Determining functionals for random partial differential equations

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Abstract

Determining functionals are tools to describe the finite dimensional long-term dynamics of infinite dimensional dynamical systems. There also exist several applications to infinite dimensional *random* dynamical systems. In these applications the convergence condition of the trajectories of an infinite dimensional random dynamical system with respect to a finite set of linear functionals is assumed to be either in mean or *exponential* with respect to the convergence almost surely. In contrast to these ideas we introduce a convergence concept which is based on the convergence in probability. By this ansatz we get rid of the assumption of exponential convergence. In addition, setting the random terms to zero we obtain usual deterministic results.
We apply our results to the 2D Navier - Stokes equations forced by a white noise.

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1 Introduction

The question of the number of parameters that are necessary for the description of the long-term behaviour of solutions to nonlinear partial differential equations was first discussed by Foias and Prodi [14] and by Ladyzhenskaya [20] for the deterministic 2D Navier-Stokes equations. They proved that the asymptotic behaviour of the solutions is completely determined by the dynamics of the first N Fourier modes, if N is sufficiently large. After [14] and [20] similar results were obtained for other parameters and other deterministic equations and a general approach to the problem of the existence of a finite number of determining parameters was developed (see [7, 8, 16] and the literature quoted therein).

Assume that we have a dynamical system with the phase state H and the evolution operator S_t. Roughly speaking the general problem on the existence of finite sets of determining parameters (functionals) can be stated (cf. [7, 8]) as follows: find the conditions on a finite set $\{l_j : j = 1, ..., N\}$ of functionals on H which guarantees that the convergence (in certain sense)

$$\max_j |l_j(S_t u_1 - S_t u_2)| \to 0 \quad \text{when} \quad t \to +\infty,\ u_1, u_2 \in H$$

implies that $S_t u_1 - S_t u_2 \to 0$ in some topology of the phase space H. Besides from applied point of view it is also important to find bounds for the number N of determining functionals (in the sense above) and to describe families of functionals with minimal N.

The deterministic theory of determining functionals was developed by many authors (see, e.g. [7, 8] and the references therein). Similar problems for stochastic systems were also discussed in [6, 4, 12, 9]. In papers [6, 4, 12] ω-wise approach to construction of determining functionals were developed. However in these papers it was assumed that either (see [6, 4, 12]) the nonlinear term in the equation is a globally Lipschitz mapping or (see [4, 12]) one of initial data belongs to the random attractor. The mode of convergence for functionals and trajectories is the convergence almost surely in these papers. Moreover in the papers [4, 12] the authors assume that the functionals of the difference of two solutions go to zero exponentially fast. Then they prove that some norm of the difference of these solutions tends to zero with exponential speed which is less than the speed of convergence of the functionals. On the other hand the approach presented in [9] does not assume these conditions, and it relies on some estimates exponential moments of solutions and deals with convergence in mean. The speed of convergence to zero of the functionals and of the norms are the same as in [9].

In contrast to [6, 4, 12, 9] we consider determinig functionals with respect to the convergence in probability. Using such determining functionals we can avoid the assumption that the images (under linear functionals) of the trajectories converge exponentially fast. In particular, our approach recovers the deterministic results when we remove the stochastic terms. Another advantage is that we do not have to assume that one trajectory must be contained in the random attractor. Finally, we mention that the convergence in probability is quite natural for RDS, see [11, 2].

In Section 2 we consider an abstract setting of random dissipative systems and prove two existence theorem of finite sets of determining functionals in the sense of the definition given below for arbitrary initial data. These theorems show two different approaches to the construction of determining functionals. In Section 3 we apply the results of Section 2 to 2D Navier - Stokes equations subject to additive white noise. We prove the existence of determining functionals for this problem *without* the assuming that one of solutions belongs to the attractor.

2 The existence of determining functionals

We consider a *random dynamical system* (REDS) which consists of two components. The first component is a *metric dynamical system* $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ as a model for a noise, where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and θ is a $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}), \mathcal{F}$ measurable flow: we have

$$\theta_0 = \mathrm{id}, \qquad \theta_{t+\tau} = \theta_t \circ \theta_\tau =: \theta_t \theta_\tau$$

for $t, \tau \in \mathbb{R}$. The measure $\mathbb{P}$ is supposed to be ergodic with respect to θ. The second component of a random dynamical system is a $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F} \otimes \mathcal{B}(H), \mathcal{B}(H)$-measurable mapping φ satisfying the *cocycle* property

$$\varphi(t+\tau, \omega, x) = \varphi(t, \theta_\tau \omega, \varphi(\tau, \omega, x)), \qquad \varphi(0, \omega, x) = x,$$

where the phase space H is a separable metric space and x is chosen arbitrarily in H. We will denote this RDS by symbol φ.
A standard model for such a noise θ is the twosided *Brownian motion*: Let U be a separable Hilbert space. We consider the probability space

$$(C_0(\mathbb{R}, U), \mathcal{B}(C_0(\mathbb{R}, U)), \mathbb{P})$$

where $C_0(\mathbb{R}, U)$ is the Fréchet space of continuous functions on $\mathbb{R}$ which are zero at zero and $\mathcal{B}(C_0(\mathbb{R}, U))$ is the corresponding Borel σ-algebra. Suppose that we have a covariance operator Q on U. Then $\mathbb{P}$ denotes the *Wiener measure* with respect to Q. Note that $\mathbb{P}$ is ergodic with respect to the flow

$$\theta_t \omega = \omega(\cdot + t) - \omega(t), \qquad \text{for } \omega \in C_0(\mathbb{R}, U).$$

For detailed presentation of random dynamical systems we refer to the monograph by L. Arnold [1].

On a $V \subset H \subset V'$ rigged Hilbert space with compact embedding $V \subset H$ and duality mapping $\langle \cdot, \cdot \rangle$ we investigate RDS φ generated by the evolution equation

$$\frac{du}{dt} + Au = F(u, \theta_t \omega), \quad u(0) = x, \tag{1}$$

over some metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$. Here A is a positive self-adjoint operator in H such that $D(A^{1/2}) = V$, where $D(B)$ denotes the domain of the operator B. We suppose that V is equipped with the norm $\|\cdot\|_V = \|A^{1/2}\cdot\|_H$. We also assume that the nonlinear mapping F from $V \times \Omega$ into H is such that $F(u, \omega)$ is measurable for any fixed $u \in V$ and subordinate (in the sense of (3)) to the operator A. We suppose that the solution $u(t, \omega)$ of the problem (1) is unique and depends measurably on (t, ω, x). Then the operator

$$(t, \omega, x) \to u(t, \omega), \qquad u(0, \omega) = x$$

defines a random dynamical system (cocycle) φ. In addition, this random dynamical system is supposed to be continuous which means that

$$x \to \varphi(t, \omega, x)$$

is continuous for any (t, ω). The trajectories of this random dynamical system has to be contained in $L_{2,loc}(0, \infty; V) \cap C([0, \infty); H)$.

In the following we assume that φ is *dissipative*. It means that there exists a *compact* random set $B \subset V$ which is forward invariant:

$$\varphi(t, \omega, B(\omega)) \subset B(\theta_t \omega),\ t > 0,$$

and which is absorbing: for any $\varepsilon > 0$ and for any random variable $x(\omega) \in H$ there exists a $t_{\varepsilon,x} > 0$ such that if $t \geq t_{\varepsilon,x}$

$$\varphi(t, \omega, x(\omega)) \in B(\theta_t \omega)$$

with probability $1 - \varepsilon$. Note that B is absorbing with probability one, due to the forward invariance.
A random variable $x \geq 0$ is called tempered if

$$\lim_{t \to \pm\infty} \frac{\log^+ x(\theta_t \omega)}{|t|} = 0 \quad \text{a.s.}$$

Note that the only alternative to this property is that

$$\limsup_{t \to \pm\infty} \frac{\log^+ x(\theta_t \omega)}{|t|} = \infty \quad \text{a.s.},$$

see Arnold [1], page 164 f. We also assume that B is tempered which means that the mapping

$$\omega \to \sup_{x \in B(\omega)} \|x\|_H$$

is tempered.

