LIFSHITZ TAILS FOR RANDOM SCHRÖDINGER OPERATORS WITH NEGATIVE SINGULAR POISSON POTENTIAL

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ABSTRACT. We develop a new method of asymptotic study of the integrated density of states $N(E)$ of a random Schrödinger operator with a non-positive (attractive) Poisson potential. The method is based on the periodic approximations of the potential instead of the Dirichlet-Neumann bracketing used before. This allow us to derive more precise bounds for the rate of approximations of the IDS by the IDS of respective periodic operators and to obtain rigorously for the first time the leading term of $\log N(E)$ as $E \to -\infty$ for the Poisson random potential with a singular single-site (impurity) potential, in particular, for the screened Coulomb impurities, dislocations, etc.

RÉSUMÉ. Nous développons une méthode nouvelle pour létude de l'asymptotique de la densité d'états intégrée $N(E)$ d'un opérateur de Schrödinger aléatoire avec potentiel de Poisson attractif. Cette nouvelle méthode est fondée sur les approximations périodiques. Pour ces approximations, la convergence de la densité d'états intégrée vers la densité d'états de l'opérateur aléatoire est exponentielle. En conséquence, nous obtenons l'asymptotique de $\log N(E)$ quand $E \to -\infty$ pour des potentiels aléatoires de Poisson avec un potentiel simple site singulier, comme par exemple, le potentiel de Coulomb ecranté, des potentiels de dislocation, etc.

0. INTRODUCTION: PROBLEMS AND HISTORY

The Integrated Density of States (IDS) is one of the simplest but quite important characteristics of the random Schrödinger operator. Among numerous problems related to the IDS, the problem of its asymptotic behavior near the spectral edges (i.e. the edges of the spectrum) is well known and fairly studied. Results of these studies can be summarized as follows. One has to distinguish two types of spectral edges: stable and fluctuational (see e.g. [7, 18, 21]). The latter are special for shortly correlated random potentials. In the simplest case of the lower edge of the spectrum, they are determined by the absolute minimum of the potential since the spectrum in a neighborhood of this edge exists only because of the (arbitrarily large) fluctuations of the potential arbitrarily close to the minimum. By using the quantum mechanical terminology one can call potential wells these portions of the realizations.

A heuristic derivation of the fluctuational asymptotics was proposed by I.Lifshitz in the early 60's ([16, 17]). The asymptotics is given by the probability to have a potential well whose ground state energy is close enough to the spectral edge.

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Since the probability of these realizations having the form of very broad or deep potential wells (and known as optimal fluctuations) is usually exponentially small, one has to deal here with a version of the large deviation technique in the spectral context. In particular, to determine the asymptotic formula for the IDS one has to be able to give rather detailed description of the statistics of these special realizations. This is why precise and explicit asymptotic formulae are known only for comparatively restricted classes of random potentials.

One of the widely studied random potentials is the Poisson potential having the form

$$V_\omega(x) = \sum_j V(x - x_j) \tag{0.1}$$

where $\{x_j\}$ is the Poisson point field of the density μ in $\mathbb{R}^d$ and $V(x)$, the one-site (or single site) potential, is a function decaying sufficiently fast at infinity. The Poisson potential is of considerable interest both, in the spectral theory and in theoretical physics of disordered systems. It possesses a number of nontrivial asymptotic regimes, only part of which has been studied so far. One has to mention first the case of the nonnegative one-site potential $V(x)$ of a compact support. In this case $E = 0$ is a fluctuational edge and according to I.Lifshitz

$$N(E) \simeq \exp(-const \cdot E^{-d/2}), \quad E \to +0. \tag{0.2}$$

The right hand side of this formula is just the probability to have a well (a region of $\mathbb{R}^d$ free of x_j''s) of width $L \simeq E^{-1/2}$; the latter relation is due to the uncertainty principle. In other words, the asymptotics of the IDS in this case is determined by an optimization procedure, balancing the competition of the quantum and the probabilistic components of the problem. This is why formula (0.2) is often called the quantum Lifshitz tail. Rigorous derivations of various versions of (0.2) (e.g. its logarithmic or even its the double logarithmic forms) have required a number of rather sophisticated probabilistic and spectral techniques (see e.g. [30, 31, 3, 21, 27, 12] for results and references). Here and below we use the symbol "$\simeq$" to denote the asymptotic equivalence without indicating explicitly the order of the remainder and respective constants.

Other asymptotic regimes of the IDS for the potential (0.1) correspond to the case when the one-site potential has a non positive part, i.e. $\inf V(x) < 0$, so that the lower edge of the spectrum is $E = -\infty$. In this case one has to distinguish the two asymptotic regimes, called usually the quantum and the classical one. We will present respective asymptotic formulae by using a version of Lifshitz's arguments adapted to this case.

Recall the definition of the IDS (implicit also in [17, 16]): it is the limit as $L \to +\infty$ of the expectation of the normalized counting function of eigenvalues of the Schrödinger operator H_{Λ_L}, where H_{Λ_L} is the restriction of $H_\omega = -\Delta + V_\omega$ to $L^2(\Lambda_L)$ (see e.g. [21, 7]). Here Λ_L is the cube of the $\mathbb{R}^d$ of center zero and of side length L. The definition shows that the IDS can be regarded as the probability to find an eigenvalue of H_{Λ_L} lying below a given energy E. For $E \to -\infty$ these eigenvalues are produced by very deep potential wells, created by large clusters of k Poisson points x_j's, confined to a sufficiently small regions of the space, say a cube of the side length $l << L$. The respective probability is

$$e^{-\mu l^d}(\mu l^d)^k / k! \tag{0.3}$$

Assume first that l is independent of k. This corresponds to a single site potential that is varying relatively slowly with respect to the exponential decay of the single site ground state. Then, the IDS has to be asymptotically equivalent to the probability (0.3) in which k is determined by the equation $E(k) = E$ where $E(g)$ is the ground state energy of the well $gV(x)$. In other words, for a non positive one-site potential $V(x)$ optimal fluctuations are potential wells whose form is roughly $kV(x)$. Thus, we have to find the ground state energy $E(g)$ of the well $gV(x)$ as the function of its coupling constant g and then the asymptotic form of the IDS will be given by (0.3) in which $k = g(E)$ where $g(E)$ is the asymptotic inverse of $-E(g)$ for $E \to -\infty$, $g \to +\infty$. The leading contribution is given by the factor $1/k!$. Thus the asymptotic formula should have the form

$$N(E) \simeq \exp(-g(E)\log g(E)), \quad E \to -\infty \tag{0.4}$$

or

$$\log N(E) \simeq -g(E)\log g(E), \quad E \to -\infty \tag{0.5}$$

The simplest case when this formula can be made rigorous is when $V(x)$ is continuous in its (negative) absolute minimum. In this case

$$E(g) = gV(0)(1 + o(1)), \quad E \to -\infty \tag{0.6}$$

The respective rigorous formula

$$\log N(E) = \frac{E}{V(0)} \log|E|(1 + o(1)), \quad E \to -\infty \tag{0.7}$$

was proved in [20] using a combination of variational and Wiener integral techniques. The formula is obviously classical because it includes only the characteristics of the potential but not the Laplacian. In addition it can be easily shown that the r.h.s. of (0.4) is the asymptotic form of the logarithm of $P(V_\omega(x) < E)$. This provides one more evidence of the classical nature of the formula.

This paper is devoted to quantum versions of (0.4), i.e. to singular one-site potentials $V(x)$. Despite a considerable amount of physical results in the question (see e.g. [18, 4]) rigorous results are almost absent, except the one dimensional case of point one-site potentials

$$V(x) = -g_0\delta(x), \quad g_0 > 0 \tag{0.8}$$

where $\delta(x)$ is the Dirac delta-function. In this case by using the special Markov processes technique one can show (see [18]) an asymptotic of the form

$$\log N(E) = -2\sqrt{\frac{E}{E_0}}\log\sqrt{\frac{E}{E_0}}(1 + o(1)), \quad E \to -\infty$$

where $E_0 = -g_0^2/4$. We see that formula (0.5) is only valid up to a factor. 2.

This difference is the result of a tunneling phenomenon related to the question how close to one another should k potentials (0.8) be in order to be regarded as the potential $k^*(k)\delta(x)$ i.e. the potential of same shape as (0.8) and of amplitude $k^*(k)$. In other words, in this case contrary to the classical case, the radius of the exponential decay of the single site ground state is much larger than the width of the single site potential. Thus the optimal cluster should be much smaller (i.e. its radius should tend to 0 sufficiently fast) in order to be modeled by a single site potential of some effective amplitude $k^*(k)$. We shall see below that in many interesting cases $l \simeq k^{-\alpha^*}$ for some $\alpha^* > 0$. Because of that, the factor $k^{-\alpha^* dk}$ in (0.3) will also contribute to the asymptotic formula of the IDS. The study of this phenomenon is one of the topics dealt with in the present paper.

In this paper we study the case of singular one-site potential (mainly with power-law singularities) following in essence the scheme outlined above. We find the precise form of $k^*(k)$ (see Theorem 1.4 and 1.6 below). This became possible owing to an improving of one of the technique in the field, based on approximations of the Schrödinger operator in the whole space by the operator with the same potential but defined in a finite box whose size is properly chosen as a function of energy. Previous versions of this technique were based on the so called Neumann-Dirichlet bracketing where the boxes with the Neumann and Dirichlet boundary conditions were used to construct the upper and the lower bounds for the IDS. The error in these bounds is of the order $O(L^{-1})$ where L is the size of the box. This precision is not sufficient to treat the quantum case.

In the present paper we approximate the IDS of the random model by the IDS of some well chosen periodic Schrödinger operators and obtain much more precise bounds (see section 2 and more precisely Lemmas 2.1 and 2.3)). This method was proposed in [12] and has been used to solve several problems in the field ([11, 10]). We obtain the once logarithmic versions of (0.4), i.e. (0.5) with explicit $g(E)$ and constants in front of $g(E)\log g(E)$ (see Theorems 1.5 and 1.6).

Let us now give just one example of the results given in section 1.2; let V the single site potential be the 3-dimensional attractive screened Coulomb potential $V(x) = -\dfrac{e^{-|x|}}{|x|}$. In this case we prove

$$\log N(E) = -2\sqrt{|E|}\log|E|(1+o(1)), \quad E \to -\infty,$$

(see Theorem 1.6 and the discussion following it).

The role of the IDS in the spectral theory and theoretical physics of disordered systems is well known and appreciated. However, for us there is one more reason to study this quantity. Since the pioneering papers of I.Lifshitz, the study of the IDS has been providing a first important step in the understanding and study of more complicated properties and quantities in a respective version of the strong localization regime. In particular, the IDS is the first moment (see formula (1.4)) of the spectral kernel of the Schrödinger operator. From the mathematical physics point of view, the IDS determines the equilibrium properties of disordered systems, i.e. the ideal gas of elementary excitation (electrons, phonons, spin waves etc.) in the random environment. The study of the kinetic properties of this gas and the interaction effects requires the knowledge of higher moments of the spectral kernel, the second moments first of all. The knowledge of these correlators allows one to answer a number of relevant questions concerning the existence and the nature of the localization and behavior of related quantities. In particular, in subsequent paper ([13]), we use the technique developed in this paper in order to find the large deficit asymptotic behavior of the interband light absorption coefficient.

The paper is organized as follows. In section 1 we define the framework of our study and give a brief account of our results. We also present several examples at the end of the section. In section 2 we first prove the basic relation (1.4) expressing the IDS in terms of the spectral family of the random Schrödinger operator. Then we construct our main technical tool, the periodic approximations of the IDS. Sections 3 and 4 are devoted to the derivation of the lower and upper bounds for the IDS using the periodic approximations. Section 5 contains auxiliary facts on the statistics of the Poisson field, on random Schrödinger operators and on the structure of the ground state of the Schrödinger operator with a singular single site potential.

Contents

1. The assumptions and the results

Let $V : \mathbb{R}^d \to \mathbb{R}$ be a function such that

$$V = V_1 + V_2$$

where

H1 for some $C > 0$ and any $x \in \mathbb{R}^d$, $|V_1(x)| \leq Ce^{-|x|/C}$.

H2 V_2 is a compactly supported function satisfying $V_2 \in L^p(\mathbb{R}^d)$ where $p > p(d)$ and $p(d) = 2$ if $d \leq 2$ and $p(d) = d/2$ if $d \geq 3$.

H3 for some set of positive measure $\mathcal{E}$, $V\big|_{\mathcal{E}} < 0$.

Define the random potential

$$V_\omega(x) = \int_{\mathbb{R}^d} V(x-y)m(\omega, dy) \tag{1.1}$$

where $m(\omega, dy)$ is a random Poisson measure of concentration μ. V_ω is an ergodic random field on $\mathbb{R}^d$. Consider the random Schrödinger operator

$$H_\omega = -\Delta + V_\omega. \tag{1.2}$$

One has

Theorem 1.1 ([7]). *Under the assumptions made above, H_ω is essentially self-adjoint on $\mathcal{C}_0^\infty(\mathbb{R}^d)$ ω-almost surely.*

Under our assumptions on V, we know that the almost sure spectrum of H_ω is $\Sigma = \mathbb{R}$ ([21, 7]).

1.1. **The integrated density of states.** Let Λ be a cube centered at 0 in $\mathbb{R}^d$. We define $H^D_{\omega,\Lambda}$ to be the Dirichlet restriction of H_ω to Λ. Pick $E \in \mathbb{R}$. Consider the quantity

$$N_{\omega,\Lambda}(E) = \frac{1}{\mathrm{Vol}(\Lambda)} \sharp\{\text{eigenvalues of } H^D_{\omega,\Lambda} \text{ smaller than or equal to } E\}. \tag{1.3}$$

Then one has

Theorem 1.2 ([7]). *Under the assumptions made above, there exists a non random, non decreasing, non negative, right continuous function $N(E)$ such that, ω-almost surely, for all $E \in \mathbb{R}$, E a continuity point of N, $N_{\omega,\Lambda}(E)$ converges to $N(E)$ as Λ exhausts $\mathbb{R}^d$.*

$N(E)$ is the *integrated density of states* (IDS) of H_ω. As N is non decreasing, one can define its distributional derivative dN. It is a positive measure and is supported on the almost sure spectrum of H_ω (see [7, 21]). One has the following result

Theorem 1.3. *For $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$, we have*

$$(\varphi, dN) = \mathbb{E}(tr(\mathbf{1}_{C(0,1)}\varphi(H_\omega)\mathbf{1}_{C(0,1)})). \tag{1.4}$$

where $C(0,1)$ is the cube of center 0 and side length 1.

Formula (1.4) is well known under more restrictive assumptions on the potential V_ω i.e. for a less singular single site potential V (see [21]).

1.2. **The asymptotics of the IDS.** To describe the asymptotic behavior of $N(E)$ near $-\infty$, we will need to define an auxiliary operator. For $g \in \mathbb{R}$, define

$$H(g) = -\Delta + gV. \tag{1.5}$$

Under our assumptions on V, V is relatively form bounded with respect to $-\Delta$ with relative bound 0. Hence $H(g)$ admits a unique self-adjoint extension. Let $\sigma(H(g))$ denote its spectrum. It is lower semi-bounded. The infimum of $\sigma(H(g))$ i.e. the ground state energy of $H(g)$ will be denoted by $E(g)$. Let φ_g be the respective ground state i.e. the unique positive normalized eigenfunction of $H(g)$ associated to energy $E(g)$ ([22, 26]). In the sequel it will often be more convenient to work with $E_-(g) = -E(g)$ instead of $E(g)$ itself. From assumption H3, one easily infers that $E_-(g) \to +\infty$ when $g \to +\infty$ (see section 5). Moreover E_- is strictly increasing in a neighborhood of $+\infty$. Let g be an inverse of E_- in a neighborhood of $+\infty$. g is strictly increasing.

In the regular (classical) case, it was found that g is governing the first term asymptotic of $\log N$ (cf [21, 20]). In the singular (quantum) case, the singular set of V will play a special part in the asymptotics. To measure this role, we introduce the notion of *asymptotic ground state* i.e.

Definition 1.1. Let $g \in (1, +\infty) \mapsto \psi_g \in H^1(\mathbb{R}^d)$. We will say that ψ_g is an *asymptotic ground state* if and only if

- the vector ψ_g is normalized.
- $\exists g_0 > 1$, $l_0 > 0$ such that $\forall g \geq g_0$, $\operatorname{supp} \psi_g \subset C(0, l_0)$ (where $C(x, l)$ denotes the cube of center x and side length l).
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$$\frac{|\langle (H(g) - E(g))\psi_g, \psi_g\rangle|}{|E(g)|} \to 0 \text{ as } g \to +\infty. \tag{1.6}$$

In Lemma 5.6, we prove the existence of an asymptotic ground state. For $a \in \mathbb{R}^d$, we define the translation τ_a by $\tau_a V(x) = V(x - a)$ and we define

$$\mathcal{A}_{\psi_g} = \left\{ \alpha > 0;\ \lim_{g \to +\infty} \sup_{|a| \leq g^{-\alpha}} \left[\frac{g|\langle (\tau_a V - V)\psi_g, \psi_g\rangle|}{E_-(g)} \right] = 0 \right\}$$

If $\mathcal{A}_{\psi_g} \neq \emptyset$, then we define $\alpha^*(\psi_g) := \inf \mathcal{A}_{\psi_g}$. Moreover, we define $\mathcal{A}$ to be the union of all $\mathcal{A}_{\psi_g}$. By Lemma 5.6, we know that $\mathcal{A} \neq \emptyset$. We define

$$\alpha^* := \inf \mathcal{A}. \tag{1.7}$$

Roughly speaking, the dependence of the radius of the exponential decay of the single site potential ground state on the coupling constant g is of the form $g^{-\alpha^*}$. This determines the characteristic size l of the optimal cluster.

Then we prove

Theorem 1.4. *Under the assumptions H1, H2 and H3, for sufficiently large E, one has*

$$-(1 + \alpha^* d) g(E) \log g(E)(1 + o(1)) \leq \log N(-E) \leq -g(E) \log g(E)(1 + o(1)). \tag{1.8}$$

One may complain that Theorem 1.4 is somewhat imprecise in that it only gives a two sided estimate. But, as we will see below, this is in some way unavoidable as the true asymptotic does not only depend on g but also on the singular set of the negative part of V. More precisely, as can be seen from Theorem 1.6 (and from the proof of Theorem 1.4), the asymptotics of the IDS depends on the way the eigenfunction associated to the lowest eigenvalue for the operator $-\Delta + gV$ concentrates near the singular set of the negative part of V as g becomes large. In general the correction also depends on the geometry of the singular set. For example, if the singular set is a segment (e.g. a dislocation), using the techniques developed in section 3, one can see that neither the lower nor the upper bound given by Theorem 1.4 are sharp. The two sided estimate (1.8) can be made more precise if we know more on V.

The first and simplest example we give is the case when V is bounded from below, reaches its minimum at a single point, say0, and is continuous near 0. Then one easily proves that $\alpha^* = 0$ and the upper and lower bounds in (1.8) coalesce to give (0.7).

We will now give other results that, we think, enclose most of the physically relevant examples.

Let v_- be the essential infimum of V and assume that V is bounded from below, say

H1' $-\infty < v_- < 0$.

It is easy to show that $g(E) \sim E/|v_-|$ when $E \to +\infty$ (see Lemma 5.5). We obtain

Theorem 1.5. *Under the assumptions H1, H2 and H1', one has*

$$\log N(-E) \underset{E\to+\infty}{\sim} -g(E)\log g(E) \underset{E\to+\infty}{\sim} \frac{E}{v_-}\log E. \tag{1.9}$$

Here and in the rest of the paper, $a \sim b$ will always mean $a = b(1+o(1))$.

This result extends (0.7) removing the continuity assumption near the minimum.

Consider now an example a bit more general that could be regarded as a model of line defect (charged dislocation). In this case, $d = 2$ and

$$V_2(x) = \log_- |x|, \quad x \in \mathbb{R}^2$$

where, for $a \geq 0$, $\log_- a = \min\{\log a, 0\}$. Using the inequality $\log_- |x| + \log R \leq \log_- R|x| \leq \log |x|$ for $0 < R < 1$ and the variational principle for the ground state energy, one shows that, in this case, $E_-(g) \underset{g\to+\infty}{\sim} g/2\log g$ hence $g(E) \underset{E\to+\infty}{\sim} 2E/\log E$. One also shows that $\alpha^* = 0$ for this single site potential. Therefore, Theorem 1.4 tells us that

$$\log N(-E) \underset{E\to+\infty}{\sim} -g(E)\log g(E) \underset{E\to+\infty}{\sim} -2E.$$

Hence asymptotic (0.5) is also valid for certain mildly singular potentials.

