde{\delta}_0 + i[I - JIJ, \cdot], \qquad I = \pi_\rho(V),$$

with $D(\tilde{\delta}) = D(\tilde{\delta}_0)$. Note that

$$\begin{aligned} I &= \pi_\rho(V) = (Y \otimes 1) \otimes (a(g_1) + a^*(g_1)), \\ JIJ &= \pi_\rho^\sharp(V) = (1 \otimes \bar{Y}) \otimes (-1)^N (a(g_2) - a^*(g_2)), \end{aligned}$$

where

$$\begin{aligned} g_1 = g_1^1 \oplus g_1^2, &\quad g_2 = g_2^1 \oplus g_2^2, \\ g_1^1 = (1-\rho)^{\frac{1}{2}} f, &\quad g_1^2 = \rho^{\frac{1}{2}} \bar{f}, \\ g_2^1 = \rho^{\frac{1}{2}} f, &\quad g_2^2 = (1-\rho)^{\frac{1}{2}} \bar{f}. \end{aligned} \tag{10}$$

The restriction of $\tilde{\alpha}_t$ to $\mathcal{O}$ is $\alpha_t$.

On the other hands, the perturbed Liouville operator $L$ is defined by

$$L \equiv L_f + I - JIJ.$$

$L$ is a self-adjoint operator, and $\mathrm{e}^{itL}$ is an unitary operator. By perturbation theory, we have

$$\mathrm{e}^{itL} x \mathrm{e}^{-itL} = \tilde{\alpha}_t(x) \qquad x \in \mathcal{M}. \tag{11}$$

As the perturbation $I - JIJ$ satisfies

$$J(I - JIJ)J = -(I - JIJ), \tag{12}$$

and $[\mathrm{e}^{itL_0}, J] = 0$, we have

$$\left[\mathrm{e}^{itL}, J\right] = 0. \tag{13}$$

So we get

$$\begin{aligned} &\mathrm{e}^{itL}\mathcal{P} \subset \mathcal{P}, \\ &\mathrm{e}^{itL}\xi(\omega) = \xi(\tilde{\alpha}_{-t}(\omega)). \end{aligned} \tag{14}$$

Unitary operator satisfying (11),(13),(14) is unique by Theorem 3.2, and $L$ is the Liouville operator of $\tilde{\alpha}_t$.

Hence by Section 3, the problem of investigating the stability of $\omega$ under $\alpha_t$ is translated to the problem of investigating the spectral property of $L$;

$$\begin{aligned} L &= (H_S \otimes 1) \otimes 1 - (1 \otimes H_S) \otimes 1 + 1 \otimes d\Gamma\left(\tilde{h}\right) \\ &+ \lambda (Y \otimes 1) \otimes (a(g_1) + a^*(g_1)) - \lambda \left(1 \otimes \bar{Y}\right) \otimes (-1)^N (a(g_2) - a^*(g_2)) \end{aligned} \tag{15}$$

on $\mathcal{H}_\omega$, where

$$\tilde{h} = h \oplus -h. \tag{16}$$

# 5 Rescaling Group

In order to construct the positive commutator for the lattice system, we introduce a strongly continuous one parameter group of unitaries. We first transform the variable $k \in [-\pi, \pi)$ to $t \in \mathbb{R}$ by

$$t = \tan \frac{k}{2}.$$

By this variable transformation, we have

$$\mathfrak{h} = L^2\left([-\pi, \pi), \mathrm{d}k\right) = L^2\left([-\infty, \infty), \mathrm{d}\mu\right),$$

where $\mu$ is the positive measure on $\mathbb{R}$ given by

$$\mathrm{d}\mu\left(t\right) = \frac{2}{t^2+1}\mathrm{d}t.$$

The multiplication operator $m(k)$ is transformed as $m(k) \rightarrow m(k(t))$. In particular, the single particle energy $h$ is transformed to

$$h\left(k\right) = \cos k - \gamma \quad \rightarrow \quad h\left(k\left(t\right)\right) = \frac{2}{t^2+1} - 1 - \gamma.$$

Now we introduce a unitary operator $u(\theta)$ on $\mathfrak{h}$ defined by

$$\left(u(\theta)f\right)(t) \equiv \mathrm{e}^{\frac{\theta}{2}}\sqrt{\frac{t^2+1}{\mathrm{e}^{2\theta}t^2+1}}f\left(\mathrm{e}^{\theta}t\right), \quad \theta \in \mathbb{R}, \quad f \in L^2([-\infty, \infty), d\mu).$$

By an elementary calculation, we have the following lemma;

**Lemma 5.1** *$u(\theta)$ is a strongly continuous one parameter group of unitaries.*

From this lemma, $u(\theta)$ is written as $u(\theta) = \mathrm{e}^{i\theta p}$ with a selfadjoint operator $p$. The action of $p$ on $C_0^\infty\left(\mathbb{R}\right)$ is

$$\left(pf\right)\left(t\right) = \frac{1}{2}f\left(t\right) - \frac{t^2}{t^2+1}f\left(t\right) + tf'\left(t\right) \quad f \in C_0^\infty\left(\mathbb{R}\right).$$

Let $\tilde{u}(\theta)$ be a unitary operator on $\mathfrak{h} \oplus \mathfrak{h}$ defined by

$$\tilde{u}(\theta) \equiv u(\theta) \oplus u(-\theta).$$

The generator of $\tilde{u}(\theta)$ is $\tilde{p} \equiv p \oplus -p$. We define a unitary operator $U(\theta)$ on $\mathcal{F}(\mathfrak{h} \oplus \mathfrak{h})$ by the second quantization of $\tilde{u}(\theta)$:

$$U\left(\theta\right) \equiv \Gamma\left(\tilde{u}\left(\theta\right)\right).$$

Since $U(\theta)$ is a strongly continuous one parameter unitary group, it is written as $U(\theta) = \mathrm{e}^{\theta A_0}$, where $A_0$ is an anti-selfadjoint operator.

Now we introduce the commutator of $L_f$ and $A_0$ as a quadratic form on the dense set $(D(N)\cap D(A_0)) \times (D(N)\cap D(A_0))$. By straight forward calculation, the action of $\mathrm{Ad}\tilde{u}(\theta)$ on a multiplication operator $m_1 \oplus m_2$ is

$$(\tilde{u}(-\theta)(m_1 \oplus m_2)\tilde{u}(\theta))(t) = m_1\left(\mathrm{e}^{-\theta}t\right) \oplus m_2\left(\mathrm{e}^{\theta}t\right).$$

Then, the action of $\mathrm{Ad}U(\theta)$ on the second quantization $d\Gamma(m_1 \oplus m_2)$ of $m_1 \oplus m_2$ is

$$U(-\theta)\,\mathrm{d}\Gamma(m_1 \oplus m_2)\,U(\theta) = \mathrm{d}\Gamma\left(m_1^{-\theta} \oplus m_2^{\theta}\right), \tag{17}$$

where

$$\left(m_i^{\theta} f\right)(t) = m_i\left(\mathrm{e}^{\theta}t\right) f(t).$$

We have the following proposition:

**Proposition 5.2** *Suppose that $m_1, m_2$ are bounded multiplication operators on $L^2(\mathbb{R}, d\mu)$ which are differentiable and satisfy*

$$\sup_{t\in\mathbb{R}} |qm_i(t)| < \infty,$$

*where*

$$qm_i(t) \equiv -tm_i'(t).$$

*Then we have*

$$\langle \mathrm{d}\Gamma(m_1 \oplus m_2)\varphi, A_0\psi\rangle + \langle A_0\varphi, \mathrm{d}\Gamma(m_1 \oplus m_2)\psi\rangle = \langle \varphi, \mathrm{d}\Gamma(qm_1 \oplus -qm_2)\psi\rangle, \tag{18}$$

*for all $\varphi, \psi \in D(N)\cap D(A_0)$.*

*Proof*
As $U(\theta)$ preserves number, we have $U(\theta)D(N) \subset D(N)$ and have the following relation:

$$\begin{aligned}
&\lim_{\theta\to 0}\left\langle \varphi \frac{\mathrm{d}\Gamma\left(m_1^{-\theta} \oplus m_2^{\theta}\right) - \mathrm{d}\Gamma(m_1 \oplus m_2)}{\theta}\psi \right\rangle \\
&= \lim_{\theta\to 0}\left\langle \mathrm{d}\Gamma(m_1 \oplus m_2)U(\theta)\varphi, \frac{U(\theta)-1}{\theta}\psi\right\rangle + \left\langle \frac{U(\theta)-1}{\theta}\varphi, \mathrm{d}\Gamma(m_1 \oplus m_2)\psi\right\rangle.
\end{aligned} \tag{19}$$

Using dominated convergence theorem, we have the following statements:

1. For each $\psi \in D(N)$ and bounded operator $m$ on $\mathfrak{h} \oplus \mathfrak{h}$,

$$\mathrm{d}\Gamma(m)\,U(\theta)\psi \to \mathrm{d}\Gamma(m)\psi, \quad \theta \to 0,$$

2. For each $\psi \in D(A_0)$,

$$\lim_{\theta\to 0} \frac{U(\theta)-1}{\theta}\psi = A_0\psi,$$

3. For each $\psi \in D(N)$,

$$\lim_{\theta\to 0}\left\| \left( \frac{\mathrm{d}\Gamma\left(m_1^{-\theta}\oplus m_2^{\theta}\right)-\mathrm{d}\Gamma\left(m_1\oplus m_2\right)}{\theta} - \mathrm{d}\Gamma\left(qm_1\oplus -qm_2\right)\right)\psi\right\| = 0.$$

Substituting these to (19), we obtain the statement of the proposition. □

Substituting $m_1 = h$, $m_2 = -h$ to (18), we obtain

$$\langle L_f\varphi, A_0\psi\rangle + \langle A_0\varphi, L_f\psi\rangle = \langle \varphi, S_1\psi\rangle .$$

Here $S_1$ is the second quantization $d\Gamma\left(s_1\oplus s_1\right)$ of $s_1\oplus s_1$ on $\mathfrak{h}\oplus\mathfrak{h}$, with $s_1$ defined by

$$\left(s_1 f\right)(t) = s_1(t)f(t), \quad s_1(t) = \frac{4t^2}{(1+t^2)^2}.$$

Note that $S_1$ is positive. This is the main point of this commutator.

Similarly, we have

$$\begin{aligned}\langle S_1\varphi, A_0\psi\rangle + \langle A_0\varphi, S_1\psi\rangle &= \langle \varphi, S_2\psi\rangle ,\\ \langle S_2\varphi, A_0\psi\rangle + \langle A_0\varphi, S_2\psi\rangle &= \langle \varphi, S_3\psi\rangle ,\end{aligned}$$

where

$$\begin{aligned}&S_2 = \mathrm{d}\Gamma\left(s_2\oplus -s_2\right),\\ &\left(s_2 f\right)(t) = s_2(t)f(t), \quad s_2(t) = \frac{8\left(1-t^2\right)t^2}{\left(1+t^2\right)^3},\\ &S_3 = \mathrm{d}\Gamma\left(s_3\oplus s_3\right),\\ &\left(s_3 f\right)(t) = s_3(t)f(t), \quad s_3(t) = \frac{16\left(t^2-4t^4+t^6\right)}{\left(1+t^2\right)^4}.\end{aligned}$$

Note that $S_1, S_2, S_3$ are all $N$-bounded.

The commutator of $A_0$ with the interaction term can also be considered. Suppose that $g_1, g_2 \in D(\tilde{p}^3)$. We denote $I - JIJ$ by $\lambda I_0$, i.e,

$$I_0 \equiv \Big( \left(Y\otimes 1\right)\otimes\left(a\left(g_1\right)+a^*\left(g_1\right)\right) - \left(1\otimes\bar{Y}\right)\otimes\left(-1\right)^N\left(a\left(g_2\right)-a^*\left(g_2\right)\right)\Big). \tag{20}$$

We have

$$\langle \varphi, I_0A_0\psi\rangle + \langle A_0\varphi, I_0\psi\rangle = \langle \varphi, I_1\psi\rangle ,$$

on $D(A_0) \times D(A)$. Here $I_1$ is defined by

$$I_1 \equiv (Y \otimes 1) \otimes (a(-i\tilde{p}g_1) + a^*(-i\tilde{p}g_1)) - \left(1 \otimes \bar{Y}\right) \otimes (-1)^N (a(-i\tilde{p}g_2) - a^*(-i\tilde{p}g_2)).$$

Similarly, we have

$$\langle \varphi, I_1 A_0 \psi \rangle + \langle A_0 \varphi, I_1 \psi \rangle = \langle \varphi, I_2 \psi \rangle, \quad \langle \varphi, I_2 A_0 \psi \rangle + \langle A_0 \varphi, I_2 \psi \rangle = \langle \varphi, I_3 \psi \rangle,$$

with

$$I_2 \equiv (Y \otimes 1) \otimes \left(a\left(-\tilde{p}^2 g_1\right) + a^*\left(-\tilde{p}^2 g_1\right)\right) - \left(1 \otimes \bar{Y}\right) \otimes (-1)^N \left(a\left(-\tilde{p}^2 g_2\right) - a^*\left(-\tilde{p}^2 g_2\right)\right),$$
$$I_3 \equiv (Y \otimes 1) \otimes \left(a\left(i\tilde{p}^3 g_1\right) + a^*\left(i\tilde{p}^3 g_1\right)\right) - \left(1 \otimes \bar{Y}\right) \otimes (-1)^N \left(a\left(i\tilde{p}^3 g_2\right) - a^*\left(i\tilde{p}^3 g_2\right)\right).$$

# 6 Cut Off of the Interaction and Eigenvector

In case the dispersion relation of the field $L_f$ is $\omega(k) = |k|$, $A_0$ is taken as the generator of the the shift operator. And the commutator is given by the number operator $N : [L_f, A_0] = N$. Note that the dispersion relation of $N$ has a strictly positive spectral gap $1 > 0$. However in our case, the dispersion relation is $\omega(k) = \cos(k) - \gamma$, and the commutator is $[L_f, A_0] = S_1$. Note that the dispersion relation of $S_1$ is positive, but it does not have a spectral gap: it attains zero at $t = 0$ and $t = \pm\infty$. The existence of the spectral gap is essential in application of positive commutator method as seen in the arguments in [10]. So this situation causes a problem. To overcome this difficulty, we assume the Assumption 2.1 on the interaction.

**Assumption2.1** *We assume that $g_i^j$ $(i, j = 1, 2)$ are in the domain $D(p^3)$, and the support of $g_i^j$ satisfy*

$$\mathrm{supp} g_i^j \subset \Lambda_v,$$

*for some* $0 < v < 1$.