We now give our basic definition:

Definition 2.1 *A set $\mathcal{L} = \{l_j,\ j = 1, \cdots, k\}$ of linear continuous and linearly independent functionals on V is called asymptotically determining in probability if*

$$(\mathbb{P}) \lim_{t\to\infty} \int_t^{t+1} \max_j |l_j(\varphi(\tau,\omega,x_1) - \varphi(\tau,\omega,x_2))|^2 d\tau \to 0$$

for two initial conditions x_1, $x_2 \in H$ implies

$$(\mathbb{P}) \lim_{t\to\infty} \|\varphi(t,\omega,x_1) - \varphi(t,\omega,x_2)\|_H \to 0.$$

As in [7, 8] we use the concept of the *completeness defect* for a description of sets of determining functionals. Assume that X and Y are Banach spaces and X continuously and densely embedded into Y. Let $\mathcal{L} = \{l_j : j = 1, ..., k\}$ be a finite set of linearly independent continuous functionals on X. We define the completeness defect $\varepsilon_{\mathcal{L}}(X,Y) \equiv \varepsilon_{\mathcal{L}}$ of the set $\mathcal{L}$ with respect to the pair of the spaces X and Y by the formula

$$\varepsilon_{\mathcal{L}} = \sup\{\|w\|_Y\ : w \in X,\ l_j(w) = 0,\ l_j \in \mathcal{L},\ \|w\|_X \leq 1\}.$$

The value $\epsilon_{\mathcal{L}}$ is proved to be very useful for characterization of sets of determining functionals (see, e.g., [7, 8] and the references therein). One can show that the completeness defect $\varepsilon_{\mathcal{L}}(X,Y)$ is the best possible global error of approximation in Y of elements $u \in X$ by elements of the form $u_{\mathcal{L}} = \sum_{j=1}^k l_j(u)\varphi_j$, where $\{\varphi_j : j = 1, \ldots, k\}$ is an arbitrary set in X. The smallness of $\varepsilon_{\mathcal{L}}(X,Y)$ is the main condition (see the results presented below) that guarantee the property of a set of functionals to be asymptotically determining. The so-called modes, nodes and local volume averages (the description of these functionals can be found in [8], for instance) are the main examples of sets of functionals with a small completeness defect. For further discussions and for other properties of the completeness defect we refer to [7, 8]. Here we only point out the following estimate

$$\|u\|_Y \leq \varepsilon_{\mathcal{L}} \cdot \|u\|_X + C_{\mathcal{L}} \cdot \max_{j=1,\ldots,k} |l_j(u)|, \quad u \in X, \tag{2}$$

where $C_{\mathcal{L}} > 0$ is a constant depending on $\mathcal{L}$.

We are now in a position to prove the first main theorem for systems introduced in (1). To do this we will use the completeness defect $\varepsilon_{\mathcal{L}} \equiv \varepsilon_{\mathcal{L}}(X,Y)$ with $H = Y$, $V = X$.

Theorem 2.2 *Let $\mathcal{L} = \{l_j : j = 1, ..., k\}$ be a set of linear continuous and linearly independent functionals on V. We assume that we have a forward absorbing and forward invariant set B in V such that $\sup_{x\in B(\omega)} \|x\|_V^2$ is bounded by a tempered random variable and $t \to \sup_{x\in B(\theta_t\omega)} \|x\|_V^2$ is locally integrable. Suppose there exist a constant $c > 0$ and a measurable function $l \geq 0$ such that for $x_1(\omega)$, $x_2(\omega) \in B(\omega)$ we have*

$$\langle -A(x_1 - x_2) + F(x_1,\omega) - F(x_2,\omega), x_1 - x_2\rangle \tag{3}$$

$$\leq -c\|x_1 - x_2\|_V^2 + l(x_1, x_2, \omega)\|x_1 - x_2\|_H^2.$$

Assume that

$$\frac{1}{m}\mathbb{E}\left\{\sup_{x_1,x_2\in B(\omega)}\int_0^m l(\varphi(t,\omega,x_1),\varphi(t,\omega,x_2),\theta_t\omega)dt\right\} < c\varepsilon_{\mathcal{L}}^{-2} \tag{4}$$

for some $m > 0$. Then $\mathcal{L}$ is a set of asymptotically determining functionals in probability for RDS (θ, φ).

Proof. 1) Without loss of generality, we only consider the case $m = 1$. That is, we assume that (4) is fulfilled for $m = 1$. Since we intend to prove convergence in probability we can suppose that the random variables $x_1(\omega)$, $x_2(\omega)$ are contained in $B(\omega)$. Such random variables exist because B is a random set, see Caistaing and Valadier [5] Chapter III. Indeed, B is *forward* absorbing such that $\varphi(t,\omega,x_i(\omega)) \in B(\theta_t\omega)$ with probability $1-\varepsilon$ for any $\varepsilon > 0$ if t is sufficiently large.

Let $w(t,\omega)$ be defined by $\varphi(t,\omega,x_1(\omega)) - \varphi(t,\omega,x_2(\omega))$. Since $\|\cdot\|_V = \|A^{1/2}\cdot\|_H$, we obtain by (3):

$$\frac{d\|w\|_H^2}{dt} + 2c\|w\|_V^2 \leq 2l(\varphi(t,\omega,x_1(\omega)),\varphi(t,\omega,x_2(\omega)),\theta_t\omega)\|w\|_H^2 .$$

We have by (2)

$$\|w\|_V^2 \geq (1+\delta)^{-1}\varepsilon_{\mathcal{L}}^{-2}\|w\|_H^2 - C_{\delta,\mathcal{L}}\max_{j=1,\cdots,k}|l_j(w)|^2.$$

for any $\delta > 0$ with appropriate positive constant $C_{\delta,\mathcal{L}}$. This allows us to write the following inequality:

$$\|w(t)\|_H^2 \leq \|w(0)\|_H^2 e^{\int_0^t q(s,\omega)ds} + C_{\delta,\mathcal{L}}\cdot\int_0^t e^{\int_s^t q(\tau,\omega)d\tau}\eta_{\mathcal{L}}(s,\omega)ds\ , \tag{5}$$

where $\eta_{\mathcal{L}}(s,\omega) = \max_j |l_j(w(s,\omega))|^2$ and

$$q(t,\omega) = 2l(\varphi(t,\omega,x_1(\omega)),\varphi(t,\omega,x_2(\omega)),\theta_t\omega) - 2c(1+\delta)^{-1}\varepsilon_{\mathcal{L}}^{-2}.$$

Let

$$Q(\omega) = \sup_{x_1,x_2\in B(\omega)}\int_0^1 2l(\varphi(t,\omega,x_1),\varphi(t,\omega,x_2),\theta_t\omega)dt - 2\,c(1+\delta)^{-1}\varepsilon_{\mathcal{L}}^{-2}$$

with $\delta > 0$ chosen such that $\mathbb{E}Q < 0$. This is possible because of (4).

2) Since B is forward invariant and $q(t,\omega) \geq -2c(1+\delta)^{-1}\varepsilon_{\mathcal{L}}^{-2}$, the first term on the right hand side of (5) can be estimated by

$$\|w(0)\|_H^2 e^{\sum_{j=0}^{[t]} Q(\theta_j\omega)} e^{2c(1+\delta)^{-1}\varepsilon_{\mathcal{L}}^{-2}}.$$

Since $\mathbb{E}Q < 0$, by the ergodic theorem we have for $t \to \infty$

$$\sum_{j=0}^{[t]} Q(\theta_j\omega) \sim ([t]+1)\mathbb{E}Q \to -\infty \tag{6}$$

which shows the convergence assertion for the first term.
We now investigate the second term in (5). Since B is forward invariant, this term can be estimated by

$$\begin{aligned} C_{\delta,\mathcal{L}} \cdot \int_0^{[t]+1} & e^{\int_{[s]}^{[t]+1} q(\tau,\omega)d\tau} \eta_{\mathcal{L}}(s,\omega)ds\, e^{4c(1+\delta)^{-1}\varepsilon_{\mathcal{L}}^{-2}} \\ & \leq C_{\delta,\mathcal{L}} \cdot e^{4c(1+\delta)^{-1}\varepsilon_{\mathcal{L}}^{-2}} \sum_{j=0}^{[t]} e^{\sum_{j'=j}^{[t]} Q(\theta_{j'}\omega)} \int_0^1 \eta_{\mathcal{L}}(s+j,\omega)ds\ . \end{aligned}$$