Another case where one can find an asymptotic for $\log N$ is when V has only power law singularities. Let q be a positive integer and pick q positive exponents $(\nu_i)_{i=1,\dots,q}$ and q functions $(h_i(\theta))_{i=1,\dots,q}$ continuous on the sphere $\mathbb{S}^{d-1}$. For $1 \leq i \leq q$, consider the potentials

$$V_i(x) = \frac{h_i(\theta(x))}{|x|^{\nu_i}} \text{ where } \theta(x) = \frac{x}{|x|} \tag{1.10}$$

Assume that

$$0 < \nu_i < \begin{cases} 2 \text{ if } d = 1, 2, \\ d/2 \text{ if } d \geq 3. \end{cases} \tag{1.11}$$

Then V_i is relatively form bounded with respect to $-\Delta$ with relative bound 0 and we can consider the operators $H^i = -\Delta + V_i$ with form domain $H^1(\mathbb{R}^d)$. For $1 \leq i \leq q$, E_i denotes the ground state energy of H^i. Now we assume that

H1"
- there exists q distinct points $(x_i)_{i=1,\dots,q}$ in $\mathbb{R}^d$ and q continuous compactly supported function $(W_i)_{i=1,\dots,q}$ such that $W_i(0) = 1$ such that

$$V_2(x) = \sum_{i=1}^{q} W_i(x - x_i)V_i(x - x_i) = \sum_{i=1}^{q} \tau_{x_i}(W_iV_i)(x) \tag{1.12}$$

  where the exponents $(\nu_i)_{1\leq i\leq q}$ satisfy (1.11).
- for some $1 \leq i_0 \leq q$, we have $E_{i_0} < 0$.

Notice that assumption H1" implies assumptions H2 and H3. Define

$$\nu^\dagger = \sup\{\nu_i;\ 1 \le i \le q \text{ such that } E_i < 0\}, \tag{1.13}$$

$$E_- = \sup\left\{|E_i|;\ 1 \le i \le q \text{ such that } E_i < 0 \text{ and } \nu_i = \nu^\dagger\right\}, \tag{1.14}$$

$$\alpha^\dagger = \frac{1}{2-\nu^\dagger}. \tag{1.15}$$

Then, by Lemma 5.7, we know that $E_-(g) \sim E_- g^{2\alpha^\dagger}$ when $g \to +\infty$. In addition, using the periodic approximation scheme, in this case, we can find an upper bound having the same form as the lower bound in (1.8) where α^* is replaced by $\alpha^\dagger$ (Proposition 4.4). On the other hand, we prove that $\alpha^* \le \alpha^\dagger$ (Lemma 3.1). From these two facts and from Theorem 1.4, we deduce $\alpha^* = \alpha^\dagger$ and

Theorem 1.6. *Under the assumptions H1 and H1", one has*

$$\begin{aligned} \log N(-E) &\underset{E\to+\infty}{\sim} -(1+\alpha^\dagger d) g(E) \log g(E) \\ &\underset{E\to+\infty}{\sim} -\left(1+\frac{d-\nu^\dagger}{2}\right)\left(\frac{E}{E_-}\right)^{1-\nu^\dagger/2} \log\left(\frac{E}{E_-}\right) \end{aligned} \tag{1.16}$$

We see that the lower bound in (1.8) is rather universal. There exist other methods to obtain this bound, for instance a version of the variational method of [18, 21] (also [13]. This method also gives a sharp lower bound for practically all asymptotic formulae for $\log N(-E)$ known so far. This is true in particular for the Poisson potential with a non-negative single site potential. Thus it is the upper bound that has been requiring different techniques depending on the specific potential (see e.g. [3, 27, 30, 12])

Assumptions H1 and H1" include most physically interesting cases as, for example, the 3-dimensional attractive screened Coulomb potential $V(x) = -\dfrac{e^{-|x|}}{|x|}$. In this case we have $\alpha^* = \alpha^\dagger = 1$ and $E_- = 1$ (see [15]); thus

$$\log N(-E) \underset{E\to+\infty}{\sim} -2\sqrt{E}\log E.$$

There is another physically interesting case that has not been discussed here: it is the case of point potentials. Such potentials will be studied in a sequel to this paper [13].

2. Periodic approximations

2.1. **A general approximation result.** In this subsection, we will show how to adapt the periodic approximation scheme developed in [12] to the case of unbounded potentials. The main new technical problems will come from the fact that, in the present case, the negative part of the random potential grows much faster at infinity than in the case dealt with in [12] or when the single site potential is bounded.

In the sequel $C(x,l)$ will denote the cube of center x and side length l. For $y \in \mathbb{R}^d$ and f a function on $\mathbb{R}^d$, f_y will denote the translate of f by y i.e. $f_y(x) = f(x-y)$.
Fix ω; the support of the Poisson measure $\mu(\omega, dy)$ is a set of discrete points that we

denote by $(x_k(\omega))_{k\in\mathbb{N}}$. We then define the following periodic potential

$$V_{\omega,n} = \sum_{\beta\in n\mathbb{Z}^d} \left(\sum_{k;\ \omega_k\in C(0,n)} V(x-\beta-x_k(\omega)) \right)$$

and the corresponding periodic Schrödinger operator

$$H_{\omega,n} = -\Delta + V_{\omega,n}. \tag{2.1}$$

By our regularity assumptions on V, ω-almost surely, $H_{\omega,n}$ is relatively form bounded with respect to $-\Delta$ (see [2]). It is essentially self-adjoint on $\mathcal{C}_0^\infty(\mathbb{R}^d)$. We can define the integrated density of states $N_{\omega,n}(E)$ of this periodic operator by the same limit procedure as in the case of the random operator (see [22, 14]). We will now compare $N(E)$ and $\mathbb{E}(N_{\omega,n}(E))$. We prove

Lemma 2.1. *For any $\alpha\in(0,1)$, there exists $C>1$ and $\rho>0$ (depending only on d) such that, for any $\varphi\in\mathcal{C}_0^\infty(\mathbb{R})$, for $k\in\mathbb{N}^*$ and $n\in\mathbb{N}^*$, we have*

$$|\mathbb{E}((\varphi,dN_{\omega,n}))-(\varphi,dN)| \leq Ce^{C\mu}n^{-(1-\alpha)k}e^{Ck\log k}\sup_{\substack{x\in\mathbb{R}\\ 0\leq j\leq k+\rho}}\left|(|x|+C)^{\rho+k}\frac{d^j\varphi}{d^jx}(x)\right|. \tag{2.2}$$

Remark 2.1. The proof shows that the constant C obtained in Lemma 2.1 is independent of the concentration μ of the Poisson process.

Before starting the proof of Lemma 2.1, let us recall some basic facts about the density of states of a periodic Schrödinger operator. Let $\mathbb{T}_n^* = \mathbb{R}^d/(2\pi n\mathbb{Z}^d)$. For $\theta\in\mathbb{T}_n^*$, we can consider $H_{\omega,n,\theta}$ the unique self-adjoint operator defined by the quadratic form $\|\nabla\varphi\|^2 + \langle V_\omega\varphi,\varphi\rangle$ on $L^2_{\theta,\mathrm{loc}}$ (i.e. the set of L^2_{loc}-functions that satisfy the boundary conditions $\varphi(x+n\gamma)=e^{in\theta\gamma}\varphi(x)$ that is endowed with the usual scalar product on $L^2(C(0,n))$). We know that $H_{\omega,n,\theta}$ has a compact resolvent (see [23]); hence its spectrum is discrete. Let us denote its eigenvalues by

$$E_0(\theta,\omega,n)\leq E_1(\theta,\omega,n)\leq\cdots\leq E_n(\theta,\omega,n)\leq\dots$$

The functions $(\theta\mapsto E_n(\theta,\omega,n))_{n\in\mathbb{N}}$ are Lipschitz continuous in θ and one has

$$E_n(\theta,\omega,n)\to+\infty \text{ as } n\to+\infty \text{ (uniformly in } \theta).$$

One proves that the IDS of $H_{\omega,n}$ satisfies

$$N_{\omega,n}(E) = \frac{1}{(2\pi)^d}\sum_{n\in\mathbb{N}}\int_{\{\theta\in\mathbb{T}_n^*;\ E_n(\theta,\omega,n)\leq E\}} d\theta \tag{2.3}$$

and

$$(\varphi,dN_{\omega,n})) = \mathrm{tr}(\mathbf{1}_{C(0,n)}\varphi(H_{\omega,n})\mathbf{1}_{C(0,n)}).$$

for any $\varphi\in\mathcal{C}_0^\infty(\mathbb{R})$ (see [24, 23] or [28]).
The rest of this subsection will be devoted to the proof of Lemma 2.1. To prove this result we will need the formula given in Theorem 1.3. The proof of this formula will be given at the end of the section. Let us proceed with the proof of Lemma 2.1. Fix $\varphi\in\mathcal{C}_0^\infty(\mathbb{R})$.

We want to estimate $|\mathbb{E}((\varphi, dN_{\omega,n})) - (\varphi, dN)|$. The computation done in the proof of Theorem 5.1 in [12] gives

$$\mathbb{E}((\varphi, dN_{\omega,n})) = \mathbb{E}(\mathrm{tr}(\mathbf{1}_{C(0,1)}\varphi(H_{\omega,n})\mathbf{1}_{C(0,1)})).$$

Here we used the fact that the Poisson process is $\mathbb{Z}^d$-homogeneous. Notice that our regularity assumptions on $V_{\omega,n}$ are weaker than the one needed in [12] and [22]. Indeed ω almost surely, $V_{\omega,n}$ is only relatively form bounded with respect to $-\Delta$ with relative bound 0 (see Lemma 5.1). Nevertheless the proofs of the relevant results in these papers extend easily to the case of relatively form bounded perturbations of $-\Delta$.
Now we only have to estimate $|\mathbb{E}(\mathrm{tr}(\mathbf{1}_{C(0,1)}(\varphi(H_{\omega,n}) - \varphi(H_\omega))))|$. This is done with an integral representation of $\varphi(H)$ using an almost analytic extension of φ. Pick $\varphi \in \mathcal{S}(\mathbb{R})$ (the Schwartz space of rapidly decreasing functions). An almost analytic extension of φ is a function $\tilde{\varphi}: \mathbb{C} \to \mathbb{C}$ satisfying

1. for $z \in \mathbb{R}$, $\tilde{\varphi}(z) = \varphi(z)$,
2. $\mathrm{supp}(\tilde{\varphi}) \subset \{z \in \mathbb{C};\ |\mathrm{Im}(z)| < 1\}$,
3. $\tilde{\varphi} \in \mathcal{S}(\{z \in \mathbb{C};\ |\mathrm{Im}(z)| < 1\})$,
4. The family of functions $x \mapsto \dfrac{\partial\tilde{\varphi}}{\partial\bar{z}}(x+iy)\cdot|y|^{-n}$ (for $0<|y|<1$) is bounded in $\mathcal{S}(\mathbb{R})$ for any $n \in \mathbb{N}$.

Such extensions always exist for $\varphi \in \mathcal{S}$ (see [19]) and, one has the following estimates: for $n \geq 0$, $\alpha \geq 0$, $\beta \geq 0$, there exists $C_{n,\alpha,\beta} > 0$ such that

$$\sup_{0<|y|\leq 1}\sup_{x\in\mathbb{R}}\left|x^\alpha\frac{\partial^\beta}{\partial x^\beta}\left(|y|^{-n}\cdot\frac{\partial\tilde{\varphi}}{\partial\bar{z}}(x+iy)\right)\right| \leq C_{n,\alpha,\beta}\sup_{\substack{\beta'\leq n+\beta+2\\ \alpha'\leq\alpha}}\sup_{x\in\mathbb{R}}\left|x^{\alpha'}\frac{\partial^{\beta'}\varphi}{\partial x^{\beta'}}(x)\right|. \tag{2.4}$$

Let $\tilde{\varphi}$ be an almost analytic extension of $(i+x)^q\varphi(x)$ for some $q > d/2$, q integer. Then, by [5] and [8], we know that, for any n and $\omega \in \Omega$, the following formula holds,

$$\varphi(H_{\omega,n}) = \frac{i}{2\pi}\int_{\mathbb{C}}\frac{\partial\tilde{\varphi}}{\partial\bar{z}}(z)\cdot(i+H_{\omega,n})^{-q}(z-H_{\omega,n})^{-1}dz\wedge d\bar{z}. \tag{2.5}$$

For $q > d/2$, $\mathbf{1}_{C(0,1)}(i+H_{\omega,n})^{-q}(z-H_{\omega,n})^{-1}$ is trace-class and we have

$$\begin{aligned}(2.6)\quad &\mathrm{tr}\left(\mathbf{1}_{C(0,1)}\varphi(H_{\omega,n})\mathbf{1}_{C(0,1)}\right)\\ &\qquad= \frac{i}{2\pi}\int_{\mathbb{C}}\frac{\partial\tilde{\varphi}}{\partial\bar{z}}(z)\cdot\mathrm{tr}\left(\mathbf{1}_{C(0,1)}(i+H_{\omega,n})^{-q}(z-H_{\omega,n})^{-1}\mathbf{1}_{C(0,1)}\right)dz\wedge d\bar{z}.\end{aligned}$$

By Lemma 5.1 and section B.12 in [26], we know that, ω-almost surely H_ω is essential self-adjoint on $\mathcal{C}_0^\infty(\mathbb{R}^d)$ and that $\mathbf{1}_{C(0,1)}(i+H_\omega)^{-q}(z-H_\omega)^{-1}$ is trace-class. Hence (2.5) and (2.6) also hold for H_ω.

We are now going to use Lemma 5.1. We pick $p' \in (p(d), p)$ and $b = 1$, and compute

$$
\begin{aligned}
&|\mathbb{E}(\mathrm{tr}(\mathbf{1}_{C(0,1)}(\varphi(H_{\omega,n}) - \varphi(H_\omega))))| \\
&\quad \leq \frac{1}{2\pi} \int_{\mathbb{C}} \left| \frac{\partial \tilde{\varphi}}{\partial \bar{z}}(z) \right| \mathbb{E}\left(\left\| \mathbf{1}_{C(0,1)} \left((i + H_{\omega,n})^{-q} (z - H_{\omega,n})^{-1} \right.\right.\right. \\
&\qquad\qquad \left.\left.\left. - (i + H_\omega)^{-q} (z - H_\omega)^{-1} \right) \mathbf{1}_{C(0,1)} \right\|_{\mathrm{tr}} \right) dxdy \\
&\quad \leq \sum_{k \geq 1} \frac{1}{2\pi} \int_{\mathbb{C}} \left| \frac{\partial \tilde{\varphi}}{\partial \bar{z}}(z) \right| \mathbb{E}\left(\mathbf{1}_{\{\omega;\ V_\omega \in \Omega_k(\alpha,1,p') \setminus \Omega_{k-1}(\alpha,1,p')\}} K(z,\omega) \right) dxdy
\end{aligned} \tag{2.7}
$$

where

$$
K(z,\omega) = \left\| \mathbf{1}_{C(0,1)} \left((i + H_{\omega,n})^{-q} (z - H_{\omega,n})^{-1} - (i + H_\omega)^{-q} (z - H_\omega)^{-1} \right) \mathbf{1}_{C(0,1)} \right\|_{\mathrm{tr}}
$$

Here $\|\cdot\|_{\mathrm{tr}}$ denotes the trace-class norm.
We need to estimate $\mathbf{1}_{C(0,1)}((i + H_{\omega,n})^{-q}(z - H_{\omega,n})^{-1} - (i + H_\omega)^{-q}(z - H_\omega)^{-1})\mathbf{1}_{C(0,1)}$ under the assumption $V_\omega \in \Omega_k(\alpha, 1, p')$. Therefore we imitate the method used in [12]. We write

$$
\left\| \mathbf{1}_{C(0,1)}((i + H_{\omega,n})^{-q}(z - H_{\omega,n})^{-1} - (i + H_\omega)^{-q}(z - H_\omega)^{-1})\mathbf{1}_{C(0,1)} \right\|_{\mathrm{tr}} \leq A + B \tag{2.8}
$$

where

$$
\begin{aligned}
A &= \left\| \mathbf{1}_{C(0,1)} \left((z - H_{\omega,n})^{-1} - (z - H_\omega)^{-1} \right) (i + H_\omega)^{-q} \mathbf{1}_{C(0,1)} \right\|_{\mathrm{tr}} \\
&= \left\| \mathbf{1}_{C(0,1)} \left((z - H_{\omega,n})^{-1} (V_{\omega,n} - V_\omega) (z - H_\omega)^{-1} \right) (i + H_\omega)^{-q} \mathbf{1}_{C(0,1)} \right\|_{\mathrm{tr}}
\end{aligned}
$$

and

$$
\begin{aligned}
B &= \left\| \mathbf{1}_{C(0,1)} (z - H_{\omega,n})^{-1} \left((i + H_{\omega,n})^{-q} - (i + H_\omega)^{-q} \right) \mathbf{1}_{C(0,1)} \right\|_{\mathrm{tr}} \\
&= \left\| \mathbf{1}_{C(0,1)} (z - H_{\omega,n})^{-1} \sum_{l=1}^{q-1} (i + H_{\omega,n})^{l-q} (V_{\omega,n} - V_\omega) (i + H_\omega)^{-l} \mathbf{1}_{C(0,1)} \right\|_{\mathrm{tr}}.
\end{aligned}
$$

The estimates for A and B being obtained essentially in the same way, we will write the details for A only. We have

$$
\begin{aligned}
A \leq \sum_{\gamma' \in \mathbb{Z}^d} &\left\| \mathbf{1}_{C(0,1)} (z - H_{\omega,n})^{-1} (V_{\omega,n} - V_\omega) \mathbf{1}_{C(\gamma',1)} \right\|_{\mathcal{L}(L^2)} \cdot \\
&\qquad \cdot \left\| \mathbf{1}_{C(\gamma',1)} (z - H_\omega)^{-1} (i + H_\omega)^{-q} \mathbf{1}_{C(0,1)} \right\|_{\mathrm{tr}}
\end{aligned}
$$

By Lemma 5.4 applied to H_ω and to $H_{\omega,n}$, for $V_\omega \in \Omega_k(\alpha, 1, p')$, we know that, for some $\epsilon > 0$, $\rho > 0$ and $C > 0$, for all $\gamma' \in \mathbb{Z}^d$, we have

$$
\begin{aligned}
\left\| \mathbf{1}_{C(0,1)} (z - H_{\omega,n})^{-1} (V_{\omega,n} - V_\omega) \mathbf{1}_{C(\gamma',1)} \right\|_{\mathcal{L}(L^2)} &\leq C \frac{(1 + |\gamma'|)^\alpha}{\eta(z,K)^\rho} e^{-\epsilon \cdot \eta(z,K) |\gamma'|^{1-\alpha}}, \\
\left\| \mathbf{1}_{C(\gamma',1)} (z - H_\omega)^{-1} (i + H_\omega)^{-q} \mathbf{1}_{C(0,1)} \right\|_{\mathrm{tr}} &\leq C (1 + |\gamma'|)^\alpha \frac{e^{-\epsilon \cdot \eta(z,K) |\gamma'|^{1-\alpha}}}{\eta(z,K)^\rho}.
\end{aligned}
$$

where $K = Ck^{\frac{p}{p-p'}}$, $\eta(z,K) = \eta(z,K,1) = \dfrac{|\mathrm{Im} z|}{|z| + K + C}$ and ρ depends only on d and q.
On the other hand, due to the exponential decay of V, for $|\gamma'| \leq n/2$ and some $C > 0$,

we have

$$\left\|\mathbf{1}_{C(0,1)}(z-H_{\omega,n})^{-1}(V_{\omega,n}-V_\omega)\mathbf{1}_{C(\gamma',1)}\right\|_{\mathcal{L}(L^2)}\leq\frac{e^{-n/C}}{|\mathrm{Im}z|}.$$

Hence, if we sum up these estimates, we get that

$$A\leq C\frac{n^{(d+3)\alpha}}{\eta(z,K)^\rho}e^{-\epsilon\cdot\eta(z,K)n^{1-\alpha}}. \tag{2.9}$$