Now we decompose $\tilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathfrak{h}$ with respect to $\Lambda_v$;

$$\tilde{\mathfrak{h}} = \tilde{\mathfrak{h}}_{\Lambda_v} \oplus \tilde{\mathfrak{h}}_{\Lambda_v^c},$$
$$\tilde{\mathfrak{h}}_{\Lambda_v} = L^2(\Lambda_v, d\mu) \oplus L^2(\Lambda_v, d\mu), \quad \tilde{\mathfrak{h}}_{\Lambda_v^c} = L^2(\Lambda_v^c, d\mu) \oplus L^2(\Lambda_v^c, d\mu),$$

where $\Lambda_v^c$ is the complement of $\Lambda_v$ in $\mathbb{R}$. Note that $\tilde{h}$ (16) is decomposed into $\tilde{h} = \tilde{h}_{\Lambda_v} \oplus \tilde{h}_{\Lambda_v^c}$, with respect to this decomposition of the Hilbert space. Let $N_{\Lambda_v}$ be

$$N_{\Lambda_v} \equiv d\Gamma(1 \oplus 0),$$

with respect to this decomposition. Below we denote by $\mathcal{U}(\mathfrak{h}')$ a set of unitary operators on a Hilbert space $\mathfrak{h}'$. We have the following proposition:

**Proposition 6.1** *There is a unitary operator* $U : \mathcal{F}(\tilde{\mathfrak{h}}_{\Lambda_v} \oplus \tilde{\mathfrak{h}}_{\Lambda_v^c}) \to \mathcal{F}(\tilde{\mathfrak{h}}_{\Lambda_v}) \otimes \mathcal{F}(\tilde{\mathfrak{h}}_{\Lambda_v^c})$ *which satisfies the following conditions;*

1. *For any* $u_{\Lambda_v} \in \mathcal{U}(\tilde{\mathfrak{h}}_{\Lambda_v})$ *and* $u_{\Lambda_v^c} \in \mathcal{U}(\tilde{\mathfrak{h}}_{\Lambda_v^c})$,
$$U\Gamma(u_{\Lambda_v} \oplus u_{\Lambda_v^c})U^* = \Gamma(u_{\Lambda_v}) \otimes \Gamma(u_{\Lambda_v^c}),$$

2. *For any* $f \in \tilde{\mathfrak{h}}_{\Lambda_v}$ *and* $g \in \tilde{\mathfrak{h}}_{\Lambda_v^c}$
$$Ua(f \oplus g)U^* = a(f) \otimes 1 + (-1)^{N_{\Lambda_v}} \otimes a(g).$$

By this proposition, we have

$$U\Gamma(\mathrm{e}^{it\tilde{h}_{\Lambda_v}} \oplus \mathrm{e}^{it\tilde{h}_{\Lambda_v^c}})U^* = \Gamma(\mathrm{e}^{it\tilde{h}_{\Lambda_v}}) \otimes \Gamma(\mathrm{e}^{it\tilde{h}_{\Lambda_v^c}}).$$

Recall that $L$ is given by (15). As the Assumption 2.1 implies $g_1, g_2 \in \tilde{\mathfrak{h}}_{\Lambda_v}$, we obtain the following unitary equivalence;

$$ULU^* = L_S \otimes 1 \otimes 1 + 1 \otimes \mathrm{d}\Gamma_{\Lambda_v}(\tilde{h}_{\Lambda_v}) \otimes 1 + 1 \otimes 1 \otimes \mathrm{d}\Gamma_{\Lambda_v^c}(\tilde{h}_{\Lambda_v^c})$$
$$+\lambda\,(Y \otimes 1)\otimes(a\,(g_1) + a^*\,(g_1))\otimes 1 - \lambda \left(1 \otimes \bar{Y}\right)\otimes(-1)^{N_{\Lambda_v}}\,(a\,(g_2) - a^*\,(g_2))\otimes(-1)^{N_{\Lambda_v^c}}\,.$$

By this equivalence, we have the following lemma,

**Lemma 6.2** *$L$ and $N_\Lambda^c$ strongly commute.*

*Proof*
By the above arguments, it suffices to show that

$$L = (L_S \otimes 1) \otimes 1 + \left(1 \otimes \mathrm{d}\Gamma_{\Lambda_v}(\tilde{h}_{\Lambda_v})\right) \otimes 1 + (1 \otimes 1) \otimes \mathrm{d}\Gamma_{\Lambda_v^c}(\tilde{h}_{\Lambda_v^c}) + V_1 \otimes 1 + V_2 \otimes (-1)^{N_{\Lambda_v^c}},$$

and $1 \otimes N_\Lambda^c$ strongly commute, where $V_1, V_2$ are bounded operators. First note that

$$\mathrm{e}^{itL_0} = \left(\mathrm{e}^{itL_S} \otimes \Gamma_{\Lambda_v}\left(\mathrm{e}^{it\tilde{h}_{\Lambda_v}}\right)\right) \otimes \Gamma_{\Lambda_v^c}\left(\mathrm{e}^{it\tilde{h}_{\Lambda_v^c}}\right),$$

and

$$1 \otimes \mathrm{e}^{isN_{\Lambda_v^c}},$$

commute for all $t, s$:

$$\left[\mathrm{e}^{itL_0}, 1 \otimes \mathrm{e}^{isN_{\Lambda_v^c}}\right] = 0. \tag{21}$$

This means $L_0$ and $N_{\Lambda_v^c}$ strongly commute [16]. Next we show that $L$ and $N_\Lambda^c$ strongly commute. By the perturbation theory, we have

$$\mathrm{e}^{iLt} = \mathrm{e}^{iL_0 t} + \sum_{n=1}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n V(t_n) \cdots V(t_1)\mathrm{e}^{itL_0},$$
$$V(t) = \mathrm{e}^{itL_0}\left(V_1 \otimes 1 + V_2 \otimes (-1)^{N_{\Lambda_v^c}}\right)\mathrm{e}^{-itL_0},$$

where the right hand side converges in norm because of the boundedness of $V$. Using (21), we have

$$\left[V(t), 1\otimes \mathrm{e}^{isN_{\Lambda_v^c}}\right] = \mathrm{e}^{itL_0}\left[V_1\otimes 1 + V_2\otimes(-1)^{N_{\Lambda_v^c}}, 1\otimes \mathrm{e}^{isN_{\Lambda_v^c}}\right]\mathrm{e}^{-itL_0} = 0.$$

Substituting this to the perturbation expansion, we obtain

$$\left[\mathrm{e}^{iLt}, 1\otimes \mathrm{e}^{isN_{\Lambda_v^c}}\right] = 0.$$

This shows that $L$ and $N_{\Lambda_v^c}$ strongly commute. $\square$

Similarly, $N_{\Lambda_v^c}$ strongly commutes with $L_S$, $L_f$, $L_0$, $\lambda I$, $\lambda JIJ$, $N$, $S_1$. Using this fact, we can derive the form of $L$'s eignvector;

**Proposition 6.3** *Suppose $\psi$ is an eigenvector of $L$. Then $\psi$ satisfies*

$$P\left(N_{\Lambda_v^c} = 0\right)\psi = \psi,$$

*where $P\left(N_{\Lambda_v^c} = 0\right)$ is the spectral projection of $N_{\Lambda_v^c}$ corresponding to $N_{\Lambda_v^c} = 0$.*

*Proof*

Let $P_e$ (resp.$P_o$) be the spectral projection of $N_{\Lambda_v^c}$ onto $N_{\Lambda_v^c} =$ even (resp.$N_{\Lambda_v^c} =$ odd), and let $E_n$ be the spectral projection onto the interval $[-n, n]$ of $L$. Let us decompose the Hilbert space $\mathcal{H}$ as

$$\mathcal{H} = P_e\mathcal{H} \oplus P_o\mathcal{H}.$$

Then $L$ is decomposed into

$$L = P_eL \oplus P_oL, \tag{22}$$

with respect to the subspaces $P_e\mathcal{H}$ and $P_o\mathcal{H}$. This is derived by the following consideration: for all $\varphi \in D(L)$, we have

$$\lim_{n\to\infty} LE_n\varphi = \lim_{n\to\infty} E_nL\varphi = L\varphi.$$

By the strong commutativity of $L$ and $N_{\Lambda_v^c}$,

$$P_eL\varphi = P_e\lim_{n\to\infty} LE_n\varphi = \lim_{n\to\infty} P_eLE_n\varphi = \lim_{n\to\infty} LE_nP_e\varphi.$$

On the other hands, we have

$$\lim_{n\to\infty} E_nP_e\varphi = P_e\varphi.$$

Hence, as $L$ is closed, we have $P_e\varphi \in D(L)$ and

$$LP_e\varphi = P_eL\varphi.$$

Similarly, we have $P_o\varphi \in D(L)$ and

$$LP_o\varphi = P_oL\varphi.$$

Hence we obtain (22). In particular, if $\psi$ is an eigenvector of $L$, $P_e\psi$ is an eigenvector of $P_eL$ and $P_o\psi$ is an eigenvector of $P_oL$. In the representation on $\mathcal{H}_S \otimes \mathcal{F}(\tilde{\mathfrak{h}}_{\Lambda_v}) \otimes \mathcal{F}(\tilde{\mathfrak{h}}_{\Lambda_v^c})$, $P_eL$ on $P_e\mathcal{H}$ is

$$P_eL|_{P_e\mathcal{H}} = \left((L_S \otimes 1) + 1 \otimes \mathrm{d}\Gamma_{\Lambda_v}\left(\tilde{h}_{\Lambda_v}\right) + V_1 + V_2\right) \otimes 1 + (1 \otimes 1) \otimes \mathrm{d}\Gamma_{\Lambda_v^c}\left(\tilde{h}_{\Lambda_v^c}\right)|_{\mathcal{H}_e}$$

because

$$P_e = 1 \otimes P(N_{\Lambda_v^c} = even).$$

On the other hand, $P_oL$ on $\mathcal{H}_o$ is

$$P_oL|_{\mathcal{H}_o} = \left(L_S \otimes 1 + 1 \otimes \mathrm{d}\Gamma_{\Lambda_v}\left(\tilde{h}_{\Lambda_v}\right) + V_1 - V_2\right) \otimes 1 + (1 \otimes 1) \otimes \mathrm{d}\Gamma_{\Lambda_v^c}(\tilde{h}_{\Lambda_v^c})|_{\mathcal{H}_o}$$

because

$$P_o = 1 \otimes P(N_{\Lambda_v^c} = odd).$$

Note that $\mathrm{d}\Gamma_{\Lambda_v^c}(\tilde{h}_{\Lambda_v^c})$ and $N_{\Lambda_v^c}$ strongly commute.So $\mathrm{d}\Gamma_{\Lambda_v^c}(\tilde{h}_{\Lambda_v^c})$ is decomposed into $P_e\mathrm{d}\Gamma_{\Lambda_v^c}(\tilde{h}_{\Lambda_v^c}) \oplus P_o\mathrm{d}\Gamma_{\Lambda_v^c}(\tilde{h}_{\Lambda_v^c})$. On the other hand, $\mathrm{d}\Gamma_{\Lambda_v^c}(\tilde{h}_{\Lambda_v^c})$ in $\mathcal{F}(\tilde{\mathfrak{h}}_{\Lambda_v^c})$ has unique eigenvector $\Omega_{\Lambda_v^c}$ which is the vacuum vector of $\mathcal{F}(\tilde{\mathfrak{h}}_{\Lambda_v^c})$. So $\mathrm{d}\Gamma_{\Lambda_v^c}\big|_{P_e\mathcal{F}(\tilde{\mathfrak{h}}_{\Lambda_v^c})}$ has the unique eigenvector $\Omega_{\Lambda_v^c}$, while $\mathrm{d}\Gamma_{\Lambda_v^c}\big|_{P_o\mathcal{F}(\tilde{\mathfrak{h}}_{\Lambda_v^c})}$ has no eigenvector. Hence by Theorem A.1, if there exists an eigenvector $\psi_e$ of $P_eL$, it is of the form

$$\psi_e = \varphi \otimes \Omega_{\Lambda_v^c},$$

and $P_oL$ has no eigenvector. Hence if $\psi$ is the eigen vector of $L$, we have

$$P_e\psi = \varphi \otimes \Omega_{\Lambda_v^c}, \quad P_o\psi = 0.$$

That is,

$$\psi = \varphi \otimes \Omega_{\Lambda_v^c}.$$

This means

$$P(N_{\Lambda_v^c} = 0)\psi = \varphi \otimes P(N_{\Lambda_v^c} = 0)\Omega_{\Lambda_v^c} = \varphi \otimes \Omega_{\Lambda_v^c} = \psi. \qquad \square$$

# 7 Positive Commutator

In this section, we construct strictly positive commutator. First we define the operator $[L, A_0]$ on $D([L, A_0]) \equiv D(N)$ by

$$[L, A_0] \equiv S_1 + \lambda I_1.$$

Note that this is defined as an operator of the right hand side, and not as a commutator $LA_0 - A_0L$. However the arguments in Section 5 guarantees

$$\langle L\varphi, A_0\psi\rangle + \langle A_0\varphi, L\psi\rangle = \langle \varphi, S_1 + \lambda I_1\psi\rangle = \langle \varphi, [L, A_0]\,\psi\rangle\,,$$

for all $\psi, \varphi \in D(A_0) \cap D(N)$. Next we introduce the modification term. Let $b(e)$ be a bounded operator defined by

$$b = b(e) = \theta\lambda(\bar{Q}R_\epsilon^2 I_0 Q - QI_0 R_\epsilon^2 \bar{Q}),$$
$$R_\epsilon = \left((L_0 - e)^2 + \epsilon^2\right)^{-\frac{1}{2}},$$

where $Q \equiv P(L_S = e) \otimes P_\Omega$, $\bar{Q} \equiv 1 - Q$ and $I_0$ is defined at (20). The parameters $\theta$ and $\epsilon$ will be determined later. We define $[L, b(e)]$ on $D(N)$ as $[L, b(e)] \equiv Lb(e) - b(e)L$. Note that $L_S b(e) - b(e)L_S$ and $\lambda I_0 b(e) - b(e)\lambda I_0$ are bounded. For all $\psi \in D(N)$,

$$[L_f, b(e)]\psi = \theta\lambda\left(\bar{Q}R_\epsilon^2\left[L_f, I_0\right]Q - Q\left[L_f, I_0\right]R_\epsilon^2\bar{Q}\right)\psi = \theta\lambda\left(\bar{Q}R_\epsilon^2\tilde{I}_1 Q - Q\tilde{I}_1 R_\epsilon^2\bar{Q}\right)\psi,$$

where

$$\tilde{I}_1 \equiv (Y \otimes 1) \otimes \left(-a\left(\tilde{h}g_1\right) + a^*\left(\tilde{h}g_1\right)\right) + \left(1 \otimes \bar{Y}\right) \otimes (-1)^N\left(a\left(\tilde{h}g_2\right) + a^*\left(\tilde{h}g_2\right)\right).$$

As $\tilde{I}_1$ is bounded, it can be extended to the whole $\mathcal{H}$. We denote the extension by the same symbol $[L, b(e)]$. We define $[L, A]$ on $D(N)$ by

$$[L, A] = [L, A_0] + [L, b] = S_1 + \lambda I_1 + [L, b].$$

**Theorem 7.1** *Let $\Delta$ be the interval of $\mathbb{R}$ whose interior includes an eigenvalue $e$ of $L_S$. Suppose no other eigenvalue is included in $\bar{\Delta}$, i.e, $\sigma(L_S) \cap \bar{\Delta} = \{e\}$. Let $h \in C_0^\infty(\mathbb{R})$ be a smooth function s.t.$h = 1$ on $\Delta$ and $\operatorname{supp} h \cap \sigma(L_S) = \{e\}$. Let $\theta$ and $\epsilon$ be*

$$\epsilon = \lambda^{\frac{44}{100}}, \quad \theta = \lambda^{\frac{26}{100}}.$$

*Suppose that Assumption 2.1 and 2.2 are satisfied. Then there exists $\gamma_e > 0$ such that*

$$Ph(L)\left[L, A\right]h(L)P \geq \frac{1}{2} \cdot \lambda^{\frac{182}{100}} \cdot \gamma_e Ph(L)\left(1 - 14\tilde{P}_e\right)h(L)P,$$

*for sufficiently small $\lambda$. Here $P$ is the spectral projection of $N_{\Lambda_v^c}$ onto $\{0\}$, and $\tilde{P}_e$ is*

$$\tilde{P}_e = P(L_S = e) \cdot \mathrm{Ker}\Gamma(e) \otimes P_\Omega,$$

*with $P_\Omega$, the projection onto $\Omega$.*

We carry out the proof in two steps. The first one is to show the strict positivity of $B$ defined by

$$B \equiv [L, A] - \frac{1}{10}S_1 = \frac{9}{10}S_1 + \lambda I_1 + [L, b],$$

with respect to the spectral localization in $L_0$. The second step is to derive the positivity of $[L, A]$ with respect to the spectral localization in $L$. The division of $[L, A]$ into $B$ and $\frac{1}{10}S_1$ is to recover the difference of the spectral projection of $L_0$ and $L$.

