Limit law for transition probabilities and moderate deviations for Sinai's random walk in random environment*

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Abstract

We consider a one-dimensional random walk in random environment in the Sinai's regime. Our main result is that logarithms of the transition probabilities, after a suitable rescaling, converge in distribution as time tends to infinity, to some functional of the Brownian motion. We compute the law of this functional when the initial and final points agree. Also, among other things, we estimate the probability of being at time t at distance at least z from the initial position, when z is larger than $\ln^2 t$, but still of logarithmic order in time.

Keywords: random environment, Sinai's regime, elevation, moments of return, t-stable points, spectral gap, metastability

1 Introduction and notations

Suppose that for all $x \in \mathbb{Z}$, we are given two positive numbers, ω_x^+, ω_x^-. Define the Markov process $\xi^x = (\xi_t^x)_{t\geq 0}$ on $\mathbb{Z}$ starting at x, such that, if currently at site y, it jumps to $y+1$

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with rate ω_y^+, and to $y-1$ with rate ω_y^-. Then, the transition probabilities are determined by

$$\begin{aligned} \mathtt{P}_\omega[\xi^x_{t+h} = y \pm 1 \mid \xi^x_t = y] &= \omega_y^\pm h + o(h) \, , \\ \mathtt{P}_\omega[\xi^x_{t+h} = y \mid \xi^x_t = y] &= 1 - (\omega_y^+ + \omega_y^-)h + o(h), \qquad \text{as } h \searrow 0. \end{aligned}$$

Now, we suppose that $\omega = (\omega_x^+, \omega_x^-)_{x\in\mathbb{Z}}$ is a fixed realization of an i.i.d. sequence of positive random variables. We refer to ω as *the environment*, and to ξ^x as the random walk in the random environment ω. This model has been much studied recently in its discrete version. The discrete random walk in random environment is the Markov chain embedded in the present continuous time process. In this paper we study only the case of *Sinai's regime*, which means that the following condition is satisfied:

Condition S. We have

$$\mathbb{E}\ln\frac{\omega_0^+}{\omega_0^-} = 0 \, , \quad \sigma^2 = \mathbb{E}\ln^2\frac{\omega_i^-}{\omega_i^+} \in (0, +\infty) \, .$$

We will use $\mathbb{P}, \mathbb{E}$ to denote probability and expectation with respect to ω, keeping $\mathtt{P}_\omega, \mathtt{E}_\omega$ for (so-called "quenched") probability and expectation for random walks in the fixed environment ω.

Solomon [21] proved that the random walk of individual particle is recurrent if and only if the first part of Condition S is satisfied. In addition to Condition B below, the second part of Condition S ensures non-degenerate randomness of the medium, i.e., our random walk in random environment is not a time-change of a simple random walk. Sinai [20] proved that abnormal diffusion takes place: Under Condition S, ξ_t^0 is of order $\ln^2 t$. Due to strong disorder and traps, this model has many other remarkable features: ultra-slow relaxation, aging phenomena [13], complex large deviation properties [22]...For surveys on the subject, the reader is referred to the lecture notes of a recent course by Zeitouni [22], to the very complete book of Hughes [10], and to the stochastic calculus approach by Shi [19].

In addition to Condition S, for technical reasons we need also the following

Condition B. There exists a positive number κ such that

$$\kappa^{-1} \le \omega_0^+ \le \kappa, \quad \kappa^{-1} \le \omega_0^- \le \kappa \ \text{ a.s.}$$

As the transitions occur only between nearest neighbors, one can write down the reversible measure θ by solving the detailed balanced conditions $\theta_x\omega_x^+ = \theta_{x+1}\omega_{x+1}^-$:

$$\theta_x = \begin{cases} \prod_{i=0}^{x-1} \frac{\omega_i^+}{\omega_{i+1}^-}, & x > 0, \\ 1, & x = 0, \\ \prod_{i=x}^{-1} \frac{\omega_{i+1}^-}{\omega_i^+}, & x < 0. \end{cases}$$

Reversibility is a strong tool for estimating probabilities, we will heavily use it all through the paper. Note the following, elementary consequence of reversibility: for every x, y, t it holds that

$$\theta_x \mathtt{P}_\omega[\xi_t^x = y] = \theta_y \mathtt{P}_\omega[\xi_t^y = x] .$$

In this paper, we will obtain sharp estimates on large time transition probabilities for the random walk in random environment. To give a flavor of our results, let us briefly indicate our basic statement, Theorem 2.1: The transition probability is essentially a (random) power of t, i.e., for x, y at distance at most $\ln^2 t$,

$$\mathtt{P}_\omega[\xi_t^x = y] = \exp\{-\alpha_t(x, y) + o(\ln t)\} ,$$

and

$$\exp\{-\alpha_t(x' \ln^2 t, y' \ln^2 t)\} \stackrel{\text{law}}{=} t^{-\alpha_e(x',y')} ,$$

where $\alpha_t(x, y)$ is identified in terms of the environment (the symbol "$\stackrel{\text{law}}{=}$" stands for the equality in law). This positive process has nice scaling properties, and we also compute its law when $x = y$.

Estimating the quantity $\mathtt{P}_\omega[\xi_t^x = y]$ is quite natural, but our main motivation is that it comes as a crucial ingredient when studying systems of random walks in random environment. Results in this direction are left for a forthcoming paper. Our estimates below will be much in the spirit of Freidlin and Wentzell theory [4] for small random perturbation of a differential equation in the gradient case, and of symmetric Markov processes with rare transitions. The main difference is that, here as well as in [14], the primary estimate is available on the invariant measure itself, and not directly on the probabilities of paths. In this paper, we will use the sharp spectral estimates obtained in [15] for general birth and death processes on tree-like graphs. These will allow us to approximate, in some specific situations, the true law of the walk at time t with the invariant probability of some reflected random walk in random environment (reflected in a suitable domain depending on t). Our approach seems to be original and powerful, it is of probabilistic nature compared to the analytic approach of metastability recently developed in [1].

The paper is organized as follows. The main results are formulated in Section 2. More specifically, after introducing the notion of t-stable point (which is central in this paper), in Section 2.1 we state the result about the existence of the limit law for the logarithms of transition probabilities for points which are at distance of order $\ln^2 t$ from each other. For a particular case (the probabilities of t-step transitions from 0 to 0), we give the density of the law. In Section 2.2, we study the probability that at time t the particle deviates more than $\ln^2 t$ from its initial position. The moments of first return to the origin are studied in Section 2.3. In Section 3 we formulate and prove some technical results needed afterwards. The contents of that section is the following: first we recall an elementary lemma about hitting probabilities. Then, in Section 3.1 we identify, via estimates on the spectral gap, the speed of convergence to equilibrium for a finite Markov chain obtained by placing reflecting barriers at the endpoints of a finite interval. In Section 3.2, it is shown, by using the results of the previous section, that until the moment t the particle will hit one of the two neighboring t-stable points with very large probability. In Section 3.3 we study the (small) probability of escaping a t-stable well in a given direction. The proof of the limit law for the transition probabilities is given in Section 4, and in Section 5 we deal with the results related to moderate deviations and moments of return. Finally, we placed into Section 6 all the arguments related to explicit calculations for the functionals of the Brownian motion; that is, in that section we evaluate two parameters needed in the course of the proof of the theorem about moderate deviations, and also we calculate the exact limit law for the probabilities of t-step transitions from 0 to 0 there.

2 Results

First of all, we introduce some notations. Given the realization of the random environment ω, define for $x \in \mathbb{Z}$

$$V(x) = \begin{cases} \sum_{i=0}^{x-1} \ln \frac{\omega_i^-}{\omega_i^+}, & x > 0, \\ 0, & x = 0, \\ \sum_{i=x+1}^{0} \ln \frac{\omega_i^+}{\omega_i^-}, & x < 0. \end{cases}$$

Note that Condition B implies that

$$K_1 e^{-V(x)} \le \theta_x \le K_2 e^{-V(x)} \tag{1}$$

for all $x \in \mathbb{Z}$. Here and in the whole paper, $K_1, K_2, \dots$ denote absolute constants, which may change from line to line.

In this paper, we focus on estimating the quenched law of the walk, for typical configurations ω of the environment. This enables us to approximate V by Brownian motion, but it is most convenient to use the well-known Komlós-Major-Tusnády [12] strong approximation theorem, as remarked by Hu and Shi [9]. Indeed, it allows to relate the features of long time behavior for the walk to Brownian functionals directly built in the model, simplifying much the proof of limit properties such as strict positivity: In a possibly enlarged probability space there exist a version of our environment process ω and a two-sided Brownian motion $(W(x), x \in \mathbb{R})$ with diffusion constant σ (i.e., $\operatorname{Var} W(x) = \sigma^2|x|$), such that for some $\hat{K} > 0$

$$\mathbb{P}\Big[\limsup_{x\to\pm\infty} \frac{|V(x) - W(x)|}{\ln|x|} \le \hat{K}\Big] = 1. \tag{2}$$

The next definitions are central in the construction. They apply to numerical functions W defined on $\mathbb{Z}$ or to continuous ones defined on $\mathbb{R}$, not only to Brownian motion.

Definition 2.1 *Let $t > 1$ and, for $m \in \mathbb{R}$, denote by $l = l(t,m)$ the largest $x < m$ such that $W(x) \geq W(m) + \ln t$ (with the convention $l = -\infty$ if no such x exists), and by $r = r(t,m)$ the smallest $x > m$ such that $W(x) \geq W(m) + \ln t$ (with the similar convention).*

*We say that m is a t-*stable *point if m is the smallest point on the interval $[l,r]$ which satisfies $W(m) = \min_{x\in[l,r]} W(x)$.*

Note that t-stable points are similar to bottoms of valleys of depth t in [20], except that they are defined in terms of W. We will see below that, typically for large t, the walker spends most of the time interval $[0,t]$ in sites which are close to such t-stable points.

The fact that $\limsup_{x\to\pm\infty} W(x) = -\liminf_{x\to\pm\infty} W(x) = \infty$ implies that the set $\mathcal{S}_t$ of t-stable points is infinite $\mathbb{P}$-a.s., as well as its traces $\mathcal{S}_t \cap (x,+\infty)$ and $\mathcal{S}_t \cap (-\infty,x)$. Also, for $m \neq m'$ elements of $\mathcal{S}_t$, the intervals $(l(m),r(m))$ and $(l(m'),r(m'))$ are non-intersecting. For any point x we consider the two t-stable points which surround x,

$$m_t^{+,x} = \inf\{y \geq x : y \in \mathcal{S}_t\}, \quad m_t^{-,x} = \sup\{y < x : y \in \mathcal{S}_t\}. \tag{3}$$

We also define the main passes $h_t^{\pm,x}$ between x and $m_t^{\pm,x}$, i.e., the smallest points $h_t^{+,x} \in [x, m_t^{+,x}]$ and $h_t^{-,x} \in [m_t^{-,x}, x]$ such that

$$W(h_t^{+,x}) = \max_{y\in[x,m_t^{+,x}]} W(y), \quad W(h_t^{-,x}) = \max_{y\in[m_t^{-,x},x]} W(y). \tag{4}$$

Since the open interval $(m_t^{-,x}, m_t^{+,x})$ contains no t-stable points, we observe that

$$\min\{W(h_t^{-,x}); W(h_t^{+,x})\} - \min_{x\in[h_t^{-,x},h_t^{+,x}]} W(x) < \ln t, \tag{5}$$

which will ensure that the confinement in space interval $[h_t^{-,x}, h_t^{+,x}]$ up to time t is extremely unlikely.