For B, we get an estimate analogous to (2.9); only the constants change. Plugging this into (2.8) and (2.7), summing over k using the estimate (5.1) for the probability of $\Omega_k(\alpha,1,p')$, we get

$$|\mathbb{E}(\mathrm{tr}\,(\mathbf{1}_{C(0,1)}(\varphi(H_{\omega,n})-\varphi(H_\omega))))|\leq Cn^{(d+3)\alpha}\frac{1}{2\pi}\int_{\mathbb{C}}\left|\frac{\partial\tilde{\varphi}}{\partial\overline{z}}(z)\right|S(z,n)dxdy \tag{2.10}$$

where

$$S(z,n):=\sum_{k\geq1}\frac{(C\mu)^k}{k!}\left(\frac{|z|+Ck^{\frac{p}{p-p'}}+C}{|\mathrm{Im}z|}\right)^\rho e^{-\epsilon\frac{|\mathrm{Im}z|}{|z|+Ck^{\frac{p}{p-p'}}+C}n^{1-\alpha}}.$$

As $\mathrm{supp}\tilde{\varphi}\subset\{z\in\mathbb{C};\ |\mathrm{Im}(z)|<1\}$, using the notation $z=x+iy$, for $l\in\mathbb{N}^*$, we estimate $S(z,n)$ for $|y|<1$ by

$$\begin{aligned}S(z,n)&=\sum_{k\geq1}\frac{(C\mu)^k}{k!}\left(\frac{(|x|+C)k^{\frac{p}{p-p'}}}{|y|}\right)^\rho e^{-\epsilon\frac{|y|}{(|x|+C)k^{\frac{p}{p-p'}}}n^{1-\alpha}}\\&=n^{-l(1-\alpha)}\left(\frac{|x|+C}{|y|}\right)^{\rho+l}\sum_{k\geq1}\frac{(C\mu)^k}{k!}k^{\frac{p}{p-p'}\rho}\left(\frac{|y|n^{1-\alpha}}{(|x|+C)}\right)^l e^{-\epsilon\frac{n^{1-\alpha}|y|}{(|x|+C)}k^{-\frac{p}{p-p'}}}\\&=n^{-l(1-\alpha)}\left(\frac{|x|+C}{|y|}\right)^{\rho+l}f_l(t)\end{aligned} \tag{2.11}$$

where $t=\dfrac{|y|n^{1-\alpha}}{|x|+C}$ and

$$\begin{aligned}f_l(t)&:=\sum_{k\geq1}\frac{(C\mu)^k}{k!}k^{\frac{p}{p-p'}\rho}t^le^{-\epsilon tk^{\frac{p}{p'-p}}}\leq\sum_{k\geq1}\frac{(C\mu)^k}{k!}k^{\rho\frac{p}{p-p'}}\left(\frac{lk^{\frac{p}{p-p'}}}{\epsilon}\right)^le^{-l}\\&\leq\left(\frac{l}{\epsilon}\right)^le^{-l}\sum_{k\geq1}\frac{(C\mu)^k}{k!}k^{(\rho+l)\frac{p}{p-p'}}\leq\left(\frac{l}{\epsilon}\right)^le^{-l}L!(e^{C\mu e}-e^{C\mu}),\end{aligned} \tag{2.12}$$

where L denotes the smallest integer larger than $(\rho+l)\dfrac{p}{p-p'}$. Here we used Stirling's formula and the identity

$$\sum_{l\geq1}\frac{1}{l!}\left(\sum_{k\geq1}\frac{(C\mu)^k}{k!}k^l\right)=\sum_{k\geq1}\frac{(C\mu)^k}{k!}\left(\sum_{l\geq1}\frac{k^l}{l!}\right)=\sum_{k\geq1}\frac{(C\mu)^k}{k!}(e^k-1)=e^{C\mu e}-e^{C\mu}$$

Hence, for some $C > 0$ (independent of l, n, z and μ), we have

$$S(z,n) \le Ce^{C\mu}n^{-(1-\alpha)l}e^{Cl\log l}\left(\frac{|x|+C}{|y|}\right)^{\rho+l}. \tag{2.13}$$

Plugging this into (2.10) and using estimate (2.4) for almost analytic extensions, we get (2.2) and end the proof of Lemma 2.1.

□

Remark 2.2. In the proof of Lemma 2.1, we only have used the fact that the space of realization Ω could be written as $\Omega = \cup_n \Omega_n$ where the probability $P(\Omega_n)$ was decreasing fast enough, and that in these subsets, we had uniform estimates on the quantity we want to compute. Obviously, to get such a decomposition, one does not need to have a Poisson potential but only a homogeneous random field with suitable bounds at infinity. This idea is applicable to many other random Schrödinger operators.

Proof of Theorem 1.3. By [7], we know that, for $\phi \in \mathcal{C}_0^\infty(\mathbb{R})$ and for almost every ω, we have

$$\begin{aligned}\langle \phi, dN\rangle &= \lim_{\Lambda\to\mathbb{R}^d}\frac{1}{\mathrm{Vol}(\Lambda)}\mathrm{tr}(\phi(H^D_{\omega,\Lambda}))\\ &= \lim_{\Lambda\to\mathbb{R}^d}\frac{1}{\mathrm{Vol}(\Lambda)}\mathbb{E}(\mathrm{tr}(\phi(H^D_{\omega,\Lambda})))\\ &= \lim_{\Lambda\to\mathbb{R}^d}\frac{1}{\mathrm{Vol}(\Lambda)}\sum_{\gamma\in\Lambda\cap\mathbb{Z}^d}\mathbb{E}(\mathrm{tr}(\mathbf{1}_{C(\gamma,1)}\phi(H^D_{\omega,\Lambda})\mathbf{1}_{C(\gamma,1)})).\end{aligned} \tag{2.14}$$

On the other hand, as H_ω is homogeneous, for any $\gamma \in \mathbb{Z}^d$, we have

$$\mathbb{E}(\mathrm{tr}(\mathbf{1}_{C(\gamma,1)}\phi(H_\omega)\mathbf{1}_{C(\gamma,1)})) = \mathbb{E}(\mathrm{tr}(\mathbf{1}_{C(0,1)}\phi(H_\omega)\mathbf{1}_{C(0,1)})).$$

So that

$$\mathbb{E}(\mathrm{tr}(\mathbf{1}_{C(0,1)}\phi(H_\omega)\mathbf{1}_{C(0,1)})) = \frac{1}{\mathrm{Vol}(\Lambda)}\sum_{\gamma\in\Lambda\cap\mathbb{Z}^d}\mathbb{E}(\mathrm{tr}(\mathbf{1}_{C(\gamma,1)}\phi(H_\omega)\mathbf{1}_{C(\gamma,1)})).$$

Hence, by (2.14), to get equation (1.4), we just have to prove that

$$\lim_{\Lambda\to\mathbb{R}^d}\frac{1}{\mathrm{Vol}(\Lambda)}\left|\sum_{\gamma\in\Lambda\cap\mathbb{Z}^d}\mathbb{E}\left(\mathrm{tr}(\mathbf{1}_{C(\gamma,1)}\phi(H^D_{\omega,\Lambda})\mathbf{1}_{C(\gamma,1)}) - \mathrm{tr}(\mathbf{1}_{C(\gamma,1)}\phi(H_\omega)\mathbf{1}_{C(\gamma,1)})\right)\right| = 0. \tag{2.15}$$

To shorten the notations, let $\Lambda_L = C(0,L)$ be the cube of center 0 and side length L. Pick $\epsilon > 0$. To prove (2.15), one could try to prove that, for any $\gamma \in \mathbb{Z}^d$, one has

$$\lim_{L\to\infty} A(\gamma,\Lambda_L) := \lim_{\Lambda_L\to\mathbb{R}^d}\mathbb{E}\left(\mathrm{tr}(\mathbf{1}_{C(\gamma,1)}\phi(H^D_{\omega,\Lambda_L})\mathbf{1}_{C(\gamma,1)}) - \mathrm{tr}(\mathbf{1}_{C(\gamma,1)}\phi(H_\omega)\mathbf{1}_{C(\gamma,1)})\right) = 0 \tag{2.16}$$

in some uniform way. However, this may be difficult as, because of the Dirichlet boundary conditions, $A(\gamma,\Lambda_L)$ may have some non uniform behavior for γ close to the boundary of Λ_L. So we are going to split the difficulty into two parts; on the one hand, we will show that, for any $\epsilon > 0$ and for $\gamma \in \Lambda_L \setminus \Lambda_{(1-\epsilon)L}$, we will show that $A(\gamma,\Lambda_L)$ stays bounded (uniformly in L, ω). As there are only very few such terms, this part of the sum tends to

0. On the other hand, for $\gamma \in \Lambda_{(1-\epsilon)L}$, we will show that $A(\gamma, \Lambda_L)$ tends to 0 uniformly in L and ω. As in the proof of Lemma 2.1, uniformity in ω cannot be achieved over the whole set of realization, but only over subsets whose measure we control (see Lemma 5.1). This will suffice.

To estimate $A(\gamma, \Lambda_L)$, we will use (2.5) to compute $\phi(H_\omega)$ and $\phi(H^D_{\omega,\Lambda_L})$. Hence, we see that we only to estimate the following expression (cf (2.8))

$$\mathrm{tr}(\mathbf{1}_{C(\gamma,1)}(z - H^D_{\omega,\Lambda_L})^{-1}(i + H^D_{\omega,\Lambda_L})^{-q}\mathbf{1}_{C(\gamma,1)}) - \mathrm{tr}(\mathbf{1}_{C(\gamma,1)}(z - H_\omega)^{-1}(i + H_\omega)^{-q}\mathbf{1}_{C(\gamma,1)})$$

where $q > d/2$ is an even integer.
To do this we need a way to compare the resolvent of the Dirichlet problem on Λ_L with the resolvent of H_ω over the whole space. We use the following resolvent identity (see e.g. [1]): let $\chi \in \mathcal{C}^2_0(\overset{\circ}{\Lambda}_L)$ then, for $\mathrm{Im} z \neq 0$, we have

$$\chi(z - H_\omega)^{-1} = (z - H^D_{\omega,\Lambda_L})^{-1}\chi + (z - H^D_{\omega,\Lambda_L})^{-1}[-\Delta, \chi](z - H_\omega)^{-1}. \tag{2.17}$$

Pick α, k and p' are taken as in Lemma 2.1. We will use the following lemma

Lemma 2.2. *Assume that $q > d/2$ is an even integer and that ω is such that $V_\omega \in \Omega_k(\alpha, 1, p')$. Then, there exists $C_q > 0$ such that, for any $\Lambda' \subset \Lambda_L$ (Λ' measurable), we have*

$$\|(i + H^D_{\omega,\Lambda_L})^{-1}\|_{\mathcal{T}_q} \leq C_q k^{\frac{pd}{2q(p-p')}} L^{\frac{d}{q}(1+\frac{\alpha}{2})} \tag{2.18}$$

$$\|(1 - \Delta^D_{\Lambda_L})^{-1}\mathbf{1}_{\Lambda'}\|_{\mathcal{T}_q} \leq C_q|\Lambda'|^{1/q}. \tag{2.19}$$

Here $|\Lambda'|$ denotes the measure of Λ' and

$$\|\cdot\|_{\mathcal{T}_q} = (tr(|\cdot|^q))^{1/q}.$$

Let us postpone the proof of this result to finish the proof of Theorem 1.3. Pick $\epsilon > 0$ to be chosen precisely later. For ω such that $V_\omega \in \Omega_k(\alpha, 1, p')$. Hence, by Lemma 5.1, we can decompose $V_\omega = V^{(i)}_\omega + V^{(r)}_\omega$ where

- $V^{(i)}_\omega \in L^{p'}_{\mathrm{loc,unif}}(\Lambda_L)$ and $\sup_{x\in\Lambda_L} \|V^{(i)}_\omega\|_{L^{p'}(C(x,1))} \leq 1$.
- for $x \in \Lambda_L$, $|V^{(r)}_\omega(x)| \leq Ck^{\frac{p}{p-p'}} L^\alpha$.

Hence, combining the two estimates of Lemma 2.2, we get

$$\|(i + H^D_{\omega,\Lambda_L})^{-1}\mathbf{1}_{\Lambda_L\setminus\Lambda_{(1-2\epsilon)L}}\|_{\mathcal{T}_q} \leq C\epsilon^{\frac{1}{q}} k^{\frac{(p+2q)d}{2q(p-p')}} L^\alpha L^{\frac{d}{q}(1+\frac{\alpha}{2})+\frac{d}{q}} \tag{2.20}$$

as

$$(i + H^D_{\omega,\Lambda_L})^{-1} = \left[1 + (i + H^D_{\omega,\Lambda_L})^{-1}(1 - i + V_\omega)\right](1 - \Delta^D_{\Lambda_L})^{-1}.$$

So, by formula (2.5), we get that, for ω such that $V_\omega \in \Omega_k(\alpha, p', 1)$, we have

$$|\mathrm{tr}(\phi(H^D_{\omega,\Lambda_L})\mathbf{1}_{\Lambda_L\setminus\Lambda_{(1-2\epsilon)L}})| \leq Ck^{\frac{(p+2q)d}{2q(p-p')}} \epsilon^{\frac{1}{q}} L^{\alpha+\frac{d}{q}+\frac{d(q-1)}{q}(1+\frac{\alpha}{2})}.$$

Hence, taking the expectation in ω and summing over all k, we get that

$$\left|\mathbb{E}\left[\mathrm{tr}(\phi(H^D_{\omega,\Lambda_L})\mathbf{1}_{\Lambda_L\setminus\Lambda_{(1-2\epsilon)L}})\right]\right| \leq C\epsilon^{\frac{1}{q}} L^{\alpha+\frac{d}{q}+\frac{d(q-1)}{q}(1+\frac{\alpha}{2})}$$

Finally, we obtain

$$\frac{1}{\operatorname{Vol}(\Lambda_L)}\left|\sum_{\substack{\gamma\in\Lambda_L\setminus\Lambda_{(1-2\epsilon)L}\\ \gamma\in\mathbb{Z}^d}}\mathbb{E}\left[\operatorname{tr}(\mathbf{1}_{C(\gamma,1)}\phi(H^D_{\omega,\Lambda_L})\mathbf{1}_{C(\gamma,1)})\right]\right| \tag{2.21}$$
$$= \frac{1}{\operatorname{Vol}(\Lambda_L)}\left|\mathbb{E}\left[\operatorname{tr}(\phi(H^D_{\omega,\Lambda_L})\mathbf{1}_{\Lambda_L\setminus\Lambda_{(1-2\epsilon)L}})\right]\right| \leq C\epsilon^{\frac{1}{q}}L^{\alpha(1+\frac{d(q-1)}{2q})}.$$

On the other hand, let χ be a $\mathcal{C}^\infty_0$-function supported in Λ_L that is constant equal to 1 in $\Lambda_{(1-\epsilon)L}$. For $\gamma\in\Lambda_{(1-2\epsilon)L}$, we use the resolvent formula (2.17) to write

$$\begin{aligned}(i+H^D_{\omega,\Lambda_L})^{-q}\mathbf{1}_{C(\gamma,1)} &= (i+H^D_{\omega,\Lambda_L})^{-q}\chi\mathbf{1}_{C(\gamma,1)}\\ &= (i+H^D_{\omega,\Lambda_L})^{-(q-1)}\chi(i+H_\omega)^{-1}\mathbf{1}_{C(\gamma,1)}\\ &\qquad - (i+H^D_{\omega,\Lambda_L})^{-(q-1)}\left[\Delta,\chi\right](i+H_\omega)^{-1}\mathbf{1}_{C(\gamma,1)}\\ &= \chi(i+H_\omega)^{-q}\mathbf{1}_{C(\gamma,1)} - \sum_{j=1}^{q}(i+H^D_{\omega,\Lambda_L})^{-j}\left[\Delta,\chi\right](i+H_\omega)^{-q+j-1}\mathbf{1}_{C(\gamma,1)}.\end{aligned}$$

Hence

$$(i+H^D_{\omega,\Lambda_L})^{-q}\mathbf{1}_{C(\gamma,1)} = (i+H_\omega)^{-q}\mathbf{1}_{C(\gamma,1)} + R(L) \tag{2.22}$$

where

$$R(L) = -(1-\chi)(i+H_\omega)^{-q}\mathbf{1}_{C(\gamma,1)} - \sum_{j=1}^{q}(i+H^D_{\omega,\Lambda_L})^{-j}\left[\Delta,\chi\right](i+H_\omega)^{-q+j-1}\mathbf{1}_{C(\gamma,1)}.$$

By the support properties of $1-\chi$ and $[\Delta,\chi]$, by Lemma 5.4 and by the assumption on ω, we have that, for some $C>0$,

$$\|R(L)\|_{\mathcal{T}_1} \leq Ck^{\frac{p}{p-p'}}e^{-(\epsilon L)^{1-\alpha}/(Ck^{p/(p-p')})}.$$

Plugging this into (2.5), we get that, for some $C>0$,

$$|\operatorname{tr}(\mathbf{1}_{C(\gamma,1)}\phi(H^D_{\omega,\Lambda_L})\mathbf{1}_{C(\gamma,1)}) - \operatorname{tr}(\mathbf{1}_{C(\gamma,1)}\phi(H_\omega)\mathbf{1}_{C(\gamma,1)})| \leq Ck^{\frac{p}{p-p'}}e^{-(\epsilon L)^{1-\alpha}/(Ck^{p/(p-p')})}. \tag{2.23}$$

Taking the expectation of (2.23), summing in k and using (5.1) in the same way as in (2.13), we get that

$$\mathbb{E}\left(|\operatorname{tr}(\mathbf{1}_{C(\gamma,1)}\phi(H^D_{\omega,\Lambda_L})\mathbf{1}_{C(\gamma,1)}) - \operatorname{tr}(\mathbf{1}_{C(\gamma,1)}\phi(H_\omega)\mathbf{1}_{C(\gamma,1)})|\right) \leq C(\epsilon L)^{-(1-\alpha)}. \tag{2.24}$$

Then, by (2.21) and (2.24), we have that

$$\frac{1}{\operatorname{Vol}(\Lambda_L)}\sum_{\gamma\in\Lambda_L\cap\mathbb{Z}^d}\mathbb{E}\left(\operatorname{tr}(\mathbf{1}_{C(\gamma,1)}\phi(H^D_{\omega,\Lambda_L})\mathbf{1}_{C(\gamma,1)})\right) =$$
$$= \frac{1}{\operatorname{Vol}(\Lambda_L)}\sum_{\gamma\in\Lambda_L\cap\mathbb{Z}^d}\mathbb{E}\left(\operatorname{tr}(\mathbf{1}_{C(\gamma,1)}\phi(H_\omega)\mathbf{1}_{C(\gamma,1)})\right) + Q(L)$$

where

$$|Q(L)| \leq C(\epsilon L)^{-(1-\alpha)} + C\epsilon^{\frac{1}{q}} L^{\alpha(1+\frac{d(q-1)}{2q})} + CL^{-1}.$$

If we choose $\beta = \alpha(q + d(q-1)/2)$, set $\epsilon = L^{-\beta}$ and pick $0 < \alpha < 1/3$ small enough so that $1 - \beta > 1/2$, we get that $Q(L) \to 0$ as $L \to +\infty$. This completes the proof of Theorem 1.3.

□

Remark 2.3. Remark 2.2 applies also for the proof of Theorem 1.3.

Proof of Lemma 2.2. Under our assumptions on ω, we have $V_\omega \in \Omega_k(\alpha, 1, p')$. Hence, by Lemma 5.1, we can decompose $V_\omega = V_\omega^{(i)} + V_\omega^{(r)}$ where

- $V_\omega^{(i)} \in L^{p'}_{\text{loc,unif}}(\Lambda_L)$ and $\sup_{x\in\Lambda_L} \|V_\omega^{(i)}\|_{L^{p'}(C(x,1))} \leq 1$.
- for $x \in \Lambda_L$, $|V_\omega^{(r)}(x)| \leq Ck^{\frac{p}{p-p'}} L^\alpha$.