We use here that $l(x_1, x_2, \omega)$ is a nonnegative function. Thus we have to prove that

$$(\mathbb{P}) \lim_{k\to\infty} \sum_{j=0}^{k} \left(e^{\sum_{j'=j}^{k} Q(\theta_{j'}\omega)} \int_0^1 \eta_{\mathcal{L}}(s+j,\omega)ds \right) = 0. \tag{7}$$

We now replace ω by $\theta_{-k}\omega$ in the relation under the limit sign. It gives

$$\sum_{j=-k}^{0} e^{\sum_{j'=j}^{0} Q(\theta_{j'}\omega)} \int_0^1 \eta_{\mathcal{L}}(s+k+j,\theta_{-k}\omega)ds,$$

which is equal to

$$\sum_{j=-\infty}^{0} e^{\sum_{j'=j}^{0} Q(\theta_{j'}\omega)} \chi_k(j) \int_0^1 \eta_{\mathcal{L}}(s+k+j,\theta_{-k}\omega)ds$$

where $\chi_k(j) = 1$ if $j \geq -k$ and 0 otherwise.
3) Since we can assume that $x_i(\omega) \in B(\omega)$ there exists a tempered random variable b such that $\eta_{\mathcal{L}}(s,\omega) \leq b(\theta_s\omega)$ where $s \to b(\theta_s\omega)$ is locally integrable. Since $s \to b(\theta_s\omega)$ is tempered

$$j \to \chi_k(j) \int_0^1 \eta_{\mathcal{L}}(s+k+j,\theta_{-k}\omega) \leq \int_0^1 b(\theta_{s+j}\omega)ds$$

has a subexponential growth for any $k \geq 0$. We consider the finite measure $\mu^\omega(j) = e^{\frac{1}{2}\sum_{j'=j}^{0} Q(\theta_{j'}\omega)}\delta_j$ on $\mathbb{Z}^-$ where δ_j are Dirac measures on j. Set

$$f(k,j,\omega) := \chi_k(j) e^{\frac{1}{2}\sum_{j'=j}^{0} Q(\theta_{j'}\omega)} \int_0^1 \eta_{\mathcal{L}}(s+k+j,\theta_{-k}\omega)ds.$$

Since $j \to \int_0^1 \eta_{\mathcal{L}}(s+k+j, \theta_{-k}\omega)ds$ has a subexponential growth and

$$j \to e^{\frac{1}{2}\sum_{j'=j}^{0} Q(\theta_{j'}\omega)}$$

goes to zero exponentially fast (see (6)), there exists a constant n depending only on ω such that

$$f(k,j,\omega) \le n(\omega) \qquad \text{for any} \quad -j,\, k \in \mathbb{Z}^+. \tag{8}$$

The term $\int_0^1 \eta_{\mathcal{L}}(s+k+j,\omega)ds$ tends to zero in probability for $k \to \infty$ and fixed j. Hence, also $\int_0^1 \eta_{\mathcal{L}}(s+k+j,\theta_{-k}\omega)ds$ tends to zero in probability for $k \to \infty$ and fixed j. Let $\lambda(j) = e^{\frac{1}{4}\mathbb{E}Qj}\delta_j$, $j \in \mathbb{Z}^-$ be a finite measure on $\mathbb{Z}^-$ and $\Xi_N(\omega)$ be the indicator function of the set

$$\{\omega : e^{\frac{1}{2}\sum_{j'=j}^{0} Q(\theta_{j'}\omega)} \le N e^{\frac{1}{4}\mathbb{E}Qj} \text{ for } j \in \mathbb{Z}^-\},$$

where Ξ_N tends increasingly to one for $N \to \infty$. We set $\mu_N^\omega = \Xi_N(\omega)\mu^\omega$. For the asserted convergence we consider

$$\begin{aligned}
&\mathbb{P}(\int f(k,j,\omega)d\mu^\omega(j) > 3\delta)\\
&= \mathbb{P}(\int (f(k,j,\omega)\wedge N)d\mu_N^\omega(j) + \int f(k,j,\omega) - (f(k,j,\omega)\wedge N)d\mu_N^\omega(j)\\
&\qquad + \int f(k,j,\omega)d(\mu^\omega - \mu_N^\omega)(j) > 3\delta)\\
&\le \mathbb{P}(\int (f(k,j,\omega)\wedge N)d\mu_N^\omega(j) > \delta)\\
&\qquad + \mathbb{P}(\int f(k,j,\omega) - (f(k,j,\omega)\wedge N)d\mu^\omega(j) > \delta)\\
&\qquad + \mathbb{P}(\int f(k,j,\omega)d(\mu^\omega - \mu_N^\omega)(j) > \delta).
\end{aligned}$$

Note that by (8) the second term on the right hand side is less than ε uniformly in k if N is sufficiently large uniformly in k. The integral in the third term can be estimated by

$$\int f(k,j,\omega)d(\mu^\omega - \mu_N^\omega)(j) \le n(\omega)(\mu^\omega - \mu_N^\omega)(\mathbb{Z}^-) = n(\omega)(1 - \Xi_N(\omega))\mu^\omega(\mathbb{Z}^-).$$

Therefore this third term is less than ε for large N. To see that the first term tends to zero in probability for $k \to \infty$ and any N, we note that by the definition of the metric of the convergence in probability

$$\begin{aligned}
&\mathbb{E}\frac{\int (f(k,j,\omega)\wedge N)d\mu_N^\omega(j)}{1+\int (f(k,j,\omega)\wedge N)d\mu_N^\omega(j)} \le \mathbb{E}\int (f(k,j,\omega)\wedge N)d\mu_N^\omega(j)\\
&\quad = \int \mathbb{E}(f(k,j,\omega)\wedge N)d\mu_N^\omega(j) \le N(N+1)\int \frac{\mathbb{E}(f(k,j,\omega)\wedge N)}{1+\mathbb{E}(f(k,j,\omega)\wedge N)}d\lambda(j),
\end{aligned}$$

where the right hand side tends to zero for any $N \geq 0$ by Lebesgue's theorem. Thus the asserted convergence (7) follows. $\square$

Now we present another version of the theorem on the existence of finite number of determining functionals which can be easily applied to the random squeezing property introduced by Flandoli and Langa [12].

Theorem 2.3 *Let φ be RDS whose phase space is a Banach space H with the norm $\|\cdot\|$. Suppose that this RDS is dissipative in H with a forward invariant absorbing random set $B(\omega)$ such that the random variable $\rho(\omega) = \sup_{x \in B(\omega)} \|x\|$ is tempered and $\rho(\theta_t \omega) \in L^p_{loc}(\mathbb{R})$ for some $p \geq 1$ and all $\omega \in \Omega$. Assume that for each $\omega \in \Omega$ RDS φ possesses the following properties:*

$$\|\varphi(t, \omega, x_1) - \varphi(t, \omega, x_2)\| \leq M(\omega) \|x_1 - x_2\| \tag{9}$$

for all $t \in [0, 1]$, $x_1, x_2 \in B(\omega)$ and

$$\begin{aligned} \|\varphi(1, \omega, x_1) - \varphi(1, \omega, x_2)\| \quad &\leq \quad \mathcal{N}(\varphi(1, \omega, x_1) - \varphi(1, \omega, x_2)) \\ &+ \quad e^{\int_0^1 r(\theta_\tau \omega) d\tau} \cdot \|x_1 - x_2\| \end{aligned} \tag{10}$$

for all $x_1, x_2 \in B(\omega)$. Here $M(\omega)$ is a tempered and finite almost surely random variable, $\mathcal{N}(\cdot)$ is a positive continuous scalar function on H such that $\mathcal{N}(x) \leq C \cdot (1 + \|x\|^p)$ and $r(\omega)$ is a random variable with finite expectation such that $\mathbb{E} r < 0$. Then the condition

$$(\mathbb{P}) \lim_{n \to +\infty} \mathcal{N}(\varphi(n, \omega, x_1) - \varphi(n, \omega, x_2)) = 0 \tag{11}$$

for some $x_1, x_2 \in H$ implies that

$$(\mathbb{P}) \lim_{t \to +\infty} \|\varphi(t, \omega, x_1) - \varphi(t, \omega, x_2)\| = 0. \tag{12}$$