Define at last $\hat{m}_t^x$ by

$$\hat{m}_t^x = \begin{cases} m_t^{+,x}, & \text{if } W(h_t^{+,x}) < W(h_t^{-,x}), \\ m_t^{-,x}, & \text{otherwise.} \end{cases} \tag{6}$$

A celebrated result of Sinai [20] states that, as $t \to \infty$,

$$\frac{\xi_t^0 - \hat{m}_t^0}{\ln^2 t} \to 0$$

in $\mathbb{P} \otimes \mathtt{P}_\omega$-probability. Usually, this result is stated in the discrete time case, but it is easy to check, using Condition B, that it still applies in our case. It has been refined by Golosov [5], who proved that the numerator itself has a distributional limit. Note that, from Brownian scaling, the law of $\hat{m}_t^0/\ln^2 t$ does not depend on t.

All through the paper, we will use the notation $(r)^+ = \max\{r, 0\}$ for the positive part of $r \in \mathbb{R}$; The reader will make no confusion with superscripts in $m_t^{+,x}, h_t^{+,x}$, etc. Also, we stress that these quantities are random variables depending on W, but we do not indicate this dependency in our notations.

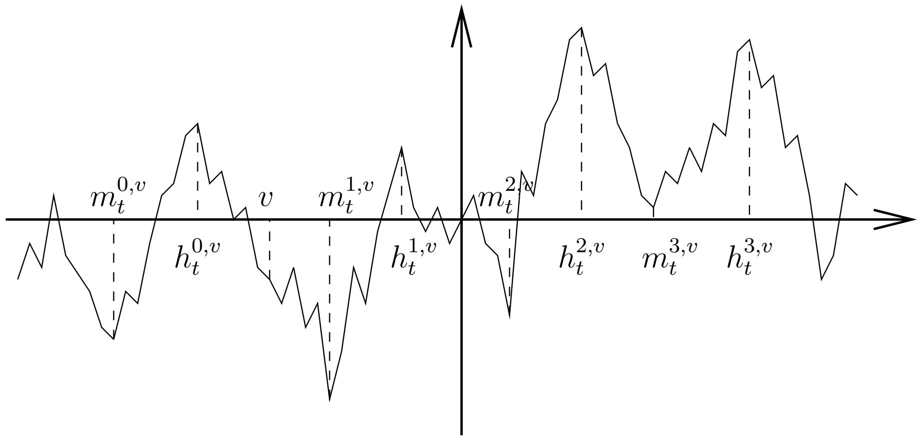

Figure 1: On the definition of $m_t^{i,v}$ and $h_t^{i,v}$

2.1 A limit law for the transition probabilities

Our main result gives the leading order of the transition probability $\mathtt{P}_\omega[\xi_t^v = z]$ for any two points v, z which are at distance of order $\ln^2 t$ from the origin. Let

$$\mathcal{S}_t = \{\ldots, m_t^{-1,v}, m_t^{0,v}, m_t^{1,v}, \ldots\} ,$$

where $m_t^{i,v} < m_t^{i+1,v}$ for all $i \in \mathbb{Z}$, be the set of all t-stable points labeled in such a way that $m_t^{0,v} < v \leq m_t^{1,v}$, and let $h_t^{i,v}$ be the smallest point such that

$$W(h_t^{i,v}) = \max_{x \in [m_t^{i,v}, m_t^{i+1,v}]} W(x)$$

for all $i \in \mathbb{Z}$ (see Figure 1). Note that we have $m_t^{0,v} = m_t^{-,v}$, $m_t^{1,v} = m_t^{+,v}$ in the notation introduced above, and that one of the points $h_t^{-,v}, h_t^{+,v}$ is equal to $h_t^{0,v}$.

Definition 2.2 *We will refer to the interval $[h_t^{i-1,v}, h_t^{i,v}]$ as the t-stable well of $m_t^{i,v}$.*

Next, we define the stochastic process $\alpha_t(v,z) = \alpha_t(v,z)(W), t > 1$ (recall that we want $\ln t > 0$), postponing explanations and comments on this definition until Theorem 2.1:

1. For $z \in [m_t^{0,v}, m_t^{1,v}]$, define

$$\alpha_t(v,z) = \min\{S^{(0)}, S^{(1)}, S^{(01)}, S^{(10)}\} ,$$

with

$$\begin{aligned} S^{(0)} &= W(z) - W(m_t^{0,v}) + \Big(W(h_t^{-,v}) - W(h_t^{+,v})\Big)^+ + \Big(W(h_t^{-,z}) - W(h_t^{+,z})\Big)^+ , \\ S^{(1)} &= W(z) - W(m_t^{1,v}) + \Big(W(h_t^{+,v}) - W(h_t^{-,v})\Big)^+ + \Big(W(h_t^{+,z}) - W(h_t^{-,z})\Big)^+ , \end{aligned}$$

$$
\begin{aligned}
S^{(01)} &= W(z) - W(m_t^{1,v}) + \Big(W(h_t^{-,v}) - W(h_t^{+,v})\Big)^+ \\
&\quad + \Big(W(h_t^{+,z}) - W(h_t^{-,z})\Big)^+ + \Big(W(h_t^{0,v}) - W(m_t^{0,v}) - \ln t\Big) , \\
S^{(10)} &= W(z) - W(m_t^{0,v}) + \Big(W(h_t^{+,v}) - W(h_t^{-,v})\Big)^+ \\
&\quad + \Big(W(h_t^{-,z}) - W(h_t^{+,z})\Big)^+ + \Big(W(h_t^{0,v}) - W(m_t^{1,v}) - \ln t\Big) .
\end{aligned}
$$

2. When $z \in [m_t^{k,v}, h_t^{k,v}]$ for some $k \geq 1$, define

$$
\begin{aligned}
\alpha_t(v,z) &= \min\Big\{\Big(W(h_t^{+,v}) - W(h_t^{-,v})\Big)^+ ; W(h_t^{0,v}) - W(m_t^{0,v}) - \ln t\Big\} \\
&\quad + \sum_{i=1}^{k-1} \Big(W(h_t^{i,v}) - W(m_t^{i,v}) - \ln t\Big) + W(z) - W(m_t^{k,v}).
\end{aligned}
$$

3. When $z \in [h_t^{k,v}, m_t^{k+1,v}]$ for some $k \geq 1$, define

$$
\begin{aligned}
\alpha_t(v,z) &= \min\Big\{\Big(W(h_t^{+,v}) - W(h_t^{-,v})\Big)^+ ; W(h_t^{0,v}) - W(m_t^{0,v}) - \ln t\Big\} \\
&\quad + \sum_{i=1}^{k-1} \Big(W(h_t^{i,v}) - W(m_t^{i,v}) - \ln t\Big) + W(z) - W(m_t^{k,v}) \\
&\quad + \min\Big\{\Big(W(h_t^{-,z}) - W(h_t^{+,z})\Big)^+ ; W(h_t^{-,z}) - W(m_t^{k+1,v}) - \ln t\Big\}.
\end{aligned}
$$

4. When $z < v$ and $z \notin [m_t^{0,v}, m_t^{1,v}]$, define $\alpha_t(z,v)$ by the symmetric construction.

One readily checks that $\alpha_t(v,z)$ is a.s. positive for $t > 1$ and, from Brownian scaling, that the process $\alpha_t(v,z)$ itself has scaling properties, precisely, for all $\lambda > 0$,

$$
\Big(\frac{\alpha_{s^\lambda}(\lambda^2 v, \lambda^2 z)}{\lambda}\Big)_{s>1} \overset{\text{law}}{=} (\alpha_s(v,z))_{s>1}. \tag{7}
$$

Now, we state the main result of this section.

Theorem 2.1 *Fix an arbitrary $M > 0$. With $\alpha_t(v,z)$ defined above, it holds that*

$$
\sup_{|v|,|z| \leq M \ln^2 t} \Big| \frac{\ln \mathsf{P}_\omega[\xi_t^v = z] + \alpha_t(v,z)}{\ln t} \Big| \longrightarrow 0 \qquad \textit{in } \mathbb{P}\textit{-probability.}
$$

Moreover, for any fixed $v, z \in \mathbb{R}$

$$
\frac{\ln \mathsf{P}_\omega[\xi_t^{v \ln^2 t} = z \ln^2 t]}{\ln t} \longrightarrow -\alpha_e(v,z)
$$

in law as $t \to \infty$. (In this formula, $v \ln^2 t$ and $z \ln^2 t$ stand for their integer part.)

We now comment the definition of $\alpha_t(v,z)$, starting with the case $z \in [m_t^{0,v}, m_t^{1,v}]$. Note first that, if $W(h_t^{+,v}) - W(h_t^{-,v}) \geq 0$ and $W(h_t^{+,z}) - W(h_t^{-,z}) \geq 0$, then $S^{(0)} = W(z) - W(m_t^{0,v})$ is larger than $S^{(01)}$ and $S^{(10)}$. More generally, when v and z are attracted (at time horizon t) by the same point in $\mathcal{S}_t$, i.e. when $\hat{m}_t^v = \hat{m}_t^z$ (or, equivalently, when $[W(h_t^{+,v}) - W(h_t^{-,v})][W(h_t^{+,z}) - W(h_t^{-,z})] \geq 0$), then the definition of α_t reduces to $\alpha_t(v,z) = \min\{S^{(0)}, S^{(1)}\}$. In particular, $\alpha_t(v,v)$ reduces to this simple formula, see (8) below. The cases 2) and 3) in the definition of $\alpha_t(v,z)$ above are the formal extension of the case 1) to positive k. In fact, we will see from the proof of Theorem 2.1, that each one of the four terms $S^{(0)}, S^{(1)}, S^{(01)}, S^{(10)}$, corresponds to a different strategy for the walker starting from v to be in z at time t. The term $S^{(0)}$ comes from such paths, but which also hit $m_t^{0,v}$ before time t. Let us depict this strategy as $v \mapsto m_t^{0,v} \mapsto z$. Then, the term $S^{(1)}$ corresponds to the strategy $v \mapsto m_t^{1,v} \mapsto z$, $S^{(01)}$ to $v \mapsto m_t^{0,v} \mapsto m_t^{1,v} \mapsto z$ (paths which hit $m_t^{0,v}$ first, then $m_t^{1,v}$, before time t), and $S^{(10)}$ to $v \mapsto m_t^{1,v} \mapsto m_t^{0,v} \mapsto z$.

The case $v = z = 0$ is of special interest. Introduce the shorthand notations $m_t^{\pm} := m_t^{\pm,0}$, $h_t^{\pm} := h_t^{\pm,0}$, and consider $\hat{\alpha}_t = \alpha_t(0,0)$, i.e. the random process

$$\hat{\alpha}_t = \min\{2(W(h_t^+) - W(h_t^-))^+ - W(m_t^+); 2(W(h_t^-) - W(h_t^+))^+ - W(m_t^-)\}. \tag{8}$$

Clearly, $\hat{\alpha}_t$ is a.s. strictly positive for $t > 1$. The process $\hat{\alpha}$ is piecewise constant, non-decreasing with limit 0 as $t \to 0+$ and has finitely many jumps on compact intervals of $(0,\infty)$. An immediate consequence of Theorem 2.1 is the following

Corollary 2.1 *With $\hat{\alpha}_t$ from (8) it holds*

$$\frac{\ln \mathtt{P}_\omega[\xi_t^0 = 0] + \hat{\alpha}_t}{\ln t} \longrightarrow 0, \qquad \text{in } \mathbb{P}\text{-probability.}$$

In particular,

$$\frac{\ln \mathtt{P}_\omega[\xi_t^0 = 0]}{\ln t} \longrightarrow -\hat{\alpha}_e$$

in law as $t \to \infty$.