## 7.1 *Positive commutator with respect to the spectral localization in $L_0$*

In this subsection, we take the first step. Here is the main theorem of this subsection;

**Theorem 7.2** *Let $\Delta$ be an interval of $\mathbb{R}$, containing exactly one eigenvalue $e$ of $L_S$ and let $E^0_\Delta$ be the spectral projection of $L_0$ onto $\Delta$. Let $P$ be the spectral projection of $N_{\Lambda_v^c}$ onto $\{0\}$ and define an operator $\Gamma(e)$ on the space $P(L_S = e) \cdot \mathcal{H}_S$ by*

$$
\begin{aligned}
\Gamma(e) \equiv & \int_{-\pi}^{\pi} dk\, m(k,1)^* P\left(L_S \neq e\right) \delta\left(\omega\left(k\right) + L_S - e\right) m(k,1) \\
& + \int_{-\pi}^{\pi} dk\, m(k,2)^* P\left(L_S \neq e\right) \delta\left(-\omega\left(k\right) + L_S - e\right) m(k,2) \\
& \qquad m(k,i) \equiv Y \otimes 1 \cdot g_1^i(k) - 1 \otimes \bar{Y} \cdot g_2^i(k) \quad i = 1,2,
\end{aligned}
\tag{23}
$$

*where $g_i^j$ are defined by (10). Let $\gamma_e$ be a positive constant such that*

$$\Gamma(e) \geq \gamma_e \cdot \left(\mathrm{Ker}\Gamma(e)\right)^{\perp}$$

*where $\mathrm{Ker}\Gamma(e)$ indicates the projection onto the kernel of $\Gamma(e)$ in $P(L_S = e)\mathcal{H}_S$. Suppose that Assumption 2.1 and 2.2 are satisfied. Then there exists $0 < t_i$, $i = 1, \cdots 5$ such that, if $\lambda$, $\epsilon$, $\theta$ satisfy*

$$\lambda + \theta\lambda^2\epsilon^{-2} < t_1, \quad \theta < t_2, \quad \theta\lambda^2\epsilon^{-1} < t_3, \quad \epsilon < t_4, \quad \theta^{-1}\epsilon + \theta\lambda^2\epsilon^{-3} < t_5,$$

*then*

$$PE^0_\Delta B E^0_\Delta P \geq \frac{\theta\lambda^2}{\epsilon}\gamma_e \left(1 - 7\tilde{P}_e\right) E^0_\Delta P,$$

*where $\tilde{P}_e$ is*

$$\tilde{P}_e = P(L_S = e) \cdot \mathrm{Ker}\Gamma(e) \otimes P_\Omega,$$

*with $P_\Omega$, the projection onto $\Omega$.*

**Remark 7.3** *By the definition, $\Gamma(e)$ is positive. As $\gamma_e$ is the minimal eigenvalue above 0, it is strictly positive if $\Gamma(e) \neq 0$.*

The proof goes parallel to [10]. The difference emerges from the difference of the commutator $[L_0, A_0]$: in [10], it was $[L_0, A_0] = N$, while we have $[L_0, A_0] = S_1$. Whereas $N$ has a spectral gap, $S_1$ has no spectral gap. Since positive commutator method makes use of the finite spectral gap, the lack of the gap causes a difficulty. However, if we restrict ourselves to the subspace $P\mathcal{H}$ where $P = P(N_v^c = 0)$, $S_1|_{P\mathcal{H}}$ has a spectral gap. That is why we introduced the cut off of the interaction in Section 6. To prove the theorem, we use the Feshbach map theorem [13];

**Theorem 7.4** *Let $H$ be a closed operator densely defined on a Hilbert space $\mathcal{H}$, and let $P$ be a projection operator such that* $\mathrm{Ran}P \subset D(H)$. *We set $\bar{P} = 1 - P$. Then $H_{\bar{P}} \equiv \bar{P}H\bar{P}$ is a densely defined operator on $\bar{P}\mathcal{H}$. Let $z$ be an element in the resolvent set $\rho(H_{\bar{P}})$ of $H_{\bar{P}}$ on $\bar{P}\mathcal{H}$. Assume*

$$\left\| PH\bar{P}\left(H_{\bar{P}} - z\right)^{-1}\bar{P} \right\| < \infty, \quad \left\| \bar{P}\left(H_{\bar{P}} - z\right)^{-1}\bar{P}HP \right\| < \infty.$$

*Then the Feshbach map*

$$f_P(H - z) \equiv P(H - z)P - PH\bar{P}(H_{\bar{P}} - z)^{-1}\bar{P}HP$$

*is well defined on $P\mathcal{H}$. $f_P(H - z)$ and $H$ have the isospectral property in the sense that*

$$z \in \sigma(H) \iff 0 \in \sigma(f_P(H - z)), \quad z \in \sigma_{PP}(H) \iff 0 \in \sigma_{PP}(f_P(H - z)).$$

*Here $\sigma$ and $\sigma_{PP}$ represent spectrum and pure point spectrum respectively.*

To apply the theorem, we introduce

$$\chi_\nu \equiv P(N \le \nu).$$

Projections $\tilde{P}_e, \chi_\nu, P, Q, E^0_\Delta$ are in relation that

$$\tilde{P}_e \le \chi_\nu, \quad \tilde{P}_e \le P, \quad \tilde{P}_e \le Q, \quad Q \le E^0_\Delta, \quad Q \le P.$$

Hence we have the followings;

$$\bar{Q}\tilde{P}_e = 0, \quad \bar{Q}\tilde{P}_e^\perp = \bar{Q}, \quad E^0_\Delta P\chi_\nu\tilde{P}_e = \tilde{P}_e.$$

As $L_0$, $N_{\Lambda_v^c}$, $N$, $L_S$ strongly commute, the projections $E^0_\Delta$, $Q$, $P$, $\chi_\nu$ commute each other. So we can define the following projections

$$Q_1 \equiv QE^0_\Delta P\chi_\nu, \quad Q_2 \equiv \bar{Q}E^0_\Delta P\chi_\nu, \tag{24}$$

which satisfy

$$Q_1\tilde{P}_e = \tilde{P}_e, \quad Q_2\tilde{P}_e = 0, \quad Q_2\tilde{P}_e^\perp = Q_2. \tag{25}$$

The relation $NP = N_{\Lambda_v}P$ implies

$$N_{\Lambda_v}Q_2 = N_{\Lambda_v}P\bar{Q}E^0_\Delta\chi_\nu = NP\bar{Q}E^0_\Delta\chi_\nu = NQ_2.$$

As $P(N = 0) = 1 \otimes P_\Omega$, and $\Delta \cap \sigma(L_0) = \{e\}$, we have

$$P(N = 0)\bar{Q} = (1 - P(L_S = e)) \otimes P_\Omega = \sum_{f \ne e} P(L_S = f) \otimes P_\Omega \le P(L_0 \subset \Delta^c),$$

which implies

$$P(N = 0)Q_2 = P(N = 0)\bar{Q}E^0_\Delta P\chi_\nu = 0.$$

It follows that

$$NQ_2 = N_{\Lambda_v} Q_2 \geq Q_2. \tag{26}$$

*Proof of Theorem 7.2*
We apply Theorem 7.4 to the operator $B'$ defined by

$$B' \equiv B - \delta_e \cdot \tilde{P}_e^{\perp}.$$

$\delta_e$ will be defined later. We define operators $B'_{ij}$ by

$$B'_{i,j} \equiv Q_i B' Q_j \quad i,j = 1,2.$$

If $\vartheta \in \rho(B'_{22})$, then

$$\varepsilon(\vartheta) \equiv B'_{11} - B'_{12}(B'_{22} - \vartheta)^{-1} B'_{21},$$

is well-defined and

$$\varepsilon(\vartheta) = f_{Q_1}(B' - \vartheta) + \vartheta Q_1.$$

First we investigate the lower bound of $B'_{22}$ in order to study the resolvent set $\rho(B'_{22})$.

**Proposition 7.5** *We have*

$$B'_{22} \geq \left( \frac{9}{10} v - \delta_e - C_1 \left( \lambda + \theta \lambda^2 \epsilon^{-2} \right) \right) \cdot Q_2,$$

*where*

$$C_1 = \|I_1\| + 2 \cdot \|I_0\|^2.$$

*Proof*
We estimate the each term of

$$B'_{22} = Q_2 ( \frac{9}{10} S_1 + \lambda I_1 + [L, b] - \delta_e \cdot \tilde{P}_e^{\perp} ) Q_2.$$

From the inequality $S_1 \geq v \cdot N_{\Lambda_v}$ and the inequality (26), the first term is bounded below as $\frac{9}{10} Q_2 S_1 Q_2 \geq \frac{9}{10} v Q_2$. The lower bound of the second term is given by $\lambda I_1 \geq -\lambda \cdot \|I_1\|$, as $I_1$ is bounded. Let us estimate the norm of the third term. Substituting $L = L_0 + \lambda I_0$, $L_0 Q = Q L_0$ and $Q_2 Q = 0$, we obtain $Q_2 [L, b] Q_2 = -\theta \lambda^2 Q_2 (I_0 Q I_0 R_\epsilon^2 \bar{Q} + \bar{Q} R_\epsilon^2 I_0 Q I_0) Q_2$. Then it is estimated as $\|Q_2 [L, b] Q_2\| \leq 2 \cdot \|I_0\|^2 \theta \lambda^2 \epsilon^{-2}$, scince we have $\|R_\epsilon^2\| \leq \epsilon^{-2}$. The fourth term is $Q_2 \tilde{P}_e^{\perp} Q_2 = Q_2$ by (25). Hence we obtain

$$B'_{22} \geq \left( \frac{9}{10} v - C_1 \left( \lambda + \theta \lambda^2 \epsilon^{-2} \right) - \delta_e \right) Q_2$$

where $C_1 = \|I_1\| + 2 \cdot \|I_0\|^2$. $\square$
Suppose $\lambda, \theta, \epsilon$ satisfies

$$\lambda + \theta\lambda^2\epsilon^{-2} < t_1,$$

where $t_1 = \frac{v}{5C_1}$. Then if $\vartheta$ satisfies $\vartheta \leq \frac{1}{2}v - \delta_e$, it is in the resolvent set $\rho(B'_{22})$ of $B'_{22}$ by this proposition.

Next we estimate the lower bound of $\varepsilon(\vartheta)$.

**Proposition 7.6** *Suppose $\lambda, \theta, \epsilon$ satisfies*

$$\lambda + \theta\lambda^2\epsilon^{-2} < t_1,$$

*where $t_1 = \frac{v}{5C_1}$. Then if $\vartheta$ satisfies $\vartheta \leq \frac{1}{2}v - \delta_e$, it is in the resolvent set $\rho(B'_{22})$ of $B'_{22}$ and the following inequality holds;*

$$\varepsilon(\vartheta) \geq 2\theta\lambda^2 Q_1 \left[ I_0 \bar{R}_\epsilon^2 I_0 - \frac{1}{2\theta\lambda^2}\delta_e \cdot \tilde{P}_e^\perp \right] Q_1 - C_2 \left( \theta^2\lambda^2 Q_1 I_0 \bar{R}_\epsilon^2 I_0 Q_1 + (\lambda^2 + \theta^2\lambda^4\epsilon^{-4}) Q_1 \right)$$

*where $C_2 = 10v^{-1}\|I_1\|^2 + 20v^{-1} + 80v^{-1}\|I_0\|^4$.*

*Proof*
First $B'_{11}$ is given by

$$B'_{11} = Q_1 \left( \frac{9}{10} S_1 + \lambda I_1 + [L, b] - \delta_e \cdot \tilde{P}_e^\perp \right) Q_1.$$

As $S_1 Q = 0$ and $aQ = Qa^* = 0$, the first and the second term vanish. Using $\bar{Q}Q_1 = 0$, $L_0 Q = Q L_0$, we obtain

$$B'_{11} = 2\theta\lambda^2 Q_1 \left[ I_0 \bar{R}_\epsilon^2 I_0 - \frac{1}{2\theta\lambda^2}\delta_e \cdot \tilde{P}_e^\perp \right] Q_1. \tag{27}$$

Second, $Q_2 \left(B'_{22} - \vartheta\right)^{-1} Q_2$ is evaluated as follows;

**Lemma 7.7** *Suppose $\lambda + \theta\lambda^2\epsilon^{-2} < t_1$ where $t_1 = \frac{v}{5(\|I_1\| + 2\|I_0\|^2)}$. Then for all $\vartheta \leq \frac{1}{2}v - \delta_e$, $\vartheta$ is in the resolvent set $\rho(B'_{22})$ of $B'_{22}$ and*

$$0 \leq \left\langle \psi, Q_2 \left(B'_{22} - \vartheta\right)^{-1} Q_2 \psi \right\rangle \leq 5\|S_1^{-\frac{1}{2}} Q_2 \psi\|^2, \quad \forall \psi \in \mathcal{H}.$$

*Proof*
As $S_1 Q_2 \geq v N_{\Lambda_v} Q_2 \geq v Q_2$, $S_1$ is invertible on $Q_2\mathcal{H}$. The proof is similar to that of [10]. In facts, it is easier because we are considering Fermion system, whose interaction terms are bounded. We omit the details.$\square$
Third, $\left\| S_1^{-\frac{1}{2}} B'_{21} \psi \right\|$ is estimated as follows;

**Lemma 7.8**

$$\left\|S_1^{-\frac{1}{2}}B'_{21}\psi\right\|^2 \le C_3\left(\theta^2\lambda^2\|\bar{R}_\epsilon I_0 Q_1\psi\|^2 + \theta^2\lambda^4\epsilon^{-4}\|\psi\|^2 + \lambda^2\|\psi\|^2\right)$$

*where* $C_3 = 2v^{-1}\|I_1\|^2 + 4v^{-1} + 16v^{-1}\|I_0\|^4$.