Proof. As above we can assume that $x_i(\omega) \in B(\omega)$. Using the cocycle property $\varphi(m, \omega) = \varphi(1, \theta_{m-1}\omega, \varphi(m-1, \omega))$ and relation (10) we obtain that

$$d_m(\omega) \leq \mathcal{N}(m, \omega) + e^{\int_{m-1}^m r(\theta_\tau \omega) d\tau} \cdot d_{m-1}(\omega),$$

where

$$\mathcal{N}(n, \omega) = \mathcal{N}(\varphi(n, \omega, x_1(\omega)) - \varphi(n, \omega, x_2(\omega)))$$

and

$$d_t(\omega) = \|\varphi(t, \omega, x_1(\omega)) - \varphi(t, \omega, x_2(\omega))\|.$$

After iterations we find that

$$d_m(\omega) \leq d_0(\omega) e^{\int_0^m r(\theta_\tau \omega) d\tau} + \sum_{j=0}^{m-1} \mathcal{N}(m-j, \omega) e^{\int_{m-j}^m r(\theta_\tau \omega) d\tau}.$$

Applying now the same arguments as in the proof of Theorem 2.2 we find that (11) implies that $(\mathbb{P})\ \lim_{m\to+\infty} d_m(\omega) = 0$. From (9) we have that $d_t(\omega) \le M(\theta_{[t]}\omega)d_{[t]}(\omega)$. Thus we should prove that $(\mathbb{P})\ \lim_{n\to+\infty} M(\theta_n\omega)d_n(\omega) = 0$. It follows from $(\mathbb{P})\ \lim_{n\to+\infty} M(\omega)d_n(\theta_{-n}\omega) = 0$. The last relation follows from the convergence $(\mathbb{P})\ \lim_{m\to+\infty} d_m(\theta_{-m}\omega) = 0$ and the properties of $M(\omega)$. □

Now following Flandoli and Langa [12] we introduce the concept of random squeezing property.

Definition 2.4 *Let φ be RDS whose phase space is a separable Hilbert space H. We say that RDS (θ, φ) satisfies a* random squeezing property *(RSP) on the random set $B(\omega)$ if there exist a finite-dimensional projector P and a random variable $r(\omega)$ with finite expectation such that $\mathbb{E}r < 0$ and for almost all $\omega \in \Omega$ we have either*

$$\|(I-P)\varphi(1,\omega,x_1) - \varphi(1,\omega,x_2)\| \le \|P\varphi(1,\omega,x_1) - \varphi(1,\omega,x_2)\|$$

or

$$\|\varphi(1,\omega,x_1) - \varphi(1,\omega,x_2)\| \le e^{\int_0^1 r(\theta_\tau\omega)d\tau} \cdot \|x_1 - x_2\|$$

for all $x_1, x_2 \in B(\omega)$.

In deterministic case a similar property is well-known for dissipative systems with finite-dimensional long-time behaviour (see, e.g. [24] and the references therein). Flandoli and Langa [12] have proved random squeezing property for a class of stochastic reaction-diffusion equations and for stochastic $2D$ Navier - Stokes equations with periodic boundary condition.

Now we are in position to state corollaries from Theorem 2.3.

Corollary 2.5 *Assume that RDS φ with Hilbert phase space H is dissipative with a forward invariant absorbing random set $B(\omega)$ satisfying the hypotheses of Theorem 2.3. Suppose that φ possesses property (9) and satisfies RSP with an orthogonal projector P. Then the property*

$$(\mathbb{P}) \lim_{n\to+\infty} \{(\varphi(n,\omega,x_1), e_i)_H - (\varphi(n,\omega,x_2), e_i)_H\} = 0, \quad i = 1, 2, \ldots, d,$$

for some $x_1, x_2 \in H$ implies (12). Here $\{e_i : i = 1, \ldots, d\}$ is a basis in the subspace PH.

Proof. It is clear that RSP implies (10) with $\mathcal{N}(u) = 2\|Pu\|$. Thus we can apply Theorem 2.3. □

This result on determining modes extends in some sense the result by Flandoli and Langa [12] for the case $k = 0$.

Corollary 2.6 *Assume that RDS φ satisfies the hypotheses of Corollary 2.5. Suppose that there exists a Banach space W such that H continuously and densely embedded into W and the projector P can be extended to continuous operator from W into H such that $\|Pu\|_H \le a_0\|u\|_W$ with a positive constant a_0. Let $\mathcal{L} = \{l_j : j = 1, ..., k\}$ be a set of linearly independent continuous functionals on H with the completeness defect $\varepsilon_{\mathcal{L}}(H, W)$ with respect to the pair of the spaces H and W. If*

$$2a_0\varepsilon_{\mathcal{L}}(H,W) < 1 \quad \textit{and} \quad \mathbb{E}r + \log\frac{1}{1-2a_0\varepsilon_{\mathcal{L}}(H,W)} < 0,$$

then the property

$$(\mathbb{P}) \lim_{n\to+\infty} \{l_i(\varphi(n,\omega,x_1)) - l_i(\varphi(n,\omega,x_2))\} = 0 \quad i = 1,2,\ldots,k,$$

for some $x_1, x_2 \in H$ implies (12).

Proof. As above RSP implies (10) with $\mathcal{N}(u) = 2\|Pu\|_H$. However using (2) with $X = H$ and $Y = W$ we have

$$2\|Pu\|_H \le 2a_0\|u\|_W \le 2a_0\varepsilon_{\mathcal{L}}(H,W)\cdot\|u\|_H + C_{\mathcal{L}}\bar{\eta}(u),$$

where $\bar{\eta}_{\mathcal{L}}(u) = \max\{|l_j(u)| \, : j = 1,\ldots,k\}$. Therefore from (10) we have

$$\begin{aligned}\|\varphi(1,\omega,x_1) \;-\; \varphi(1,\omega,x_2)\|_H &\le \frac{C_{\mathcal{L}}}{1-2a_0\varepsilon_{\mathcal{L}}(H,W)}\cdot\bar{\eta}(\varphi(1,\omega,x_1)-\varphi(1,\omega,x_2)) \\ &+ \exp\left\{\int_0^1 r(\theta_\tau\omega)d\tau + \log\frac{1}{1-2a_0\varepsilon_{\mathcal{L}}(H,W)}\right\}\cdot\|x_1-x_2\|_H.\end{aligned}$$

Thus we can apply Theorem 2.3. $\square$

Remark 2.7 The space W with the properties listed in Corollary 2.6 can be easily constructed in the following situation. Assume that A is a positive self-adjoint operator in H with compact resolvent. Let $0 < \lambda_1 \le \lambda_2 \le \ldots$ be the corresponding eigenvalues. If P is orthogonal projector on the first k eigenvectors of A, then we have $\|Pu\|_H \le \lambda_{k+1}^s\|A^{-s/2}u\|_H$ for any $s > 0$. Thus we can choose W as a completion of H with respect to the norm $\|A^{-s/2}\cdot\|_H$ for some positive s.

Remark 2.8 We point out the essential difference between Theorem 2.2 and Corollary 2.6. This corollary relies on the random squeezing property. For problems like (1) this property is usually proved in the main space H. Therefore the corollary mentioned is applied to functionals on H only. However in the case of Theorem 2.2 the functionals are defined on V with $V \subset H$. Thus Theorem 2.2 admits more singular functionals in comparison with Corollary 2.6. On the other hand Corollary 2.6 requires convergence of functionals on the discrete sequence of times $t_n = n$. We note that as in the deterministic case (see [8]) it is also possible to consider more general sequences $\{t_n\}$.