For $\varphi \in \mathcal{C}_0^\infty(\Lambda_L)$, this yields

$$\begin{aligned}\langle H^D_{\omega,\Lambda_L}\varphi,\varphi\rangle &= \langle -\Delta^D_{\Lambda_L}\varphi,\varphi\rangle + \langle V^i_\omega\varphi,\varphi\rangle + \langle V^r_\omega\varphi,\varphi\rangle \\ &\geq \frac{1}{2}\langle -\Delta^D_{\Lambda_L}\varphi,\varphi\rangle - C\left(k^{\frac{p}{p-p'}} L^\alpha + 1\right)\|\varphi\|^2\end{aligned} \tag{2.25}$$

Equation (2.25) and the variational principle for eigenvalues immediately imply that

$$\lambda_j(H^D_{\omega,\Lambda_L}) \geq \frac{1}{2}\lambda_j(-\Delta^D_{\Lambda_L}) - C\left(k^{\frac{p}{p-p'}} L^\alpha + 1\right)$$

where $\lambda_j(H)$ denotes the j-th eigenvalue of H (ordered increasingly counting multiplicity). Hence

(2.26)

$$\begin{aligned}\|(i + H^D_{\omega,\Lambda_L})^{-1}\|^q_{\mathcal{T}_q} &= \sum_{j\in\mathbb{N}} \frac{1}{\left(1 + (\lambda_j(H^D_{\omega,\Lambda_L}))^2\right)^{q/2}} \\ &\leq \sum_{\substack{j\in\mathbb{N}\\ \lambda_j(-\Delta^D_{\Lambda_L}) < 4C(k^{\frac{p}{p-p'}}L^\alpha+1)}} 1 + \\ &\quad + \sum_{\substack{j\in\mathbb{N}\\ \lambda_j(-\Delta^D_{\Lambda_L}) \geq 4C(k^{\frac{p}{p-p'}}L^\alpha+1)}} \frac{1}{\left(1 + (\lambda_j(-\Delta^D_{\Lambda_L}) - C(k^{\frac{p}{p-p'}}L^\alpha + 1))^2\right)^{q/2}}.\end{aligned}$$

Now the eigenvalues of $-\Delta^D_{\Lambda_L}$ are known explicitly: for $j=(j_1,\dots,j_d)\in(\mathbb{N}\setminus\{0\})^d$, one has $\lambda_j(-\Delta^D_{\Lambda_L})=\dfrac{\pi^2}{L^2}\sum_{1\le p\le d} j_p^2$. Hence equation (2.26) tells us that

$$
\begin{aligned}
\|(i+H^D_{\omega,\Lambda_L})^{-1}\|^q_{\mathcal{T}_q} \le{}& \left(4CL^2(k^{\frac{p}{p-p'}}L^\alpha+1)\right)^{d/2}+\\
&+\sum_{|j|\ge 2\sqrt{C}L(k^{\frac{p}{p-p'}}L^\alpha+1)^{1/2}} \frac{L^{2q}}{\left(L^4+(|j|^2-CL^2(k^{\frac{p}{p-p'}}L^\alpha+1))^2\right)^{q/2}}\\
\le{}& \left(4CL^2(k^{\frac{p}{p-p'}}L^\alpha+1)\right)^{d/2}+\\
&+\sum_{|j|\ge 2\sqrt{C}L(k^{\frac{p}{p-p'}}L^\alpha+1)^{1/2}} \frac{L^{2q}}{(L^4+|j|^4/2)^{q/2}}\\
\le{}& \left(4CL^2(k^{\frac{p}{p-p'}}L^\alpha+1)\right)^{d/2}+C_qL^d\\
\le{}& C_q k^{\frac{pd}{2(p-p')}}L^{d(1+\frac{\alpha}{2})}.
\end{aligned}
$$

This completes the proof of (2.18). The proof of (2.19) will follow from an explicit computation. As a corollary of the Golden-Thompson inequalities (see [25]), we have

$$
\begin{aligned}
\|(1-\Delta^D_{\Lambda_L})^{-1}\mathbf{1}_{\Lambda'}\|^q_{\mathcal{T}_q} &\le \|\mathbf{1}_{\Lambda'}(1-\Delta^D_{\Lambda_L})^{-q/2}\|^2_{\mathcal{T}_2}\\
&=\sum_{j\in(\mathbb{N}\setminus\{0\})^d}\langle(1-\Delta^D_{\Lambda_L})^{-q/2}\mathbf{1}_{\Lambda'}(1-\Delta^D_{\Lambda_L})^{-q/2}\varphi_j,\varphi_j\rangle\\
&=\sum_{j\in(\mathbb{N}\setminus\{0\})^d}\frac{1}{(1+\lambda_j)^q}\langle\mathbf{1}_{\Lambda'}\varphi_j,\varphi_j\rangle
\end{aligned}
\tag{2.27}
$$

where, for the sake of simplicity, we denoted the eigenvalues of $-\Delta^D_{\Lambda_L}$ by $(\lambda_j)_j$ and the associated eigenvector by $(\varphi_j)_j$. The vector φ_j being explicitly known, one easily computes

$$\langle\mathbf{1}_{\Lambda'}\varphi_j,\varphi_j\rangle\le\left(\frac{2}{L}\right)^d|\Lambda'|$$

Plugging this into equation (2.27), we get

$$\|(1-\Delta^D_{\Lambda_L})^{-1}\mathbf{1}_{C(\gamma,1)}\|^q_{\mathcal{T}_q}\le\left(\frac{2}{L}\right)^d\sum_{j\in(\mathbb{N}\setminus\{0\})^d}\frac{|\Lambda'|}{(1+\lambda_j)^q}\le C_q|\Lambda'|$$

using the explicit formula given above for λ_j.

$\square$

2.2. **Application to the estimation of the integrated density of states.** We prove

Lemma 2.3. *Pick $\nu > 0$. There exists $\beta > 0$ and $E_\nu > 0$ such that, for $E > E_\nu$ and for $n \geq E^\beta$, one has*

$$\mathbb{E}(N_{\omega,n}(-E-1)) - e^{-E^\nu} \leq N(-E) \leq \mathbb{E}(N_{\omega,n}(-E+1)) + e^{-E^\nu}. \tag{2.28}$$

Proof of Lemma 2.3. Pick $\nu > 0$ arbitrary. By equation (5.13), the a-priori estimate on N given in Lemma 5.2, we know that, for some $\tau > 0$

$$N(-E^\tau) \leq \frac{1}{4}e^{-E^\nu}. \tag{2.29}$$

Hence we just have to estimate $N(-E) - N(-E^\tau)$. Therefore introduce two functions $\varphi_\pm$ defined by

$$\varphi_\pm = \mathbf{1}_{[-E^\tau \mp \frac{1}{2}, -E \pm \frac{1}{2}]} * \varphi_0$$

where $\varphi_0 \in \mathcal{C}_0^\infty(\mathbb{R})$ is a non negative Gevrey class function of Gevrey exponent $\rho > 1$ such that $\varphi_0 \equiv 1$ on $[-\frac{1}{4}, \frac{1}{4}]$ and $\text{supp}\varphi_0 \subset [-\frac{1}{2}, \frac{1}{2}]$. The functions $\varphi_\pm$ are then Gevrey class of Gevrey exponent ρ and $\text{supp}\varphi_\pm \subset [-E^\tau - 1, -E + 1]$ (see e.g. [6]). Then one has

$$\langle \varphi_-, dN \rangle \leq N(-E) - N(-E^\tau) \leq \langle \varphi_+, dN \rangle. \tag{2.30}$$

Using the Gevrey estimates for the derivatives and the estimates on the support of $\varphi_\pm$, by Lemma 2.1, we have that, for some $C > 0$, and for all $n \in \mathbb{N}^*$ and all $k \in \mathbb{N}^*$,

$$|\langle \varphi_\pm, dN \rangle - \mathbb{E}(\langle \varphi_\pm, dN_{\omega,n} \rangle)| \leq Cn^{-(1-\alpha)k} e^{Ck \log k} (E^\tau + C)^{\rho+k} (\rho + k)^{\eta(\rho+k)}. \tag{2.31}$$

We optimize the right hand side of (2.31) in k to get

$$|\langle \varphi_\pm, dN \rangle - \mathbb{E}(\langle \varphi_\pm, dN_{\omega,n} \rangle)| \leq e^{-(\eta+C)(n^{1-\alpha}(E^\tau+C))^{\frac{1}{\eta+C}}}.$$

Hence, for some $\beta > 0$, if $n \geq E^\beta$, we have

$$|\langle \varphi_\pm, dN \rangle - \mathbb{E}(\langle \varphi_\pm, dN_{\omega,n} \rangle)| \leq \frac{1}{4}e^{-E^\nu}. \tag{2.32}$$

Thus

$$\begin{aligned} \mathbb{E}(N_{\omega,n}(-E-1) - N_{\omega,n}(-E^\tau + 1)) &\leq \mathbb{E}(\langle \varphi_-, dN_{\omega,n} \rangle) \\ &\leq \mathbb{E}(\langle \varphi_+, dN_{\omega,n} \rangle) \leq \mathbb{E}(N_{\omega,n}(-E+1) - N_{\omega,n}(-E^\tau - 1)). \end{aligned} \tag{2.33}$$

Using (5.14) to estimate $N_{\omega,n}(-E^\tau \pm 1)$ and summing equations (2.29), (2.30) and (2.33), we end the proof of Lemma 2.3.

□

Remark 2.4. Notice that we could have estimated $N(-E)$ with $\mathbb{E}(N_{\omega,n}(-E \pm \epsilon))$ (for ϵ small). The price to pay to get an error of the same size as in (2.28) would have been to take n of size $E^\beta \epsilon^{-\zeta}$ for some $\zeta > 0$. This was used in [10] to get precise asymptotics for N at high energy.

3. The lower bounds

In this section, we will prove the asymptotic lower bound on the approximated density of states $dN_{\omega,n}$ defined in section 2.1 in the different cases considered in the introduction.

3.1. **The general case.** We prove the following general bound

Proposition 3.1. *Under the assumptions H1, H2 and H3, there exist* $\beta_0 > 0$ *such that, for any* $\beta > \beta_0$, *there exists* $E_0 >$ *such that, for* $E > E_0$ *and* $n = [E^\beta]$, *we have*

$$\log\left(\mathbb{E}(N_{\omega,n}(-E-1))\right) \geq -(1+\alpha^* d)g(E)\log g(E)(1+o(1)). \tag{3.1}$$

Here $[\cdot]$ *denotes the integer part of* $\cdot$.

Proof. The strategy used to prove the lower bound is quite obvious: we construct a normalized vector φ such that $\langle (H+V_\omega)\varphi,\varphi\rangle \leq -E-1$, this with a sufficiently large probability. The right candidate will be an asymptotic ground state for $H(g)$ for g chosen properly (see section 1.2).

Pick $n = [E^\beta]$ and $l = [|\log E|^\beta]$. Pick $\rho > 1$ large, $1 > \varepsilon > 0$ small and $k > 1$ large. Let ψ_g be an asymptotic ground state for $H(g)$ such that $\alpha^*(\psi_g) < \alpha^*(1+\varepsilon)$ (see section 1.2). Define

$$\Omega_{k,E} = \Omega^1_{k,E} \cap \Omega^2_E$$

where

$$\Omega^1_{k,E} = \{\omega :\ k(1+2\varepsilon) \geq m(\omega, C(0,k^{-\alpha^*(1+\varepsilon)})) = m(\omega, C(0,l)) \geq k(1+\varepsilon)\} \tag{3.2}$$

$$\Omega^2_E = \{\omega :\ \forall \gamma \in r_0\mathbb{Z}^d,\ m(\omega, C(\gamma, r_0)) < E^\rho(|\gamma|^{\alpha'}+1)\}. \tag{3.3}$$

Here r_0 is chosen such that $\mathrm{supp}V_2 \subset C(0,r_0)$. We minorize the probability of $\Omega^1_{k,E}$ by the probability that $m(\omega, C(0,k^{-\alpha^*(1+\varepsilon)})) = k(1+\varepsilon)$ and that $m(\omega, C(0,l)\setminus C(0,k^{-\alpha^*(1+\varepsilon)})) = 0$. Using (5.2) to majorize the probability of Ω^2_E, for E sufficiently large, we get that the probability of $\Omega_{k,E}$ is estimated by

$$P(\Omega_{k,E}) \geq \frac{1}{C} e^{-C\mu l^d} \frac{(\mu k^{-d\alpha^*(1+\varepsilon)})^{k(1+\varepsilon)}}{\Gamma(k(1+\varepsilon))} - \frac{C}{\Gamma(E^\rho/2)}. \tag{3.4}$$

where $C > 0$ is a constant independent of l, k and ε, and Γ is the Euler Γ-function. For $\omega \in \Omega_{k,E}$, we have

$$\langle(-\Delta + V_{\omega,n})\chi_l\varphi_k, \chi_l\varphi_k\rangle = \langle(-\Delta + V^{(i)}_{\omega,n})\chi_l\varphi_k, \chi_l\varphi_k\rangle + \langle V^{(e)}_{\omega,n}\chi_l\varphi_k, \chi_l\varphi_k\rangle$$

where

$$V^{(i)}_\omega = \int_{C(0,l)} V(x-y)m(\omega,dy) \text{ and } V^{(e)}_\omega = \int_{\mathbb{R}^d\setminus C(0,l)} V(x-y)m(\omega,dy)$$

i.e. they are the parts of V_ω with centers in $C(0,l)$ or outside of $C(0,l)$, and $V^{(i,e)}_{\omega,n}$ is built from these in the same way as $V_{\omega,n}$ is from V_ω (see equation (2.1)).
As V_2 is of compact support, as V_1 is exponentially decaying and as $\omega \in \Omega^2_{k,E}$, using the support properties of χ_l, we get that

$$|\langle V^{(e)}_{\omega,n}\psi_k, \psi_k\rangle| \leq CE^\rho e^{-|\log E|^\beta/C}. \tag{3.5}$$

On the other hand, if we set $m(\omega)=m(\omega,B(0,k^{-\alpha^*(1+\varepsilon)}))$ and define $(x_i(\omega))_i$ to be the points supporting $m(\omega,dx)$ in $B(0,k^{-\alpha^*(1+\varepsilon)})$, we have

$$\begin{aligned}
&\langle(-\Delta+V_\omega^{(i)})\psi_k,\psi_k\rangle\\
&=\langle(-\Delta+m(\omega)V)\psi_k,\psi_k\rangle+\sum_{i=1}^{m(\omega)}\langle(\tau_{x_i(\omega)}V-V)\psi_k,\psi_k\rangle\\
&=\langle(-\Delta+kV)\psi_k,\psi_k\rangle+\frac{m(\omega)-k}{k}\langle kV\psi_k,\psi_k\rangle+\sum_{i=1}^{m(\omega)}\langle(\tau_{x_i(\omega)}V-V)\psi_k,\psi_k\rangle\\
&\leq E(k)+o(E(k))+\frac{m(\omega)-k}{k}(E(k)+o(E(k)))+\sum_{i=1}^{m(\omega)}|\langle(\tau_{x_i(\omega)}V-V)\psi_k,\psi_k\rangle|\\
&\leq E(k)+o(E(k))+\varepsilon(E(k)+o(E(k)))+\sum_{i=1}^{m(\omega)}|\langle(\tau_{x_i(\omega)}V-V)\psi_k,\psi_k\rangle|.
\end{aligned}\tag{3.6}$$

We used the fact that ψ_k is an asymptotic ground state and the fact that $k(1+\varepsilon)\leq m(\omega)$. As $m(\omega)\leq k(1+2\varepsilon)$ and $\alpha^*(\psi_g)<\alpha^*(1+\varepsilon)$, the definition of $\alpha^*(\psi_g)$ tells us that

$$\frac{1}{E_-(k)}\left|\sum_{i=1}^{m(\omega)}\langle(\tau_{x_i(\omega)}V-V)\varphi_k,\varphi_k\rangle\right|\to 0 \text{ as } k\to+\infty.$$

Plugging all this into (3.6), we get

$$\langle(-\Delta+V_\omega^{(i)})\psi_k,\psi_k\rangle\leq E(k)(1+o(1)+\varepsilon)\text{ when } k\to+\infty.\tag{3.7}$$

If we now chose

$$k=g(E)(1+\varepsilon)\tag{3.8}$$

then, for sufficiently large k, we get

$$\langle(-\Delta+V_\omega^{(i)})\psi_k,\psi_k\rangle\leq-(E+1).\tag{3.9}$$

But, as an asymptotic ground state, ψ_k vanishes outside some fixed cube independent of k; hence for E sufficiently large, it vanishes in a neighborhood of the boundary of the cube $C(0,n)$ and can be continued so as to satisfy any quasi-periodic boundary conditions on $C(0,n)$ (see section 2.1). This implies that there exist $C>0$ independent of n and E such that for $\omega\in\Omega_{k,E}$ and k given by (3.8), we have.

$$N_{\omega,n}(-E-1)\geq n^{-d}/C.\tag{3.10}$$

Taking into account the probability estimate (3.4), if ρ is large enough and as $k\geq E^\delta$ for some $\delta>0$ by (5.29), we get

$$\mathbb{E}(N_{\omega,n}(-E-1))\geq\frac{1}{C}E^{-\beta d}e^{-C\mu l^d}\frac{\left(\mu k^{-d\alpha^*(1+\varepsilon)}\right)^{k(1+\varepsilon)}}{\Gamma(k(1+\varepsilon))}.$$

So that, as E tends to $+\infty$, we get

$$\log[\mathbb{E}(N_{\omega,n}(-E-1))]\geq-(1+\varepsilon+d\alpha^*(1+\varepsilon))g(E)\log g(E)(1+8\varepsilon).$$

As we can choose ε as small as we please, this ends the proof of Proposition 3.1. $\square$

By Lemma 5.5, we know that, for some $C>0$ and $\alpha>1$, $g(E)\leq CE$ for some $C>0$ and sufficiently large E. Thus, if $\beta>1$ is large enough and ν, given by equation (2.28), satisfies $\nu>1$, Lemma 2.3 and Proposition 3.1 immediately imply

Proposition 3.2. *Under the assumptions H1, H2 and H1', there exists $E_0>$ such that, for $E>E_0$, we have*

$$\log N(E)\geq -(1+\alpha^* d)g(E)\log g(E)(1+o(1)).$$

3.2. **The case when V is bounded from below.** We now assume that V is bounded from below i.e. that V_- is bounded. We then prove

Proposition 3.3. *Under the assumptions H1, H2 and H3, there exist $\beta_0>0$ such that, for any $\beta>\beta_0$, there exists $E_0>$ such that, for $E>E_0$ and $n=E^\beta$, we have*

$$\log\left(\mathbb{E}(N_{\omega,n}(-E-1))\right)\geq -g(E)\log g(E)(1+o(1)). \tag{3.11}$$

Proof. The strategy of the proof is the one used in the proof of Proposition 3.1. Recall that v_- is the essential infimum of V. Then, for any $\varepsilon>0$, we can find $\chi\in\mathcal{C}_0^\infty(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d}V(x)\chi^2(x)dx\leq v_-+\varepsilon/2 \text{ and } \int_{\mathbb{R}^d}\chi^2(x)dx=1.$$

Recall that $\tau_a\chi(x)=\chi(x-a)$. As $V\in L^1(\mathbb{R}^d)$, there exists $\delta>0$ such that

$$\sup_{|a|\leq\delta}\int_{\mathbb{R}^d}|V(x)|\cdot|\tau_a\chi^2(x)-\chi^2(x)|dx\leq\varepsilon/2.$$

Pick now n and l as in the proof of Proposition 3.1; pick k large and define

$$\Omega_{\delta,E}=\Omega^1_{\delta,E}\cap\Omega^2_E$$

where Ω^2_E is defined in (3.3) and (see (3.2))

$$\Omega^1_{\delta,E}=\{\omega;\ k(1+2\varepsilon)\geq m(\omega,C(0,\delta))=m(\omega,C(0,l))\geq k(1+\varepsilon)\}.$$

Using (5.2), we get that the probability of $\Omega_{\delta,E}$ is estimated by

$$P(\Omega_{\delta,E})\geq\frac{1}{C}e^{-C\mu l^d}\frac{(\mu\delta)^{kd(1+\varepsilon)}}{\Gamma(k(1+\varepsilon))}-\frac{C}{\Gamma(E^\rho/2)}. \tag{3.12}$$

Pick $\omega\in\Omega_{\delta,E}$. An argument similar to that used in the proof of Proposition 3.1 in which $k^{-\alpha^*(1+\varepsilon)}$ is replaced by δ yields (cf (3.6))

$$\begin{aligned}\langle(-\Delta+V_{\omega,n})\chi,\chi\rangle&=\|\nabla\chi\|^2+\sum_{i=1}^{m(\omega)}\int\tau_{x_i(\omega)}(V\chi^2)+\\&\quad+\sum_{i=1}^{m(\omega)}\int[\tau_{x_i(\omega)}V](\tau_{x_i(\omega)}\chi^2-\chi^2)+\langle V^{(e)}_{\omega,n}\chi,\chi\rangle\\&\leq k(1+\varepsilon)(v_-+\varepsilon)+C+O(1/E)\end{aligned} \tag{3.13}$$

Hence, if we take $k = E/|v_-|(1+\varepsilon/|v_-|)$, for E sufficiently large, we get (cf (3.9))

$$\langle(-\Delta + V_{\omega,n})\chi, \chi\rangle \leq -E - 1.$$

This inequality and the probability estimate (3.12) give, for some $C > 0$ independent of E and ε

$$\log\left(\mathbb{E}(N_{\omega,n}(-E-1))\right) \geq -\frac{E}{|v_-|}(1+C\varepsilon)\log E.$$

This ends the proof of Proposition 3.3.