*Proof*

The proof is the same as that of [10].□

Now let us complete the proof of Proposition 7.6. Combining Lemma 7.7 and 7.8, we obtain the following evaluation: if $\lambda, \theta, \epsilon$ satisfies $\lambda + \theta\lambda^2\epsilon^{-2} < t_1$, then for all $\vartheta \le \frac{1}{2}v - \delta_e$, we have $\vartheta \in \rho(B'_{22})$ and

$$0 \le \left\langle \psi, B'_{12}\left(B'_{22} - \vartheta\right)^{-1} B'_{21}\psi\right\rangle \le C_2\left(\theta^2\lambda^2\left\|\bar{R}_\epsilon I_0 Q_1\psi\right\|^2 + (\lambda^2 + \theta^2\lambda^4\epsilon^{-4})\|\psi\|^2\right)$$

where $C_2 = 10v^{-1}\|I_1\|^2 + 20v^{-1} + 80v^{-1}\|I_0\|^4$. Combining this with (27), we obtain the required estimation. □

Now let us complete the proof of Theorem 7.2. Suppose that $\lambda + \theta\lambda^2\epsilon^{-2} < t_1$. Then by Proposition 7.6, for all $\vartheta \le \frac{1}{2}v - \delta_e$, we have $\vartheta \in \rho(B'_{22})$ and

$$\varepsilon(\vartheta) \ge D,$$

where

$$D = 2\theta\lambda^2 Q_1\left[I_0\bar{R}_\epsilon^2 I_0 - \frac{1}{2\theta\lambda^2}\delta_e \cdot \tilde{P}_e^\perp\right]Q_1 - C_2\left(\theta^2\lambda^2 Q_1 I_0\bar{R}_\epsilon^2 I_0 Q_1 + (\lambda^2 + \theta^2\lambda^4\epsilon^{-4})Q_1\right).$$

Note that $D$ is $\vartheta$-independent. Let $\vartheta_0 \equiv \inf\sigma(B'|_{(Q_1\oplus Q_2)\mathcal{H}})$. Here we have two cases,

1. $\vartheta_0 \ge \frac{1}{2}v - \delta_e$.
2. $\frac{1}{2}v - \delta_e > \vartheta_0$. In this case, $\vartheta_0 \in \rho(B'_{22})$ and $\varepsilon(\vartheta_0)$ is well-defined. Then $f_{Q_1}(B' - \vartheta_0)$ satisfies

$$f_{Q_1}(B' - \vartheta_0) = \varepsilon(\vartheta_0) - \vartheta_0 \cdot Q_1 \ge (\inf\sigma(D) - \vartheta_0)\cdot Q_1.$$

   On the other hand, note that $\vartheta_0 \in \sigma(B'|_{(Q_1\oplus Q_2)\mathcal{H}})$. By Feshbach Theorem 7.4, this implies $0 \in \sigma(f_{Q_1}(B' - \vartheta_0))$. Hence $f_{Q_1}(B' - \vartheta_0)$ is not invertible, and we get

$$\vartheta_0 \ge \inf\sigma(D).$$

So we have

$$\vartheta_0 \ge \min\{\frac{1}{2}v - \delta_e, \inf\sigma(D)\},$$

i.e.,

$$B'|_{(Q_1\oplus Q_2)\mathcal{H}} \ge \min\{\frac{1}{2}v - \delta_e, \inf\sigma(D)\}. \tag{28}$$

The next problem is to investigate the lower bound of $D$. For the purpose, we estimate $Q_1 I_0\bar{R}_\epsilon^2 I Q_1$. Under the Assumption 2.2 and using pull through formula, we obtain the following bound;

**Proposition 7.9** *Under Assumption 2.2 we have*

$$Q_1 I_0 \bar{R}_\epsilon^2 I_0 Q_1 \geq \frac{\pi}{\epsilon} Q_1 \left[ \Gamma(e) - C_3(\epsilon^{\frac{1}{4}}) \right] Q_1$$

*where $C_3$ is a positive constant.*

Assume that $\theta, \lambda, \epsilon$ satisfies the followings.

**Assumption 7.1**

$$\lambda + \theta\lambda^2\epsilon^{-2} < t_1, \quad \theta < t_2, \quad \theta\lambda^2\epsilon^{-1} < t_3, \quad \epsilon < t_4, \quad \theta^{-1}\epsilon + \theta\lambda^2\epsilon^{-3} < t_5.$$

*where*

$$t_2 \equiv \frac{1}{C_2}, \quad t_3 \equiv \frac{v}{8\pi\gamma_e}, \quad t_4 \equiv (\frac{\gamma_e}{4C_3})^4, \quad t_5 \equiv \frac{\pi\gamma_e}{4C_2}.$$

The second assumption implies $2\theta\lambda^2 - C_2\theta^2\lambda^2 > \theta\lambda^2$. Using this and the lower bound of $I_0\bar{R}_\epsilon^2 I_0$ of Proposition 7.9, we obtain

$$D \geq \theta\lambda^2 \frac{\pi}{\epsilon} Q_1 \left[ \Gamma(e) - C_3(\epsilon^{\frac{1}{4}}) \right] Q_1 - \delta_e \cdot Q_1 \tilde{P}_e^\perp Q_1 - C_2 \left( \lambda^2 + \theta^2\lambda^4\epsilon^{-4} \right) Q_1. \tag{29}$$

Recall that $\Gamma(e)$ is bounded below as $\Gamma(e) \geq \gamma_e \cdot (\mathrm{Ker}\Gamma(e))^\perp$. Substituting this to (29), we get

$$D \geq \theta\lambda^2 \frac{\pi}{\epsilon} Q_1 \left[ \left( \gamma_e - \delta_e \frac{\epsilon}{\pi\theta\lambda^2} \right) \cdot \tilde{P}_e^\perp - C_3(\epsilon^{\frac{1}{4}}) - \frac{C_2}{\pi} \left( \theta^{-1}\epsilon + \theta\lambda^2\epsilon^{-3} \right) \right] Q_1. \tag{30}$$

Now we determine $\delta_e$ as

$$\delta_e = \frac{\pi\theta\lambda^2}{\epsilon} (\gamma_e + a) \quad \text{with} \quad 0 < a < \gamma_e.$$

Then we have

$$D \geq \theta\lambda^2 \frac{\pi}{\epsilon} \left[ -a - C_3(\epsilon^{\frac{1}{4}}) - \frac{C_2}{\pi} \left( \theta^{-1}\epsilon + \theta\lambda^2\epsilon^{-3} \right) \right] Q_1.$$

Hence $\inf \sigma(D)$ is bounded below by

$$\inf \sigma(D) \geq \theta\lambda^2 \frac{\pi}{\epsilon} \left[ -a - C_3(\epsilon^{\frac{1}{4}}) - \frac{C_2}{\pi} \left( \theta^{-1}\epsilon + \theta\lambda^2\epsilon^{-3} \right) \right] Q_1.$$

Recall the lower bound (28) of $B'$. As the third condition of Assumption 7.1 implies

$$\frac{1}{2}v - \delta_e \geq \frac{1}{4}v > 0,$$

and

$$0 \geq \theta\lambda^2 \frac{\pi}{\epsilon} \left[ -a - C_3(\epsilon^{\frac{1}{4}}) - \frac{C_2}{\pi} \left( \theta^{-1}\epsilon + \theta\lambda^2\epsilon^{-3} \right) \right],$$

the lower bound is

$$B'|_{(Q_1 \oplus Q_2)\mathcal{H}} \geq \theta\lambda^2 \frac{\pi}{\epsilon} \left[ -a - C_3(\epsilon^{\frac{1}{4}}) - \frac{C_2}{\pi} \left( \theta^{-1}\epsilon + \theta\lambda^2\epsilon^{-3} \right) \right].$$

Substituting $B = B' + \delta_e \tilde{P}_e^{\perp}$, and the last two conditions of Assumption 7.1, we get

$$\begin{aligned} PE_{\Delta}^0 BE_{\Delta}^0 P &\geq \theta\lambda^2 \frac{\pi}{\epsilon} PE_{\Delta}^0 \left[ -a - C_3(\epsilon^{\frac{1}{4}}) - \frac{C_2}{\pi} \left( \theta^{-1}\epsilon + \theta\lambda^2\epsilon^{-3} \right) + (\gamma_e + a) \cdot \tilde{P}_e^{\perp} \right] E_{\Delta}^0 P \\ &= \theta\lambda^2 \frac{\pi}{\epsilon} PE_{\Delta}^0 \left[ \gamma_e - C_3(\epsilon^{\frac{1}{4}}) - \frac{C_2}{\pi} \left( \theta^{-1}\epsilon + \theta\lambda^2\epsilon^{-3} \right) - (\gamma_e + a) \cdot \tilde{P}_e \right] E_{\Delta}^0 P \\ &\geq \theta\lambda^2\epsilon^{-1} PE_{\Delta}^0 \gamma_e \left[ 1 - 7\tilde{P}_e \right] E_{\Delta}^0 P. \end{aligned}$$

Hence we obtain Theorem 7.2.□

### 7.2 Positive Commutator with respect to the spectral localization in $L$

In this subsection we complete the proof of Theorem 7.1.
Let $\Delta'$ be an interval s.t. $\Delta \subset \Delta'$, $\Delta' \cap \sigma(L_S) = \{e\}$ and supp$h \subset \Delta'$. Let $F_{\Delta'} \in C_0^{\infty}(\mathbb{R})$ be a smooth function $0 \leq F_{\Delta'} \leq 1$ which satisfies $F_{\Delta'} = 1$ on supp $h$ and supp $F_{\Delta'} \subset \Delta'$. We set $F_{\Delta'}^0 \equiv F_{\Delta'}(L_0)$, and $\bar{F}_{\Delta'}^0 \equiv 1 - F_{\Delta'}^0$. We evaluate each term of the following equation;

$$\begin{aligned} Ph\,[L, A]\,hP &= PhBhP + \frac{1}{10} PhS_1hP \\ &= PhF_{\Delta'}^0 BF_{\Delta'}^0 hP && (31) \\ &+ Ph\bar{F}_{\Delta'}^0 BF_{\Delta'}^0 hP + \frac{1}{40} PhS_1hP + h.c. && (32) \\ &+ Ph\bar{F}_{\Delta'}^0 B\bar{F}_{\Delta'}^0 hP + \frac{1}{20} PhS_1hP && (33) \end{aligned}$$

For the first part (31), we have the following lemma.

**Lemma 7.10** *Suppose that $\lambda, \epsilon, \theta$ satisfy Assumption 7.1. Then we have*

$$(31) = PhF_{\Delta'}^0 BF_{\Delta'}^0 hP \geq \frac{\theta\lambda^2}{\epsilon} \gamma_e Ph(1 - \lambda c_1 - 7\tilde{P}_e)hP,$$

*where $c_1$ is a constant which depends only on $F_{\Delta'}$ and $\|I_0\|$.*

*Proof*
The proof is the same as that of inequality (63) in [10]. □
Next we evaluate the second part (32).

**Lemma 7.11**

$$(32) = Ph\bar{F}^0_{\Delta'}BF^0_{\Delta'}hP + \frac{1}{40}PhS_1hP + h.c. \geq -c_2\frac{\theta\lambda^2}{\epsilon}(\epsilon\theta^{-1} + \epsilon + \lambda\epsilon^{-1})Ph^2P$$

*where $c_2$ is a constant which depends only on $F_{\Delta'}$ and $I_0$.*

*Proof*
We divide the second part (32) into three parts;

$$\begin{aligned}(32) = &\frac{9}{10}Ph\bar{F}^0_{\Delta'}S_1F^0_{\Delta'}hP + h.c. &(34)\\ &+\lambda Ph\bar{F}^0_{\Delta'}I_1F^0_{\Delta'}hP + \frac{1}{40}PhS_1hP + h.c. &(35)\\ &+Ph\bar{F}^0_{\Delta'}[L,b]F^0_{\Delta'}hP. + h.c. &(36)\end{aligned}$$

Using the fact that $0 \leq F^0_{\Delta'} \leq 1$, we get (34) $\geq 0$ as in the case (57) of [10]. To evaluate (35), we apply Lemma B.1 with $\varphi_1 = F^0_{\Delta'}P\psi$, and $\varphi_2 = \bar{F}^0_{\Delta'}P\psi$. As $L_0$ and $N_{\Lambda_v^c}$ strongly commute with $N$ respectively, $F^0_{\Delta'}P\psi, \bar{F}^0_{\Delta'}P\psi$ are in $D(N)$. By Lemma B.1, we have

$$\begin{aligned}&\left|\langle\psi\left|\lambda P\bar{F}^0_{\Delta'}I_1F^0_{\Delta'}P\right|\psi\rangle\right| \leq\\ &\leq 3c\|Y\|^2\left(\left\|N^{\frac{1}{2}}_{\Lambda_v}F^0_{\Delta'}P\psi\right\|^2 + \left\|N^{\frac{1}{2}}_{\Lambda_v}\bar{F}^0_{\Delta'}P\psi\right\|^2\right) + 3\lambda^2\frac{\|\tilde{p}g_1\|^2}{c}\left(\left\|F^0_{\Delta'}P\psi\right\|^2 + \left\|\bar{F}^0_{\Delta'}P\psi\right\|^2\right)\\ &+3c\|\bar{Y}\|^2\left(\left\|N^{\frac{1}{2}}_{\Lambda_v}F^0_{\Delta'}P\psi\right\|^2 + \left\|N^{\frac{1}{2}}_{\Lambda_v}\bar{F}^0_{\Delta'}P\psi\right\|^2\right) + 3\lambda^2\frac{\|\tilde{p}g_2\|^2}{c}\left(\left\|F^0_{\Delta'}P\psi\right\|^2 + \left\|\bar{F}^0_{\Delta'}P\psi\right\|^2\right)\\ &\leq 6c\left(\|Y\|^2 + \|\bar{Y}\|^2\right)\left\|N^{\frac{1}{2}}_{\Lambda_v}P\psi\right\|^2 + \lambda^2\frac{6}{c}\|P\psi\|^2\left(\|\tilde{p}g_1\|^2 + \|\tilde{p}g_2\|^2\right),\end{aligned}$$

for arbitrary $c > 0$. We used the strong commutativity of $N_{\Lambda_v}$ and $L_0$, together with the fact that $0 \leq F^0_{\Delta'} \leq 1$. Taking $c$ as $c = v/240(\|Y\|^2 + \|\bar{Y}\|^2)$, we obtain

$$\left|\langle\psi\left|\lambda P\bar{F}^0_{\Delta'}I_1F^0_{\Delta'}P\right|\psi\rangle\right| \leq \frac{v}{40}\left\|N^{\frac{1}{2}}_{\Lambda_v}P\psi\right\|^2 + c_1'\lambda^2\|P\psi\|^2\left(\|\tilde{p}g_1\|^2 + \|\tilde{p}g_2\|^2\right),$$

where $c_1' = 1440\frac{1}{v}(\|Y\|^2 + \|\bar{Y}\|^2)$. Hence we have

$$\begin{aligned}&\lambda P\bar{F}^0_{\Delta'}I_1F^0_{\Delta'}P + \frac{1}{40}PS_1P + h.c\\ &\geq \frac{1}{20}PS_1P - \frac{v}{20}PN_{\Lambda_v}P - 2c_1'\lambda^2P\left(\|\tilde{p}g_1\|^2 + \|\tilde{p}g_2\|^2\right) \geq -2c_1'\lambda^2P\left(\|\tilde{p}g_1\|^2 + \|\tilde{p}g_2\|^2\right).\end{aligned}$$