This result does not relate to the (annealed) asymptotics in [7] for the time, after t, of return to the starting point.

From the scaling properties of the Brownian motion and (8) it is elementary to conclude that the times of jumps of the process $\hat{\beta}_s := \hat{\alpha}_{e^{e^s}}$ form a stationary point process. We do not discuss here the questions related to the distribution of that point process (this is similar to so-called *aging properties* of random walks in random environment; cf. e.g. [13, 22]).

In the next theorem, we compute the law of $\hat{\alpha}_e$. As can be seen from scaling properties, this law does not depend on σ, i.e., it is the same for all the random walks in random environment satisfying Conditions S and B.

Theorem 2.2 *The positive random variable $\hat{\alpha}_e$ has density*

$$p(z) = \begin{cases} 2 - z - (z+2)e^{-2z}, & \text{if } z \in (0,1), \\ ([e^2 - 1]z - 2)e^{-2z}, & \text{if } z \geq 1, \end{cases}$$

and Laplace transform

$$\mathbb{E}e^{-s\hat{\alpha}_e} = \frac{4e^{-s} + 2(s^2 + 2s - 2)}{s^2(s+2)^2} \ . \tag{9}$$

2.2 Moderate deviations

The classical result of Sinai [20] shows that at time t the particle, which performs the random walk in random environment satisfying Condition S, should be at distance of order $\ln^2 t$ from its starting point. In [8] Hu and Shi identified the upper limit for the walk: $\limsup \xi_n^0/((\ln^2 n)(\ln\ln\ln n)) = 8/\pi^2\sigma^2$, a.s. As for the quenched probabilities of large deviations $\mathtt{P}_\omega[\xi_t^0 \geq tu]$, the present model in Sinai's regime is covered by [6, 2] and the decay is exponential in t, but also different regimes are considered when Condition S is not satisfied, leading to full variety of behaviors (see [22]). Therefore, in the case of Sinai's regime, there remains a big gap corresponding to moderate deviations, in spite of partial results in [3, 8]. We hope to partially close this gap with Theorem 2.3 below. It is strongly believed that the large deviations rate function is non-analytic at 0 (cf. [6, 2]), this makes any moderate deviation result interesting and rather hard to predict. Let us introduce some notations. Fix $M > 0$, and consider a positive increasing function $\varphi(t)$ such that $\varphi(t) \to \infty$ as $t \to \infty$. For all $t \geq e^e$, define the interval $R_t(\varphi, M)$ by

$$R_t(\varphi, M) = [\ln^2 t \times \ln\ln\ln t \times \varphi(t), \ln^{2+M} t].$$

Theorem 2.3 *For any such φ and M we have*

$$\sup_{z \in R_t(\varphi,M)} \left| \frac{2 \ln \mathtt{P}_\omega[\xi_t^0 \geq z]}{\sigma^2 z \ln^{-1} t} + 1 \right| \longrightarrow 0 \qquad \mathbb{P}\text{-}a.s. \tag{10}$$

as $t \to \infty$. The same result holds if one substitutes ξ_t^0 by $\max_{s \leq t} \xi_s^0$.

2.3 On the time of the first return to the origin

We finish this section by stating the following interesting fact.

Theorem 2.4 *Let $\hat{\tau} = \inf\{t > 0; \xi_t^0 = 0,$ there exists $s < t : \xi_s^0 \neq 0\}$ be the time of first return to the origin. Then, for all $a > 0$, it holds that $\mathtt{E}_\omega \hat{\tau}^a = \infty$ $\mathbb{P}$-a.s.*

This result (to be proved in Section 5) shows that, although the Sinai's random walk in random environment gives an example of strongly subdiffusive behaviour ($\ln^2 t$ instead of $\sqrt{t}$), it may be nevertheless very difficult to get back to 0.

We end this section with two general, important remarks.

Remark. The results of Section 2 are stated for a continuous-time random walks, but it can be shown that the same holds for the discrete-time Sinai's random walk as well (provided that the analogue of the Condition B is satisfied). We have chosen the continuous time because of the fact that the discrete-time random walk is periodic, and this provides some additional technical difficulties, especially when dealing with the spectral properties of the reflected walks.

Warning: Observe that, according to Definition 2.1, the points m, m' are not generally integers. Thus, throughout this paper, the statement "the random walk hits a t-stable point" means that it hits the site $x \in \mathbb{Z}$ which is closest to the t-stable point. As a rule all through the paper, real points $x \in \mathbb{R}$ will be replaced, if the context requires, with the closest integer, that we may still denote by the same symbol x, if no confusion can occur.

3 Some preliminary facts

First of all, we recall the following basic fact. Define

$$\tau_A(\xi^x) = \inf\{t > 0; \xi^x_t \in A\} \tag{11}$$

the hitting time of the set A for the process ξ^x (random walk in random environment which starts from x). For any integers $a < x < b$, the probability for ξ^x to reach b before a can be easily computed:

$$\mathtt{P}_\omega[\tau_b(\xi^x) < \tau_a(\xi^x)] = \frac{\sum_{y=a+1}^{x} e^{V(y)-V(a)}}{\sum_{y=a+1}^{b} e^{V(y)-V(a)}}, \tag{12}$$

see e.g. Lemma 1 in [20].

On the other hand from (2), if x is not too far away from the origin, then $V(x)$ and $W(x)$ are rather close for the vast majority of environments. Hence, it is convenient to introduce the following set of "good" environments, and to restrict our forthcoming computations to this set. Fix an arbitrary $M_0 > 0$; for any $t > e$, let

$$\Gamma_t = \Big\{\omega : |V(x) - W(x)| \le K_0 \ln\ln t\ ,\ x \in [-\ln^{M_0} t, \ln^{M_0} t]\Big\}, \tag{13}$$

where $K_0 \in (0, \infty)$ is chosen in such a way that for $\mathbb{P}$-almost all ω, it holds that $\omega \in \Gamma_t$ for all t large enough.

Before showing existence of such a K_0, we emphasize an important remark, in the spirit of the above *warning*. In the definition (13) of Γ_t, x is a real number, and, as usual, $V(x)$ is a slight abuse of notations for the value of V at the integer lattice site closest to x, in contrast with the value $W(x)$ of W at the precise $x \in \mathbb{R}$.

Clearly, there exists a finite $K_0 = K_0(M_0)$ such that $\mathbb{P}[\text{there exists } s : \omega \in \Gamma_t, t > s] = 1$, because of (2) and from the uniform continuity modulus of W on a bounded interval.

All through, we will repeatedly use that, for $\omega \in \Gamma_t$, the estimate

$$\left| \ln \mathtt{P}_\omega[\tau_b(\xi^x) < \tau_a(\xi^x)] - \Big(\max_{(a,x]} W - \max_{(a,b]} W \Big) \right| \leq (2K_0 + M_0) \ln \ln t = o(\ln t)$$

holds uniformly in $a, b \in [-\ln^{M_0} t, \ln^{M_0} t]$, $a < b$.

3.1 Spectral properties of reflected random walks in random environment

We will often consider couplings of our random walk in random environment with reflected ones. In this section, we recall some useful facts. Let $I = [a, b]$, $-\infty < a < b < +\infty$, be a finite interval in $\mathbb{Z}$. The random walk in random environment *reflected in* I is the Markov process $\hat{\xi}^I$ which has the same jump rates as ξ in the open interval (a, b), which jumps from a to $a+1$ at rate ω_a^+ and from b to $b-1$ at rate ω_b^-. This process is ergodic, with the reversible, invariant measure μ^I given by

$$\mu^I(x) = \theta_x \Big(\sum_{y \in I} \theta_y \Big)^{-1}, \quad x \in I,$$

and $\mu^I(x) = 0$ for $x \notin I$. From Condition B it follows that

$$K_1 e^{-V(x)} \Big[\sum_{y \in I} e^{-V(y)} \Big]^{-1} \leq \mu^I(x) \leq K_2 e^{-V(x)} \Big[\sum_{y \in I} e^{-V(y)} \Big]^{-1} \tag{14}$$

for all $x \in I$. Reversibility means that the generator $\mathcal{L}^I$ of $\hat{\xi}^I$ is symmetric in the space $L^2(\mu^I)$, and it defines a symmetric Dirichlet form $\mathcal{E}^I$ on this space,

$$\mathcal{E}^I(f, f) := -\langle \mathcal{L}^I f, f \rangle_{L^2(\mu^I)} = \sum_{x \in [a,b)} \Big(f(x+1) - f(x) \Big)^2 \mu^I(x) \omega_x^+.$$

The spectral gap of $\hat{\xi}^I$ is defined by

$$\lambda(I) = \inf \Big\{ \mathcal{E}^I(f, f); \sum_{x \in I} f^2(x) \mu^I(x) = 1, \sum_{x \in I} f(x) \mu^I(x) = 0 \Big\}.$$

The speed of convergence to the equilibrium relates to the spectral gap: for $x, y \in I$ and $s > 0$,

$$\left| \mathsf{P}_\omega[\hat{\xi}^I_s = x \mid \hat{\xi}^I_0 = y] - \mu^I(x) \right| \leq \left(\frac{\mu^I(x)}{\mu^I(y)} \right)^{1/2} \exp\{-\lambda(I)s\}, \tag{15}$$

see Corollary 2.1.5 in [18]. Here again, we do not indicate explicitly in our notations, the dependence of $\mu^I, \mathcal{E}^I, \mathcal{L}^I, \lambda(I)$ on ω.

The spectral gap of a general birth and death process can be precisely estimated using a result of Miclo. For any $A \subset I$ define $\mu(A) := \sum_{x \in A} \mu(x)$; letting

$$B^I_+(i) = \sup_{x>i} \Big(\sum_{y=i+1}^{x} (\mu^I(y)\omega^-_y)^{-1} \Big) \mu^I[x, b), \tag{16}$$

$$B^I_-(i) = \sup_{x<i} \Big(\sum_{y=x}^{i-1} (\mu^I(y)\omega^+_y)^{-1} \Big) \mu^I(a, x], \tag{17}$$

and

$$B^I = \min_{i \in I} (B^I_-(i) \vee B^I_+(i)),$$

we have, from Proposition 1.3 of [15], that

$$\frac{1}{4B^I} \leq \lambda(I) \leq \frac{2}{B^I} \ . \tag{18}$$

Consider an interval $I = [a, b], a < b$. The *elevation* of I (cf. [14]) is defined as the Brownian functional

$$\mathfrak{E}(I) = \max_{x,y \in I} \max_{z \in [x,y]} \{W(z) - W(x) - W(y) + \min_{v \in I} W(v)\}. \tag{19}$$

Our convention is that $[x, y]$ denotes the interval with endpoints x, y regardless of $x < y$ or $x > y$. It can be easily seen that in the definition of $\mathfrak{E}(I)$ one may assume that y is the global minimum of W on I, x is one of local minima, and z is one of local maxima of $W(\cdot)$ in $[a, b]$ (see Figure 2). Clearly, $\mathfrak{E}(I) \leq \mathfrak{E}(J)$ if $I \subset J$.