□

By the same argument as above, we get

Proposition 3.4. *Under the assumptions H1, H2 and H1', if V is bounded from below, there exists $E_0 >$ such that, for $E > E_0$, we have*

$$\log N(E) \geq -g(E)\log g(E)(1+o(1)).$$

This ends the proof of Theorem 1.5 as the asymptotic upper bound given by Theorem 1.4 coalesces with the lower bound given by Proposition 3.4.

3.3. **The case when V has power law singularities.** Let us assume that V_2 satisfies assumption H1". We will show that

Lemma 3.1. *Under the assumptions H1 and H1", we have*

$$\alpha^\dagger \geq \alpha^*.$$

Proof. To estimate α^* we will use the asymptotic ground state constructed for $H(g)$ in Lemma 5.7. Pick i_0 and $\psi_g^{(i_0)}$ as in Lemma 5.7. Let $\alpha > \alpha^\dagger$. If the support of χ (cf Lemma 5.7) is small enough, then, for $|a| \leq g^{-\alpha}$ and g large enough, we have

$$\begin{aligned}
\Phi(a) &= \frac{g\langle(\tau_a V - V)\psi_g^{(i_0)}, \psi_g^{(i_0)}\rangle}{E_-(g)} \\
&= \frac{g\langle[\tau_a(W_{i_0}V_{i_0}) - W_{i_0}V_{i_0}]\psi_g^{(i_0)}, \psi_g^{(i_0)}\rangle}{E_-(g)} + \frac{g\langle[\tau_a V_1 - V_1]\psi_g^{(i_0)}, \psi_g^{(i_0)}\rangle}{E_-(g)} \\
&= \frac{g\langle[(\tau_a W_{i_0}V_{i_0}) - W_{i_0}V_{i_0}]\psi_g^{(i_0)}, \psi_g^{(i_0)}\rangle}{E_-(g)} + o(1)
\end{aligned}$$

as V_1 is bounded and $g = o(E_-(g))$.
Hence to estimate $\Phi(a)$, it is enough to estimate the expression

$$\begin{aligned}
&\frac{g\langle[\tau_a(W_{i_0}V_{i_0}) - W_{i_0}V_{i_0}]\psi_g^{(i_0)}, \psi_g^{(i_0)}\rangle}{E_-(g)} \\
&= \frac{g}{E_-(g)}\int_{|x|<r_0}\left(W_{i_0}(x-a)V_{i_0}(x-a) - W_{i_0}(x)V_{i_0}(x)\right)\chi^2(x)|\varphi_g^{(i_0)}(x)|^2 dx \qquad (3.14) \\
&= \frac{g^{1+\nu_{i_0}\alpha^\dagger}}{E_-(g)}I(a) = \frac{g^{2\alpha^\dagger}}{E_-(g)}I(a)
\end{aligned}$$

where

$$I(a) = \int_{|x|<g^{\alpha^\dagger} r_0} \left(W_{i_0}(g^{-\alpha^\dagger}(x-b))V_{i_0}(x-b) - W_{i_0}(g^{-\alpha^\dagger}x)V_{i_0}(x) \right) \chi^2(g^{-\alpha^\dagger}x)|\varphi^{(i_0)}(x)|^2 dx$$

where $b = g^{\alpha^\dagger}a$. Notice that $b \to 0$ when $g \to +\infty$ as $\alpha > \alpha^\dagger$ and $|a| \leq g^{-\alpha}$. Moreover, by Lemma 5.7, $E_-(g) \sim Cg^{2\alpha^\dagger}$ (for some C). Hence to show that $\alpha > \alpha^*$, we just need to show that $I(a) \to 0$ when $g \to +\infty$. This is easily seen cutting $I(a)$ into two parts. In the first one, we integrate over some small neighborhood of 0 and this is small as χ and $\varphi^{(i_0)}$ are bounded and the singularities of V_{i_0} are integrable; outside of this neighborhood, the potentials V_{i_0} and $V_{i_0}(\cdot - b)$ are continuous and as $\varphi^{(i_0)}$ is in L^2, we can conclude using Lebesgue's Dominated Convergence Theorem. This implies that $\alpha \geq \alpha^*$. As it holds for any $\alpha > \alpha^\dagger$, we get the result of Lemma 3.1.

□

Combining Lemma 3.1 with Theorem 1.4, we get

Proposition 3.5. *Under the assumptions H1 and H1", for E large enough, one has*

$$-(1+\alpha^\dagger d)g(E)\log g(E)(1+o(1)) \leq \log N(-E)$$

In the section 4.3 we improve on the general upper bound given in Theorem 1.4 so that the new upper bound coalesces with the lower bound obtained here.

4. The upper bounds

In this section, we will prove the asymptotic upper bound on the approximated density of states $dN_{\omega,n}$ defined in section 2.1 in the different cases considered in the introduction.

4.1. The general case.

We prove the following general bound

Proposition 4.1. *Under the assumptions H1, H2 and H3, there exist $\beta_0 > 0$ such that, for any $\beta > \beta_0$, there exists $E_0 >$ such that, for $E > E_0$ and $n = 2[E^\beta] \cdot [|\log E|^\beta]$, we have*

$$\log\left(\mathbb{E}(N_{\omega,n}(-E+1))\right) \leq -g(E)\log g(E)(1+o(1)). \tag{4.1}$$

By Lemma 5.5, we know that, for some $C > 0$ and $\alpha > 1$, $g(E) \geq CE^{1/\alpha}$ for some $C > 0$ and E large enough. So that, if we pick $\beta > 1$ large enough so that ν given by equation (2.28) satisfies $\nu > 1$, Lemma 2.3 and Proposition 4.1 immediately imply

Proposition 4.2. *Under the assumptions H1, H2 and H3, there exists $E_0 >$ such that, for $E > E_0$, we have*

$$\log N(E) \leq -g(E)\log g(E)(1+o(1)).$$

Taking into account Proposition 3.2, this ends the proof of Theorem 1.4. We now turn to the proof of Proposition 4.1.

4.2. **Proof of Proposition 4.1.** The idea of the proof is to show that, if $H_{\omega,n,\theta}$ has some low energies, then the corresponding potential V_ω must have a very deep well i.e. the corresponding realization of the Poisson measure must put sufficiently many points inside the cube $C(0,n)$ and those points must be sufficiently close to each other. The main technical difficulty comes from the fact that our single site potential is not of finite range; so we need some a priori estimate on $N_{\omega,n}$ that tells us that the behavior of V_ω outside of $C(0,n)$ does not interact too much with the one inside, more precisely, that this interaction can be large only with a small probability. Actually we need this not only on the scale of the large cube used in the periodic approximation but also on a much smaller scale, namely the scale of the size of the cube where we want the Poisson points to pile up.

Pick $\beta > 0$ large. Set $n = 2[E^\beta] \cdot [|\log E|^\beta]$ and $l = [|\log E|^\beta]$. Pick $\rho > 0$ large and let Ω^2_E be defined as in (3.3). Then we prove

Lemma 4.1. *For some $C > 0$ and sufficiently large E, we have*

$$\mathbb{E}(N_{\omega,n}(-E+1)) \leq \mathbb{E}(N_{\omega,n}(-E+1)\mathbf{1}_{\Omega^2_E}) + \frac{C}{\Gamma(E^\rho/2)}. \tag{4.2}$$

Proof. We decompose

$$\mathbb{E}(N_{\omega,n}(-E+1)) = \mathbb{E}(N_{\omega,n}(-E+1)\mathbf{1}_{\Omega^2_E}) + A$$

where α and p' are as in Lemma 5.1,

(4.3)
$$\begin{aligned} A &= \sum_{k\geq 1} \mathbb{E}(N_{\omega,n}(-E+1)\mathbf{1}_{\Omega_k(\alpha,1,p')\setminus(\Omega^2_E\cup\Omega_{k-1}(\alpha,1,p'))}) + \mathbb{E}(N_{\omega,n}(-E+1)\mathbf{1}_{\Omega_0(\alpha,1,p')\setminus\Omega^2_E}) \\ &\leq \sum_{k\geq E^\rho} \mathbb{E}(N_{\omega,n}(-E+1)\mathbf{1}_{\Omega_k(\alpha,1,p')\setminus\Omega_{k-1}(\alpha,1,p')}) \end{aligned}$$

as, for $k < E^\rho$, by the proof of Lemma 5.1, we have $\Omega_k(\alpha,1,p') \subset \Omega^2_E$. For $\omega \in \Omega_k(\alpha,1,p')$ (see Lemma 5.1), we have

$$\|V^{(r)}_{\omega,n}\|_\infty \leq CE^{\alpha\beta}k^{\frac{p}{p-p'}}.$$

Hence, if N denotes the density of states of $-\Delta$, for $\omega \in \Omega_k(\alpha,1,p')$, we get

$$N_{\omega,n}(-E+1) \leq N\left(-E+2+CE^{\alpha\beta}k^{\frac{p}{p-p'}}\right) \leq C\left(-E+2+CE^{\alpha\beta}k^{\frac{p}{p-p'}}\right)^{d/2}.$$

Plugging this into (4.3) and computing the sum over k, we get Lemma 4.1.

□

Define

$$V^{(i)}_\omega = \int_{C(0,n+2l)} V(x-y)m(\omega,dy) \text{ and } V^{(e)}_\omega = \int_{\mathbb{R}^d\setminus C(0,n+2l)} V(x-y)m(\omega,dy) \tag{4.4}$$

and the corresponding periodized potentials $V^{(i,e)}_{\omega,n}$ (see equation (2.1)). As V_2 is compactly supported, $V^{(e)}_{\omega,n}$ is almost surely bounded for l (i.e. E) large. We will estimate its magnitude later.

The proof of Proposition 4.1 will be a consequence of the following lemmas.

Lemma 4.2. *For any $\varepsilon > 0$, there exists E_0 and k_0 such that, if $k > k_0$, $E > E_0$ and*

$$\forall x \in C(0, n+2l),\ m(\omega, C(x, 2l)) \leq k, \tag{4.5}$$

then, for any $\theta \in T_n^*$, we have

$$H_{\omega,n,\theta} \geq -E_-\left(\frac{k}{1-\varepsilon}\right) - \frac{C}{\varepsilon l^2} - \|V^{(e)}_{\omega,n}\|_\infty - C((n/2l)^d + 1) \cdot k e^{-l/C}. \tag{4.6}$$

and

Lemma 4.3. *Pick $\delta > 0$. If E is large enough and if $k \geq E^\delta$, then,*

$$\log\left[P(\{\omega;\ \exists x \in C(0, n+2l),\ m(\omega, C(x, 2l)) > k\})\right] \underset{E\to+\infty}{\sim} -k \log k. \tag{4.7}$$

Before proving Lemmas 4.2 and 4.3, we finish the proof of Proposition 4.1. Therefore, fix $\varepsilon > 0$ and set $k = g(E-2)(1-\varepsilon)$. For $\omega \in \Omega^2_E$, we have, for some $C > 0$,

$$\|V^{(e)}_{\omega,n}\|_\infty \leq E^\rho e^{-(\log E)^\beta / C}. \tag{4.8}$$

Hence, using (4.6) and the bounds known for $g(E)$ given in Lemma 5.5, we get that, for $\omega \in \Omega^2_E$ such that $\forall x \in C(0, n+2l)$, $m(\omega, C(x, 2l)) \leq k$, for any $\theta \in T_n^*$, we have

$$H_{\omega,n,\theta} \geq -E + 3/2.$$

In other words, we have

$$\begin{aligned}&\sharp\{\text{eigenvalues of } H_{\omega,n,\theta} \leq -E+1\}\\ &\qquad = \sharp\{\text{eigenvalues of } H_{\omega,n,\theta} \leq -E+1\}\mathbf{1}_{\{\omega\in\Omega^2_E:\ \exists x\in C(0,n+2l),\ m(\omega,C(\gamma,2l))>k\}}.\end{aligned}$$

By the definition of $N_{\omega,n}$, using Fubini's Theorem, we compute

$$\begin{aligned}&\mathbb{E}(N_{\omega,n}(-E+1)\mathbf{1}_{\Omega^2_E})\\ &= \int_{T_n^*} \mathbb{E}(\sharp\{\text{eigenvalues of } H_{\omega,n,\theta} \leq -E+1\}\mathbf{1}_{\Omega^2_E})\\ &= \int_{T_n^*} \mathbb{E}(\sharp\{\text{eigenvalues of } H_{\omega,n,\theta} \leq -E+1\}\mathbf{1}_{\{\omega\in\Omega^2_E:\ \exists x\in C(0,n+2l),\ m(\omega,C(\gamma,2l))>k\}}).\end{aligned} \tag{4.9}$$

On the other hand, for $\omega \in \Omega^2_E$ (Ω^2_E is defined in (3.3)), we have

$$\|V^{(i)}_{\omega,n}\|_{L^p(C(0,n))} \leq CE^{\beta(d+\alpha')+\rho}$$

hence, using Lemma 5.3, there exists $\nu > 0$ such that

$$H_{\omega,n,\theta} \geq -\frac{1}{2}\Delta_{n,\theta} - E^{(\beta(d+\alpha')+\rho)\nu}.$$

where $-\Delta_{n,\theta}$ is the Laplace operator on $C(0,n)$ with quasi-periodic boundary conditions. This implies that, for some $C > 0$,

$$\begin{aligned}&\sharp\{\text{eigenvalues of } H_{\omega,n,\theta} \leq -E+1\}\\ &\qquad \leq \sharp\{\text{eigenvalues of } -\Delta_{n,\theta} \leq CE^{(\beta(d+\alpha')+\rho)\nu} - 2E + 2\}\\ &\qquad\qquad \leq Cn^d(E^{(\beta(d+\alpha')+\rho)\nu} - E + 1)^{d/2}.\end{aligned}$$

Plugging this into (4.9), we get

$$\begin{aligned}&\mathbb{E}(N_{\omega,n}(-E+1)\mathbf{1}_{\Omega_E^2})\\&\qquad\leq C(E^{(\beta(d+\alpha')+\rho)\nu}-E+1)^{d/2}P(\{\omega;\ \exists x\in C(0,n+2l),\ m(\omega,C(\gamma,2l))>k\}.\end{aligned}$$

Notice that, by Lemma 5.5, our choice of k fulfills the assumptions of Lemma 4.3. Therefrom we deduce that, if E is large enough, then

$$\log\left(\mathbb{E}(N_{\omega,n}(-E+1))\right)\leq -g(E)\log g(E)(1-2\varepsilon).$$

This completes the proof of Proposition 4.1

□

4.2.1. *The proof of Lemma 4.2.* Consider the partition of the cube $C(0,n)$ into cubes of side length l i.e.

$$C(0,n)=\bigcup_{\substack{\gamma\in l\mathbb{Z}^d\\|\gamma|\leq n/l}}C(\gamma,l).$$

and the covering

$$C(0,n)\subset\bigcup_{\substack{\gamma\in l\mathbb{Z}^d\\|\gamma|\leq n/l}}C(\gamma,2l).$$

Set $L=2[E^\beta]$ and consider a $\mathbb{Z}^d$-periodic partition of unity of $\mathbb{R}^d$ i.e.

$$1=\sum_{\beta\in L\mathbb{Z}^d}\sum_{\gamma\in\mathbb{Z}^d\cap C(0,L)}\chi^2_{\gamma+\beta}.$$

where $\chi\in\mathcal{C}_0^\infty(\mathbb{R}^d)$ such that $\chi\equiv 1$ on $C(0,1/2)$, $0\leq\chi\leq 1$ and $\mathrm{supp}\chi\subset C(0,3/2)$. For $\gamma\in l\mathbb{Z}^d$, define $\chi_{\gamma,l}(x)=\chi((x-\gamma)/l)$. Then, one has

$$1=\sum_{\beta\in n\mathbb{Z}^d}\sum_{\gamma\in l\mathbb{Z}^d\cap C(0,n)}\chi^2_{\gamma+\beta,l}.$$

Assume (4.5) holds and consider $\varphi\in\mathcal{C}^\infty(\mathbb{R}^d)$ satisfying the quasi-periodic boundary conditions $\varphi(x+n\gamma)=e^{in\theta\gamma}\varphi(x)$ for $\gamma\in\mathbb{Z}^d$ (i.e. $\varphi\in\mathcal{C}^\infty(\mathbb{R}^d)\cap L^2_\theta(C(0,n))$) such that

$\|\varphi\| = 1$. For small positive ε, we compute

(4.10)

$$\begin{aligned}
\langle(-\Delta + V_{\omega,n})\varphi, \varphi\rangle &= \langle(-\Delta + V^{(i)}_{\omega,n})\varphi, \varphi\rangle + \langle V^{(e)}_{\omega,n}\varphi, \varphi\rangle \\
&\geq \|\nabla\varphi\|^2 + \langle V^{(i)}_{\omega,n}\varphi, \varphi\rangle - \|V^{(e)}_{\omega,n}\|_\infty \\
&= \sum_{\gamma\in l\mathbb{Z}^d\cap C(0,n+l)} \left(\|\chi_{\gamma,l}\nabla\varphi\|^2 + \langle V^{(i)}_{\omega,n}\chi_{\gamma,l}\varphi, \chi_{\gamma,l}\varphi\rangle\right) - \|V^{(e)}_{\omega,n}\|_\infty \\
&\geq (1-\varepsilon) \sum_{\gamma\in l\mathbb{Z}^d\cap C(0,n+l)} \left(\|\nabla\left(\chi_{\gamma,l}\varphi\right)\|^2 + \frac{1}{1-\varepsilon}\langle V^{(i)}_{\omega,n}\chi_{\gamma,l}\varphi, \chi_{\gamma,l}\varphi\rangle\right) \\
&\qquad + \left(1 - \frac{1}{\varepsilon}\right) \sum_{\gamma\in l\mathbb{Z}^d\cap C(0,n+l)} \|\left(\nabla\chi_{\gamma,l}\right)\varphi\|^2 - \|V^{(e)}_{\omega,n}\|_\infty \\
&\geq (1-\varepsilon) \sum_{\gamma\in l\mathbb{Z}^d\cap C(0,n+l)} \left(\|\nabla\left(\chi_{\gamma,l}\varphi\right)\|^2 + \frac{1}{1-\varepsilon}\langle V^{(i)}_{\omega,n}\chi_{\gamma,l}\varphi, \chi_{\gamma,l}\varphi\rangle\right) \\
&\qquad - \frac{C}{\varepsilon l^2} - \|V^{(e)}_{\omega,n}\|_\infty
\end{aligned}$$

Define

$$V^{(i,\gamma)}_\omega = \int_{C(\gamma,2l)} V(x-y)m(\omega,dy) \text{ and } V^{(e,\gamma)}_\omega = \int_{\mathbb{C}(0,n+2l)\setminus C(\gamma,2l)} V(x-y)m(\omega,dy)$$

and the corresponding periodized potentials $V^{((i,e),\gamma)}_{\omega,n}$ (see equation (2.1)). Then (4.5) tells us that $m(\omega, C(0,n+2l)) \leq C(n/2l)^d \cdot k$. Using this and the exponential decay of V_1, we get that, for some $C > 0$,

$$\begin{aligned}
\sup_{x\in C(\gamma,l)} |V^{(e,\gamma)}_{\omega,n}(x)| &\leq (n/2l)^d \cdot ke^{-l/C} \\
\sup_{x\in C(\gamma,l)} \left|\left(V^{(i,\gamma)}_{\omega,n} - V^{(i,\gamma)}_\omega\right)(x)\right| &\leq Cke^{-l/C}.
\end{aligned}$$

So that (4.10) gives us

$$\begin{aligned}
\text{(4.11)}\quad \langle(-\Delta + V_{\omega,n})\varphi, \varphi\rangle \geq (1-\varepsilon) &\sum_{\gamma\in l\mathbb{Z}^d\cap C(0,n+l)} \left\langle \left(-\Delta + \frac{1}{1-\varepsilon}V^{(i,\gamma)}_\omega\right)\chi_{\gamma,l}\varphi, \chi_{\gamma,l}\varphi\right\rangle \\
&- \frac{C}{\varepsilon l^2} - \|V^{(e)}_{\omega,n}\|_\infty - C((n/2l)^d + 1)\cdot ke^{-l/C}.
\end{aligned}$$

Now, if $(x_j(\omega))_{j=1,\dots,m(\omega,C(\gamma,2l))}$ denotes the support of $m(\omega,dx)$ in $C(\gamma,2l)$, we write

$$\begin{aligned}
&\left\langle \left(-\Delta+\frac{1}{1-\varepsilon}V_\omega^{(i,\gamma)}\right)\chi_{\gamma,l}\varphi,\chi_{\gamma,l}\varphi\right\rangle \\
&=\left\langle \left(-\Delta+\frac{1}{1-\varepsilon}\sum_{j=1}^{m(\omega,C(\gamma,2l))}\tau_{x_j(\omega)}V\right)\chi_{\gamma,l}\varphi,\chi_{\gamma,l}\varphi\right\rangle \\
&=\frac{1}{m(\omega,C(\gamma,2l))}\sum_{j=1}^{m(\omega,C(\gamma,2l))}\left\langle \left(-\Delta+\frac{m(\omega,C(\gamma,2l))}{1-\varepsilon}\tau_{x_j(\omega)}V\right)\chi_{\gamma,l}\varphi,\chi_{\gamma,l}\varphi\right\rangle \qquad (4.12)\\
&\geq -\frac{1}{m(\omega,C(\gamma,2l))}\sum_{j=1}^{m(\omega,C(\gamma,2l))}E_-\left(\frac{m(\omega,C(\gamma,2l))}{1-\varepsilon}\right)\|\chi_{\gamma,l}\varphi\|^2\\
&\geq -E_-\left(\frac{k}{1-\varepsilon}\right)\|\chi_{\gamma,l}\varphi\|^2.
\end{aligned}$$

Here we have used (4.5). Plugging (4.12) into (4.11), we get

$$\langle(-\Delta+V_{\omega,n})\varphi,\varphi\rangle\geq -E_-\left(\frac{k}{1-\varepsilon}\right)-\frac{C}{\varepsilon l^2}-\|V_{\omega,n}^{(e)}\|_\infty-C((n/2l)^d+1)\cdot ke^{-l/C}. \qquad (4.13)$$

Lemma 4.2 follows then from the fact that $\mathcal{C}^\infty(\mathbb{R}^d)\cap L^2_\theta(C(0,n))$ is dense in the domain of $H_{\omega,n,\theta}$.