Sandwiching this inequality with $h$, and using the commutativity of $h$ and $P$, we obtain

$$(35) = \lambda Ph\bar{F}^0_{\Delta'}I_1F^0_{\Delta'}hP + \frac{1}{40}PhS_1hP + h.c. \geq -2c_1'\lambda^2h^2P\left(\|\tilde{p}g_1\|^2 + \|\tilde{p}g_2\|^2\right). \qquad (37)$$

The last part (36) can be estimated in a manner of Proposition 5.2 in [10]:

$$(36) = Ph\bar{F}^0_{\Delta'}[L,b]F^0_{\Delta'}hP + h.c. = Ph\left(\sqrt{\bar{F}_{\Delta'}(L_0)} - \sqrt{\bar{F}_{\Delta'}(L)}\right)\sqrt{\bar{F}^0_{\Delta'}}[L,b]F^0_{\Delta'}hP + h.c.$$
$$\geq -c_2''\left(\theta\lambda^2 + \theta\lambda^3\epsilon^{-2}\right)h^2P = -c_2''\frac{\theta\lambda^2}{\epsilon}\left(\epsilon + \lambda\epsilon^{-1}\right)h^2P.$$

Hence we have

$$(32) = (34) + (35) + (36) \geq -2c_1'\lambda^2h^2P\left(\|\tilde{p}g_1\|^2 + \|\tilde{p}g_2\|^2\right) - c_2''\frac{\theta\lambda^2}{\epsilon}\left(\epsilon + \lambda\epsilon^{-1}\right)h^2P$$
$$\geq -c_2\frac{\theta\lambda^2}{\epsilon}\left(\theta^{-1}\epsilon + \epsilon + \lambda\epsilon^{-1}\right)h^2P,$$

where $c_2$ depends only on $I_0$ and $F_{\Delta'}$.□

Finally, the third part (33) can be estimated as follows.

**Lemma 7.12**

$$Ph\bar{F}^0_{\Delta'}B\bar{F}^0_{\Delta'}hP + \frac{1}{20}PhS_1hP \geq -c_3\frac{\theta\lambda^2}{\epsilon}(\theta^{-1}\epsilon + \lambda\epsilon^{-1})h^2P.$$

*where $c_3$ depends only on $I_0$ and $F_{\Delta'}$.*

*Proof*
The same as that of Proposition 5.2 in [10]. □
Now let us complete the proof of Theorem 7.1. By Lemma 7.10 to 7.12, if $\lambda, \epsilon$ and $\theta$ satisfy Assumption 7.1, we obtain

$$Ph\,[L,A]\,hP = (31) + (32) + (33) \geq \frac{\theta\lambda^2}{\epsilon}\gamma_e Ph\left(1 - 7\tilde{P}_e - C_1(\lambda + \epsilon\theta^{-1} + \epsilon + \lambda\epsilon^{-1})\right)hP.$$

Let $\epsilon$, $\theta$ be

$$\epsilon = \lambda^{\frac{44}{100}}, \quad \theta = \lambda^{\frac{26}{100}}.$$

Then

$$\epsilon\theta^{-1} = \lambda^{\frac{18}{100}}, \quad \lambda\epsilon^{-1} = \lambda^{\frac{56}{100}}, \quad \theta\lambda^2\epsilon^{-1} = \lambda^{\frac{182}{100}}, \quad \theta\lambda^2\epsilon^{-2} = \lambda^{\frac{138}{100}}, \quad \theta\lambda^2\epsilon^{-3} = \lambda^{\frac{94}{100}}$$

and if $\lambda$ is sufficiently small, Assumption 7.1 is satisfied. Further more, for $\lambda$ small enough,

$$C_1(\lambda + \epsilon\theta^{-1} + \epsilon + \lambda\epsilon^{-1}) < \frac{1}{2}.$$

is satisfied. Hence we finally obtain

$$Ph\,[L,A]\,hP \geq \frac{\theta\lambda^2}{\epsilon}\frac{\gamma_e}{2}Ph\left(1 - 14\tilde{P}_e\right)hP = \lambda^{\frac{182}{100}}\frac{\gamma_e}{2}Ph\left(1 - 14\tilde{P}_e\right)hP. \quad \square$$

# 8 Virial Theorem

In this section, we apply the new method introduced by M.Merkli [10], to investigate the eigenvectors of $L$. Our situation here is different from that of [10] in the following sense: in [10], the double commutator of $L_0$ with $A_0$ vanishes i.e., $[[L_0, A_0], A_0] = 0$, while in our case it does not $[[L_0, A_0], A_0] = S_2$. That is the reason why we still need the scaling factor $n$ (see below) although our interaction is bounded.

Assume that $\psi$ is a normalized eigenvector of $L$ with eigenvalue $e$. Let $g \in C_0^\infty(\mathbb{R})$ be a smooth function such that supp$g \subset [-2, 2]$, $0 \le g \le 1$, and $g = 1$ on $[-1, 1]$. Let $f \in C_b^\infty(\mathbb{R})$ be a smooth function which satisfies the following conditions;

$$f' \ge 0, \quad f'(0) = 1, \quad \|f^{(k)}\|_{k-4} < \infty, \quad k = 0, \cdots, 5, \quad \|f^{(l)}\|_{l-3} < \infty, \quad l = 0, \cdots, 4.$$

where we used the notation

$$\|f\|_m \equiv \int dx \langle x\rangle^m |f(x)|, \quad \langle x\rangle = (1 + x^2)^{\frac{1}{2}}.$$

Furthermore, we assume that $h = \sqrt{f'}$ satisfies that

$$\begin{aligned} h^{(k)} \in C_b^\infty(\mathbb{R}), \quad k = 0, 1, \quad \|h^{(l)}\|_{l-2} < \infty, \quad l = 0, \cdots, 3, \\ \|h^{(m)}\|_{m-3} < \infty, \quad m = 0, \cdots, 4, \quad \sup_{x\in\mathbb{R}} |x| \cdot |h(x)| < \infty. \end{aligned}$$

Now let $\alpha, \nu$ be positive parameters. We define the following operators;

$$f_\alpha \equiv f(i\alpha A_0), \quad f'_\alpha \equiv f'(i\alpha A_0), \quad f''_\alpha \equiv f''(i\alpha A_0), \quad h_\alpha \equiv \sqrt{f'_\alpha}, \quad g_\nu \equiv g(\nu N).$$

By the condition that $\sup_{x\in\mathbb{R}} |x| \cdot |h(x)| < \infty$, we have $h_\alpha \mathcal{H} \subset D(A_0)$.

For simplicity, below, we use the notation $K \equiv [L, A_0] = S_1 + I_1^\lambda$, $\langle A\rangle_\psi \equiv \langle \psi, A\psi\rangle$, and $\psi_{\alpha,\nu} \equiv h_\alpha g_\nu \psi$. We also use the notation $I_j^\lambda \equiv \lambda I_j$. We first show the following proposition:

**Proposition 8.1** *Let $\nu = \alpha^{\frac{3}{2}}$. If $\psi$ is an eigenvector of $L$, then*

$$\lim_{\alpha\to 0} \langle K\rangle_{\psi_{\alpha\nu}} = \lim_{\alpha\to 0} \langle \psi \,| g_\nu h_\alpha \,[L, A_0]\, h_\alpha g_\nu |\, \psi\rangle = 0.$$

*Proof*
Let $\psi$ be a unit eigenvector of $L$ with eigenvalue $e$. We have the following lemma.

**Lemma 8.2**

$$|\langle \psi \,| g_\nu f_\alpha (L - e) g_\nu |\, \psi\rangle| \le C_1 \cdot \nu.$$

*where $C_1$ is a positive constant which is independent of $\nu$ and $\alpha$.*

*Proof*
As $L_0$ and $N$ strongly commute and $\psi \in D(L) = D(L_0)$, we have

$$(L_0 - e)g_\nu \psi = g_\nu (L_0 - e)\psi.$$

As $g \in C_0^\infty(\mathbb{R})$, using operator calculous, we obtain the bound

$$\|[I_0^\lambda, g_\nu]\| < C_1 \cdot \nu,$$

where $C_1$ is a constant which is independent of $\nu$ and $\alpha$. Hence, we have

$$|\langle \psi | g_\nu f_\alpha (L - e) g_\nu | \psi \rangle| = \left| \left\langle \psi \left| g_\nu f_\alpha [I_0^\lambda, g_\nu] \right| \psi \right\rangle \right| \leq C_1 \cdot \nu. \quad \square$$

On the other hands, we have the following lemma.

**Lemma 8.3**

$$\begin{aligned} \left| \langle K \rangle_{\psi_{\alpha\nu}} \right| &= \left| \left\langle \psi | g_\nu h_\alpha \left( S_1 + I_1^\lambda \right) h_\alpha g_\nu \psi \right\rangle \right| \\ &\leq 2\alpha^{-1} \left| \mathrm{Im} \langle \psi | g_\nu f_\alpha (L - e) g_\nu | \psi \rangle \right| + \alpha^2 \nu^{-1} \cdot b_0 + \alpha \cdot \tilde{b}_0. \end{aligned}$$

*where $b_0$ and $\tilde{b}_0$ are positive constants which depend only on $f$ and $I_0$.*

*Proof*
First we consider the field term: $\mathrm{Im} \langle \psi | g_\nu f_\alpha (L_f - e) g_\nu | \psi \rangle$. Observe that

$$\begin{aligned} 2\mathrm{Im} \langle \psi | g_\nu f_\alpha (L_f - e) g_\nu | \psi \rangle &= -i \langle \psi | g_\nu f_\alpha (L_f - e) g_\nu \psi \rangle + i \langle (L_f - e) g_\nu \psi | f_\alpha g_\nu \psi \rangle \\ &= \langle g_\nu \psi | [L_f, i f_\alpha] g_\nu \psi \rangle = \mathrm{Re} \langle \psi | g_\nu [L_f, i f_\alpha] g_\nu \psi \rangle. \end{aligned} \tag{38}$$

By operator calculous, we obtain

$$\begin{aligned} [L_f, i f(i\alpha A_0)] g_\nu \psi &= -\alpha f'(i\alpha A_0) S_1 g_\nu \psi - i \frac{\alpha^2}{2} f''(i\alpha A_0) S_2 g_\nu \psi + \alpha^3 R_3 g_\nu \psi, \\ R_3 &= \int d\tilde{f}(z) (z - i\alpha A_0)^{-3} S_3 (z - i\alpha A_0)^{-1}. \end{aligned}$$

Below, we estimate each term of

$$\begin{aligned} \mathrm{Re} \langle \psi | g_\nu [L_f, i f_\alpha] g_\nu \psi \rangle = &- \alpha \mathrm{Re} \langle \psi | g_\nu f'(i\alpha A_0) S_1 g_\nu \psi \rangle & (39) \\ &+ \frac{\alpha^2}{2} \mathrm{Im} \left\langle \psi | g_\nu f''(i\alpha A_0) S_2 g_\nu \psi \right\rangle & (40) \\ &+ \alpha^3 \mathrm{Re} \langle \psi | g_\nu R_3 g_\nu \psi \rangle. & (41) \end{aligned}$$

First we estimate (39). Note that

$$\mathrm{Re} \langle \psi | g_\nu f'(i\alpha A_0) S_1 g_\nu \psi \rangle = \mathrm{Re} \langle \psi | g_\nu h_\alpha [h_\alpha, S_1] g_\nu \psi \rangle + \langle \psi | g_\nu h_\alpha S_1 h_\alpha g_\nu \psi \rangle. \tag{42}$$

The first term

$$\mathrm{Re} \langle \psi | g_\nu h_\alpha [h_\alpha, S_1] g_\nu \psi \rangle = \frac{1}{2} \langle \psi | g_\nu [h_\alpha, [h_\alpha, S_1]] g_\nu \psi \rangle \tag{43}$$

is estimated as follows. By operator calculous, we have

$$[h_\alpha, S_1]\varphi = [h(i\alpha A_0), S_1]\varphi = -i\alpha h'(i\alpha A_0)S_2\varphi + \alpha^2 R_2\varphi, \tag{44}$$

for all $\varphi \in D(N)$. Here $R_2$ is a $N$-bounded operator and we have $\|R_2 g_\nu\| \leq \nu^{-1} \cdot \frac{b_1''}{2}$. On the other hand, we have $\|\alpha\,[h_\alpha, h'_\alpha S_2]\,g_\nu\| \leq \alpha^2\nu^{-1}\cdot b'_2$. Hence we obtain

$$|(43)| = |\mathrm{Re}\,\langle\psi|g_\nu h_\alpha[h_\alpha, S_1]g_\nu\psi\rangle| = \frac{1}{2}\,|\langle\psi|g_\nu\,[h_\alpha,[h_\alpha,S_1]]\,g_\nu\psi\rangle| \leq \alpha^2\nu^{-1}\cdot b_1. \tag{45}$$

where $b_1$ is a constant which depends only on the function $h$.
Next let us estimate (40). The imaginary part is derived by the commutator,

$$2\mathrm{Im}\left\langle\psi\left|g_\nu f''_\alpha S_2 g_\nu\psi\right.\right\rangle = i\left\langle\psi\left|g_\nu\left[S_2 g_\nu, f''_\alpha\right]\psi\right.\right\rangle.$$

which is bounded using operator calculous by $\left\|\left[S_2 g_\nu, f''_\alpha\right]\right\| \leq 4\alpha\nu^{-1}\cdot b_2$. Then we get

$$|(40)| \leq \alpha^3\nu^{-1}\cdot b_2. \tag{46}$$

Finally, the last term (41) is estimated as

$$|(41)| = \left|\alpha^3\mathrm{Re}\,\langle\psi|g_\nu R_3 g_\nu\psi\rangle\right| \leq \alpha^3\nu^{-1}\cdot b_3. \tag{47}$$

Note that from (38) to (42),

$$\langle\psi|g_\nu h_\alpha S_1 h_\alpha g_\nu\psi\rangle + 2\alpha^{-1}\mathrm{Im}\,\langle\psi\,|g_\nu f_\alpha(L_f - e)g_\nu|\,\psi\rangle = \alpha^{-1}(40) + \alpha^{-1}(41) - (43).$$