3 Application to the 2D stochastic Navier-Stokes equations

We consider the stochastic Navier-Stokes equations

$$dv = (-\nu Av + \tilde{F}(v))dt + dw, \tag{13}$$

where

$$A = -\frac{1}{2}\Delta, \quad \tilde{F}(v) = -\frac{\nu}{2}\Delta - (v, \nabla)v + f,$$

as an evolution equation on the rigged Hilbert space $V \subset H \subset V'$ where $V = \{u \in \overset{\circ}{W}{}_2^1(D), \operatorname{div} u = 0\}$ and $H = \overline{V}^{L_2(D)}$ is the closure of V in $L^2(D)$. Here D is a bounded domain with sufficiently smooth boundary ∂D in $\mathbb{R}^2$, $f \in H$ and $\nu > 0$ is a constant. We supplement the Navier-Stokes equations with no-slip or zero Dirichlet boundary condition $v|_{\partial D} = 0$. We suppose w is a Wiener process in the space H^2 with covariance Q such that $\mathrm{tr}_{H^2} Q < \infty$. Here and below we denote by H^s the domain of the operator $A^{s/2}$, $s > 0$. We obviously have $V = H^{1/2}$. In the space V we will use the norm $\|\cdot\|_V := \|\nabla \cdot\|_H = \sqrt{2}\|A^{1/2}\cdot\|_H$.
For different ideas to treat this problem, one can find in [11], [10], [15], [3], [25].
We now transform this stochastic equation to a random equation as in (1). To do this we need a stationary Ornstein - Uhlenbeck process z. This process will be generated by the stochastic differential equation

$$dz + 2(k+1)\nu Az\, dt = dw \tag{14}$$

for a positive sufficiently large constant k. It is known (see, e.g. Da Prato and Zabczyk [21] Chapter 5) that there exists a tempered random variable z in V such that

$$\mathbb{R} \ni t \to z(\theta_t\omega)$$

solves (14). This Ornstein - Uhlenbeck process $z(\theta_t\omega)$ has trajectories in the space $L^2_{loc}(\mathbb{R}; H^3)$. The constant k may be considered as a control parameter. We now consider the nonautonomous differential equation

$$\frac{du}{dt} + \nu A\, u = F(u, \theta_t\omega), \quad u(0) = x \in H, \tag{15}$$

where

$$\begin{aligned} F(u,\omega) = -\nu Au - (u\cdot\nabla)u - (z(\omega)\cdot\nabla)u - (u\cdot\nabla)z(\omega) \\ -(z(\omega)\cdot\nabla)z(\omega) + 2\nu kAz(\omega) + f. \end{aligned}$$

The idea of this transformation can be found in Crauel and Flandoli [11]. Since the coefficients of this equation have similar properties as the coefficients of the original 2D Navier - Stokes equations, this equation has a unique solution. More precisely, we have

Lemma 3.1 *The solution of (15) defines a continuous random dynamical system φ with respect to θ on H. Let*

$$x \to T(\omega, x) := x - z(\omega) \tag{16}$$

be a random homeomorphism on H. Then $T^{-1}(\theta_t\omega, \varphi(t, \omega, T(\omega, x))) =: \tilde{\varphi}(t, \omega, x)$ defines a random dynamical system with respect to θ. In particular,

$$t \to \tilde{\varphi}(t, \omega, x)$$

solves (13).

Since $z(\omega) \in V$ the mapping T can be considered as a homeomorphism on V.
It is well known that the random dynamical system φ has a random compact absorbing forward invariant set B in V, see for instance Crauel and Flandoli [11]. We now formulate a version of these results and will prove some additional properties.

Lemma 3.2 *The random dynamical system φ has a compact forward invariant absorbing set B in H. This absorbing set is contained in the closed ball in H with center zero and with square radius*

$$R^2(\omega) = (1 + \varepsilon) \int_{-\infty}^{0} m(\theta_\tau\omega) e^{\nu\lambda_1\tau + \frac{8}{\nu}\int_\tau^0 \|z(\theta_s\omega)\|_V^2 ds} d\tau, \tag{17}$$

where $\varepsilon > 0$ is arbitrary, λ_1 is the first eigenvalue of the operator $-\Delta$ with the Dirichlet boundary condition,

$$m(\omega) = \frac{4}{\nu}\left(\frac{2}{\lambda_1}\|z(\omega)\|_V^4 + k^2\nu^2\|z(\omega)\|_V^2 + \|f\|_{V'}^2\right) \tag{18}$$

and the parameter k in (14) is chosen such that

$$\lambda_1 > \frac{4\,\mathrm{tr}_H Q}{(k+1)\nu^3}. \tag{19}$$

In addition, B is tempered and $t \to \sup_{x\in B(\theta_t\omega)} \|x\|_H^2$ is a locally integrable stationary process.

Proof. We sketch the proof of this lemma. We obtain by Temam [23] Lemma III.3.4

$$2\langle F(u, \theta_t\omega), u\rangle \le \frac{8}{\nu}\|u\|_H^2\|z(\theta_t\omega)\|_V^2 + \frac{8}{\nu\lambda_1}\|z(\theta_t\omega)\|_V^4 + \frac{4}{\nu}\|f\|_{V'}^2 + 4k^2\nu\|z(\theta_t\omega)\|_V^2.$$

Let R_0^2 be the stationary solution of the random affine one-dimensional differential equation

$$\frac{d\rho}{dt} + \nu\lambda_1\rho = \frac{8}{\nu}\rho\|z(\theta_t\omega)\|_V^2 + \frac{8}{\nu\lambda_1}\|z(\theta_t\omega)\|_V^4 + \frac{4}{\nu}\|f\|_{V'}^2 + 4k^2\nu\|z(\theta_t\omega)\|_V^2. \tag{20}$$

This stationary solution $R_0^2(\omega)$ exists and it is exponentially attracting provided

$$\frac{8}{\nu}\lim_{\tau\to-\infty}\frac{1}{|\tau|}\int_\tau^0 \|z(\theta_s\omega)\|_V^2 ds \equiv \frac{8}{\nu}\mathbb{E}\|z\|_V^2 < \nu\lambda_1.$$

A Simple calculation shows that this relation is equivalent to (19). Moreover $R^2(\omega) = (1+\varepsilon)R_0^2(\omega)$ has the form (17). The temperedness of R^2 follows from Flandoli and Schmalfuß [13] Lemma 7.2. Since the solution $R_0^2(\theta_t\omega)$ of the above equation is continuous, the mapping

$$t \to R^2(\theta_t\omega)$$

is locally integrable. In addition, a comparison argument yields that the random ball $B(0,R(\omega))$ is forward invariant and forward absorbing. Finally, we note that

$$B(\omega) := \overline{\varphi(1,\theta_{-1}\omega, B(0,R(\theta_{-1}\omega)))} \subset B(0,R(\omega)) \tag{21}$$

is a compact forward invariant and forward absorbing set by the regularization property of φ. □

However, there are other compact absorbing sets defined by a ball $B(0,R(\omega))$ with random radius $R(\omega)$, see for instance Flandoli and Langa [12]. In the following we propose another method to calculate moments of (17). This technique is based on the standard density of the Girsanov theory.

Lemma 3.3 *Let R^2 be defined by (17) then if we choose a k such that*

$$\lambda_1 > \frac{16\,\mathrm{tr}_H Q}{(k+1)\nu^3}, \qquad \lambda_1 \ge \frac{256\,\mathrm{tr}_H Q}{(k+1)^2\nu^3}, \tag{22}$$

we have $\mathbb{E}R^8 < \infty$.

Proof. We rewrite

$$R^2 = (1+\varepsilon)\int_{-\infty}^0 m(\theta_\tau\omega)e^{\nu\lambda_1\tau + c\int_\tau^0\|z\|_V^2}d\tau$$

for $c = \frac{8}{\nu}$ and some $\varepsilon > 0$. We obtain by the Cauchy - Schwarz inequality for an appropriate $c_1 > 0$

$$\begin{aligned}\mathbb{E}R^8 &\le c_1(\mathbb{E}m^8)^{\frac{1}{2}}\left(\mathbb{E}\int_{-\infty}^0 e^{4\nu\lambda_1\tau + 8c\int_\tau^0\|z\|_V^2}d\tau\right)^{\frac{1}{2}}\\ &= c_1(\mathbb{E}m^8)^{\frac{1}{2}}\left(\int_0^\infty e^{-4\nu\lambda_1\tau}\cdot\mathbb{E}e^{8c\int_0^\tau\|z\|_V^2}d\tau\right)^{\frac{1}{2}}.\end{aligned}$$