It follows from (19) and Definition 2.1 that for a t-stable point m,

$$\mathfrak{E}[l, r] < \ln t \tag{20}$$

with $l = l(t, m), r = r(t, m)$. The following result shows the interest of the quantity defined by (19).

Proposition 3.1 *Fix an arbitrary $M > 0$. Then*

$$\lim_{t \to \infty} \sup_{[a,b] \subset [-\ln^M t, \ln^M t]} \frac{|\ln \lambda[a, b] + \mathfrak{E}[a, b]|}{\ln t} = 0 \qquad \mathbb{P}\text{-}a.s.$$

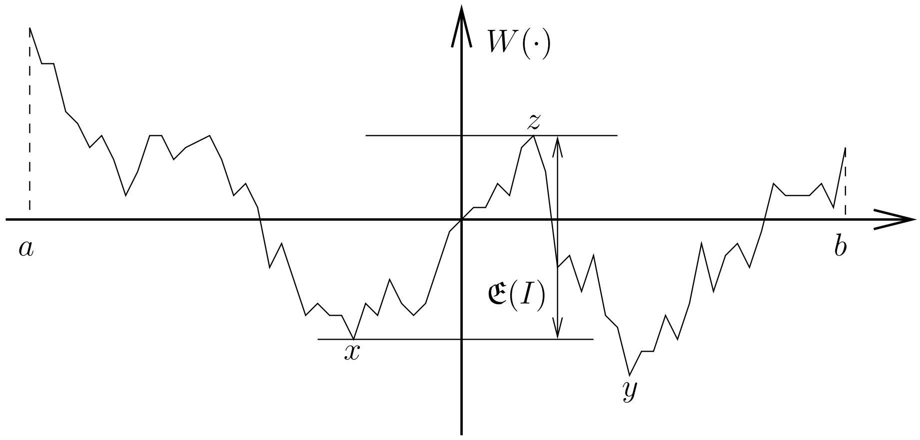

Figure 2: On the definition of elevation $\mathfrak{E}(I)$ on the interval $I = [a, b]$

Proof. For fixed $a, b \in [-\ln^M t, \ln^M t]$ abbreviate $I = [a, b]$. Using (1) and (14), we estimate the spectral constants $B^I_\pm(i)$ defined in the formulae above (16)–(17): for $i \in I$,

$$\begin{aligned} K_1 \exp\Big\{ \max_{j>i, j\in I} [\max_{k\in[i,j]} V(k) - \min_{l\in I, l\geq j} V(l)]\Big\} & \\ \leq \ & B^I_+(i) \\ \leq \ & K_2 |I|^2 \exp\Big\{ \max_{j>i, j\in I} [\max_{k\in[i,j]} V(k) - \min_{l\in I, l\geq j} V(l)]\Big\}, \end{aligned}$$

and similarly for B^I_-. Using now (13) and (18), we see, denoting

$$\mathcal{H}(i,j) = \begin{cases} [j, b], & \text{if } j > i, \\ [a, j], & \text{if } j < i, \end{cases}$$

that

$$\Big(\ln \lambda(I) + \min_{i\in I} \Big\{ \max_{j\in I} [\max_{k\in[i,j]} W(k) - \min_{l\in\mathcal{H}(i,j)} W(l)]\Big\}\Big) = o(\ln t) \tag{21}$$

uniformly in $I = [a, b] \subset [-\ln^M t, \ln^M t]$, $\mathbb{P}$-a.s. Taking now x to be the global minimum of W on I, y and w local minima and z the maximum of W on $[x, y]$, it is then easy to see that

$$\min_{i\in I} \Big\{ \max_{j\in I} [\max_{k\in[i,j]} W(k) - \min_{l\in\mathcal{H}(i,j)} W(l)]\Big\} = \mathfrak{E}(I),$$

and we finish the proof of Proposition 3.1 by using (21). ∎

3.2 Upper bound on the probability of confinement

Let $m < m'$ be two neighboring t-stable points (i.e., $(m,m')\cap\mathcal{S}_t = \emptyset$, where $\mathcal{S}_t$ is the set of all t-stable points). Our first goal in this section is to show that the random walk starting somewhere in between m and m' will hit, with extremely high probability, at least one of the points m, m' before time t (Lemma 3.1). Suppose also that $m, m' \in [-\ln^{M_0} t, \ln^{M_0} t]$, with M_0 from (13).

Let $h \in [m, m']$ be such that

$$W(h) = \max_{y\in[m,m']} W(y), \tag{22}$$

and fix some number $\kappa > 0$ (in the sequel we will generally suppose that κ is a constant, but this technique can be used with $\ln\kappa = o(\ln t)$ as $t \to \infty$).

We study here the probability of confinement $\mathtt{P}_\omega[\tau_{\{m',m\}}(\xi^x) > t/\kappa]$, making use of the excursions of the random walk from the point h.

Let $\hat{\xi}^x_t$ be the reflected random walk in random environment on the interval $[m, m']$ starting from $x \in [m, m']$; clearly, $\tau_{\{m',m\}}(\xi^x)$ has the same distribution as $\tau_{\{m',m\}}(\hat{\xi}^x)$.

We need to consider two other processes, which also are reflected versions of our random walk in random environment: Let $\hat{\xi}^{x,+}$ be the reflected random walk in random environment on $I^+ := [h, m']$ starting from h, and let $\hat{\xi}^{x,-}$ be the reflected random walk in random environment on $I = I^- := [m, h]$ starting from x (without restricting of generality we suppose that $x \in [m, h]$). The process $\hat{\xi}^{x,+}$ has the same jump rates as $\hat{\xi}^x$ on $(h, m']$, but jumps from h to $h+1$ at rate ω_h^+, while $\hat{\xi}^{x,-}$ has the same jump rates as $\hat{\xi}^x$ on $[m, h)$, but jumps from h to $h-1$ at rate ω_h^-.

It follows from the excursion theory for Markov processes that our reflected random walk in random environment can be obtained by mingling these two walks. Indeed, let $S_n^\pm, n \geq 1$ be the n-th excursion time from h for $\hat{\xi}^{x,\pm}$, let $\ell^{h,\pm}(\cdot)$ be its local time at point h, and $\zeta_n^\pm$ be its n-th excursion. It is well known that the excursion processes

$$\mathcal{N}^\pm = \sum_{n\geq 1} \delta_{(\ell^{h,\pm}(S_n^\pm),\zeta_n^\pm)}$$

are Poisson point processes with intensity measure $ds \otimes \omega_h^\pm \nu^\pm$ (see e.g. [17], VI–43). Here $\nu^\pm$ is the probability distribution (the so-called excursion law) of our reflected random walk in random environment starting at $h\pm 1$ and killed at h. Considering the excursion process is equivalent to considering the Markov process itself, and $\mathcal{N}^+, \mathcal{N}^-$ are independent. The superposition $\mathcal{N}^+ + \mathcal{N}^-$ is still a Poisson point process with intensity $ds\otimes(\omega_h^+\nu^+ + \omega_h^-\nu^-) = ds\otimes(\omega_h^+ + \omega_h^-)\nu$, where ν stands for the excursion law of $\hat{\xi}^x$. It then follows that $\mathcal{N}^+ + \mathcal{N}^-$ is the excursion process of our random walk in random environment $\hat{\xi}^x$ reflected in $[m, m']$. (Assuming $x \in [m, h]$ as above, we need first to patch $\hat{\xi}^{x,-}$ until it reaches h.) Therefore, for $\kappa \geq 1$,

$$\mathtt{P}_\omega[\tau_{\{m',m\}}(\hat{\xi}^x) > t/\kappa] \;\leq\; \mathtt{P}_\omega[\{\tau_{m'}(\hat{\xi}^{x,+}) > t/(2\kappa)\} \cup \{\tau_m(\hat{\xi}^{x,-}) > t/(2\kappa)\}]$$

$$\leq \ \mathtt{P}_\omega[\tau_{m'}(\hat{\xi}^{x,+}) > t/(2\kappa)] + \mathtt{P}_\omega[\tau_m(\hat{\xi}^{x,-}) > t/(2\kappa)].$$

Let n be a positive integer. Since $\tau_{m'}(\hat{\xi}^{x,+}) > t/(2\kappa)$ implies $\hat{\xi}^{x,+}_s \neq m'$, for $s = kt/(2\kappa n), k = 1, 2, \dots, n$, we can use the Markov property and (15) to get

$$\begin{aligned}
&\mathtt{P}_\omega[\tau_{m'}(\hat{\xi}^{x,+}) > t/(2\kappa)] \\
&\leq \sup_{y \in I^+ \setminus \{m'\}} \mathtt{P}_\omega[\hat{\xi}^{y,+}_{t/(2\kappa n)} \neq m']^n \\
&\leq \Big(1 - \mu_{I^+}(m') + \Big[\frac{\mu_{I^+}(m')}{\inf_{x \in I^+ \setminus \{m'\}} \mu_{I^+}(x)}\Big]^{1/2} e^{-\lambda(I^+)t/(2\kappa n)}\Big)^n \\
&\leq \exp\Big\{-n\Big(\mu_{I^+}(m') - \Big[\frac{\mu_{I^+}(m')}{\inf_{I^+ \setminus \{m'\}} \mu_{I^+}(\cdot)}\Big]^{1/2} e^{-\lambda(I^+)t/(2\kappa n)}\Big)\Big\}. \qquad (23)
\end{aligned}$$

Now, denote $\gamma_1 = \max_{x \in [m,m']} V(x) - \min_{x \in [m,m']} V(x)$, $\Delta_1 = m' - m$, and take n as the integer part of $t^{\frac{1}{2}(1 - \mathfrak{E}(I^+)\ln^{-1} t)}$, which is strictly positive, see (20) for a similar fact. As, by (13), $\mu_{I^+}(m') \geq K_1(\Delta_1 \ln^{2K_0} t)^{-1}$, with this choice of n the formula (23) implies that

$$\begin{aligned}
\mathtt{P}_\omega[\tau_{m'}(\hat{\xi}^{x,+}) > t/(2\kappa)] \ \leq \ \exp\Big\{ &- t^{\frac{1}{2}(1 - \mathfrak{E}(I^+)\ln^{-1} t)}\Big(K_1(\Delta_1 \ln^{2K_0} t)^{-1} \\
&- K_2 e^{\gamma_1/2} \exp\Big\{-\frac{\lambda(I^+) e^{\mathfrak{E}(I^+)} t^{\frac{1}{2}(1 - \mathfrak{E}(I^+)\ln^{-1} t)}}{2\kappa}\Big\}\Big)\Big\}.
\end{aligned}$$

We have obviously a similar estimate for $\hat{\xi}^{x,-}$. Hence, we have proved the following

Lemma 3.1 *Let $t > 1$, and $I^+ := [h, m']$ and $I^- := [m, h]$ (which depend on t). For all $x \in [m, m']$, it holds on Γ_t that*

$$\begin{aligned}
\mathtt{P}_\omega[\tau_{\{m',m\}}(\xi^x) > t/\kappa] \ \leq \ &\exp\Big\{ - t^{\frac{1}{2}(1 - \mathfrak{E}(I^+)\ln^{-1} t)}\Big(K_1(\Delta_1 \ln^{2K_0} t)^{-1} \\
&\qquad - K_2 e^{\gamma_1/2} \exp\Big\{ - \frac{\lambda(I^+) e^{\mathfrak{E}(I^+)} t^{\frac{1}{2}(1 - \mathfrak{E}(I^+)\ln^{-1} t)}}{2\kappa}\Big\}\Big)\Big\} \\
&+ \exp\Big\{ - t^{\frac{1}{2}(1 - \mathfrak{E}(I^-)\ln^{-1} t)}\Big(K_1(\Delta_1 \ln^{2K_0} t)^{-1} \\
&\qquad - K_2 e^{\gamma_1/2} \exp\Big\{ - \frac{\lambda(I^-) e^{\mathfrak{E}(I^-)} t^{\frac{1}{2}(1 - \mathfrak{E}(I^-)\ln^{-1} t)}}{2\kappa}\Big\}\Big)\Big\},
\end{aligned}$$

where K_1 and K_2 are absolute constants.