Hence by evaluations (45),(46), and (47), we have the estimate

$$\left|\langle\psi|g_\nu h_\alpha S_1 h_\alpha g_\nu\psi\rangle + 2\alpha^{-1}\mathrm{Im}\,\langle\psi\,|g_\nu f_\alpha(L_f - e)g_\nu|\,\psi\rangle\right| \leq \alpha^2\nu^{-1}\cdot b_0.$$

Similarly, we obtain the bound

$$\left|\left\langle\psi|g_\nu h_\alpha I_1^\lambda h_\alpha g_\nu\psi\right\rangle + 2\alpha^{-1}\mathrm{Im}\left\langle\psi\left|g_\nu f_\alpha(I_0^\lambda - e)g_\nu\right|\psi\right\rangle\right| \leq \alpha\cdot\tilde{b}_0.$$

Hence we finally obtain

$$\left|\left\langle\psi|g_\nu h_\alpha\left(S_1 + I_1^\lambda\right)h_\alpha g_\nu\psi\right\rangle\right| \leq 2\alpha^{-1}\left|\mathrm{Im}\,\langle\psi\,|g_\nu f_\alpha(L - e)g_\nu|\,\psi\rangle\right| + \alpha^2\nu^{-1}\cdot b_0 + \alpha\cdot\tilde{b}_0. \qquad \square$$

Now let us complete the proof of Proposition 8.1. By Lemma 8.2 and Lemma 8.3, we obtain

$$\left|\left\langle\psi|g_\nu h_\alpha\left(S_1 + I_1^\lambda\right)h_\alpha g_\nu\psi\right\rangle\right| \leq 2C_1\cdot\alpha^{-1}\nu + \alpha^2\nu^{-1}\cdot b_0 + \alpha\cdot\tilde{b}_0.$$

Now let $\nu$ be $\nu = \alpha^{\frac{3}{2}}$. Then we have

$$\left|\left\langle\psi|g_\nu h_\alpha\left(S_1 + I_1^\lambda\right)h_\alpha g_\nu\psi\right\rangle\right| \leq 2C_1\cdot\alpha^{\frac{1}{2}} + \alpha^{\frac{1}{2}}\cdot b_0 + \alpha\cdot\tilde{b}_0.$$

Taking $\alpha \to 0$, we obtain

$$\lim_{\alpha\to 0} \langle K\rangle_{\psi_{\alpha,\nu}} = \lim_{\alpha\to 0} \left\langle \psi | g_\nu h_\alpha \left(S_1 + I_1^\lambda\right) h_\alpha g_\nu \psi \right\rangle = 0. \qquad \square$$

Next we calculate the lower bound of $\langle K\rangle_{\psi_{\alpha,\nu}}$. We use Theorem 7.1. For $e \in \mathbb{R}$, let $\tilde{e} \equiv e$, if $e$ is an eigenvalue of $L_S$, and let $\tilde{e}$ be an eigenvalue of $L_S$ which is next to $e$, in case that $e$ is not an eigenvalue of $L_S$.

**Proposition 8.4** *Let $\nu \equiv \alpha^{\frac{3}{2}}$ and suppose the Assumption 2.1 and 2.2 are satisfied. Then for sufficiently small $\lambda$, we have*

$$\lim_{\alpha\to 0} \langle K\rangle_{\psi_{\alpha,\nu}} \geq b_{\tilde{e}} \left( \|\psi\|^2 - 14 \|\tilde{P}_{\tilde{e}}\psi\|^2 \right),$$

*where $b_{\tilde{e}} = \frac{1}{2} \cdot \lambda^{\frac{182}{100}} \cdot \gamma_{\tilde{e}}$.*

*Proof*
Let $\chi \in C_0^\infty(\mathbb{R})$ be a smooth function such that

$$0 \leq \chi \leq 1 \qquad \text{and} \qquad \begin{cases} \chi(t) = 1 & |t| \leq \dfrac{1}{2} \\ \chi(t) = 0 & 1 \leq |t|. \end{cases},$$

Suppose that the function $0 \leq \bar{\chi} \leq 1$ defined by

$$\chi^2 + \bar{\chi}^2 = 1,$$

belongs to $C_b^\infty(\mathbb{R})$. Let $\chi_n \equiv \chi(\frac{N}{n})$ and $\bar{\chi}_n \equiv \bar{\chi}(\frac{N}{n})$. In the following, we assume that $n < \nu^{-1}$. Recall that $g \in C_0^\infty(\mathbb{R})$ is a smooth function such that supp $g \subset [-2, 2]$, $0 \leq g \leq 1$, and $g = 1$ on $[-1, 1]$. Then we have $g_\nu \chi_n = \chi_n$. First we prove the following Lemma.

**Lemma 8.5** *$\langle K\rangle_{\psi_{\alpha,\nu}}$ is bounded below by*

$$\langle K\rangle_{\psi_{\alpha,\nu}} \geq \langle K\rangle_{\chi_n P\psi_{\alpha,\nu}} - \|I_1^\lambda\| \cdot \|\bar{P}\psi_{\alpha,\nu}\|^2 - \|I_1^\lambda\| \cdot \|\bar{\chi}_n \psi_{\alpha,\nu}\|^2 - n^{-2} \cdot C_2$$

*where $C_2$ is a positive constant which depends only on $I_0^\lambda$ and $\chi$.*

*Proof*
As $\chi_n D(N) \subset D(N)$, $\bar{\chi}_n D(N) \subset D(N)$ and $K$ is $N$-bounded we have

$$\langle K\rangle_{\psi_{\alpha,\nu}} = \left\langle \chi_n K \chi_n + \bar{\chi}_n K \bar{\chi}_n + \frac{1}{2}[\chi_n, [\chi_n, K]] + \frac{1}{2}[\bar{\chi}_n, [\bar{\chi}_n, K]] \right\rangle_{\psi_{\alpha,\nu}}. \tag{48}$$

By the same arguments as we had in Lemma 6.2, $K$ and $N_{\Lambda_v^c}$ strongly commute. Now we estimate each term of (48).

Let us decompose the first term of (48) as

$$\begin{aligned} \langle \chi_n K \chi_n\rangle_{\psi_{\alpha,\nu}} = {} & \langle \psi_{\alpha,\nu}, P\chi_n K \chi_n P\psi_{\alpha,\nu}\rangle + \left\langle \psi_{\alpha,\nu}, \bar{P}\chi_n K \chi_n P\psi_{\alpha,\nu}\right\rangle \\ & + \left\langle \psi_{\alpha,\nu}, P\chi_n K \chi_n \bar{P}\psi_{\alpha,\nu}\right\rangle + \left\langle \psi_{\alpha,\nu}, \bar{P}\chi_n K \chi_n \bar{P}\psi_{\alpha,\nu}\right\rangle. \end{aligned}$$

By the strong commutativity of $N_{\Lambda_v^c}$ with and $K$ and $N$, we have

$$\left\langle \psi_{\alpha,\nu}, \bar{P}\chi_n K \chi_n P \psi_{\alpha,\nu} \right\rangle = \left\langle \psi_{\alpha,\nu}, \bar{P} P \chi_n K \chi_n \psi_{\alpha,\nu} \right\rangle = 0.$$

Similarly $\left\langle \psi_{\alpha,\nu}, P\chi_n K \chi_n \bar{P} \psi_{\alpha,\nu} \right\rangle = 0$. As $S_1 \geq 0$, we have

$$\begin{aligned} \left\langle \psi_{\alpha,\nu}, \bar{P}\chi_n K \chi_n \bar{P} \psi_{\alpha,\nu} \right\rangle &= \left\langle \psi_{\alpha,\nu}, \bar{P}\chi_n (S_1 + I_1^\lambda) \chi_n \bar{P} \psi_{\alpha,\nu} \right\rangle \\ &\geq \left\langle \psi_{\alpha,\nu}, \bar{P}\chi_n I_1^\lambda \chi_n \bar{P} \psi_{\alpha,\nu} \right\rangle \geq -\|I_1^\lambda\| \cdot \|\bar{P}\psi_{\alpha,\nu}\|^2 \end{aligned}$$

Hence we obtain

$$\langle \chi_n K \chi_n \rangle_{\psi_{\alpha,\nu}} \geq \langle K \rangle_{\chi_n P \psi_{\alpha,\nu}} - \|I_1^\lambda\| \cdot \|\bar{P}\psi_{\alpha,\nu}\|^2.$$

The estimation of the second term of (48) is the same as above;

$$\langle \bar{\chi}_n K \bar{\chi}_n \rangle_{\psi_{\alpha,\nu}} = \left\langle \bar{\chi}_n (S_1 + I_1^\lambda) \bar{\chi}_n \right\rangle_{\psi_{\alpha,\nu}} \geq \left\langle \bar{\chi}_n I_1^\lambda \bar{\chi}_n \right\rangle_{\psi_{\alpha,\nu}} \geq -\|I_1^\lambda\| \cdot \|\bar{\chi}_n \psi_{\alpha,\nu}\|^2.$$

Using operator calculous, the third and the forth term of (48) is bounded as

$$\|[\chi_n, [\chi_n, K]]\| \leq \frac{c_{11}}{n^2}, \quad \|[\bar{\chi}_n, [\bar{\chi}_n, K]]\| \leq \frac{c_{12}}{n^2}.$$

Hence we obtain

$$\langle K \rangle_{\psi_{\alpha,\nu}} \geq \langle K \rangle_{\chi_n P \psi_{\alpha,\nu}} - \|I_1^\lambda\| \cdot \|\bar{P}\psi_{\alpha,\nu}\|^2 - \|I_1^\lambda\| \cdot \|\bar{\chi}_n \psi_{\alpha,\nu}\|^2 - n^{-2} \cdot C_2,$$

$C_2$ is a constant which depends only on $\hat{I}_1^\lambda$ and $\chi$. □

Furthermore we estimate the lower bound of $\langle K \rangle_{\chi_n P \psi_{\alpha,\nu}}$. Let $\Delta$ be an interval which contains the eigenvalue $e$ of $L$ corresponding to $\psi$. Suppose that $\Delta$ contains exactly one eigenvalue $\tilde{e}$ of $L_S$ i.e., $\Delta \cap \sigma(L_S) = \{\tilde{e}\}$, and $\tilde{e}$ belongs to the interior of $\Delta$. Let $\chi_\Delta \in C_0^\infty(\mathbb{R})$ be a smooth function such that

$$0 \leq \chi_\Delta \leq 1, \qquad \chi_\Delta(t) = 1 \quad \text{for} \quad t \in \Delta, \qquad \mathrm{supp}\chi_\Delta \cap \sigma(L_S) = \{\tilde{e}\},$$

and suppose that the function $0 \leq \bar{\chi}_\Delta \leq 1$ defined by

$$\chi_\Delta^2 + \bar{\chi}_\Delta^2 = 1$$

belongs to $C_b^\infty(\mathbb{R})$. We set $\chi_\Delta \equiv \chi_\Delta(L)$ and $\bar{\chi}_\Delta \equiv \bar{\chi}_\Delta(L)$. Then by Theorem 7.1, if we take $\epsilon$, $\theta$ and $\lambda$ suitably, we have

$$P\chi_\Delta(L)\,[L, A]\,\chi_\Delta(L)P \geq b_{\tilde{e}} P\chi_\Delta(L) \left(1 - 14\tilde{P}_{\tilde{e}}\right) \chi_\Delta(L)P. \tag{49}$$

for some $b_{\tilde{e}} > 0$. We have the following lemma.

**Lemma 8.6** *Suppose the Assumption 2.1 and 2.2 are satisfied. Then for sufficiently small $\lambda$ we have*

$$\begin{aligned} \langle K \rangle_{\chi_n P \psi_{\alpha,\nu}} \geq b_{\tilde{e}} &\left( \|\chi_\Delta P \chi_n \psi_{\alpha\nu}\|^2 - 14\|\tilde{P}_{\tilde{e}} \chi_\Delta P \chi_n \psi_{\alpha\nu}\|^2 \right) \\ &- \|I_1^\lambda\| \cdot \|P\bar{\chi}_\Delta \chi_n \psi_{\alpha\nu}\|^2 - C_3(n^{-1} + \alpha + \alpha n + \nu). \end{aligned}$$

*Proof*
We have $\chi_\Delta D(N) \subset D(N)$ and the following decomposition is well-defined:

$$\begin{aligned}\langle K\rangle_{\chi_n P\psi_{\alpha,\nu}} &= \langle \chi_\Delta\left(K+[L,b]\right)\chi_\Delta\rangle_{\chi_n P\psi_{\alpha,\nu}} &(50)\\ &+ \langle \bar{\chi}_\Delta K \bar{\chi}_\Delta\rangle_{\chi_n P\psi_{\alpha,\nu}} &(51)\\ &+ \left\langle \frac{1}{2}[\chi_\Delta,[\chi_\Delta,K]] + \frac{1}{2}[\bar{\chi}_\Delta,[\bar{\chi}_\Delta,K]]\right\rangle_{\chi_n P\psi_{\alpha,\nu}} &(52)\\ &- \langle \chi_\Delta[L,b]\chi_\Delta\rangle_{\chi_n P\psi_{\alpha,\nu}} &(53)\end{aligned}$$

The first part (50) is estimated by (49) as

$$\begin{aligned}(50) = \langle \chi_\Delta\left(K+[L,b]\right)\chi_\Delta\rangle_{\chi_n P\psi_{\alpha,\nu}} &= \langle \chi_n\psi_{\alpha,\nu}\left|P\chi_\Delta[L,A]\chi_\Delta P\right|\chi_n\psi_{\alpha,\nu}\rangle\\ &\geq b_{\tilde{e}}\left(\|\chi_\Delta P\chi_n\psi_{\alpha\nu}\|^2 - 14\|\tilde{P}_{\tilde{e}}\chi_\Delta P\chi_n\psi_{\alpha\nu}\|^2\right).\end{aligned}$$

The estimation of the second part (51) is due to $S_1 \geq 0$;

$$(51) = \left\langle \bar{\chi}_\Delta(S_1+I_1^\lambda)\bar{\chi}_\Delta\right\rangle_{\chi_n P\psi_{\alpha,\nu}} \geq \left\langle \bar{\chi}_\Delta I_1^\lambda \bar{\chi}_\Delta\right\rangle_{\chi_n P\psi_{\alpha,\nu}} \geq -\|I_1^\lambda\|\|P\bar{\chi}_\Delta\chi_n\psi_{\alpha,\nu}\|^2$$

Now we estimate the third part (52). Note that

$$\left\langle \frac{1}{2}[\bar{\chi}_\Delta,[\bar{\chi}_\Delta,K]]\right\rangle_{\chi_n P\psi_{\alpha,\nu}} = \frac{1}{2}\langle \bar{\chi}_\Delta[\bar{\chi}_\Delta,K] - [\bar{\chi}_\Delta,K]\bar{\chi}_\Delta\rangle_{\chi_n P\psi_{\alpha,\nu}} = \mathrm{Re}\,\langle \bar{\chi}_\Delta[\bar{\chi}_\Delta,K]\rangle_{\chi_n P\psi_{\alpha,\nu}} \tag{54}$$