The first factor is finite, since z is a Gaussian random variable. We now restrict ourselves to calculate $\mathbb{E}\exp\{8c\int_0^\tau \|z\|_V^2\}$. Ito's formula applied to $\|\cdot\|_H^2$ for $z(\theta_t\omega)$ yields:

$$\|z(\theta_\tau\omega)\|_H^2 + 2(k+1)\nu\int_0^\tau \|z(\theta_s\omega)\|_V^2 ds = \|z(\omega)\|_H^2 + 2\int_0^\tau (z,dw)_H + \tau\, \mathrm{tr}_H Q.$$

Hence we can derive that

$$e^{16c\int_0^\tau \|z\|_V^2} \leq e^{\frac{8c}{(k+1)\nu}\|z\|_H^2} \cdot e^{\frac{8c}{(k+1)\nu}\tau\, \mathrm{tr}_H Q} \cdot e(\tau) \cdot e^{\frac{256c^2}{(k+1)^2\nu^2}\int_0^\tau (Qz,z)},$$

where

$$e(\tau) \equiv e(\tau,\omega) = \exp\left\{\frac{16c}{(k+1)\nu}\int_0^\tau (z,dw)_H - \frac{256c^2}{(k+1)^2\nu^2}\int_0^\tau (Qz,z)\right\}.$$

From (22) we have that $\frac{256c^2 \mathrm{tr}_H Q}{(k+1)^2\nu^2\lambda_1} \leq 8c$. Therefore using the Cauchy - Schwarz inequality and

$$(Qz,z) \leq \mathrm{tr}_H Q\|z\|_H^2 \leq \frac{\mathrm{tr}_H Q}{\lambda_1}\|z\|_V^2$$

we have

$$\mathbb{E}e^{8c\int_0^\tau \|z\|_V^2} \leq \left(\mathbb{E}e^{\frac{16c}{(k+1)\nu}\|z\|_H^2}\right)^{\frac{1}{2}} \left(\mathbb{E}e(\tau)^2\right)^{\frac{1}{2}} e^{\frac{8c}{(k+1)\nu}\tau\, \mathrm{tr}_H Q}. \tag{23}$$

We can use the standard arguments (see, e.g., [19] and [18]) to find the estimate $\mathbb{E}[e(\tau)^2] \leq 1$ for the mean value of Girsanov's density $e(\tau)^2$. The value $z(\omega)$ is a Gaussian variable in H with the zero mean and with the covariance

$$\mathbb{E}\langle z,h_1\rangle\langle z,h_2\rangle = \langle \tilde{Q}h_1,h_2\rangle, \quad h_1,\, h_2 \in H,$$

where

$$\tilde{Q} = \int_0^\infty e^{-2t(k+1)\nu A} Q e^{-2t(k+1)\nu A}\, dt.$$

Therefore simple calculation (see, e.g. Kuo [17]) Page 105 shows that the first factor in the right hand side of (23) is finite provided $\frac{16c}{(k+1)\nu} < \frac{1}{2\mathrm{tr}_H \tilde{Q}}$. Moreover

$$\mathbb{E}e^{\frac{16c}{(k+1)\nu}\|z\|_H^2} \leq \exp\left\{\frac{16c\mathrm{tr}_H\tilde{Q}}{(k+1)\nu - 32c\mathrm{tr}_H\tilde{Q}}\right\}.$$

However it is easy to see that $\mathrm{tr}_H\tilde{Q} \leq \frac{\mathrm{tr}_H Q}{2\lambda_1(k+1)\nu}$. Therefore from the second assumption of (22) we have that

$$\mathbb{E}e^{\frac{16c}{(k+1)\nu}\|z\|_H^2} \leq \exp\left\{\frac{8c\mathrm{tr}_H Q}{\lambda_1(k+1)^2\nu^2 - 16c\mathrm{tr}_H Q}\right\} < \infty. \tag{24}$$

Since from (22) we also have $4\nu\lambda_1 > \frac{8c\mathrm{tr}_H Q}{(k+1)\nu}$, the expectation of R^8 is finite. □

The following lemma allows us to conclude the existence of a set $\mathcal{L}$ of determining functionals for the random dynamical system $\tilde{\varphi}$ generated by (13) if the random dynamical system φ generated by (15) has the set of determining functionals $\mathcal{L}$. This lemma is formulated for more general transformations than (16).

Lemma 3.4 *Suppose that the random dynamical systems $\tilde{\varphi}$ and φ are conjugated by a random homeomorphism T on H, i.e. $\tilde{\varphi}(t,\omega,\tilde{x}(\omega)) = T^{-1}(\theta_t\omega, \varphi(t,\omega,x(\omega))$, where $x(\omega) = T(\omega,\tilde{x}(\omega))$. Suppose that φ has a compact absorbing set and forward invariant set B. Then $\tilde{\varphi}(t,\omega,\tilde{x}_1(\omega)) - \tilde{\varphi}(t,\omega,\tilde{x}_2(\omega))$ tends to zero in probability for $t\to\infty$ if and only if $\varphi(t,\omega,x_1(\omega)) - \varphi(t,\omega,x_2(\omega))$ tends to zero in probability for $t\to\infty$. Here $x_i(\omega) = T(\omega,\tilde{x}_i(\omega))$.*

Proof. Suppose that $\varphi(t,\omega,x_1(\omega)) - \varphi(t,\omega,x_2(\omega))$ tends to zero in probability for $t\to\infty$. By the absorbing property of B we can assume that $x_1(\omega)$, $x_2(\omega)\in B(\omega)$. For any $\varepsilon>0$ there exists a compact set C_ε such that $C_\varepsilon\supset B(\omega)$ with probability bigger than $1-\varepsilon$. Indeed, this follows by the regularization property of φ and by the construction of B in (21). $T^{-1}(\omega)$ is uniformly continuous on C_ε: for any $\omega\in\Omega$, $\mu>0$, y_1, $y_2\in C_\varepsilon$ there exists a $\delta(\omega)>0$ such that if $\|y_1-y_2\|_H<\delta(\omega)$ then $\|T^{-1}(\omega,y_1)-T^{-1}(\omega,y_2)\|_H<\mu$. On the other hand since $\delta(\omega)>0$ there exists a $\delta_\varepsilon>0$:

$$\mathbb{P}(\delta_\varepsilon<\delta(\omega))>1-\varepsilon.$$

Hence for sufficiently large t_ε we have

$$\begin{aligned}&\mathbb{P}(\|\varphi(t,\omega,x_1(\omega))-\varphi(t,\omega,x_2(\omega))\|_H>\delta(\theta_t\omega))\\&\qquad\le\varepsilon+\mathbb{P}(\|\varphi(t,\omega,x_1(\omega))-\varphi(t,\omega,x_2(\omega))\|_H>\delta_\varepsilon)<2\varepsilon\end{aligned}$$

if $t\ge t_\varepsilon$. Hence

$$\begin{aligned}&\|\tilde{\varphi}(t,\omega,\tilde{x}_1(\omega))-\tilde{\varphi}(t,\omega,\tilde{x}_2(\omega))\|_H\\&\qquad=\|T^{-1}(\theta_t\omega,\varphi(t,\omega,x_1(\omega)))-T^{-1}(\theta_t\omega,\varphi(t,\omega,x_2(\omega)))\|_H<\mu\end{aligned}$$

with probability bigger than $1-3\varepsilon$ for $t>t_\varepsilon$.
$\tilde{B}=T^{-1}(B)$ is a compact forward invariant absorbing set for (13) if and only if B is a compact forward invariant absorbing set for (15). Therefore we can show the second direction similarly as the proof above for the first direction. □

Corollary 3.5 *The set of linear functionals $\mathcal{L}$ on V is determining in probability for the random dynamical system $\tilde{\varphi}$ generated by (13) if and only if $\mathcal{L}$ is determining in probability for φ defined by (15).*

Proof. The proof is based on the fact that for some $l\in\mathcal{L}$ the limit in probability for $t\to\infty$ of $l(\tilde{\varphi}(t,\omega,\tilde{x}_1(\omega))-\tilde{\varphi}(t,\omega,\tilde{x}_2(\omega)))$ is zero if and only if $l(\varphi(t,\omega,x_1(\omega))-$

$\varphi(t,\omega,x_2(\omega)))$ tends to zero in probability for $t\to\infty$ which follows from the particular shape of T. On the other hand, we can also apply the last lemma. $\square$

For the following we need two a priori estimates for φ:

Lemma 3.6 *The random dynamical system φ satisfies the following a priori estimate*

$$\nu \sup_{x\in B(\omega)}\int_0^t \|\varphi(\tau,\omega,x)\|_V^2 d\tau \le R^2(\omega)+\frac{8}{\nu}\int_0^t \|z(\theta_\tau\omega)\|_H^2\|z(\theta_\tau\omega)\|_V^2 d\tau$$
$$+\frac{4}{\nu}t\|f\|_{V'}^2+\nu k^2\int_0^t \|z(\theta_\tau\omega)\|_V^2 d\tau + c_E M\int_0^t \|z(\theta_\tau\omega)\|_{H^3}^2 d\tau$$
$$+\frac{c_E}{M}\int_0^t R^4(\theta_\tau\omega)d\tau.$$

where M is an arbitrarily positive number and c_E is the norm of the embedding operator of $H^3=D(A^{3/2})$ into the space $W_\infty^1(D)$ of two-dimensional functions v such that $v,\nabla v\in L^\infty(D)$. Similarly, for an appropriate polynomial p

$$2\nu \sup_{x\in B(\omega)}\int_0^t \|\varphi(\tau,\omega,x)\|_H^2\|\varphi(\tau,\omega,x)\|_V^2 d\tau \le R^4(\omega)$$
$$+\int_0^t p(\|z(\theta_\tau\omega)\|_H^2,\|z(\theta_\tau\omega)\|_V^2,\|z(\theta_\tau\omega)\|_H^3,\|f\|_{V'}^2)d\tau+\int_0^t R^8(\theta_\tau\omega)d\tau\ .$$

This a priori estimate is based on the calculation of $\|u(t)\|_H^2$ for (15). The term $\langle (u\cdot\nabla)z),u\rangle$ arising in the calculation can be estimated by the Sobolev lemma:

$$|\langle (u\cdot\nabla)z),u\rangle|\le c_E\|z\|_{H^3}\|u\|_H^2\le \frac{c_E M}{2}\|z\|_{H^3}^2+\frac{c_E}{2M}R^4$$

because $x\in B$. The second estimate follows similarly for $\|u(t)\|_H^4$.

Lemma 3.7 *Under conditions (22) the following estimate holds:*

$$\begin{aligned}\Sigma_k \quad &\equiv \limsup_{m\to\infty}\tfrac{1}{m}\mathbb{E}\left\{\sup_{x\in B(\omega)}\int_0^m \|\varphi(\tau,\omega,x)\|_V^2 d\tau\right\} \\ &\le \left(\tfrac{4}{\nu^2}\|f\|_{V'}^2+g_k(\nu,\lambda_1,Q)\right)\cdot\left(1+h_k(\nu,A,Q)\right),\end{aligned}\tag{25}$$

where

$$g_k(\nu,\lambda_1,Q)=a_0\frac{\mathrm{tr}_H Q}{\nu}\cdot\left(k+\frac{a_1\,\mathrm{tr}_H Q}{(k+1)^2\lambda_1\nu^3}\right)\tag{26}$$

and

$$h_k(\nu,A,Q)=2c_E\left(\frac{[\mathrm{tr}_H QA^2]^2}{\nu^3\lambda_1^3(k+1)\cdot[\nu^3\lambda_1(k+1)-16\mathrm{tr}_H Q]}\right)^{1/4}.\tag{27}$$

Here a_0 and a_1 are some absolute constants and c_E is the same as in Lemma 3.6.

Proof. It follows from Lemma 3.6 that

$$\Sigma_k \le \frac{4}{\nu^2}\|f\|^2_{V'} + \frac{8}{\nu^2}\mathbb{E}\left(\|z\|^2_H\|z\|^2_V\right) + k^2\mathbb{E}\|z\|^2_V + \frac{c_E}{\nu}M\mathbb{E}\|z\|^2_{H^3} + \frac{c_E}{\nu M}\mathbb{E}R^4.$$

If we choose $M = \left(\mathbb{E}\|z\|^2_{H^3}\right)^{-1/2}\cdot\left(\mathbb{E}R^4\right)^{1/2}$, then we obtain

$$\begin{aligned}\Sigma_k \quad &\le \tfrac{4}{\nu^2}\|f\|^2_{V'} + \tfrac{8}{\nu^2}\left(\mathbb{E}\|z\|^4_H\right)^{1/2}\cdot\left(\mathbb{E}\|z\|^4_V\right)^{1/2}\\ &+k^2\mathbb{E}\|z\|^2_V + \tfrac{2c_E}{\nu}\left(\mathbb{E}\|z\|^2_{H^3}\right)^{1/2}\cdot\left(\mathbb{E}R^4\right)^{1/2}.\end{aligned}$$

Using the definition of z it is easy to find that for any positive $\alpha\in[0,3/2]$ we have

$$\mathbb{E}\|A^\alpha z\|^2_H = \frac{1}{4(k+1)\nu}\mathrm{tr}_H(QA^{2\alpha-1}). \tag{28}$$

Furthermore it is clear that

$$\mathbb{E}\|A^\alpha z\|^{2l}_H \le c_l\left(\mathbb{E}\|A^\alpha z\|^2_H\right)^l, \quad l=1,2,\ldots, \tag{29}$$

with appropriate constants c_l. Therefore we have

$$\Sigma_k \le \frac{4}{\nu^2}\|f\|^2_{V'} + \frac{c_0[\mathrm{tr}_HQ]^2}{\nu^4(k+1)^2\lambda_1} + \frac{k}{2\nu}\cdot\mathrm{tr}_HQ + \frac{c_E[\mathrm{tr}_HQA^2]^{1/2}}{\nu^{3/2}(k+1)^{1/2}}\cdot\left(\mathbb{E}R^4\right)^{1/2}.$$

with some absolute constant c_0. Now we estimate $\left(\mathbb{E}R^4\right)^{1/2}$. We use the idea of the proof of Lemma 3.3. It is clear that

$$\left(\mathbb{E}R^4\right)^{1/2} \le (1+\varepsilon)\left(\frac{3}{2\nu\lambda_1}\right)^{3/4}\left(\mathbb{E}m^4\right)^{1/4}\left(\mathbb{E}\int_{-\infty}^0 e^{2\nu\lambda_1\tau+4c\int_\tau^0\|z(\theta_s\omega)\|^2_V ds}d\tau\right)^{1/4},$$

where $m(\omega)$ is given by (18) and $c=8/\nu$. Using Girsanov's trick and (23) and (24) with $c:=c/2$ we have

$$\mathbb{E}e^{4c\int_0^\tau\|z\|^2_V} \le \exp\left\{\frac{2c\mathrm{tr}_HQ}{\lambda_1(k+1)^2\nu^2-8c\mathrm{tr}_HQ}\right\}\cdot\exp\left\{\frac{4c}{(k+1)\nu}\tau\,\mathrm{tr}_HQ\right\}$$

under conditions (22). However (22) implies that $\lambda_1(k+1)^2\nu^2\ge 32c\mathrm{tr}_HQ$. Therefore

$$\mathbb{E}e^{4c\int_0^\tau\|z\|^2_V} \le \exp\left\{\frac{1}{12}+\frac{4c}{(k+1)\nu}\tau\,\mathrm{tr}_HQ\right\} \tag{30}$$

under conditions (22). From (30) we have

$$\left(\mathbb{E}R^4\right)^{1/2} \le (1+\varepsilon)e^{1/48}\left(\frac{3}{2\nu\lambda_1}\right)^{3/4}\cdot\left(2\nu\lambda_1-\frac{32}{(k+1)\nu^2}\mathrm{tr}_HQ\right)^{-1/4}\left(\mathbb{E}m^4\right)^{1/4}.$$

Now we estimate $\left(\mathbb{E}m^4\right)^{1/4}$. It is clear from (18) that

$$\left(\mathbb{E}m^4\right)^{1/4} \le \frac{4}{\nu}\left\{\frac{2}{\lambda_1}\left(\mathbb{E}\|z(\omega)\|_V^{16}\right)^{1/4} + k^2\nu^2\left(\mathbb{E}\|z(\omega)\|_V^{8}\right)^{1/4} + \|f\|_{V'}^2\right\}.$$

Therefore using (29) we obtain

$$\left(\mathbb{E}m^4\right)^{1/4} \le \frac{4}{\nu}\left\{\|f\|_{V'}^2 + \frac{c_1[\mathrm{tr}_H Q]^2}{\nu^2(k+1)^2\lambda_1} + c_2 k\nu\cdot \mathrm{tr}_H Q\right\},$$

where c_1 and c_2 are absolute constants. Put all these estimates together we obtain the upper bound (25) for Σ_k. □

We have seen that B, defined in (21), is bounded in V, and hence it is a compact set in H which is tempered with respect the H norm. We now prove that B is also tempered and locally integrable in V.