Now we finish this section by considering another typical situation, for which the probability of confinement is extremely small. Consider an interval $[\hat{h}, h] \subset [-\ln^{M_0} t, \ln^{M_0} t]$ (M_0 is from (13)), and let m be such that

$$W(m) = \min_{x \in [\hat{h}, h]} W(x);$$

Suppose also that

$$W(\hat{h}) = \max_{x\in[\hat{h},m]} W(x), \qquad W(h) = \max_{x\in[m,h]} W(x).$$

Now, we assume that

$$\min\{W(\hat{h}), W(h)\} - W(m) < (1-\varepsilon_1)\ln t$$

for some fixed $\varepsilon_1 > 0$, implying that $[\hat{h}, h]$ is *not* a t-stable well. Assume also that

$$\mathfrak{E}[\hat{h}, h] < (1-\varepsilon_2)\ln t$$

for some $\varepsilon_2 > 0$. For any $x \in [\hat{h}, h]$, we are going to study the probability of confinement $\mathsf{P}_\omega[\tau_{\{\hat{h},h\}}(\xi^x) > t/\kappa]$, and show that it is extremely unlikely for the walk to stay in the interval $[\hat{h}, h]$ up to time of order t.

Without restricting of generality, suppose that $W(h)-W(m) < (1-\varepsilon_1)\ln t$. It is a fact that the distribution of the random variable $\tau_{\{\hat{h},h\}}(\xi^x)$ depends only on the environment on the interval $[\hat{h}+1, h-1]$. So, the first idea would be to consider the random walk in random environment with reflection in $\hat{h}, h$, and use the same method as in the proof of Lemma 3.1. This, however, does not work because of the fact that $\mu(h)$ can be rather small. To get around that difficulty, we use the following construction. Consider a random walk in random environment $\bar{\xi}^x$ on the interval $[\hat{h}, 2h-m]$, defined as follows:

- there is reflection in $\hat{h}$ and in $2h-m$;
- on $[\hat{h}, h]$, this random walk has the same transition rates as ξ^x;
- for $i = 1, \dots, h-m-1$, we define the jump rates $\bar{\omega}^+_{h+i} := \omega^-_{h-i}$, $\bar{\omega}^-_{h+i} := \omega^+_{h-i}$, and $\bar{\omega}^-_h = \bar{\omega}^+_h := \omega^-_h$, so the part of the potential $V(\cdot)$ on $[m, h]$ is "reflected" around h onto $[h, 2h-m]$.

We use symbols with "$^-$" to refer to this new random walk in random environment $\bar{\xi}^x$, e.g. $\bar{V}$ denotes its potential on $[\hat{h}, 2h-m]$. From this construction and the definition (19) it follows that $\bar{\mathfrak{E}}[\hat{h}, 2h-m] \le (1-\varepsilon_3)\ln t$, where $\varepsilon_3 = \varepsilon_1 \wedge \varepsilon_2$. By symmetry, $\bar{V}(2h-m) = \bar{V}(m)$, and then $\bar{\mu}(2h-m) \ge K\bar{\mu}(m)$ by (14). Now, observe that $\tau_{\{\hat{h},h\}}(\bar{\xi}^x) \le \tau_h(\bar{\xi}^x) < \tau_{2h-m}(\bar{\xi}^x)$ for any $x \in [\hat{h}+1, h-1]$. Using this observation, we proceed analogously to (23) and we get the following estimate:

Lemma 3.2 *Let $\gamma_2 = \max\{V(\hat{h}), V(h)\} - V(m)$, $\Delta_2 = 2h-m-\hat{h}$, and ε_3 defined above. For any $x \in [\hat{h}+1, h-1]$ it holds on Γ_t that*

$$\mathsf{P}_\omega[\tau_{\{\hat{h},h\}}(\xi^x) > t/\kappa] \le \exp\Big\{-t^{\varepsilon_3/2}\Big(K_1(\Delta_2 \ln^{2K_0} t)^{-1} - K_2 e^{\gamma_2/2} \exp\Big\{-\frac{\bar{\lambda}(I)e^{\bar{\mathfrak{E}}(I)}t^{\varepsilon_3/2}}{2\kappa}\Big\}\Big)\Big\},$$

where $I := [\hat{h}, 2h-m]$.

To conclude this section, we note that the upper bounds in Lemmas 3.2 and 3.1 are stretched exponential as $t \to \infty$, and therefore are negligible compared to any negative power of t, in $\mathbb{P}$-probability.

3.3 Cost of escaping from a t-stable well

First, we need the following fact:

Lemma 3.3 *(i) For some $K_1 \in (0, \infty)$, we have for all $s > 0, x, y \in \mathbb{Z}$*

$$\mathtt{P}_\omega[\tau_y(\xi^x) < s] \leq K_1 \int_0^{s+1} \mathtt{P}_\omega[\xi_u^x = y] du.$$

(ii) Also, for some $K_2 \in (0, \infty)$, we have for all $s > 0, x < y \in \mathbb{Z}$

$$\mathtt{P}_\omega[\tau_y(\xi^x) < s] \geq K_2 \mathtt{P}_\omega[\tau_y(\xi^{y-1}) \geq s] \int_0^s \mathtt{P}_\omega[\xi_u^x = y] du$$

(the extension to the case $x > y$ is straightforward).

This lemma is of interest for us, since it is easier to estimate $\mathtt{P}_\omega[\xi_s^x = y]$ using reversibility, than to estimate the quantity $\mathtt{P}_\omega[\tau_y(\xi^x) < s]$ of interest.

Proof of Lemma 3.3. Indeed, if $N_y(\xi^x, t)$ denotes the number of visits of the process ξ^x at y before time t, we have by Fubini's theorem

$$\begin{aligned} \int_0^{s+1} \mathtt{P}_\omega[\xi_u^x = y] du &= \mathtt{E}_\omega \int_0^{s+1} \mathbf{1}_{\{\xi_u^x = y\}} du \\ &\geq K_3 \mathtt{E}_\omega N_y(\xi^x, s) \\ &\geq K_3 \mathtt{P}_\omega[N_y(\xi^x, s) \geq 1] \\ &= K_3 \mathtt{P}_\omega[\tau_y(\xi^x) < s], \end{aligned}$$

where $K_3 = K_3(y) \in (0, 1)$ is the expectation of the minimum $S \wedge 1$ between an exponential random variable S with rate $\omega_y^+ + \omega_y^-$ and 1. Using Condition B we see that $K_3(y)$ is bounded from below by some universal constant, which concludes the proof of the the part (i).

As for the part (ii), first of all, let us place a reflecting (to the right) barrier at site y; clearly this does not change the distribution of the random variable $\tau_y(\xi^x)$. Now, we use Fubini's theorem, Condition B, and domination by a geometric random variable to get that

$$\begin{aligned} \int_0^s \mathtt{P}_\omega[\xi_u^x = y] du &= \mathtt{E}_\omega \int_0^s \mathbf{1}_{\{\xi_u^x = y\}} du \\ &\leq K_4 \mathtt{E}_\omega N_y(\xi^x, s) \quad \leq \quad \frac{K_5 \mathtt{P}_\omega[\tau_y(\xi^x) < s]}{1 - \mathtt{P}_\omega[\tau_y(\xi^{y-1}) < s]}, \end{aligned}$$

which proves the part (ii). ■

Again, let $m < m'$ be two neighboring t-stable points and h, defined by (22), is the highest pass between them. In this section we study the cost of escaping the t-stable well of m in the given direction (all results are stated for the escaping to the right; their extension for the other case is immediate). Throughout this section we suppose that all the t-stable points in question are within the interval $[-\ln^{M_0} t, \ln^{M_0} t]$ (with M_0 from (13)), and that ω is from Γ_t.

Lemma 3.4 *There is a constant $K > 0$ such that for any $x > h$ and any s we have*

$$\mathtt{P}_\omega[\tau_x(\xi^m) < s] \leq K(s+1)e^{-V(h)+V(m)}.$$

Proof. Combining Lemma 3.3 (i) with reversibility and using (1) we obtain

$$\begin{aligned}\mathtt{P}_\omega[\tau_x(\xi^m) < s] &\leq \mathtt{P}_\omega[\tau_h(\xi^m) < s] \\ &\leq K_1 \int_0^{s+1} \mathtt{P}_\omega[\xi_u^m = h]du \\ &= K_1 \int_0^{s+1} \frac{\theta_h}{\theta_m} \mathtt{P}_\omega[\xi_u^h = m]du \\ &\leq K_6(s+1)e^{-V(h)+V(m)}, \end{aligned} \tag{24}$$

which proves the lemma. ■

To provide an analogous lower bound, first we need to estimate the probability that a particle which started from a given t-stable point will be in this point after a given time. To this end, let $\tilde{m} = \sup\{x \in \mathcal{S}_t : x < m\}$ and denote by $\tilde{h}$ the point such that $W(\tilde{h}) = \max_{y\in[\tilde{m},m]} W(y)$. Define, for a fixed $K > 0$ and any $s \geq 0$

$$\varphi_{s,K}(m, \tilde{h}, h) = (h - \tilde{h})^{-1} \ln^{-2K_0} t - e^{-\lambda[\tilde{h},h]s} - Kse^{-\min\{V(\tilde{h}),V(h)\}+V(m)} \tag{25}$$

(the constant K_0 is from (13)).

Lemma 3.5 *There exists a positive constant K such that, for any $s > 0$, we have on Γ_t*

$$\mathtt{P}_\omega[\xi_s^m = m] \geq \varphi_{s,K}(m, \tilde{h}, h) .$$

Proof. Let $\tilde{\xi}$ be the reflected random walk in random environment on $[\tilde{h}, h]$ starting at m. Then, by a coupling argument,

$$\mathtt{P}_\omega[\xi_s^m = m] \geq \mathtt{P}_\omega[\tilde{\xi}_s = m] - \mathtt{P}_\omega[\tau_{\{\tilde{h},h\}}(\xi^m) < s]. \tag{26}$$

For the first term in right-hand side of the above display we use (15), and obtain

$$\mathtt{P}_\omega[\tilde{\xi}_s = m] \geq \mu_{[\tilde{h},h]}(m) - e^{-\lambda[\tilde{h},h]s}. \tag{27}$$

Analogously to Lemma 3.4 we can write

$$\mathtt{P}_\omega[\tau_{\tilde h}(\xi^m) < s] \le K_1(s+1)e^{-V(\tilde h)+V(m)}.$$

Therefore, the last term in (26) can be bounded by

$$\begin{aligned}\mathtt{P}_\omega[\tau_{\{\tilde h,h\}}(\xi^m) < s] &\le \mathtt{P}_\omega[\tau_{\tilde h}(\xi^m) < s] + \mathtt{P}_\omega[\tau_h(\xi^m) < s]\\ &\le K_2 s e^{-\min\{V(\tilde h),V(h)\}+V(m)}. \qquad (28)\end{aligned}$$

Note that, by (13), $\mu_{[\tilde h,h]}(m) \ge (h-\tilde h)^{-1}\ln^{-2K_0} t$. So, inserting (27) and (28) into (26), we finish the proof of Lemma 3.5. ∎

Now we give a lower bound for the probability of escaping from a t-stable well.