As $N_{\Lambda_v^c}$ strongly commutes with $N$, $L$, and $K$, we have

$$\langle \bar{\chi}_\Delta[\bar{\chi}_\Delta,K]\rangle_{\chi_n P\psi_{\alpha,\nu}} = \langle \chi_n P\psi_{\alpha,\nu}\left|\bar{\chi}_\Delta[\bar{\chi}_\Delta,K]\right|\chi_n P\psi_{\alpha,\nu}\rangle = \langle \chi_n\psi_{\alpha,\nu}\left|\bar{\chi}_\Delta P[\bar{\chi}_\Delta,K]\right|\chi_n\psi_{\alpha,\nu}\rangle,$$

and get the bound

$$\left|\langle \bar{\chi}_\Delta[\bar{\chi}_\Delta,K]\rangle_{\chi_n P\psi_{\alpha,\nu}}\right| \leq \|\bar{\chi}_\Delta\chi_n\psi_{\alpha,\nu}\|\cdot\|[\bar{\chi}_\Delta,K]\chi_n\psi_{\alpha,\nu}\|.$$

Let us estimate $\|\bar{\chi}_\Delta\chi_n\psi_{\alpha,\nu}\|$. As $L\psi = e\psi$ and $\bar{\chi}_\Delta(e) = 0$, we have $\bar{\chi}_\Delta\psi = 0$. So we get

$$\bar{\chi}_\Delta\chi_n\psi_{\alpha,\nu} = [\bar{\chi}_\Delta,\chi_n]h_\alpha g_\nu\psi + \chi_n[\bar{\chi}_\Delta,h_\alpha]g_\nu\psi + \chi_n h_\alpha[\bar{\chi}_\Delta,g_\nu]\psi.$$

By operator calculous, we have

$$\|[\bar{\chi}_\Delta,\chi_n]\| \leq n^{-1}\cdot c_2',\quad \|\chi_n[\bar{\chi}_\Delta,h_\alpha]\| \leq c_3'\cdot\alpha(1+n),\quad \|[\bar{\chi}_\Delta,g_\nu]\| \leq \nu\cdot c_4', \tag{55}$$

and hence we obtain

$$\|\bar{\chi}_\Delta\chi_n\psi_{\alpha,\nu}\| \leq c_5'(n^{-1}+\alpha+\alpha n+\nu),$$

where $c_5'$ is a positive constant.

On the other hand, $\|[\bar{\chi}_\Delta, K]\chi_n\psi_{\alpha,\nu}\|$ is estimated as

$$\|[\bar{\chi}_\Delta, K]\chi_n\psi_{\alpha,\nu}\| \leq c_6'.$$

Hence we get

$$\begin{aligned}\left\langle \frac{1}{2}[\bar{\chi}_\Delta, [\bar{\chi}_\Delta, K]] \right\rangle_{\chi_n P\psi_{\alpha,\nu}} &= \mathrm{Re}\, \langle \bar{\chi}_\Delta[\bar{\chi}_\Delta, K]\rangle_{\chi_n P\psi_{\alpha,\nu}} \geq - \|\bar{\chi}_\Delta\chi_n\psi_{\alpha,\nu}\| \cdot \|[\bar{\chi}_\Delta, K]\chi_n\psi_{\alpha,\nu}\| \\ &\geq - \left| \langle \bar{\chi}_\Delta[\bar{\chi}_\Delta, K]\rangle_{\chi_n P\psi_{\alpha,\nu}} \right| \geq -c_7'(n^{-1} + \alpha + \alpha n + \nu).\end{aligned}$$

The same argument is adaptable for $\left\langle \frac{1}{2}[\chi_\Delta, [\chi_\Delta, K]]\right\rangle$, and the third part (52) is bounded by

$$(52) = \left\langle \frac{1}{2}[\chi_\Delta, [\chi_\Delta, K]] + \frac{1}{2}[\bar{\chi}_\Delta, [\bar{\chi}_\Delta, K]] \right\rangle_{\chi_n P\psi_{\alpha,\nu}} \geq -C_3' \cdot (n^{-1} + \alpha + \alpha n + \nu).$$

The fourth term (53)is bounded by

$$\begin{aligned}\left| \langle \chi_\Delta[L, b]\chi_\Delta\rangle_{\chi_n P\psi_{\alpha,\nu}} \right| &= \left| \langle \chi_\Delta[L - e, b]\chi_\Delta\rangle_{\chi_n P\psi_{\alpha,\nu}} \right| \\ &\leq 2\|(L - e)\chi_\Delta\chi_n\psi_{\alpha,\nu}\| \cdot \|b\chi_\Delta\chi_n P\psi_{\alpha,\nu}\|\end{aligned}$$

We estimate $\|(L - e)\chi_\Delta\chi_n\psi_{\alpha,\nu}\|$. We have

$$(L - e)\chi_\Delta\chi_n\psi_{\alpha,\nu} = \chi_\Delta[L, \chi_n]\psi_{\alpha,\nu} + \chi_\Delta\chi_n[L, h_\alpha]g_\nu\psi + \chi_\Delta\chi_n h_\alpha[L, g_\nu]\psi.$$

By operator calculous we get

$$\|[L, \chi_n]\| \leq n^{-1}c_1'', \quad \|[L, g_\nu]\| \leq \nu c_2'', \quad \|\chi_n[L, h_\alpha]\| \leq c_3''(\alpha \cdot n + \alpha),$$

and obtain

$$\|(L - e)\chi_\Delta\chi_n P\psi_{\alpha,\nu}\| \leq c_4''(n^{-1} + \nu + \alpha \cdot n + \alpha).$$

On the other hand, we have $\|b\chi_\Delta\chi_n P\psi_{\alpha,\nu}\| \leq c_5''$, since $b$ is bounded. Hence we obtain

$$\left| \langle \chi_\Delta[L, b]\chi_\Delta\rangle_{\chi_n P\psi_{\alpha,\nu}} \right| \leq C_3''(n^{-1} + \nu + \alpha \cdot n + \alpha),$$

and get

$$(53) = - \langle \chi_\Delta[L, b]\chi_\Delta\rangle_{\chi_n P\psi_{\alpha,\nu}} \geq -C_3''(n^{-1} + \nu + \alpha \cdot n + \alpha)$$

Gathering the estimations, we obtain

$$\begin{aligned}&\langle K\rangle_{\chi_n P\psi_{\alpha,\nu}} = (50) + (51) + (52) + (53) \\ &\quad \geq b_{\tilde{e}}\left(\|\chi_\Delta P\chi_n\psi_{\alpha\nu}\|^2 - 14\|\tilde{P}_{\tilde{e}}\chi_\Delta P\chi_n\psi_{\alpha\nu}\|^2\right) - \|I_1^\lambda\|\|P\bar{\chi}_\Delta\chi_n\psi_{\alpha,\nu}\|^2 \\ &\qquad\qquad - C_3(n^{-1} + \nu + \alpha \cdot n + \alpha). \qquad \square\end{aligned}$$

Now let us complete the proof of Proposition 8.4. By substituting $\nu = \alpha^{\frac{3}{2}}$ and $n = \alpha^{-\frac{1}{2}}$ into Lemma 8.5 and 8.6, we have

$$\langle K \rangle_{\psi_{\alpha,\nu}} \geq b_{\tilde{e}} \left( \|\chi_\Delta P \chi_n \psi_{\alpha\nu}\|^2 - 14 \|\tilde{P}_{\tilde{e}} \chi_\Delta P \chi_n \psi_{\alpha\nu}\|^2 \right) - \|I_1^\lambda\| \cdot \|P \bar{\chi}_\Delta \chi_n \psi_{\alpha\nu}\|^2$$
$$- C_3(\alpha^{\frac{1}{2}} + \alpha + \alpha^{\frac{1}{2}} + \alpha^{\frac{3}{2}}) - \|I_1^\lambda\| \cdot \|\bar{P} \psi_{\alpha,\nu}\|^2 - \|I_1^\lambda\| \cdot \|\bar{\chi}_n \psi_{\alpha,\nu}\|^2 - \alpha \cdot C_2.$$

As

$$\lim_{\alpha \to 0, \nu \to 0, n \to \infty} \chi_n \psi_{\alpha\nu} = \psi, \qquad \lim_{\alpha \to 0, \nu \to 0, n \to \infty} \bar{\chi}_n \psi_{\alpha\nu} = 0, \qquad \lim_{\alpha \to 0, \nu \to 0} \psi_{\alpha\nu} = \psi,$$

we obtain

$$\lim_{\alpha \to 0} \langle K \rangle_{\psi_{\alpha,\nu}} \geq b_{\tilde{e}} \left( \|\chi_\Delta P \psi\|^2 - 14 \|\tilde{P}_{\tilde{e}} \chi_\Delta P \psi\|^2 \right) - \|I_1^\lambda\| \cdot \|P \bar{\chi}_\Delta \psi\|^2 - \|I_1^\lambda\| \cdot \|\bar{P} \psi\|^2.$$

Substituting $\chi_\Delta \psi = \psi$, $\bar{\chi}_\Delta \psi = 0$ and $P\psi = \psi$, we obtain the required result

$$\lim_{\alpha \to 0} \langle K \rangle_{\psi_{\alpha,\nu}} \geq b_{\tilde{e}} \left( \|\psi\|^2 - 14 \|\tilde{P}_{\tilde{e}} \psi\|^2 \right). \qquad \square$$

By Proposition 8.1 and 8.4, we have

$$0 = \lim_{\alpha \to 0} \langle K \rangle_{\psi_{\alpha,\nu}} \geq b_{\tilde{e}} \left( \|\psi\|^2 - 14 \|\tilde{P}_{\tilde{e}} \psi\|^2 \right).$$

If $\tilde{P}_{\tilde{e}} \psi = 0$, this is a contradiction. Then we have the following Theorem:

**Theorem 8.7** *For $e \in \mathbb{R}$, let $\tilde{e} \equiv e$, if $e$ is an eigenvalue of $L_S$, and let $\tilde{e}$ be an eigenvalue of $L_S$ which is next to $e$, in case that $e$ is not an eigenvalue of $L_S$. Suppose that Assumption 2.1 and 2.2 are satisfied. Then if $\lambda$ is small enough, there is no eigenvector with eigenvalue $e$, which is orthogonal to $\tilde{P}_{\tilde{e}}$. In particular, if $\tilde{P}_{\tilde{e}} = 0$, $e$ is not an eigenvalue of $L$.*

# 9 The Stability of the NESS

In this section we investigate our physical interest: the stability of the NESS. Recall that the NESS of the free Fermion model is given by $n$-point functions (3) with a distribution function (4). We show if the NESS is far from equilibrium, i.e., the inverse temperature $\beta_-$ and $\beta_+$ are different, it is macroscopically unstable (Theorem 2.2). Strictly speaking, we show the following: we can choose a suitable finite system such that the NESS is destroyed macroscopically when we add it with suitable interaction. This results from the lack of detailed balance of the NESS far from equilibrium. On the other hand, for a class of interaction, we show "return to equilibrium".

## 9.1 Instability of the NESS

First let us investigate the instability of NESS under interaction with suitable small system. By Theorem 8.7 and Proposition 3.4, it suffices to show $\tilde{P}_e = \mathrm{Ker}\Gamma(e) \otimes P_\Omega = 0$ for all eigenvalue $e$ of $L_S$. For the purpose, we study the kernel of $\Gamma(e)$. Recall the definition of $\varphi_n, E_n, E_{nm}, N_l^{(j)}, N_r^{(i)}, N_l, N_r$, given in Assumption 2.4. As $H_S$ is finite dimensional, $N_r^c \neq \varnothing$ and $N_l^c \neq \varnothing$ for $e \neq 0$. By a straight forward calculation, we obtain

$$\Gamma(e) = \int_{-\pi}^{\pi} dk \sum_{E_{n,m}\neq e} F_{n,m}^{1*}(k) F_{n,m}^{1}(k) \delta\left(\omega\left(k\right) + E_{n,m} - e\right) + F_{n,m}^{2*}(k) F_{n,m}^{2}(k) \delta\left(-\omega\left(k\right) + E_{n,m} - e\right) \tag{56}$$

where

$$F_{n,m}^{i}(k) = p_n Y p_{N_l^{(m)}} \otimes p_m g_1^i(k) - p_n \otimes p_m \bar{Y} p_{N_r^{(n)}} g_2^i(k).$$

First let us consider $e \neq 0$ case. Note that each term of (56) is positive for each $n, m$. So we take the sum over the following subset

$$(n,m) \in N_l \times N_r^c \,\dot{\cup}\, N_l^c \times N_r.$$

and ignore the other contributions to estimate the lower bound. Note that $N_l, N_r, N_l^c$ and $N_r^c$ are all non-empty in case $e \neq 0$. We have the following relations: if $n \in N_l$, then $N_r^{(n)} \neq \varnothing$, if $m \in N_r$, then $N_l^{(m)} \neq \varnothing$, if $n \in N_l^c$, then $N_r^{(n)} = \varnothing$, and if $m \in N_r^c$, then $N_l^{(m)} = \varnothing$. Especially, $E_{n,m} \neq e$ for $(n,m) \in N_l \times N_r^c \,\dot{\cup}\, N_l^c \times N_r$. Hence for all $(n,m) \in N_l \times N_r^c$, we have

$$F_{n,m}^{i}(k) = -p_n \otimes p_m \bar{Y} p_{N_r^{(n)}} g_2^i(k),$$

and for $(n,m) \in N_l^c \times N_r$, we have

$$F_{n,m}^{i}(k) = p_n Y p_{N_l^{(m)}} \otimes p_m g_1^i(k).$$

Then under Assumption 2.3 and 2.4, we have

$$\begin{aligned}\Gamma(e) &\geq 2a \left( \sum_{n\in N_l} p_n \otimes p_{N_r^{(n)}} \bar{Y} p_{N_r^c} \bar{Y} p_{N_r^{(n)}} + \sum_{m \in N_r} p_{N_l^{(m)}} Y p_{N_l^c} Y p_{N_l^{(m)}} \otimes p_m \right) \\ &\geq 2a \cdot P(L_S = e)\delta_0 = \gamma_e \cdot P(L_S = e),\end{aligned}$$

with $\gamma_e = 2a\delta_0 > 0$. Hence for $e \neq 0$, we have $\mathrm{Ker}\Gamma(e) = \{0\}$, and $\tilde{P}_e = 0$.