Lemma 3.8 *The random variable* $\sup_{x\in B(\omega)}\|x\|_V^2$ *is tempered and the mapping* $t\to \sup_{x\in B(\theta_t\omega)}\|x\|_V^2$ *is locally integrable.*

Proof. To obtain an estimate in V we use the standard method which is based on the formula

$$\frac{d}{dt}(t\|u(t)\|_V^2) = \|u(t)\|_V^2 + t\frac{d\|u(t)\|_V^2}{dt}.$$

for $t=1$ by Temam [23] Lemma III.3.8 and

$$|((u\cdot\nabla)v, w)_V| \le c\|u\|_H^{\frac12}\|u\|_V^{\frac12}\|v\|_V^{\frac12}\|Av\|_H^{\frac12}\|Aw\|_H$$

for sufficiently regular u, v, w and $c>0$ which allows us to write by (15)

$$\begin{aligned}\frac{d}{dt}(t\|u(t)\|_V^2) \le\ & K(\|u(t)\|_H^2\|u(t)\|_V^2 + \|u(t)\|_H^2 + \|z(\theta_t\omega)\|_H^2\|z(\theta_t\omega)\|_V^2)\\ \times\ & (t\|u(t)\|_V^2)\\ +\ & p(\|f\|_{V'}, \|z(\theta_t\omega)\|_H, \|z(\theta_t\omega)\|_V, \|Az(\theta_t\omega)\|_H) + \|u(t)\|_V^2\end{aligned}$$

where p is an appropriate polynomial and K an appropriate positive constant. Consequently, by the Gronwall lemma

$$\begin{aligned}\sup_{x\in B}\|\varphi(1,\omega,x)\|_V^2 \le\ & \exp\left(K\sup_{x\in B}\int_0^1 \|\varphi(\tau,\omega,x)\|_H^2 d\tau\right)\\ \times\ & \exp\left(K\sup_{x\in B}\int_0^1 \|\varphi(\tau,\omega,x)\|_H^2\|\varphi(\tau,\omega,x)\|_V^2 d\tau\right)\\ \times\ & \exp\left(K\int_0^1 \|z(\theta_\tau\omega)\|_V^2\|z(\theta_\tau\omega)\|_H^2 d\tau\right)\\ \times\ & \left(\int_0^1 p(\tau)d\tau + \sup_{x\in B}\int_0^1 \|\varphi(\tau,\omega,x)\|_V^2 d\tau\right).\end{aligned}$$

Note that a product of random variables is tempered if each factor is tempered. To see that the first factor of the right hand side is tempered we use the estimate

$$\mathbb{E}\sup_{\substack{x\in B\\ s\in[0,1]}}\int_0^1\|\varphi(\tau,\theta_s\omega,x)\|_H^2d\tau\le\mathbb{E}\int_0^2R^2(\theta_s\omega)ds=2\mathbb{E}R^2<\infty$$

by Lemma 3.3 and the forward invariance of B, see Arnold [1] Proposition 4.1.3. Similarly, we get for the next factor

$$\mathbb{E}\sup_{\substack{x\in B\\ s\in[0,1]}}\int_0^1\|\varphi(\tau,\theta_s\omega,x)\|_H^2\|\varphi(\tau,\theta_s\omega,x)\|_V^2d\tau<\infty$$

which follows from Lemma 3.6. However, to justify this estimate we also need that $\mathbb{E}\sup_{s\in[0,1]}R^4(\theta_s\omega)<\infty$. For this expression we obtain an estimate if we calculate in (20) $R_0^4(\theta_s\omega)=\rho^2(s)$ by the chain rule. Then we can estimate this supremum by $R^4(\omega)$ and some integrals of norms from z which have a finite expectation. The temperedness of the remaining factors follow similarly.
The local integrability follows by the continuity of $t\to R(\theta_t\omega)$ and the local integrability of the norms of z. □

We are now in a position to formulate the main theorem of this section.

Theorem 3.9 *Let $\mathcal{L}$ be a set of linear functionals on V with completeness defect $\varepsilon_{\mathcal{L}}$. Assume that for some k satisfying (22) the completeness defect $\varepsilon_{\mathcal{L}}$ possesses the property*

$$\frac{4}{\nu}\cdot\left(\frac{4}{\nu^2}\|f\|_{V'}^2+g_k(\nu,\lambda_1,Q)\right)\cdot(1+h_k(\nu,A,Q))+\frac{2}{(k+1)\nu}\mathrm{tr}_HQ<\nu\varepsilon_{\mathcal{L}}^{-2},\quad(31)$$

where $g_k(\nu,\lambda_1,Q)$ and $h_k(\nu,A,Q)$ are given by (26) and (27). Then $\mathcal{L}$ is a system of determining functionals in probability for the 2D stochastic Navier-Stokes equation (13).

Proof. We are going to apply Theorem 2.2. The temperedness and local integrability of $\sup_{x\in B(\theta_t\omega}\|x\|_V^2$ follow by the last lemma. Then we get the assertion if we choose m sufficiently large. Indeed, in the case of large m we can reduce the influence of $\mathbb{E}R^2$. By Corollary 3.5 it is sufficient to show that $\mathcal{L}$ is a set of determining functionals for φ generated by (15). The properties of F allow us to estimate the Lipschitz constant

$$l(x_1,x_2,\omega)=\frac{2}{\nu}(\|x_1\|_V^2+\|z(\omega)\|_V^2).$$

The measurability of l follows straightforwardly. We should also take $c=\frac{\nu}{2}$ in (3). Therefore we can apply Theorem 2.2 if

$$\frac{4}{\nu m}\mathbb{E}\left\{\sup_{x\in B(\omega)}\int_0^m\|\varphi(\tau,\omega,x)\|_V^2d\tau\right\}+\frac{4}{\nu}\mathbb{E}\|z\|_V^2<\nu\varepsilon_{\mathcal{L}}^{-2}\quad(32)$$

for some m. We can find m with the property (32), if

$$\frac{4}{\nu}\Sigma_k + \frac{4}{\nu}\mathbb{E}\|z\|_V^2 < \nu\varepsilon_{\mathcal{L}}^{-2}.$$

The last relation follows from Lemma 3.7, the relation (28) and (31). $\square$

Remark 3.10 In the limit $\mathrm{tr}_H QA^2 \to 0$ relation (31) turns in the inequality

$$\varepsilon_{\mathcal{L}} < \frac{4\nu^2}{\|f\|_{V'}}. \tag{33}$$

Thus if the estimate (33) is valid, then there exists a constant $\delta_0 > 0$ such that under condition $\mathrm{tr}_H QA^2 < \delta_0$ the set $\mathcal{L}$ is a set of determining functionals in probability for the 2D stochastic Navier-Stokes equations. We also note that estimate (33) is the same order as the best known estimate for the completeness defect in the case of deterministic $2D$ Navier - Stokes equations with the periodic boundary conditions (see the survey [7] and the references therein). However in the last case relation (33) involves the completeness defect with respect to the pair $D(A)$ and H and it leads to better estimates for the number of determining functionals.

Remark 3.11 As an application of Theorem 2.2 and 2.3 we can consider the equation

$$\partial_t u = \Delta u - f(u) + \partial_t W(t,\omega)$$

in a bounded domain, where $f(u)$ is a polynomial of odd degree with positive leading coefficient. We can also consider 2D stochastic Navier-Stokes equations with multiplicative white noise $u\,dw$, where w is a scalar Wiener process. In this case we have to use the transformation $T(\omega,x) = x\,e^{-z(\omega)}$ where z defines a one dimensional stationary Ornstein-Uhlenbeck process generated by $dz + z\,dt = dw$. This equation has been investigated for instance in Schmalfuß [22] but with a little bit different transformation T.

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