Lemma 3.6 *Suppose that*

$$\mathtt{P}_\omega[\tau_{\{m,m'\}}(\xi^h) > s/4] \le \frac{(m'-m)^{-1}\ln^{-2K_0} t}{2}. \qquad (29)$$

Then there are constants $K', K'' > 0$ such that on Γ_t

$$\mathtt{P}_\omega[\tau_h(\xi^m) < s] \ge \frac{K'\varphi_{s,K}(m,\tilde h,h)se^{V(m)-V(h)}}{(m'-m)^2\ln^{4K_0} t}\left(1-K''(s+1)e^{V(m)-V(h)}\right)^+ \qquad (30)$$

with the quantity $\varphi_{s,K}(m,\tilde h,h)$ defined by (25).

Note that, for typical ω and $s \le t$, the positive part in (30) will be larger than 1/2.

Proof of Lemma 3.6. Note that the formula (12) together with Lemma 3.4 imply that for any $s \le t$

$$\begin{aligned}\mathtt{P}_\omega[\tau_h(\xi^{h-1}) \ge s] &\ge \mathtt{P}_\omega[\tau_m(\xi^{h-1}) < \tau_h(\xi^{h-1})]\mathtt{P}_\omega[\tau_h(\xi^m) \ge s]\\ &\ge K(m'-m)^{-1}(\ln^{-2K_0} t)\left(1-K''(s+1)e^{V(m)-V(h)}\right)^+ .\end{aligned}$$

By Lemma 3.3 (ii), and by reversibility,

$$\begin{aligned}\mathtt{P}_\omega[\tau_h(\xi^m) < s] &\ge \mathtt{P}_\omega[\tau_h(\xi^{h-1}) \ge s]\int_{s/2}^{s}\mathtt{P}_\omega[\xi_u^m = h]du\\ &\ge \frac{\theta_h}{2\theta_m}\mathtt{P}_\omega[\tau_h(\xi^{h-1}) \ge s]\inf_{u\in[\frac{s}{2},s]}\mathtt{P}_\omega[\xi_u^h = m]\\ &\ge \frac{\theta_h}{2\theta_m}\mathtt{P}_\omega[\tau_h(\xi^{h-1}) \ge s]\inf_{\frac{s}{4}\le u\le s}\mathtt{P}_\omega[\xi_u^m = m]\\ &\quad \times \mathtt{P}_\omega[\tau_m(\xi^h) < \tau_{m'}(\xi^h); \tau_m(\xi^h) \le s/4]. \qquad (31)\end{aligned}$$

Using (29) and (12), we get

$$\begin{aligned}\mathtt{P}_\omega[\tau_m(\xi^h) < \tau_{m'}(\xi^h); \tau_m(\xi^h) \le s/4] &\ge \mathtt{P}_\omega[\tau_m(\xi^h) < \tau_{m'}(\xi^h)] - \mathtt{P}_\omega[\tau_{m,m'}(\xi^h) > s/4] \\ &\ge \frac{1}{2}(m'-m)^{-1}\ln^{-2K_0} t.\end{aligned}$$

Applying Lemma 3.5 to the infimum in the right-hand side of (31), we then obtain the desired result. ■

4 Proof of Theorem 2.1

The present section is dedicated to the proof of the first statement in the theorem, since the last one directly follows from the scaling property (7).

For fixed ω and all $x \in \mathbb{R}$ define the "logarithmic stability index" $\mathfrak{I}(x) = \mathfrak{I}_\omega(x)$ of x, as

$$\mathfrak{I}(x) = \sup\{u > 0 : x \text{ is } e^u\text{-stable}\}\ .$$

Let $M > 0$ be as in Theorem 2.1. Recall that $\mathcal{S}_t = \{\dots, m_t^{-1,0}, m_t^{0,0}, m_t^{1,0}, \dots\}$ is the set of all t-stable points enumerated in the increasing order and such that $m_t^{0,0} \le 0 < m_t^{1,0}$, and $h_t^{i,0}$ is the highest pass between $m_t^{i,0}$ and $m_t^{i+1,0}$. It will be convenient to exclude some particular cases for the potential W, including ties and other pathologies, which happen with small probability.

Definition 4.1 *For any fixed $t > 1$, $\delta \in (0,1)$, $L > 2$, $A > M$, a particular realization of the random environment ω is called (δ, A, L, t)-good, if*

- *(i) there exist i, j such that $m_t^{i,0}, h_t^{i,0}, m_t^{i+1,0} \in (M\ln^2 t, A\ln^2 t]$ and $m_t^{j,0}, h_t^{j-1,0}, m_t^{j-1,0} \in [-A\ln^2 t, -M\ln^2 t)$;*
- *(ii) we have $|\mathcal{S}_t \cap [-A\ln^2 t, A\ln^2 t]| \le L$;*
- *(iii) for any $x \in [-A\ln^2 t, A\ln^2 t]$ it holds that $\mathfrak{I}(x) \notin [(1-\delta)\ln t, (1+\delta)\ln t]$;*
- *(iv) $\max\{|W(x)|; x \in [-A\ln^2 t, A\ln^2 t]\} \le L\ln t$.*

The following two lemmas will play a crucial role in the proof of Theorem 2.1.

Lemma 4.1 *Let ω be (δ, A, L, t)-good and let k be such that $j \le k \le i$, where i, j are from Definition 4.1 (i); abbreviate $m := m_t^{k,0}$, $\hat h := h_t^{k-1,0}$, $h := h_t^{k,0}$. Then,*

- *(i) we have $\mathfrak{E}[\hat h, h] \le (1-\delta)\ln t$,*
- *(ii) $\min\{W(\hat h), W(h)\} - W(m) \ge (1+\delta)\ln t$.*

Proof. Starting with (ii), we see that the left-hand side is not smaller than $\ln t$, by definition of $\hat{h}, h$. Set $m' = m_t^{k+1,0}$, and argue by contradiction. If $W(h) - W(m) < (1+\delta)\ln t$, then $m \notin \mathcal{S}_{t^{1+\delta}}$ or $m' \notin \mathcal{S}_{t^{1+\delta}}$ (where the first case happens if $W(m) > W(m')$, though the last one happens if $W(m) \le W(m')$), which would, in turn, imply that $\mathfrak{I}(m) \le (1+\delta)\ln t$ or $\mathfrak{I}(m') \le (1+\delta)\ln t$, a contradiction (as, clearly, $\mathfrak{I}(m) \ge \ln t$ and $\mathfrak{I}(m') \ge \ln t$). We now prove claim (i) in the lemma. By definition of $\hat{h}, h$, we have $\mathfrak{E}[\hat{h}, h] \le \ln t$. Conversely, $\mathfrak{E}([\hat{h}, h]) = \mathfrak{I}(x)$ for some $x \in [\hat{h}, h]$ and our claim follows directly from Definition 4.1 (iii). ■

Lemma 4.2 *(i) $\mathbb{P}[\omega$ is (δ, A, L, t)-good$]$ does not depend on t;*

(ii) For any fixed M and any $\varepsilon > 0$ one can choose A, L large enough and δ small enough in such a way that $\mathbb{P}[\omega$ is (δ, A, L, t)-good$] > 1 - \varepsilon$.

Proof. First, the part (i) follows directly from the scaling property of the Brownian motion. As for the part (ii), note that the set of all x which have positive index $\mathfrak{I}$ is countable. Moreover, for any $a > 0$, $0 < c < d$, the set

$$\{x \in [-a, a] : \mathfrak{I}(x) \in [c, d]\}$$

is $\mathbb{P}$-a.s. finite, and, finally, observe that $\mathbb{P}[\omega$ is $(0, \infty, \infty, t)$-good$] = 1$. Using the property (i), it is then not hard to get (ii). ■

Now, we begin proving Theorem 2.1. Fix arbitrary $\varepsilon > 0$; by Lemma 4.2 (ii), there are $A > M$, $L > 0$ and $\delta > 0$ such that

$$\mathbb{P}[\omega \text{ is } (\delta, A, L, t)\text{-good}] > 1 - \varepsilon.$$

Since the goal is to prove the convergence in $\mathbb{P}$-probability, from now on we restrict ourselves to the set of (δ, A, L, t)-good environments.

For the sake of brevity, in the rest of the proof of Theorem 2.1 we suppress the superscript "v" and the subscript "t" in m-s and h-s.

4.1 The case $z \in [m^0, m^1]$

The following decomposition is the key for our analysis:

$$\begin{aligned} \mathtt{P}_\omega[\xi_t^v = z] &= \mathtt{P}_\omega[\xi_t^v = z, \tau_{\{m^1, m^0\}}(\xi^v) > t/3] \\ &\quad + \mathtt{P}_\omega[\xi_t^v = z, \tau_{m^1}(\xi^v) \le t/3, \tau_{m^1}(\xi^v) < \tau_{m^0}(\xi^v)] \\ &\quad + \mathtt{P}_\omega[\xi_t^v = z, \tau_{m^0}(\xi^v) \le t/3, \tau_{m^0}(\xi^v) < \tau_{m^1}(\xi^v)] \\ &=: T_1 + T_2 + T_3\,, \end{aligned} \tag{32}$$

with $\tau_\cdot(\xi^\cdot)$ defined in (11).

First of all, let us show that the term T_1 is negligible. To this end, we use Lemma 3.1 and note the following:

- By Proposition 3.1, $\lambda(I)e^{\mathfrak{E}(I)} = t^{o(1)}$ as $t \to \infty$ for any $\varepsilon > 0$, where I denotes either $[m^0, h^0]$, or $[h^0, m^1]$.
- As ω is (δ, A, L, t)-good, we have that (using the notations of Section 3.2) $\Delta_1 < 2A \ln^2 t$, and, by Lemma 4.1, $1 - \mathfrak{E}(I) \ln^{-1} t \geq \delta$. Also, by definition, for (δ, A, L, t)-good environments in Γ_t, $\gamma_1 < L \ln t$.

Thus, Lemma 3.1 implies that for (δ, A, L, t)-good environments in Γ_t, we have

$$T_1 \leq \mathtt{P}_\omega[\tau_{\{m^1,m^0\}}(\xi^v) > t/3] = o(\exp\{-t^{\delta/3}\}) \tag{33}$$

as $t \to \infty$, uniformly in $v, z \in [-M \ln^2 t, M \ln^2 t]$.