Next let us consider $e = 0$ case. In this case, we can not apply the above arguments because $N_l^c = N_r^c = \varnothing$. By the first condition of Assumption 2.3, a vector in $P(L_S = 0)$ is of the form

$$\varphi = \sum_i c_i \varphi_i \otimes \varphi_i.$$

Note that Assumption 2.3 imply $N_l^{(n)} = N_r^{(n)} = \{n\}$. We have

$$
\begin{aligned}
\langle \varphi \left| \Gamma(0) \right| \varphi \rangle = \sum_{n \neq m} \left| Y_{mn} \right|^2 & \\
\left[ \left| c_m \cdot (1-\rho)^{\frac{1}{2}} (k_{nm}) - c_n \cdot \rho^{\frac{1}{2}} (k_{mn}) \right|^2 \right] \left| f(k_{mn}) \right|^2 & \int_0^\pi dk \delta \left( \omega\left(k\right) + E_{n,m} \right) \\
+ \left[ \left| c_m \cdot (1-\rho)^{\frac{1}{2}} (-k_{mn}) - c_n \cdot \rho^{\frac{1}{2}} (-k_{mn}) \right|^2 \right] \left| f(-k_{nm}^1) \right|^2 & \int_{-\pi}^0 dk \delta \left( \omega\left(k\right) + E_{n,m} \right) \\
+ \left[ \left| c_m \cdot \rho^{\frac{1}{2}} (k_{nm}) - c_n \cdot (1-\rho)^{\frac{1}{2}} (k_{nm}) \right|^2 \right] \left| f(k_{nm}) \right|^2 & \int_0^\pi dk \delta \left( -\omega\left(k\right) + E_{n,m} \right) \\
+ \left[ \left| c_m \cdot \rho^{\frac{1}{2}} (-k_{nm}) - c_n \cdot (1-\rho)^{\frac{1}{2}} (-k_{nm}) \right|^2 \right] \left| f(-k_{nm}) \right|^2 & \int_{-\pi}^0 dk \delta \left( -\omega\left(k\right) + E_{n,m} \right)
\end{aligned}
\tag{57}
$$

By Assumption 2.2 and Assumption 2.3, there exist strictly positive $\gamma_0$, such that

$$
\begin{aligned}
(57) \geq \sum_{n \neq m} \gamma_0 \Bigg\{ & \left| c_m \cdot (1-\rho)^{\frac{1}{2}} (k_{nm}) - c_n \cdot \rho^{\frac{1}{2}} (k_{mn}) \right|^2 + \left| c_m \cdot (1-\rho)^{\frac{1}{2}} (-k_{mn}) - c_n \cdot \rho^{\frac{1}{2}} (-k_{mn}) \right|^2 \\
& + \left| c_m \cdot \rho^{\frac{1}{2}} (k_{nm}) - c_n \cdot (1-\rho)^{\frac{1}{2}} (k_{nm}) \right|^2 + \left| c_m \cdot \rho^{\frac{1}{2}} (-k_{nm}) - c_n \cdot (1-\rho)^{\frac{1}{2}} (-k_{nm}) \right|^2 \Bigg\}.
\end{aligned}
$$

Hence the necessary condition for $\mathrm{Ker}\Gamma(0)$ to be non-trivial is the existence of $\{c_n\}$ which satisfies

$$
\begin{aligned}
& \left| c_m \cdot (1-\rho)^{\frac{1}{2}} (k_{nm}) - c_n \cdot \rho^{\frac{1}{2}} (k_{mn}) \right|^2 = \left| c_m \cdot (1-\rho)^{\frac{1}{2}} (-k_{mn}) - c_n \cdot \rho^{\frac{1}{2}} (-k_{mn}) \right|^2 \\
= & \left| c_m \cdot \rho^{\frac{1}{2}} (k_{nm}) - c_n \cdot (1-\rho)^{\frac{1}{2}} (k_{nm}) \right|^2 = \left| c_m \cdot \rho^{\frac{1}{2}} (-k_{nm}) - c_n \cdot (1-\rho)^{\frac{1}{2}} (-k_{nm}) \right|^2 = 0.
\end{aligned}
$$

This implies the following:

$$
\frac{c_n}{c_m} = \frac{(1-\rho)^{\frac{1}{2}} (k_{mn})}{\rho^{\frac{1}{2}} (k_{mn})} = \frac{(1-\rho)^{\frac{1}{2}} (-k_{mn})}{\rho^{\frac{1}{2}} (-k_{mn})} = \frac{\rho^{\frac{1}{2}} (k_{nm})}{(1-\rho)^{\frac{1}{2}} (k_{nm})} = \frac{\rho^{\frac{1}{2}} (-k_{nm})}{(1-\rho)^{\frac{1}{2}} (-k_{nm})}
\tag{58}
$$

for $n \neq m$. Recall that the Fermion distribution $\rho$ in the NESS is given by

$$
\rho(k) \equiv \begin{cases} \left( 1 + e^{-\beta_- (\cos(k) - \gamma)} \right)^{-1}, & k \in [0, \pi) \\ \left( 1 + e^{-\beta_+ (\cos(k) - \gamma)} \right)^{-1}, & k \in [-\pi, 0). \end{cases}
$$

The condition (58) requires

$$
e^{-\beta_- E_{nm}} = e^{-\beta_+ E_{nm}},
$$

i.e.,$\beta_- = \beta_+$. That is, $\Gamma(0)$ has non trivial kernel only if the NESS is an equilibrium state indeed. If $\beta_- \neq \beta_+$, then, we have $\tilde{P}_e = 0$ for all eigen value $e$ of $L_S$. Hence the Liouville operator $L$ corresponding to the NESS does not have an eigenvector, by Theorem 8.7. Then, by Proposition 3.4, we have the following Theorem:

**Theorem 2.2** *The NESS far from equilibrium is macroscopically unstable.*

## 9.2 Return to equilibrium

In this section we investigate the equilibrium case, i.e., $\beta_+ = \beta_-$. By the result of the previous subsection, we can show *return to equilibrium* for the class of interaction we introduced. First, we introduce the following well-known facts. The following Theorem was shown by H.Araki [17],[18]:

**Theorem 9.1** *Let $(\mathfrak{A}, \tau)$ be a $C^*$-dynamical system and let $\omega$ be a $(\beta, \tau)$-KMS state with GNS-representation $(\mathcal{H}, \pi, \Omega)$. Let $L$ be the Liouville operator corresponding to $\tau$. If $P = P^* \in \mathfrak{A}$ then $\Omega \in D(\mathrm{e}^{\beta(L+\pi(P))/2})$. Let $\tau^P$ be the perturbed automorphism group by $P$, and let $\Omega^P \equiv \mathrm{e}^{\beta(L+\pi(P))/2}\Omega$. Then the state $\omega^P$ defined by*

$$\omega^P(A) = \frac{(\Omega^P, A\Omega^P)}{(\Omega^P, \Omega^P)}$$

*is a $(\beta, \tau^P)$-KMS state.*

On the other hand, the following theorem concerning *return to equilibrium* is known [9]:

**Proposition 9.2** *Let $(\mathfrak{A}, \tau)$ be a $C^*$-dynamical system and let $\omega$ be a $(\beta, \tau)$-KMS state with the GNS-representation $(\mathcal{H}, \pi, \Omega)$. Assume that the Liouville operator $L$ of $\tau$ has a simple eigenvalue $0$ corresponding to the eigenvector $\Omega$, and that the rest of the spectrum of $L$ is continuous. Then for any $\omega$-normal state $\rho$, we have the return to equilibrium in an ergodic mean sense:*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \rho\left(\alpha_t(A)\right) dt = \omega(A).$$

Now let us return to our system. Note that $\omega_\rho$ is a $(\beta, \alpha_t^f)$-KMS state of $\mathcal{O}_f$ if $\beta_- = \beta_+ = \beta$. We denote by $\omega_\beta^S$ the $(\beta, \alpha_S)$-KMS state over $\mathcal{O}_S$. The state $\omega_\beta^S \otimes \omega_\rho$ is a $(\beta, \alpha_0)$-KMS state over $\mathcal{O}_S \otimes \mathcal{O}_f$. As our perturbation $V$ is an element of $\mathcal{O}$, the nontriviarity of Ker$L$ is guaranteed by Theorem 9.1. The fact that 0 is the simple eigenvalue of $L$ can be derived as follows : first the equation (57) implies that Ker$\Gamma(0)$ is one-dimensional and is spanned by a vector of the form

$$\varphi = \sum_i c_i \varphi_i \otimes \varphi_i,$$

with

$$c_n = {Z_\beta}^{-\frac{1}{2}} \mathrm{e}^{\frac{\beta}{2}E_n}, \quad Z_\beta = \sum_n \mathrm{e}^{\frac{\beta}{2}E_n}.$$

Let $\psi_1, \psi_2$ be eigenvectors of $L$ with eigenvalue 0. By Theorem 8.7, we have $\tilde{P}_0 \psi_i = c_i \varphi \otimes \Omega$, with $c_i \neq 0$. Then $\frac{\psi_1}{c_1} - \frac{\psi_2}{c_2}$ is an eigenvector of $L$ with eigenvalue 0, which is orthogonal to $\tilde{P}_0$. By Theorem 8.7, this entails $\psi_1 = \frac{c_1}{c_2}\psi_2$. Hence the Liouville operator $L$ has a simple eigenvalue 0. We denote the corresponding eigenvector by $\Omega_V$ and the state corresponding to $\Omega_V$ by $\omega_V$. By Theorem 9.1, $\omega_V$ is a $(\beta, \alpha)$-KMS state. On the other hand, as $\Gamma(e)$ does not have any eigenvector for $e \neq 0$, and $\tilde{P}_e = 0$. So the rest of the spectrum of $L$ is continuous. Accordingly, from the Proposition 9.2, we obtain the *return to equilibrium*:

**Theorem 2.3** *Suppose that Assumption 2.1 and 2.4 are satisfied. Then for sufficiently small $\lambda$, the system exhibits return to equilibrium in an ergodic mean sense.*

**Acknowledgement.**
The author thanks for A.Arai, F.Hiroshima, and T.Matsui for useful advices. Many thanks also go to M.Merkli for helpful arguments and comments.

# A Tensor Product of Linear Operators

About tensor product of linear operators, we have the following Theorem. It can be proved using spectral theorem given in [16].

**Theorem A.1** *Let $\mathcal{H}_i (i = 1, 2)$ be separable Hilbert spaces, and let $L_i$ be self-adjoint operators on $\mathcal{H}_i$. Let $L$ be the self-adjoint operator on $\mathcal{H}_1 \otimes \mathcal{H}_2$, defined by*

$$L \equiv L_1 \otimes 1 + 1 \otimes L_2.$$

*Suppose that $L_2$ has a unique eigenvector $\Omega$. Then every eigenvector of $L$ is of the form*

$$\varphi \otimes \Omega, \quad \varphi \in \mathcal{H}_1.$$

# B Inequality of the Second Quantized Operator

The following lemma can be derived easily from the positivity of operator $x^* x$.

**Lemma B.1** *For all $\varphi_1, \varphi_2 \in D(N)$, $g \in \mathfrak{h}_{\Lambda_v}$, and $c > 0$ we have*

$$|\langle\varphi_1|X \otimes (a(g) + a^*(g))\,\varphi_2\rangle| \leq 3c\|X\|^2 \left(\left\|N_{\Lambda_v}^{\frac{1}{2}}\varphi_1\right\|^2 + \left\|N_{\Lambda_v}^{\frac{1}{2}}\varphi_2\right\|^2\right) + 3\frac{\|g\|^2}{c}\left(\|\varphi_1\|^2 + \|\varphi_2\|^2\right)$$

$$\left|\langle\varphi_1|X \otimes (-1)^N(a(g) - a^*(g))\varphi_2\rangle\right| \leq 3c\|X\|^2 \left(\left\|N_{\Lambda_v}^{\frac{1}{2}}\varphi_1\right\|^2 + \left\|N_{\Lambda_v}^{\frac{1}{2}}\varphi_2\right\|^2\right) + 3\frac{\|g\|^2}{c}\left(\|\varphi_1\|^2 + \|\varphi_2\|^2\right)$$

# References


[1] D.Ruelle,*A variational formulation of equilibrium statistical mechanics and the Gibbs phase rule.* Commun. Math. Phys. **5**,324–329,(1967)

[2] W.Pusz, and S.L.Woronowicz, *Passive states and KMS states for general quantum systems* Commun.Math.Phys.**58**,273-290,(1978)

[3] O.Lanford and D.W.Robinson, *Statistical mechanics of quantum spin systems. III* Commun.Math.Phys.**9**,327-338,(1968)

[4] H.Araki,*On the equivalence of the KMS condition and the variational principal for quantum lattice systems* Commun.Math.Phys.**38**,1-10,(1974)

[5] R.Haag, D.Kastler, and B.Trych-Pohlmeyer, *Stability and equilibrium states.* Commun. Math. Phys. **38** (1974), 173–193.

[6] H. Araki,*On the $XY$-model on two-sided infinite chain* Publ.RIMS,Kyoto Univ, **20** 277-296 (1984).

[7] V. Jakšić, and C. A. Pillet, *On a model for quantum friction. II: Fermi's golden rule and dynamics at positive temperature*, Commun. Math. Phys. 176, 619-644 (1996).

[8] V. Jakšić, and C. A. Pillet, *On a model for quantum friction. III: Ergodic properties of the spin-boson system* Commun. Math. Phys. 178, 627-651 (1996).

[9] V.Bach, J.Fröhlich, I.M.Sigal, *Return to equilibrium.* J. Math. Phys. **41** no. 6, 3985–4060 (2000).

[10] M.Merkli,*Positive commutators in non-equilibrium quantum statistical mechanics* Commun. Math.Phys.**223** 327-362, (2001)

[11] V.Jakšić and C.A.Pillet, *Non-equilibrium steady states of finite quantum systems coupled to thermal reservoirs.* Commun. Math. Phys. **226** no.1, 131-162,(2002)

[12] V.Jakšić and C.A.Pillet, *Mathematical theory of non-equilibrium quantum statistical mechanics.* J. Statist. Phys. 108 (2002), no. 5-6, 787–829

[13] V.Bach, J.Fröhlich and I.M.Sigal, Quantum electrodynamics of confined nonrelativistic particles. Adv. Math. **137** no. 2, 299–395 (1998)

[14] O. Bratteli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics 1*, Springer-Verlag, 1986.

[15] O. Bratteli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics 2*, Springer-Verlag, 1996.

[16] M.Reed and B.Simon, *Methods of modern mathematical physics. I. Functional analysis.* Second edition. Academic Press, Inc. New York, 1980

[17] H. Araki: *Relative Hamiltonian for faithful normal states of a von Neumann algebra*, Publ. R.I.M.S., Kyoto Univ. **9** 165-209 (1973).

[18] H. Araki: *Positive cone, Radon-Nikodym theorems, relative Hamiltonians and the Gibbs condition in statistical mechanics* C*-Algebras and their Applications to Statistical Mechanics and Quantum Field Theory, edited by D. Kastler (Editrice Compositori, Bologna, 1975).

[19] T.G.Ho,andH.Araki,*Asymptotic time evolution of a partitioned infinite two-sided isotropic $XY$-chain* Proc.SteklovInst.Math,**228** 191-204 (2000).

[20] W.H. Aschbacher C.A. Pillet , *Non-Equilibrium Steady States of the $XY$ Chain* mp-arc /02-459 ,(2002).