□

4.2.2. *The proof of Lemma 4.3.* Define

$$P(n,k,l)=P(\{\omega;\ \exists x\in C(0,n+2l),\ m(\omega,C(x,2l))>k\}).$$

We will assume that k, n and l are large and that they satisfy

$$(n+l)^{d+1}e^{l^d}=o(l^{-kd}\Gamma(k)). \qquad (4.14)$$

We will prove a lower and an upper bound on $P(n,k,l)$.

We start with the lower bound. Consider the partition of the cube $C(0,n+2l)$ into cubes of side length $2l$ i.e.

$$C(0,n+2l)=\bigcup_{\substack{\gamma\in 2l\mathbb{Z}^d\\ |\gamma|\leq n/2l+1}}C(\gamma,2l).$$

Using the independence for disjoint cubes and the homogeneity of the Poisson field, we obtain that

$$\begin{aligned}
P(n,k,l)&\geq P(\{\omega;\ \exists\gamma\in 2l\mathbb{Z}^d,\ |\gamma|\leq n/2l+1;\ m(\omega,C(\gamma,2l))>k\})\\
&=1-(1-P(m(\omega,C(0,2l))>k))^{(n/2l+1)^d}
\end{aligned} \qquad (4.15)$$

By definition, $P(m(\omega,C(0,2l))>k)\geq e^{-\mu(2l)^d}\dfrac{(\mu(2l)^d)^k}{k!}$. Plugging this into (4.15), we get

$$\log P(n,k,l)\geq -k\log k(1+o(1))$$

when k, n and l tend to $+\infty$ under the assumption (4.14).

To get an upper bound, we consider the partition of the cube $C(0, n+2l)$ into cubes of side length $4l$ i.e.

$$C(0, n+2l) = \bigcup_{\substack{\gamma \in 4l\mathbb{Z}^d \\ |\gamma| \leq n/(4l)+1/2}} C(\gamma, 4l).$$

For any $x \in C(0, n+2l)$, there exists $\gamma \in 4l\mathbb{Z}^d$, $|\gamma| \leq n/(4l) + 1/2$ such that $C(\gamma, 4l) \cap C(x, 2l) = C(x, 2l)$. Hence,

$$\begin{aligned} P(\{\omega;\ \forall x \in C(0, n+2l),\ m(\omega, C(x, 2l)) \leq k\}) \geq \\ \geq P(\{\omega;\ \forall \gamma \in 4l\mathbb{Z}^d,\ |\gamma| \leq n/2l + 1/2;\ m(\omega, C(\gamma, 4l)) \leq k\}) \end{aligned}$$

that is, using the stationarity of the Poisson process,

$$\begin{aligned} P(\{\omega;\ \exists x \in C(0, n+2l),\ m(\omega, C(x, 2l)) > k\}) \\ \leq \sum_{\gamma \in 4l\mathbb{Z}^d,\ |\gamma| \leq n/2l+1/2} P(\{\omega;\quad m(\omega, C(\gamma, 4l)) > k\}) \\ \leq (n+2l)^d P(\{\omega;\quad m(\omega, C(0, 4l)) > k\}). \end{aligned}$$

On the other hand

$$P(\{m(\omega, C(0, 4l)) > k\}) = \sum_{j>k} e^{-(4l)^d \mu} \frac{[\mu(4l)^d]^j}{j!} \leq \frac{(\mu(4l)^d)^k}{k!} \tag{4.16}$$

This then implies that

$$\log P(n, k, l) \leq -k \log k (1 + o(1))$$

when k, n and l tend to $+\infty$ under assumption (4.14).

$\square$

4.3. **The case when V has power law singularities.** We now assume that H1" holds. Obviously, modifying V_1, we can assume that the functions W_i do not change sign and that the support of distinct W_i are pairwise disjoint. We then prove

Proposition 4.3. *Under the assumptions H1 and H1", there exists $\beta_0 > 0$ such that, for any $\beta > \beta_0$, there exists $E_0 > 0$ such that, for $E > E_0$ and $n = 2[E^\beta] \cdot [|\log E|^\beta]$, we have*

$$\log\left(\mathbb{E}(N_{\omega,n}(-E+1))\right) \leq -(1 + \alpha^\dagger d) g(E) \log g(E)(1 + o(1)). \tag{4.17}$$

Taking into account Lemma 5.5 and Lemma 2.3, as a corollary to Proposition 4.3, we get

Proposition 4.4. *Under the assumptions H1 and H1", there exists $E_0 >$ such that, for $E > E_0$, we have*

$$\log\left(\mathbb{E}(N(-E+1))\right) \leq -(1 + \alpha^\dagger d) g(E) \log g(E)(1 + o(1)).$$

This ends the proof of Theorem 1.6 if one takes into account Proposition 3.5.

4.3.1. *Proof of Proposition 4.3.* The idea guiding this proof is essentially the same as the one guiding the proof of Proposition 4.1. The difference comes from the fact that, as $E_-(g)$ increases faster than linearly in g, if we want to gather k single site potentials sufficiently close together so as to get the effect of having k single site potentials exactly at the same point, we need the single site potentials to be roughly at a distance less than $k^{-\alpha^\dagger}$ from each other. Hence, the scale on which we want the Poisson points to concentrate is much smaller than the one used to prove the upper bound in the general case. This leads to some supplementary technical difficulties as the single site potentials have a finite non zero range.

Recall that $(x_i)_{1\leq i\leq q}$ are the singularities of the single site potential V (see section 1.2). We define

$$K(x,r)=\bigcup_{1\leq i\leq q} C(x-x_i,r)$$

Pick $\varepsilon>0$ small and define the events

$$\begin{gathered}\tilde\Omega_1=\{\omega\in\Omega^2_E:\ \exists x\in C(0,n+2l),\ m(\omega,K(x,k^{-(\alpha^\dagger-2\varepsilon)}))>k\},\\ \tilde\Omega_2=\{\omega\in\Omega^2_E:\ \exists x\in C(0,n+2l),\ m(\omega,C(x,1))>k^{1+\varepsilon\nu^\dagger}\},\\ \text{and } \tilde\Omega=\tilde\Omega_1\cup\tilde\Omega_2\end{gathered}$$

Pick $\beta>0$ large. Set $n=2[E^\beta]\cdot[|\log E|^\beta]$ and $l=[|\log E|^\beta]$. Pick $\rho>0$ large. Taking into account Lemma 4.1, Proposition 4.3 is a direct consequence of the following two lemmas (cf Lemmas 4.2 and 4.3).

Lemma 4.4. *For any $\varepsilon>0$, there exists E_0 and k_0 such that, if $k>k_0$ and $E>E_0$, if $\omega\in\Omega^2_E$ and if*

$$\forall x\in C(0,n+2l),\ m(\omega,K(x,k^{-(\alpha^\dagger-2\varepsilon)}))\leq k \text{ and } m(\omega,C(x,1))\leq k^{1+\varepsilon\nu^\dagger} \tag{4.18}$$

then, for any $\theta\in\mathbb{T}^*_n$, we have

$$H_{\omega,n,\theta}\geq -E_-\left(\frac{k}{1-\varepsilon}\right)(1+\varepsilon)-\frac{Ck^{2(\alpha^\dagger-\nu^\dagger\varepsilon/2)}}{\varepsilon}-E^\rho e^{-(\log E)^\beta/C}-Ck^{1+\varepsilon\nu^\dagger}. \tag{4.19}$$

and

Lemma 4.5. *Pick $\delta>0$. Then, if E is large enough and if $k\geq E^\delta$, the probability of the event $\tilde\Omega$ satisfies*

$$\log P(\tilde\Omega)\underset{E\to+\infty}{\sim}-(1+(\alpha^\dagger-2\varepsilon)d)k\log k. \tag{4.20}$$

We now pick $k=[(1-\varepsilon)g(E/(1+2\varepsilon))]$, i.e. k is of order of magnitude $E^{1-\nu^\dagger/2}$ as $E\to+\infty$ (see Lemma 5.7). In that case, for sufficiently small ε, $k^{1+\varepsilon\nu^\dagger}$ and $k^{2(\alpha^\dagger-\nu^\dagger\varepsilon/2)}$ are $o(E)$. Hence Proposition 4.3 follows from Lemma 4.1, Lemma 4.4 and Lemma 4.5 in the same way as Proposition 4.1 followed from Lemmas 4.1, 4.2 and 4.3.

4.3.2. *The proof of Lemma 4.4.* We are going to proceed along the same lines as in the proof of Lemma 4.2. $V^{(i,e)}_{\omega,n}$ are defined as in (4.4). Fix $\varphi \in \mathcal{C}^\infty(\mathbb{R}^d)\cap L^2_\theta(C(0,n))$ normalized by $\|\varphi\| = 1$. Then

$$\begin{aligned}\langle H_{\omega,n,\theta}\varphi,\varphi\rangle &= \langle -\Delta\varphi,\varphi\rangle + \langle V^{(i)}_{\omega,n}\varphi,\varphi\rangle + \langle V^{(e)}_{\omega,n}\varphi,\varphi\rangle \\ &\geq \langle -\Delta\varphi,\varphi\rangle + \langle V^{(i)}_{\omega,n}\varphi,\varphi\rangle - E^\rho e^{-(\log E)^\beta/C}\end{aligned} \tag{4.21}$$

using (4.8). We can split

$$V^{(i)}_{\omega,n} = V^{(i,1)}_{\omega,n} + V^{(i,2)}_{\omega,n}$$

where

$$V^{(i,1)}_{\omega} = \int_{C(0,n+2l)} V_1(x-y)m(\omega,dy) \text{ and } V^{(i,2)}_{\omega} = \int_{C(0,n+2l)} V_2(x-y)m(\omega,dy)$$

$V^{(i,2)}_{\omega,n}$ contains all the local singularities of $V^{(i)}_{\omega,n}$ and, as V_1 is exponentially decaying, there exists $C>0$ such that for $\omega\in\Omega^2_E$ satisfying (4.18), we have

$$\|V^{(i,1)}_{\omega,n}\|_\infty \leq Ck^{1+\varepsilon\nu^\dagger}.$$

Hence (4.21) gives

$$\langle H_{\omega,n,\theta}\varphi,\varphi\rangle \geq \langle -\Delta\varphi,\varphi\rangle + \langle V^{(i,1)}_{\omega,n}\varphi,\varphi\rangle - E^\rho e^{-(\log E)^\beta/C} - Ck^{1+\varepsilon\nu^\dagger}. \tag{4.22}$$

Consider now a periodic partition of the unity of the cube $C(0,n)$ of the form

$$\mathbf{1}_{C(0,n)} = \sum_{\gamma\in\Gamma}\chi^2_\gamma.$$

where the χ_γ are supported on cells of size roughly $k^{-\alpha^\dagger+\nu^\dagger\varepsilon/2}$. These cells are centered at the points of $\Gamma = \delta(k)\mathbb{Z}^d\cap C(0,n)$ where $\delta(k) = 1/[k^{\alpha^\dagger-\nu^\dagger\varepsilon/2}]$. Here $[\cdot]$ denote the integer part. These functions can then be chosen so that, for some $C>0$, we have

$$\sup_{\gamma\in\Gamma}\|\nabla\chi_\gamma\|^2_\infty + \sup_{\gamma\in\Gamma}\|\Delta\chi_\gamma\|_\infty \leq C\delta(k)^{-2} \leq Ck^{2(\alpha^\dagger-\nu^\dagger\varepsilon/2)} \tag{4.23}$$

for some $C>1$.
We compute

$$\begin{aligned}\langle -\Delta\varphi,\varphi\rangle &= \sum_{\gamma\in\Gamma}\|\nabla(\chi_\gamma\varphi)\|^2 + \langle|\nabla\chi_\gamma|^2\varphi,\varphi\rangle - 2\mathrm{Re}(\langle\nabla(\chi_\gamma\varphi),\varphi\nabla\chi_\gamma\rangle) \\ &\geq (1+\varepsilon)\sum_{\gamma\in\Gamma}\|\nabla(\chi_\gamma\varphi)\|^2 + \left(1-\frac{1}{\varepsilon}\right)\sum_{\gamma\in\Gamma}\langle|\nabla\chi_\gamma|^2\varphi,\varphi\rangle \\ &\geq (1-\varepsilon)\sum_{\gamma\in\Gamma}\|\nabla(\chi_\gamma\varphi)\|^2 + C\left(1-\frac{1}{\varepsilon}\right)k^{2(\alpha^\dagger-\nu^\dagger\varepsilon/2)}.\end{aligned} \tag{4.24}$$

On the other hand

$$\langle V^{(i,1)}_{\omega,n}\varphi,\varphi\rangle = \sum_{\gamma\in\Gamma}\langle V^{(i,1)}_{\omega,n}\chi_\gamma\varphi,\chi_\gamma\varphi\rangle$$

Set $\tilde{V}_i = W_i V_i$. As the W_i's are of compact support, so are the $\tilde{V}_i$. Hence, for some R_0 positive, we have

$$
\begin{aligned}
\langle V^{(i,1)}_{\omega,n}\chi_\gamma\varphi,\chi_\gamma\varphi\rangle &= \sum_{\substack{1\leq j\leq q,\ u\geq 1\\ x_u(\omega)+x_j\in C(\gamma,R_0)}} \langle \tau_{x_u(\omega)+x_j}(\tilde{V}_j)\chi_\gamma\varphi,\chi_\gamma\varphi\rangle \\
&= \sum_{\substack{1\leq j\leq q,\ u\geq 1\\ x_u(\omega)+x_j\in C(\gamma,k^{2\varepsilon-\alpha^\dagger})}} \langle \tau_{x_u(\omega)+x_j}(\tilde{V}_j)\chi_\gamma\varphi,\chi_\gamma\varphi\rangle \\
&\quad + \sum_{\substack{1\leq j\leq q,\ u\geq 1\\ x_u(\omega)+x_j\in C(\gamma,R_0)\setminus C(\gamma,k^{2\varepsilon-\alpha^\dagger})}} \langle \tau_{x_u(\omega)+x_j}(\tilde{V}_j)\chi_\gamma\varphi,\chi_\gamma\varphi\rangle
\end{aligned}
\tag{4.25}
$$

where the $(x_u(\omega))_{u\geq 1}$ are the support of the Poisson measure $m(\omega,dy)$.
For $|\gamma - x_u(\omega) - x_j| \geq k^{\varepsilon-\alpha^\dagger}$ and x in the support of χ_γ, one has

$$|\tau_{x_u(\omega)+x_j}(\tilde{V}_j)(x)| \leq Ck^{\varepsilon\nu^\dagger-\alpha^\dagger\nu^\dagger}.$$

As we assumed $\forall x\in C(0,n+2l)$, $m(\omega,C(x,1))\leq k^{1+\varepsilon\nu^\dagger}$, we obtain

$$
\begin{aligned}
\sum_{\substack{1\leq j\leq q,\ u\geq 1\\ x_u(\omega)+x_j\in C(\gamma,R_0)\setminus C(\gamma,k^{2\varepsilon-\alpha^\dagger})}} \langle \tau_{x_u(\omega)+x_j}(\tilde{V}_j)\chi_\gamma\varphi,\chi_\gamma\varphi\rangle &\geq -Ck^{-2\varepsilon\nu^\dagger+\alpha^\dagger\nu^\dagger+1+\varepsilon\nu^\dagger} \\
&= -Ck^{2\alpha^\dagger-\varepsilon\nu^\dagger}
\end{aligned}
\tag{4.26}
$$

Hence, to estimate $\langle H_{\omega,n,\theta}\varphi,\varphi\rangle$ using equations (4.24), (4.25) and (4.26), we only need to estimate

$$A := \|\nabla(\chi_\gamma\varphi)\|^2 + \frac{1}{1-\varepsilon}\sum_{\substack{1\leq j\leq q,\ u\geq 1\\ x_u(\omega)\in K(\gamma,k^{2\varepsilon-\alpha^\dagger})}} \langle \tau_{x_u(\omega)+x_j}(\tilde{V}_j)\chi_\gamma\varphi,\chi_\gamma\varphi\rangle.$$

Therefore we notice that, as the $(x_j)_{1\leq j\leq q}$ are distinct, for k sufficiently large, for any ω, any γ and any u, there is at most a single $1\leq j\leq q$ such that $x_u(\omega)\in C(\gamma-x_j,k^{2\varepsilon-\alpha^\dagger})$. Hence, we get a partition

$$Q := \{x_u; x_u(\omega)\in K(\gamma,k^{2\varepsilon-\alpha^\dagger})\} = \bigcup_{1\leq j\leq q}\{x_u; x_u(\omega)\in C(\gamma-x_j,k^{2\varepsilon-\alpha^\dagger})\} =: \bigcup_{1\leq j\leq q} Q_j.$$

Set $m(\omega)=m(\omega,K(\gamma,k^{2\varepsilon-\alpha^\dagger}))$ and $q_j=\sharp Q_j$. We compute

$$\begin{aligned}
A&=\|\nabla(\chi_\gamma\varphi)\|^2+\frac{1}{1-\varepsilon}\sum_{j=1}^{q}\sum_{Q_j}\langle\tau_{x_u(\omega)+x_j}(\tilde V_j)\chi_\gamma\varphi,\chi_\gamma\varphi\rangle\\
&=\sum_{j=1}^{q}\frac{1}{m(\omega)}\left(q_j\|\nabla(\chi_\gamma\varphi)\|^2+\frac{m(\omega)}{1-\varepsilon}\sum_{Q_j}\langle\tau_{x_u(\omega)+x_j}(\tilde V_j)\chi_\gamma\varphi,\chi_\gamma\varphi\rangle\right)\\
&=\sum_{j=1}^{q}\frac{1}{m(\omega)}\sum_{Q_j}\left(\|\nabla(\chi_\gamma\varphi)\|^2+\frac{m(\omega)}{1-\varepsilon}\langle\tau_{x_u(\omega)+x_j}(\tilde V_j)\chi_\gamma\varphi,\chi_\gamma\varphi\rangle\right)\\
&\geq\left[\sum_{j=1}^{q}\frac{q_j}{m(\omega)}\sum_{Q_j}\tilde E_j\left(\frac{m(\omega)}{(1-\varepsilon)}\right)\right]\|\chi_\gamma\varphi\|^2
\end{aligned}$$

where $\tilde E_j(g)$ is the ground state of $-\Delta+g\tilde V_j$. In section 5.4, we prove that the lowest of these ground states is asymptotic to $E(g)$ as $g\to+\infty$. Hence, for k sufficiently large, we have

$$A\geq-(1+\varepsilon)E_-\left(\frac{m(\omega)}{1-\varepsilon}\right)\|\chi_\gamma\varphi\|^2\geq-(1+\varepsilon)E_-\left(\frac{k}{1-\varepsilon}\right)\|\chi_\gamma\varphi\|^2 \tag{4.27}$$

by (4.18). Combining this with equations (4.24), (4.25) and (4.26), we end the proof of Lemma 4.4.