Upper estimate for T_2. Conditioning on the σ-field $\mathcal{F}_{m^1,m^0}$ generated by ξ^v up to the stopping time $\tau_{\{m^1,m^0\}}$, we get by Markov property and using reversibility that

$$\begin{aligned}
T_2 &= \mathtt{E}_\omega \mathtt{P}_\omega[\xi^v_t = z, \tau_{m^1}(\xi^v) \leq t/3, \tau_{m^1}(\xi^v) < \tau_{m^0}(\xi^v) \mid \mathcal{F}_{m^1,m^0}] \\
&= \mathtt{E}_\omega\Big(\mathtt{P}_\omega[\xi^v_t = z \mid \mathcal{F}_{m^1,m^0}]\mathbf{1}_{\{\tau_{m^1}(\xi^v)\leq t/3, \tau_{m^1}(\xi^v)<\tau_{m^0}(\xi^v)\}}\Big) \\
&\leq \sup_{s\in[0,t]} \mathtt{P}_\omega[\xi^{m^1}_s = z] \times \mathtt{P}_\omega[\tau_{m^1}(\xi^v) \leq t/3, \tau_{m^1}(\xi^v) < \tau_{m^0}(\xi^v)] \\
&\leq \frac{\theta_z}{\theta_{m^1}} \sup_{s\in[0,t]} \mathtt{P}_\omega[\xi^z_s = m^1] \times \mathtt{P}_\omega[\tau_{m^1}(\xi^v) < \tau_{m^0}(\xi^v)].
\end{aligned} \tag{34}$$

For $s > 0$, write

$$\begin{aligned}
\mathtt{P}_\omega[\xi^z_s = m^1] &\leq \mathtt{P}_\omega[\tau_{m^1}(\xi^z) < \tau_{m^0}(\xi^z)] + \mathtt{P}_\omega[\xi^z_s = m^1, \tau_{m^0}(\xi^z) < \tau_{m^1}(\xi^z)] \\
&= \mathtt{P}_\omega[\tau_{m^1}(\xi^z) < \tau_{m^0}(\xi^z)] + \mathtt{E}_\omega\Big(\mathbf{1}_{\{\tau_{m^0}<\tau_{m^1}, \tau_{m^0}<s\}}\mathtt{P}_\omega[\xi^z_s = m^1 \mid \mathcal{F}_{m^1,m^0}]\Big) \\
&\leq \mathtt{P}_\omega[\tau_{m^1}(\xi^z) < \tau_{m^0}(\xi^z)] + \mathtt{P}_\omega[\tau_{m^0}(\xi^z) < \tau_{m^1}(\xi^z) \wedge s] \times \mathtt{P}_\omega[\tau_{m^1}(\xi^{m^0}) < s] \\
&\leq \mathtt{P}_\omega[\tau_{m^1}(\xi^z) < \tau_{m^0}(\xi^z)] + \mathtt{P}_\omega[\tau_{m^0}(\xi^z) < \tau_{m^1}(\xi^z)] \times \mathtt{P}_\omega[\tau_{h^0}(\xi^{m^0}) < s].
\end{aligned} \tag{35}$$

Finally, from (34), (35) and from Lemma 3.4 we obtain that, on Γ_t,

$$\begin{aligned}
T_2 \leq{}& \frac{\theta_z}{\theta_{m^1}} \mathtt{P}_\omega[\tau_{m^1}(\xi^v) < \tau_{m^0}(\xi^v)] \times \\
&\times \Big(\mathtt{P}_\omega[\tau_{m^1}(\xi^z) < \tau_{m^0}(\xi^z)] + \mathtt{P}_\omega[\tau_{m^0}(\xi^z) < \tau_{m^1}(\xi^z)] \Big[K_3(t+1)e^{V(m^0)-V(h^0)}\Big]\Big).
\end{aligned} \tag{36}$$

Combining this with (13) and (12), we obtain that

$$\ln T_2 \leq -\min\{S^{(1)}, S^{(10)}\} + o(\ln t) \tag{37}$$

as $t \to \infty$, uniformly in v, z, for (δ, A, L, t)-good environments $\omega \in \Gamma_t$.

Lower estimate for T_2. A calculation similar to (34) yields

$$T_2 \geq \frac{\theta_z}{\theta_{m^1}} \inf_{s\in[2t/3,t]} \mathtt{P}_\omega[\xi_s^z = m^1] \times \mathtt{P}_\omega[\tau_{m^1}(\xi^v) \leq t/3, \tau_{m^1}(\xi^v) < \tau_{m^0}(\xi^v)]. \tag{38}$$

For the last term in (38) we use the estimate

$$\begin{aligned}
&\mathtt{P}_\omega[\tau_{m^1}(\xi^v) \leq t/3, \tau_{m^1}(\xi^v) < \tau_{m^0}(\xi^v)]\\
&\quad = \mathtt{P}_\omega[\tau_{m^1}(\xi^v) < \tau_{m^0}(\xi^v)]\\
&\qquad - \mathtt{P}_\omega[\tau_{m^1}(\xi^v) < \tau_{m^0}(\xi^v), \tau_{\{m^1,m^0\}}(\xi^v) > t/3]\\
&\quad \geq \mathtt{P}_\omega[\tau_{m^1}(\xi^v) < \tau_{m^0}(\xi^v)] - \mathtt{P}_\omega[\tau_{\{m^1,m^0\}}(\xi^v) > t/3].
\end{aligned} \tag{39}$$

Analogously to (33), using Lemma 3.1 we show that the last term in (39) can be neglected for all t large enough on the set of environments which are (δ, A, L, t)-good and belong to Γ_t.

For the second term in the right-hand side of (38) we use the estimate

$$\begin{aligned}
\mathtt{P}_\omega[\xi_s^z = m^1] \;&\geq\; \mathtt{P}_\omega[\xi_s^z = m^1, \tau_{m^1}(\xi^z) < \tau_{m^0}(\xi^z), \tau_{\{m^1,m^0\}}(\xi^z) < t/3]\\
&\qquad + \mathtt{P}_\omega[\xi_s^z = m^1, \tau_{m^0}(\xi^z) < \tau_{m^1}(\xi^z), \tau_{\{m^1,m^0\}}(\xi^z) < t/3]\\
&=\; \mathtt{E}_\omega\Big(\mathbf{1}_{\{\tau_{\{m^1,m^0\}}(\xi^z)<(t/3)\wedge\tau_{m^0}(\xi^z)\}}\mathtt{P}_\omega[\xi_s^z = m^1 \mid \mathcal{F}_{m^1,m^0}]\Big)\\
&\qquad + \mathtt{E}_\omega\Big(\mathbf{1}_{\{\tau_{\{m^1,m^0\}}(\xi^z)<(t/3)\wedge\tau_{m^1}(\xi^z)\}}\mathtt{P}_\omega[\xi_s^z = m^1 \mid \mathcal{F}_{m^1,m^0}]\Big)\\
&\geq\; \inf_{u\in[t/3,t]} \mathtt{P}_\omega[\xi_u^{m^1} = m^1] \times \mathtt{P}_\omega[\tau_{m^1}(\xi^z) < \tau_{m^0}(\xi^z), \tau_{\{m^1,m^0\}}(\xi^z) < t/3]\\
&\qquad + \inf_{u\in[t/6,t]} \mathtt{P}_\omega[\xi_u^{m^1} = m^1]\\
&\qquad\qquad \times \mathtt{P}_\omega[0 < \tau_{m^1}(\xi^z) - \tau_{m^0}(\xi^z) \leq t/6, \tau_{\{m^1,m^0\}}(\xi^z) < t/3]\\
&\geq\; \inf_{u\in[t/3,t]} \mathtt{P}_\omega[\xi_u^{m^1} = m^1]\\
&\qquad\quad \times \Big(\mathtt{P}_\omega[\tau_{m^1}(\xi^z) < \tau_{m^0}(\xi^z)] - \mathtt{P}_\omega[\tau_{\{m^1,m^0\}}(\xi^z) > t/3]\Big)\\
&\qquad + \inf_{u\in[t/6,t]} \mathtt{P}_\omega[\xi_u^{m^1} = m^1] \times \mathtt{P}_\omega[\tau_{m^1}(\xi^{m^0}) \leq t/6]\\
&\qquad\qquad \times \Big(\mathtt{P}_\omega[\tau_{m^0}(\xi^z) < \tau_{m^1}(\xi^z)] - \mathtt{P}_\omega[\tau_{\{m^1,m^0\}}(\xi^z) > t/3]\Big).
\end{aligned} \tag{40}$$

Again, as before, Lemma 3.1 implies that for all t large enough on the set of (δ, A, L, t)-good environments in Γ_t, we can neglect the two terms with a negative sign in this lower bound.

To get a more explicit lower bound on T_2, we now estimate the (second) infimum in (40). We apply Lemma 3.5 to $\mathtt{P}_\omega[\xi_u^{m^1} = m^1]$ with $u \in [t/6, t]$, taking into account that, from Lemma 4.1 (i), the second term in (25) is negligible in comparison with the first one, and that, from Lemma 4.1 (ii), the third term in (25) is negligible. Hence we get that, for all t large enough and all (δ, A, L, t)-good environments $\omega \in \Gamma_t$,

$$\inf_{u\in[t/6,t]} \mathtt{P}_\omega[\xi_u^{m^1} = m^1] \geq \frac{1}{2}(2A\ln^2 t)^{-1}\ln^{-2K_0} t. \tag{41}$$

Now, coming back to (40), we first use, in the right-hand side, the first summand *only*. Applying (12) to (39) and (40) and using (41), one gets from (38) that, if ω is (δ, A, L, t)-good and $\omega \in \Gamma_t$, then

$$\begin{aligned} \ln T_2 &\geq V(m^1) - V(z) - (V(h^+) - V(h^-))^+ \\ &\quad - (V(h^{+,z}) - V(h^{-,z}))^+ + o(\ln t) \\ &= -S^{(1)} + o(\ln t) , \end{aligned} \tag{42}$$

with $h^\pm = h^{\pm,v}$, $h^{\pm,z}$ defined in (4), and $S^{(1)}$ defined before Theorem 2.1. Now, we *only* take care of the second summand in the right-hand side of (40), and we need to estimate the following quantity:

$$\begin{aligned} &\mathtt{P}_\omega[\tau_{m^1}(\xi^{m^0}) \leq t/6] \\ &\quad\geq \mathtt{P}_\omega[\tau_{h^+}(\xi^{m^0}) \leq t/12]\mathtt{P}_\omega[\tau_{m^1}(\xi^{h^+}) \leq t/12] . \end{aligned} \tag{43}$$

Since (29) holds for (δ, A, L, t)-good $\omega \in \Gamma_t$, we obtain from Lemma 3.6 that

$$\ln \mathtt{P}_\omega[\tau_{m^1}(\xi^{m^0}) \leq t/6] \geq \ln t - V(h^+) + V(m^0) + o(\ln t) , \tag{44}$$

estimating the term $\varphi_{s,K}(m, \hat{h}, h)$ in the right-hand side of (30) as in (41). Proceeding similarly to (42) from second summand in the right-hand side of (40), we derive

$$\ln T_2 \geq -S^{(10)} + o(\ln t) , \tag{45}$$

for (δ, A, L, t)-good $\omega \in \Gamma_t$.

Final step. From (37), (42) and (45), we see that for (δ, A, L, t)-good environments from Γ_t,

$$\ln T_2 = -\min\{S^{(1)}, S^{(10)}\} + o(\ln t) .$$

Clearly, a similar result holds for the term T_3, and we get from (32) and (33) that for (δ, A, L, t)-good environments from Γ_t

$$\ln \mathtt{P}_\omega[\xi_t^v = z] = \alpha(v, z) + o(\ln t) \tag{46}$$

as $t \to \infty$, uniformly in v, z, with $\alpha(v, z) = \min\{S^{(0)}, S^{(1)}, S^{(01)}, S^{(10)}\}$. Since $\mathbb{P}$-almost all ω belongs to Γ_t for large enough t, and from Lemma 4.2 (ii), the proof of this case is complete.