□

4.3.3. *The proof of Lemma 4.5.* We first prove an asymptotic lower bound for $P(\tilde\Omega_1)$. We notice that, for E large enough,

$$\begin{aligned}
&P(\{\omega\in\Omega^2_E:\ \exists x\in C(0,n+2l),\ m(\omega,K(x,k^{-(\alpha^\dagger-2\varepsilon)}))>k\})\geq\\
&\qquad\geq P(\{\omega\in\Omega^2_E:\ \exists x\in C(0,n+2l),\ m(\omega,C(x,k^{-(\alpha^\dagger-2\varepsilon)}))>k\}).
\end{aligned}$$

Hence, using the proof of the lower estimate in Lemma 4.3, we get

$$\liminf_{E\to+\infty}\frac{\log P(\tilde\Omega_1)}{(1+(\alpha^\dagger-2\varepsilon)d)k\log k}\geq-1. \tag{4.28}$$

Let us prove the asymptotic upper bound for $P(\tilde\Omega_1)$. We partition $C(0,n+3l)$ into cubes of side length $k^{-(\alpha^\dagger-2\varepsilon)}$, the cubes being indexed by $\Xi_k=k^{-(\alpha^\dagger-2\varepsilon)}\mathbb{Z}^d\cap C(0,n+3l)$. For $1\leq i\leq q$, let γ_i be the projection of x_i on the lattice $k^{-(\alpha^\dagger-2\varepsilon)}\mathbb{Z}^d$. Then, for any $x\in C(0,n+2l)$, there exists $\gamma\in\Xi_k$ such that

$$K(x,k^{-(\alpha^\dagger-2\varepsilon)})\subset\bigcup_{1\leq i\leq q}C(\gamma-\gamma_i,4k^{-(\alpha^\dagger-2\varepsilon)}).$$

Hence

$$P(\{\omega \in \Omega_E^2 :\ \exists x \in C(0, n+2l),\ m(\omega, K(x, k^{-(\alpha^\dagger - 2\varepsilon)})) > k\})$$
$$\leq \sum_{\gamma \in \Xi_k} P\left(\left\{\omega \in \Omega_E^2;\ m\left(\omega, \bigcup_{1 \leq i \leq q} C(\gamma_i + \gamma, 4k^{-(\alpha^\dagger - 2\varepsilon)})\right) > k\right\}\right)$$
$$\leq ((n+3l)k^{(\alpha^\dagger - 2\varepsilon)})^d P\left(\left\{\omega \in \Omega_E^2 :\ m\left(\omega, \bigcup_{1 \leq i \leq q} C(\gamma_i, 4k^{-(\alpha^\dagger - 2\varepsilon)})\right) > k\right\}\right).$$

As $k > E^\delta$ and as n and l are at most of polynomial size in E, this gives

$$\limsup_{E \to +\infty} \frac{\log P(\tilde{\Omega}_1)}{(1 + (\alpha^\dagger - 2\varepsilon)d)k \log k} \leq -1.$$

Combined with (4.28), we get

$$\log P(\tilde{\Omega}_1) \underset{E \to +\infty}{\sim} -(1 + (\alpha^\dagger - 2\varepsilon)d)k \log k. \tag{4.29}$$

On the other hand, Lemma 4.3 tells us that

$$\log P(\tilde{\Omega}_2) \underset{E \to +\infty}{\sim} -(1 + \varepsilon\nu^\dagger)k^{1+\varepsilon\nu^\dagger} \log k.$$

So that $P(\tilde{\Omega}_2)$ is negligible with respect to $P(\tilde{\Omega}_1)$. In view of (4.29), this completes the proof of Lemma 4.5.

□

5. Appendix

5.1. **The structure of the Poisson potential.** Let $V \in L^p(\mathbb{R}^d)$ and be compactly supported be a potential satisfying assumptions H1, H2 and H3. Let $m(\omega, dx)$ be a Poisson measure of concentration μ. Consider the potential V_ω defined by (1.1). One has

Lemma 5.1. *For any $\alpha \in (0,1)$, $p' < p$ and $0 < b < 1$, there exists $C > 0$, such that ω-almost surely,*

$$V_\omega \in \bigcup_{k \geq 1} \Omega_k(\alpha, b, p')$$

where, for $k \geq 1$, $\Omega_k(\alpha, b, p')$ is the set of measurable functions $V : \mathbb{R}^d \to \mathbb{C}$ that can be written in the form

$$V = V^{(r)} + V^{(i)}$$

where $V^{(r)}$ and $V^{(i)}$ satisfy

1. *$|V^{(r)}(x)| \leq C k^{\frac{p}{p-p'}} b^{\frac{-p'}{p-p'}} (|x|^\alpha + 1)$ for $x \in \mathbb{R}^d$,*
2. *$V^{(i)} \in L^{p'}_{loc,unif}$ and $\|V^{(i)}\|_{L^{p'}_{loc,unif}} \leq b$ (here $\|\cdot\|_{L^p_{loc,unif}} = \sup_{x \in \mathbb{R}^d} \|\cdot\|_{L^p(C(x,1))}$).*

Moreover, there exists $C > 0$ such that

$$\mathbb{P}(\{V_\omega \notin \Omega_k(\alpha, b, p')\}) \leq \frac{(C\mu)^k}{k!}. \tag{5.1}$$

Proof. Fix α, p' and b as in the lemma. Pick $0 < \alpha' < \alpha$. As V_2 is compactly supported, there exists $r_0 > 1$ such that $\mathrm{supp} V_2 \subset C(0, r_0)$. Let $\Gamma = r_0\mathbb{Z}^d$. Pick $k > 0$. Define $P_{k,\alpha'} := P(\{\exists \gamma \in \Gamma;\ m(\omega, C(\gamma, r_0)) \geq k(|\gamma|^{\alpha'} + 1)\})$ Using (4.16), we compute

$$
\begin{aligned}
P_{k,\alpha'} &\leq \sum_{\gamma\in\Gamma} P(\{\omega;\ m(\omega, C(\gamma, r_0)) \geq k(|\gamma|^{\alpha'} + 1)\} \\
&\leq \sum_{\gamma\in\Gamma} \frac{(\mu r_0^d)^{k(|\gamma|^{\alpha'}+1)}}{(k(|\gamma|^{\alpha'}+1))!} \\
&\leq C\frac{(\mu r_0^d)^k}{k!} \text{ for some } C > 0.
\end{aligned}
\tag{5.2}
$$

We write

$$V_\omega = V_{1,\omega} + V_{2,\omega} \tag{5.3}$$

where $V_{1,\omega}(x) = \int_{\mathbb{R}^d} V_1(x-y)dm(\omega, y)$ and $V_{2,\omega}(x) = \int_{\mathbb{R}^d} V_2(x-y)dm(\omega, y)$.
Now assume that ω is such that

$$\forall \gamma \in \Gamma,\ m(\omega, C(\gamma, r_0)) < k(|\gamma|^{\alpha'} + 1). \tag{5.4}$$

and $(x_n(\omega))_{n\in\mathbb{N}}$ the points in the support of $m(\omega, dx)$. We will estimate $V_{1,\omega}$ and $V_{2,\omega}$ separately. $V_{1,\omega}$ can be estimated as in [9]. One has

$$
\begin{aligned}
|V_{1,\omega}(x)| &\leq \sum_{\beta\in r_0\mathbb{Z}^d} \sum_{x_n(\omega)\in C(\beta,r_0)} |V_1(x - x_n(\omega))| \\
&\leq C \sum_{\beta\in r_0\mathbb{Z}^d} e^{-|x-\beta|/C} m(\omega, C(\gamma, r_0)) \\
&\leq C \sum_{\beta\in r_0\mathbb{Z}^d} e^{-|x-\beta|/C} k(|\beta|^{\alpha'} + 1) \\
&\leq Ck(|x|^{\alpha'} + 1).
\end{aligned}
\tag{5.5}
$$

As $V_2 \in L^p(\mathbb{R}^d)$, for $K \geq 1$, we have

$$
\begin{aligned}
\left(\int_{\mathbb{R}^d} \mathbf{1}_{|V_2(x)|\geq K} |V_2(x)|^{p'} dx\right)^{1/p'} &= \left(\int_{\mathbb{R}^d} \mathbf{1}_{|V_2(x)|\geq K} |V_2(x)|^{p} |V_2(x)|^{p'-p} dx\right)^{1/p'} \\
&\leq K^{\frac{p'-p}{p'}} \|V_2\|_{L^p}^{p/p'}.
\end{aligned}
\tag{5.6}
$$

Let us introduce a notation: take a function W and decompose it into $W = W_K^{(r)} + W_K^{(i)}$ where

$$W_K^{(r)} := \mathbf{1}_{|W(x)|\leq K} W \text{ and } W_K^{(i)} := \mathbf{1}_{|W(x)|>K} W \tag{5.7}$$

Fix a sequence of positive numbers $(K_\gamma)_{\gamma\in\Gamma}$. As $\mathrm{supp} V_2 \subset C(0, r_0)$, $V_{2,\omega}$ can be rewritten as

$$V_{2,\omega} = V_{2,\omega}^{(i)} + V_{2,\omega}^{(r)} \tag{5.8}$$

where we define

$$V_{2,\omega}^{(i,r)}(x) := \sum_{\gamma\in\Gamma} \mathbf{1}_{C(\gamma,r_0)}(x) \left(\sum_{x_n(\omega)\in C(\gamma,2r_0)} (V_2)_{K_\gamma}^{(i,r)}(x - x_n(\omega)) \right).$$

Pick $a > 0$, $K_\gamma = a(|\gamma|^\beta + 1)$ and set $\beta = \frac{p'}{p}\alpha$, $\alpha' = \frac{p-p'}{p}\alpha$. Then, for some $C > 0$, by (5.6), we have

$$\begin{aligned} \|\mathbf{1}_{C(\gamma,r_0)} V_{2,\omega}^{(i)}\|_{L^{p'}} &\le Cm(\omega, C(\gamma, 2r_0))(a(|\gamma|^\beta + 1))^{\frac{p'-p}{p'}} \\ &\le Cka^{\frac{p'-p}{p'}}(|\gamma|^{\alpha'+\beta\frac{p'-p}{p'}} + 1) \le 2Cka^{\frac{p'-p}{p'}}. \end{aligned} \tag{5.9}$$

and

$$\|\mathbf{1}_{C(\gamma,r_0)} V_{2,\omega}^{(r)}\|_\infty \le Cka(|\gamma|^{\alpha'+\beta} + 1) \le Cka(|\gamma|^\alpha + 1) \tag{5.10}$$

Pick $a = (2^{-d}b)^{\frac{p'}{p'-p}}(Ck)^{\frac{p'}{p-p'}}$ (where C is the constant appearing in (5.9)). As $r_0 > 1$, we have $\|V_{2,\omega}^{(i)}\|_{L^{p'}_{\mathrm{loc,unif}}} \le 2^d \sup_{\gamma\in\Gamma} \|\mathbf{1}_{C(\gamma,r_0)} V_{2,\omega}^{(i)}\|_{L^{p'}}$. Hence, by (5.9), we get

$$\|V_{2,\omega}^{(i)}\|_{L^{p'}_{\mathrm{loc,unif}}} \le b. \tag{5.11}$$

Moreover, by (5.10), for some $C > 0$, for $x \in \mathbb{R}^d$, we have

$$|V_{2,\omega}^{(r)}(x)| \le Ck^{\frac{p}{p-p'}} b^{\frac{-p'}{p-p'}}(|x|^\alpha + 1) \tag{5.12}$$

Now putting together equations (5.3), (5.5), (5.8), (5.11) and (5.12), we see that if ω is such that $\forall \gamma \in \Gamma$, $m(\omega, C(\gamma, r_0)) < k(|\gamma|^{\alpha'} + 1)$, then $V_\omega \in \Omega_k(\alpha, b, p')$. Taking into account equation (5.2), we get Lemma 5.1.

□

5.2. **An a-priori estimate on the density of states.** Recall that N denote the integrated density of states of H_ω. We prove

Lemma 5.2. *Under assumption H1, H2 and H3, there exists $C > 0$ such that, for $E > 1$, we have*

$$\log N(-E) \le -\frac{1}{C} E^{\frac{2p-d}{2p}} \log E, \tag{5.13}$$

and, for any $n \ge 1$ and $E > 1$, we have

$$\log \mathbb{E}(N_{\omega,n}(-E)) \le -\frac{1}{C} E^{\frac{2p-d}{2p}} \log E. \tag{5.14}$$

Proof. Equations (5.14) and (5.13) are proved along the same lines. We will only write the details for (5.13). By [7], we know that, for any cube Λ and $E > 0$, we have

(5.15)

$$\begin{aligned} N(-E) &\le \mathbb{E}\left(\frac{1}{\mathrm{Vol}(\Lambda)} \sharp\{\text{eigenvalues of } H_{\omega,\Lambda}^N \text{ smaller than or equal to } -E\} \right) \\ &= \frac{1}{\mathrm{Vol}(\Lambda)} \sum_{n\in\mathbb{N}} \mathbb{E}\left(\mathbf{1}_{\{\omega;\ m(\omega,\Lambda)=n\}} \sharp\{\text{eigenvalues of } H_{\omega,\Lambda}^N \text{ smaller than or equal to } -E\} \right) \end{aligned}$$

where $H^N_{\omega,\Lambda}$ denotes the restriction of H_ω to Λ with Neumann boundary conditions. One has the

Lemma 5.3. *Pick $p > d/2$. If $V \in L^p(\mathbb{R}^d)$ then, for $\alpha > \dfrac{d}{2p}$ and $E > 0$, $(E-\Delta)^{-\alpha/2}V(E-\Delta)^{-\alpha/2}$ is bounded on $L^2(\mathbb{R}^d)$ and for some $C_{\alpha,p} > 0$, one has*

$$\|(E-\Delta)^{-\alpha/2}V(E-\Delta)^{-\alpha/2}\|_{\mathcal{L}(L^2)} \leq C_{\alpha,p}\|V\|_{L^p(\mathbb{R}^d)}E^{-\alpha+d/2p}. \tag{5.16}$$

Proof. Using classical properties of the Fourier transform from L^p to L^q ($q^{-1}+p^{-1}=1$ and $1 \leq p \leq 2$) (see e.g. [6]), one shows $(E-\Delta)^{-\alpha/2}$ is bounded from $L^2(\mathbb{R}^d)$ to $L^r(\mathbb{R}^d)$ for $\dfrac{1}{r} < \dfrac{\alpha}{2} + \dfrac{1}{d}$ with the bound

$$\|(E-\Delta)^{-\alpha/2}\|_{L^2\to L^r} \leq C_{\alpha,r}E^{-\frac{\alpha}{2}+\frac{d(2-r)}{4r}}.$$

One concludes using duality and Hölder's inequality.

□

Then, by Lemma 5.3, there exists $C > 0$ such that, for $\phi \in H^1(\mathbb{R}^d)$ and $\epsilon > 0$, we have

$$\langle |V|\phi, \phi\rangle \leq \epsilon\|\nabla\phi\|^2_{L^2(\mathbb{R}^d)} + C\epsilon^{\frac{d}{d-2p}}\|\phi\|^2_{L^2(\mathbb{R}^d)}. \tag{5.17}$$

This identity can be carried over to $H^1(\Lambda)$ in the following way. Let $\phi \in H^1(\Lambda)$; we can extend ϕ to $\mathbb{R}^d$ and we denote this extension by $\tilde{\phi}$ (see [29, 32]). By definition, $\tilde{\phi}_{|\Lambda} = \phi$. Moreover we know that, for some $C > 0$,

$$\|\tilde{\phi}\|_{L^2(\mathbb{R}^d)} \leq C_0\|\phi\|_{L^2(\Lambda)} \text{ and } \|\nabla\tilde{\phi}\|_{L^2(\mathbb{R}^d)} \leq C_0\|\nabla\phi\|_{L^2(\Lambda)} \tag{5.18}$$

Pick $x_0 \in \mathbb{R}^d$ and define $V_{x_0}(x) = V(x-x_0)$. Then, for $\phi \in H^1(\Lambda)$ and $\epsilon > 0$, by (5.17)

$$\begin{aligned}\langle |V_{x_0}|\phi, \phi\rangle_{L^2(\Lambda)} &= \langle |V_{x_0}|\mathbf{1}_\Lambda\tilde{\phi}, \tilde{\phi}\rangle_{L^2(\mathbb{R}^d)} \\ &\leq \epsilon\|\nabla\tilde{\phi}\|^2_{L^2(\mathbb{R}^d)} + C\epsilon^{\frac{d}{d-2p}}\|\tilde{\phi}\|^2_{L^2(\mathbb{R}^d)} \\ &\leq C_0\epsilon\|\nabla\phi\|^2_{L^2(\Lambda)} + C_0C\epsilon^{\frac{d}{d-2p}}\|\phi\|^2_{L^2(\Lambda)}\end{aligned}$$

Hence, there exists $C > 0$ such that, for any $x_0 \in \mathbb{R}^d$, $\epsilon > 0$ and $\phi \in H^1(\Lambda)$, we have

$$\langle |V_{x_0}|\phi, \phi\rangle_{L^2(\Lambda)} \leq \epsilon\|\nabla\phi\|^2_{L^2(\Lambda)} + C\epsilon^{\frac{d}{d-2p}}\|\phi\|^2_{L^2(\Lambda)} \tag{5.19}$$

Now pick ω such that $m(\omega,\Lambda) = n$. By (5.19), for $\phi \in H^1(\Lambda)$, we have

$$\langle V_\omega\phi, \phi\rangle_{L^2(\Lambda)} \geq -n\epsilon\|\nabla\phi\|^2_{L^2(\Lambda)} - Cn\epsilon^{\frac{d}{d-2p}}\|\phi\|^2_{L^2(\Lambda)}.$$

Pick $\epsilon = \dfrac{1}{2n}$ to get

$$\langle(-\Delta + V_\omega)\phi, \phi\rangle_{L^2(\Lambda)} \geq \frac{1}{2}\|\nabla\phi\|^2_{L^2(\Lambda)} - Cn^{\frac{2p}{2p-d}}\|\phi\|^2_{L^2(\Lambda)}. \tag{5.20}$$

Hence

$$H^N_{\omega,\Lambda} \geq -Cn^{\frac{2p}{2p-d}}. \tag{5.21}$$

As a direct consequence of (5.20), we get that

$$
\begin{aligned}
&\sharp\{\text{eigenvalues of } H^N_{\omega,\Lambda} \text{ smaller than or equal to } -E\}\\
&\le \sharp\{\text{eigenvalues of } -\Delta^N_\Lambda \text{ smaller than or equal to } Cn^{\frac{2p}{2p-d}} - E\}\\
&\le C\mathrm{Vol}(\Lambda) n^{\frac{2p}{2p-d}}
\end{aligned}
\tag{5.22}
$$

Plugging this into (5.15), for some $C > 0$ (depending on Λ), we obtain

$$
\begin{aligned}
N(-E) &\le \frac{C}{\mathrm{Vol}(\Lambda)} \sum_{n \ge E^{\frac{2p-d}{2p}}/C} \mathbb{E}\left(\mathbf{1}_{\{\omega;\ m(\omega,\Lambda)=n\}} \mathrm{Vol}(\Lambda) n^{\frac{2p}{2p-d}}\right)\\
&\le C \sum_{n \ge E^{\frac{2p-d}{2p}}/C} \frac{n^{\frac{2p}{2p-d}}(\mu\mathrm{Vol}(\Lambda))^n}{n!}\\
&\le C \frac{E(\mu\mathrm{Vol}(\Lambda))^{E^{\frac{2p-d}{2p}}/C}}{(E^{\frac{2p-d}{2p}}/C)!}.
\end{aligned}
$$

This ends the proof of Lemma 5.2.

□

5.3. **Exponential decay estimates.** One has

Lemma 5.4. *Let $\alpha \in (0,1)$,$p > d/2$ and $q > d/2$. Then, there exists $C_{\alpha,p,q} > 0$ and $\epsilon_{\alpha,p,q} > 0$ such that, for any V, a function that satisfies*

$$V = V^i + V^r$$

where

- *$V^i \in L^p_{loc,unif}$ and $\|V^i\|_{L^p_{loc,unif}} \le b$ (for some $b > 0$).*
- *for some $K > 0$, V^r satisfies*

$$| V^r(x) | \le K(|x|^\alpha + 1) \text{ for all } x \in \mathbb{R}^d$$

Then, there exists $C_b > 0$ such that, for any $(\gamma,\gamma') \in \mathbb{Z}^d \times \mathbb{Z}^d$ and $z \in \mathbb{C} \setminus \mathbb{R}$, one has

$$\|\mathbf{1}_{C(\gamma,1)}(z - (-\Delta + V))^{-1}\chi_{C(\gamma',1)}\|_{\mathcal{T}_q} \le C_{\alpha,p,q} \frac{(1+|\gamma'|)^\alpha}{\eta(z,N,b)} e^{-\epsilon_{\alpha,p,q}\cdot\eta(z,K,b)|\delta_\alpha(\gamma)-\delta_\alpha(\gamma')|} \tag{5.23}$$

$$\text{where } \delta_\alpha(x) = (1+x^2)^{(1-\alpha)/2} \text{ and } \eta(z,K,b) = \frac{|Imz|}{|z|+K+C_b}.$$

Here $\|\cdot\|_{\mathcal{T}_q}$ denotes the norm in the q-th Schatten class and $dist(z,z')$ denotes the distance in $\mathbb{C}$.

Up to small modifications, the proof of this result is the same as the proof of Lemma 4.1 in [9].

5.4. **Some useful facts about the single site potential Hamiltonian H_g.** Recall that, for $g \in \mathbb{R}$, in section 1.2, we defined $H(g) = -\Delta + gV$ and $-E_-(g)$ to be the infimum of the spectrum of $H(g)$. It is well known that, for g large enough, $-E_-$ is a simple eigenvalue ([23, 26]); hence it is analytic in g. Moreover it is convex (by the variational principle). Its first and second derivative are positive. Let φ_g be the unique positive normalized ground state associated to $-E_-(g)$. Then the eigenvalue equation gives

$$\|\nabla\varphi_g\|^2 + g\langle V\varphi_g, \varphi_g\rangle = -E_-(g).$$

Hence $E_-'(g) = -\langle V\varphi_g, \varphi_g\rangle$; so that $E_-(g)$ satisfies

$$\|\nabla\varphi_g\|^2 + E_-(g) = gE_-'(g). \tag{5.24}$$

Let g be an inverse of E_- in a neighborhood of $+\infty$. Then, one has the following

Lemma 5.5. *Let $V \in L^p(\mathbb{R}^d)$ where p is chosen as in assumption 2. Then, there exists $C > 0$ such that, for g and E sufficiently large, one has*

$$\frac{1}{C}g \leq E_-(g) \leq Cg^{\frac{2p}{2p-d}}. \tag{5.25}$$

$$\frac{E_-(g)}{g} \text{ is bounded if and only if } V_- \text{ is bounded.} \tag{5.26}$$

In this case, we have: $E_-(g) \underset{g\to+\infty}{\sim} \|V_-\|_\infty g.$

$$\exists R > 0 \text{ such that } \int_{|x|>R} |\varphi_g|^2 dx + \int_{|x|>R} |\nabla\varphi_g|^2 dx \to 0 \text{ as } g \to +\infty. \tag{5.27}$$

$$\|\nabla\varphi_g\|^2 \leq Cg^{\frac{2p}{2p-d}}. \tag{5.28}$$

$$\frac{1}{C}E^{\frac{2p-d}{2p}} \leq g(E) \leq CE. \tag{5.29}$$

$$\exists C > 0,\ E_0 > 0 \text{ such that } \forall a > 0,\ \forall E > E_0, \text{ one has}$$

$$g(E) \leq g(E+a) \leq g(E) + Ca. \tag{5.30}$$

Proof. The two sided bound (5.25) is an immediate corollary of (5.17) and of assumption H3. The definition of g and (5.25) give (5.29). The proof of (5.26) is easy and left to the reader.

By equations (5.24) and (5.25), for g large enough, we have

$$E_-'(g) \geq \frac{1}{g}E_-(g) \geq \frac{1}{C}.$$

Integrating this relation, we get

$$E_-(g+k) - E_-(g) \geq \frac{k}{C}.$$

Hence, as g is increasing, for E large enough and $a > 0$, we have

$$g(E) \leq g(E+a)) \leq g(E_-(g(E)+Ca)) = g(E) + Ca.$$

This completes the proof of (5.30). Let us now prove (5.28). As E'_- is increasing, for $\varepsilon > 0$, we have

$$E_-(g(1+\varepsilon)) - E_-(g) \geq \varepsilon g E'_-(g) \geq \varepsilon \|\nabla \varphi_g\|^2$$

by equation (5.24). (5.28) is then an immediate consequence of (5.25).
Let us now prove (5.27). We will distinguish two cases when V_- is bounded and when it is not. Let us start with assuming V_- bounded and let v_- be its essential infimum. Then, by (5.26), as $g \to \infty$, we have

$$\|\nabla \varphi_g\|^2 + g \int V |\varphi_g|^2 dx = g v_- + o(g).$$

Hence,

$$0 \leq \int (V - v_-)|\varphi_g|^2 dx \leq o(1). \tag{5.31}$$

By assumptions H1 and H2, there exists $\delta > 0$ such that for $|x| \geq 1/\delta$, $V(x) - v_- \geq \delta$. Then, as $V - v_-$ is non negative, equation (5.31) tells us that

$$\int_{|x| \geq 1/\delta} |\varphi_g|^2 dx \leq o(1).$$

If V is not bounded from below, let χ be a $\mathcal{C}_0^\infty$ cut-off for the cube $C(0,R)$. The eigenvalue equation for φ_g gives us

$$(-\Delta - E(g))[(1-\chi)\varphi_g] = gV(1-\chi)\varphi_g + 2\nabla\chi \cdot \nabla\varphi_g + \Delta\chi\varphi_g$$

So that

$$(1-\chi)\varphi_g = a_1 + a_2 + a_3$$

where

$$\begin{aligned} a_1 &= g(-\Delta - E(g))^{-1}[V(1-\chi)\varphi_g], \\ a_2 &= 2(\Delta + E(g))^{-1}[\nabla\chi \cdot \nabla\varphi_g], \\ a_3 &= (\Delta + E(g))^{-1}[\Delta\chi\varphi_g]. \end{aligned}$$

For R large enough, V is bounded on the support of $1-\chi$; this and (5.26) implies that $\|a_1\| + \|a_3\| \to 0$ as $g \to +\infty$. We write

$$a_2 = 2\left((\Delta + E(g))^{-1}[\Delta, \nabla\chi] + \nabla\chi\right)(\Delta + E(g))^{-1}\nabla\varphi_g$$

As $(\Delta + E(g))^{-1}\nabla \to 0$ in L^2-operator norm when $g \to 0$, we have $\|a_2\| \to 0$ as $g \to +\infty$. Hence $\|(1-\chi)\varphi_g\| \to 0$ as $g \to +\infty$.

We compute

$$\begin{aligned} \|\nabla[(1-\chi)\varphi_g]\|^2 &= 2\langle (1-\chi)\nabla\varphi_g, \varphi_g\nabla\chi\rangle + \langle \Delta\chi\varphi_g, \chi\varphi_g\rangle \\ &= 2\langle \nabla[(1-\chi)\varphi_g], \varphi_g\nabla\chi\rangle - 2\langle (\nabla\chi)^2\varphi_g, \varphi_g\nabla\chi\rangle + \langle \Delta\chi\varphi_g, \chi\varphi_g\rangle. \end{aligned}$$

For R sufficiently large, by what we have just proved, the last two terms in the equation above tend to 0 as $g \to +\infty$. Using the Cauchy-Schwartz inequality, this implies

$$\|\nabla[(1-\chi)\varphi_g]\|^2 \leq 2\|\nabla[(1-\chi)\varphi_g]\| \|\varphi_g\nabla\chi\| + o(1)$$

As $\|\varphi_g \nabla\chi\| \to 0$ when $g \to +\infty$, we get that $\|\nabla[(1-\chi)\varphi_g]\| \to 0$ when $g \to +\infty$. This completes the proof of equation (5.27) hence of Lemma 5.5.

$\square$

As a corollary of this lemma, we get

Lemma 5.6. *Let $V \in L^p(\mathbb{R}^d)$ where p is chosen as in assumption 2. Then, there exists $R_0 > 0$ such that, if $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, $0 \le \chi \le 1$, $\chi(x) = 1$ if $|x| \le R_0$ and $\chi(x) = 0$ if $|x| \ge 2R_0$, then $\psi_g = \chi \cdot \varphi_g / \|\chi \cdot \varphi_g\|$ is an asymptotic ground state and we have*

$$\left\{\alpha > 0;\ \lim_{g\to+\infty} \sup_{|a|\le g^{-\alpha}} \left[\frac{g|\langle(\tau_a V - V)\psi_g, \psi_g\rangle|}{E_-(g)}\right] = 0\right\} \neq \emptyset.$$

Proof. If we pick χ as above for R_0 large enough, by Lemma 5.5, we know that

$$\|(1-\chi)\varphi_g\| + \|(1-\chi)\nabla\varphi_g\| \to 0 \text{ as } g \to +\infty \tag{5.32}$$

Then, as φ_g is the ground state of $H(g)$, we compute

$$\begin{aligned}\langle (H(g) - E(g))\psi_g, \psi_g\rangle &= \langle (H(g)-E(g))(1-\chi)\varphi_g, (1-\chi)\varphi_g\rangle \\ &= -E(g)\|(1-\chi)\varphi_g\|^2 + \langle \nabla\chi\nabla\varphi_g, (1-\chi)\varphi_g\rangle \\ &\qquad + \langle \Delta\chi\nabla\varphi_g, (1-\chi)\varphi_g\rangle.\end{aligned}$$

Equation (5.32) tells us that ψ_g is an asymptotic ground state as $\|\chi \cdot \varphi_g\| \to 1$ when $g \to +\infty$.

Define $\phi_g = (E_-(g) - \Delta)^{1/2}\varphi_g$. Then, by (5.25) and (5.28), we have

$$\|\phi_g\|^2 \le C g^{\frac{2p}{2p-d}}.$$

Pick $a \in \mathbb{R}^d$ and write

$$\langle(\tau_a V - V)\varphi_g, \varphi_g\rangle = \langle \Gamma(g)\phi_g, \phi_g\rangle$$

where

$$\Gamma(g) = (E_-(g) - \Delta)^{-1/2}(\tau_a V - V)(E_-(g)-\Delta)^{-1/2}.$$

We now estimate the norm of $\Gamma(g)$. We write

$$\begin{aligned}\Gamma(g) = (\tau_a - 1)(E_-(g)-\Delta)^{-1/2} V (E_-(g)-\Delta)^{-1/2}\tau_{-a} + \\ + (E_-(g)-\Delta)^{-1/2} V (E_-(g)-\Delta)^{-1/2}(\tau_{-a} - 1)\end{aligned}$$

Hence, for $0 < \delta$ such that $-1 + 2\delta + d/2p < 0$, by Lemma 5.3, we have

$$\begin{aligned}\|\Gamma(g)\| &\le 2\|(\tau_a - 1)(E_-(g)-\Delta)^{-\delta}\|\|(E_-(g)-\Delta)^{-1/2+\delta} V (E_-(g)-\Delta)^{-1/2}\| \\ &\le C[E_-(g)]^{-1+\delta+d/2p}\left(|a|E_-(g)\right)^\delta\end{aligned}$$

writing $(\tau_a - 1)(E_-(g)-\Delta)^{-\delta}$ as a Fourier multiplier. Hence

$$\sup_{|a|\le g^{-\alpha}} \left[\frac{g|\langle(\tau_a V - V)\varphi_g, \varphi_g\rangle|}{E_-(g)}\right] \le C g [E_-(g)]^{-2+2\delta+d/2p} g^{-\alpha\delta + \frac{2p}{2p-d}}.$$

Using (5.25) and picking α large enough, we get

$$\lim_{g\to+\infty} \sup_{|a|\leq g^{-\alpha}} \left[\frac{g|\langle(\tau_a V - V)\varphi_g, \varphi_g\rangle|}{E_-(g)}\right] = 0.$$

Now using equations (5.25), (5.32) and the fact that, for R_0 large enough, $\tau_a V - V$ is bounded outside $\{|x| \leq 2R_0\}$, we get

$$\lim_{g\to+\infty} \sup_{|a|\leq g^{-\alpha}} \left[\frac{g|\langle(\tau_a V - V)\psi_g, \psi_g\rangle|}{E_-(g)}\right] = 0.$$

This ends the proof of Lemma 5.6.

□

To end this section we describe the ground state energy of $H(g)$ and an asymptotic ground state in the case when V satisfies assumption H1".
For $1 \leq i \leq q$, define $H_g^i = -\Delta + gV_i$ where V_i is defined in equation (1.10). H_g^i is a form bounded perturbation of $-\Delta$ with relative bound 0. By the homogeneity properties of V_i, when $E_i < 0$ (E_i is the ground state energy of H_1^i), it is obvious that the ground state energy of H_g^i is given by

$$E_i(g) = E_i g^{\frac{2}{2-\nu_i}}. \tag{5.33}$$

Define $E_\dagger(g) = \inf E_i(g)$. Moreover, if φ_i is the ground state of H_1^i then $\varphi_g^{(i)}$, the ground state of H_g^i has the following form

$$\varphi_g^{(i)}(x) = g^{\frac{d}{2(2-\nu_i)}} \varphi_i\left(g^{\frac{1}{2-\nu_i}} x\right) \tag{5.34}$$

We have

Lemma 5.7. *Assume V satisfies assumptions H1 and H1". Then, we have*

$$E(g) \underset{E\to+\infty}{\sim} E_\dagger(g).$$

Moreover, if $1 \leq i_0 \leq q$ is such that $E_{i_0}(g) = E_\dagger(g)$, then, if χ_{i_0} is a $\mathcal{C}_0^\infty$ cut-off function for a sufficiently small neighborhood of x_{i_0}, $\psi_g^{(i_0)}(\cdot) = \chi_{i_0}\varphi_g^{(i_0)}(\cdot - x_{i_0})/\|\chi_{i_0}\varphi_g^{(i_0)}(\cdot - x_{i_0})\|$ is an asymptotic ground state for $H(g)$.

Proof. Fix $\varepsilon > 0$ small. As in section 4.3, at the cost of adding a term to V_1, we may assume that the functions W_i stay non-negative and smaller than $1 + \varepsilon$ and that the support of the W_i are two by two disjoint. For $1 \leq i \leq q$, let χ_i be a $\mathcal{C}_0^\infty$ cut-off function of a neighborhood of x_i such that, on the support of χ_i, the function $W_i(\cdot - x_i)$ stays larger than $1-\varepsilon$. Define $\chi_0^2 = 1 - \sum_{1\le i\le q} \chi_i^2$. Pick $\varphi \in H^1(\mathbb{R}^d)$. Then

$$
\begin{aligned}
\langle H(g)\varphi,\varphi\rangle &= \sum_{i=0}^{q} \langle -\Delta \chi_i^2 \varphi,\varphi\rangle + \sum_{i=0}^{q} g\langle V\chi_i^2\varphi,\varphi\rangle \\
&= \sum_{i=0}^{q} \|\nabla(\chi_i\varphi)\|^2 - 2\mathrm{Re}\langle \chi_i\nabla\varphi,\varphi\nabla\chi_i\rangle - \langle |\nabla\chi_i|^2\varphi,\varphi\rangle \\
&\qquad + \sum_{i=1}^{q} g\langle \tau_{x_i}(W_iV_i)\chi_i^2\varphi,\varphi\rangle + g\langle V_0\varphi,\varphi\rangle \qquad (5.35)\\
&\ge (1-\varepsilon)\sum_{i=1}^{q}\left(\|\nabla(\chi_i\varphi)\|^2 + \frac{g}{1-\varepsilon}\langle \tau_{x_i}(W_iV_i)\chi_i\varphi,\chi_i\varphi\rangle\right) \\
&\qquad + g\langle V_0\varphi,\varphi\rangle + \left(1-\frac{1}{\varepsilon}\right)\sum_{i=0}^{q}\|\nabla\chi_i\cdot\varphi\|^2
\end{aligned}
$$

where V_0 is some bounded potential (depending on ε). On the other hand, on the support of χ_i, we have

$$W_iV_i \ge V_i - \varepsilon|V|_i \text{ where } |V|_i(\cdot) := \frac{|h_i(\cdot)|}{|\cdot|^{\nu_i}}$$

Hence

$$
\begin{aligned}
&\|\nabla(\chi_i\varphi)\|^2 + \frac{g}{1-\varepsilon}\langle \tau_{x_i}(W_iV_i)\chi_i^2\varphi,\varphi\rangle \ge \\
&\qquad \ge (1-\varepsilon)\|\nabla(\chi_i\varphi)\|^2 + \frac{g}{1-\varepsilon}\langle \tau_{x_i}(V_i)\chi_i\varphi,\chi_i\varphi\rangle + \\
&\qquad\qquad + \varepsilon\left(\|\nabla(\chi_i\varphi)\|^2 - \frac{g}{1-\varepsilon}\langle \tau_{x_i}(|V|_i)\chi_i\varphi,\chi_i\varphi\rangle\right).
\end{aligned}
$$

By equation (5.35), we get that

$$\|\nabla(\chi_i\varphi)\|^2 + \frac{g}{1-\varepsilon}\langle \tau_{x_i}(W_iV_i)\chi_i^2\varphi,\varphi\rangle \ge (1+\varepsilon C)E_i\left(\frac{g}{1-\varepsilon}\right)\|\chi_i\varphi\|^2 \qquad (5.36)$$

for some constant C (independent of ε) given by the lowest eigenvalue of $-\Delta - g|V|_i$. Here we used the fact that this eigenvalue has the same growth rate in g as $E_i(g)$; indeed, in the present case, this growth rate only depends on the homogeneity properties of the potential $|V|_i$ (as can be seen by a scaling argument).
Putting equations (5.36) and (5.35) together, we obtain that, there exists $C>0$, such that, for any $\varepsilon>0$ small and $\varphi\in H^1(\mathbb{R}^d)$, we have

$$\langle H(g)\varphi,\varphi\rangle \ge \left[(1-\varepsilon)E_\dagger\left(\frac{g}{1-\varepsilon}\right) + O(g)\right]\|\varphi\|^2.$$

This proves that

$$\limsup_{g\to+\infty}\frac{E(g)}{E_\dagger(g)} \le 1.$$

Pick $1 \leq i_0 \leq q$ such that $E_{i_0}(g) = E_\dagger(g)$ and χ a $\mathcal{C}_0^\infty$ cut-off function for a sufficiently small neighborhood of x_{i_0}. By the definition of $\varphi_g^{(i_0)}$, it is immediate that $\|\chi\varphi_g^{(i_0)}\| \to 1$ as $g \to +\infty$. An immediate computations gives that

$$\lim_{g\to+\infty} \frac{\langle H(g)\chi\varphi_g^{(i_0)}, \chi\varphi_g^{(i_0)}\rangle}{E_{i_0}(g)} = 1$$

This implies that

$$\liminf_{g\to+\infty} \frac{E(g)}{E_\dagger(g)} \geq 1.$$

Hence it completes the proof of Lemma 5.7.

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