4.2 The case $z \in [m^k, m^{k+1}]$

Let us suppose that $z \in [m^k, m^{k+1}]$ for some $1 \le k \le L$. We have, by the Markov property,

$$\mathtt{P}_\omega[\xi_t^v = z] = \int_0^t \mathtt{P}_\omega[\xi_{t-s}^{m^k} = z] d\mathtt{P}_\omega[\tau_{m^k}(\xi^v) \le s]. \tag{47}$$

First, we are going to obtain an upper bound on the right-hand side of (47). The formula (47) together with the reversibility imply that

$$\begin{aligned} \mathtt{P}_\omega[\xi_t^v = z] &\le \mathtt{P}_\omega[\tau_{m^k}(\xi^v) \le t] \sup_{s \le t} \mathtt{P}_\omega[\xi_s^{m^k} = z] \\ &= \frac{\theta_z}{\theta_{m^k}} \mathtt{P}_\omega[\tau_{m^k}(\xi^v) \le t] \sup_{s \le t} \mathtt{P}_\omega[\xi_s^z = m^k]. \end{aligned} \tag{48}$$

To bound $\mathtt{P}_\omega[\tau_{m^k}(\xi^v) \le t]$ from above, let us write

$$\mathtt{P}_\omega[\tau_{m^k}(\xi^v) \le t] \le \mathtt{P}_\omega[\tau_{m^1}(\xi^v) \le t] \mathtt{P}_\omega[\tau_{m^2}(\xi^{m^1}) \le t] \dots \mathtt{P}_\omega[\tau_{m^k}(\xi^{m^{k-1}}) \le t]. \tag{49}$$

First, note that, by Lemma 3.4

$$\begin{aligned} \mathtt{P}_\omega[\tau_{m^{i+1}}(\xi^{m^i}) \le t] &\le \mathtt{P}_\omega[\tau_{h^i}(\xi^{m^i}) \le t] \\ &\le K(t+1) e^{-V(h^i)+V(m^i)}, \end{aligned} \tag{50}$$

$i = 1, \dots, k-1$. As for the first term in the right-hand side of (49), we have

$$\begin{aligned} \mathtt{P}_\omega[\tau_{m^1}(\xi^v) \le t] &\le \mathtt{P}_\omega[\tau_{m^0}(\xi^v) < \tau_{m^1}(\xi^v)] \mathtt{P}_\omega[\tau_{m^1}(\xi^{m_0}) \le t] \\ &\quad + \mathtt{P}_\omega[\tau_{m^1}(\xi^v) < \tau_{m^0}(\xi^v)], \end{aligned}$$

so, again using Lemma 3.4 to bound $\mathtt{P}_\omega[\tau_{m^1}(\xi^{m_0}) \le t]$ from above (analogously to (50)), together with (12), one gets that on Γ_t

$$\ln \mathtt{P}_\omega[\tau_{m^1}(\xi^v) \le t] \le \max\{-(V(h^+) - V(h^-))^+; \ln t + V(m^0) - V(h^0)\} + o(\ln t).$$

In view of (49) and (50) this means that

$$\begin{aligned} \ln \mathtt{P}_\omega[\tau_{m^k}(\xi^v) \le t] &\le \max\{-(V(h^+) - V(h^-))^+; \ln t + V(m^0) - V(h^0)\} \\ &\quad + \sum_{i=1}^{k-1} (\ln t + V(m^i) - V(h^i)) + o(\ln t). \end{aligned} \tag{51}$$

Now, if $z \in [m^k, h^k]$, then we simply bound the supremum in the right-hand side of (48) by 1, and use (13) to get that on Γ_t

$$\ln \mathtt{P}_\omega[\xi_t^v = z] \le \max\{-(W(h^+) - W(h^-))^+; \ln t + W(m^0) - W(h^0)\}$$

$$+\sum_{i=1}^{k-1}(\ln t+W(m^i)-W(h^i))+W(m^k)-W(z)+o(\ln t). \quad (52)$$

Suppose that $z\in[h^k,m^{k+1}]$. For $s\le t$ we bound $\mathtt{P}_\omega[\xi^z_s=m^k]$ from above analogously to what was done in (35) to get that

$$\mathtt{P}_\omega[\xi^z_s=m^k]\le \mathtt{P}_\omega[\tau_{m^k}(\xi^z)<\tau_{m^{k+1}}(\xi^z)]+\mathtt{P}_\omega[\tau_{h^k}(\xi^{m^{k+1}})<s];$$

using (12) for the first term and Lemma 3.4 for the second one, we derive

$$\begin{aligned}\ln \mathtt{P}_\omega[\xi^z_s=m^k] \;\le\; & \max\{-(V(h^{-,z})-V(h^{+,z}))^+;\\ & \qquad \ln t+V(m^{k+1})-V(h^{-,z})\}+o(\ln t). \quad (53)\end{aligned}$$

So, if $z\in[h^k,m^{k+1}]$, using (51), (53) and (13), we get from (48) that

$$\begin{aligned}\ln \mathtt{P}_\omega[\xi^v_t=z] \;\le\; & \max\{-(W(h^+)-W(h^-))^+;\ln t+W(m^0)-W(h^0)\}\\ & +\sum_{i=1}^{k-1}(\ln t+W(m^i)-W(h^i))+W(m^k)-W(z)\\ & +\max\{-(W(h^{-,z})-W(h^{+,z}))^+;\\ & \qquad \ln t+W(m^{k+1})-W(h^{-,z})\}+o(\ln t). \quad (54)\end{aligned}$$

Now, we need to find the respective lower bounds. Using reversibility, one gets from (47) that

$$\begin{aligned}\mathtt{P}_\omega[\xi^v_t=z] \;\ge\; & \mathtt{P}_\omega[\tau_{m^k}(\xi^v)\le t/2]\inf_{t/2\le s\le t}\mathtt{P}_\omega[\xi^{m^k}_s=z]\\ =\; & \frac{\theta_z}{\theta_{m^k}}\mathtt{P}_\omega[\tau_{m^k}(\xi^v)\le t/2]\inf_{t/2\le s\le t}\mathtt{P}_\omega[\xi^z_s=m^k]. \quad (55)\end{aligned}$$

Analogously to (49), we have

$$\begin{aligned}\mathtt{P}_\omega[\tau_{m^k}(\xi^v)\le t] \;\ge\; & \mathtt{P}_\omega[\tau_{m^1}(\xi^v)\le t/(2k)]\\ & \times\mathtt{P}_\omega[\tau_{m^2}(\xi^{m^1})\le t/(2k)]\dots\mathtt{P}_\omega[\tau_{m^k}(\xi^{m^{k-1}})\le t/(2k)]. \quad (56)\end{aligned}$$

We first deal with the time that it takes to go from one t-stable point to the next one. Write

$$\mathtt{P}_\omega[\tau_{m^{i+1}}(\xi^{m^i})\le t/(2k)]\ge \mathtt{P}_\omega[\tau_{h^i}(\xi^{m^i})\le t/(4k)]\mathtt{P}_\omega[\tau_{m^{i+1}}(\xi^{h^i})\le t/(4k)]. \quad (57)$$

Now, from the fact that ω is (δ,A,L,t)-good, we obtain that (29) holds on Γ_t. So, from Lemma 3.6 one gets that (the term $\varphi_{s,K}(m,\hat{h},h)$ in the right-hand side of (30) can be treated quite analogously to (41)) for (δ,A,L,t)-good environments and on Γ_t

$$\ln \mathtt{P}_\omega[\tau_{h^i}(\xi^{m^i})\le t/(4k)]\ge \ln t-V(h^i)+V(m^i)+o(\ln t). \quad (58)$$

As for the second term in the right-hand side of (57), it holds

$$\begin{aligned}
\mathtt{P}_\omega[\tau_{m^{i+1}}(\xi^{h^i}) \le t/(4k)] &\ge \mathtt{P}_\omega[\tau_{m^{i+1}}(\xi^{h^i}) < \tau_{m^i}(\xi^{h^i}); \tau_{m^{i+1}}(\xi^{h^i}) \le t/(4k)]\\
&= \mathtt{P}_\omega[\tau_{m^{i+1}}(\xi^{h^i}) < \tau_{m^i}(\xi^{h^i})] - \mathtt{P}_\omega[\tau_{\{m^i,m^{i+1}\}}(\xi^{h^i}) > t/(4k)]
\end{aligned}$$

and Lemma 3.1 shows that on Γ_t, the second term in the above display can be neglected for (δ, A, L, t)-good environments.

Since, by (12),

$$\begin{aligned}
\mathtt{P}_\omega[\tau_{m^{i+1}}(\xi^{h^i}) < \tau_{m^i}(\xi^{h^i})] \wedge \mathtt{P}_\omega[\tau_{m^i}(\xi^{h^i}) < \tau_{m^{i+1}}(\xi^{h^i})] &\ge e^{V(h^i)}\Big(\sum_{m^i<y\le m^{i+1}} e^{V(y)}\Big)^{-1}\\
&\ge K(m^{i+1}-m^i)^{-1}\ln^{-2K_0} t
\end{aligned}$$

and since $(m^{i+1} - m^i) \le 2A\ln^2 t$, we deduce from (57), (58) that for environments which are (δ, A, L, t)-good and belong to Γ_t

$$\ln \mathtt{P}_\omega[\tau_{m^{i+1}}(\xi^{m^i}) \le t/(2k)] \ge \ln t - V(h^i) + V(m^i) + o(\ln t). \tag{59}$$

Then, we write

$$\begin{aligned}
\mathtt{P}_\omega[\tau_{m^1}(\xi^v) \le t/(2k)] \ge\ & \mathtt{P}_\omega[\tau_{m^0}(\xi^v) < \tau_{m^1}(\xi^v); \tau_{m^0}(\xi^v) \le t/(4k)]\mathtt{P}_\omega[\tau_{m^1}(\xi^{m^0}) \le t/(4k)]\\
&+ \mathtt{P}_\omega[\tau_{m^1}(\xi^v) < \tau_{m^0}(\xi^v); \tau_{m^1}(\xi^v) \le t/(2k)].
\end{aligned} \tag{60}$$

Again, (12) and Lemma 3.1 show us that the last term [respectively, first term] in the right-hand side of (60) is essentially $e^{-(V(h^+)-V(h^-))^+}$ [respectively, $e^{-(V(h^-)-V(h^+))^+}$] for (δ, A, L, t)-good environments from Γ_t, and the second term can be treated quite analogously to (59), so we see that

$$\begin{aligned}
&\ln \mathtt{P}_\omega[\tau_{m^1}(\xi^v) \le t/(2k)]\\
&\quad\ge \max\{-(V(h^+) - V(h^-))^+;\\
&\qquad\qquad \ln t - (V(h^-) - V(h^+))^+ - V(h^0) + V(m^0)\} + o(\ln t)\\
&\quad= \max\{-(V(h^+) - V(h^-))^+; \ln t - V(h^0) + V(m^0)\} + o(\ln t)\ ,
\end{aligned} \tag{61}$$

observing in the last line, that the second term in the maximum is relevant only when the first one is nonzero. Now, we have to deal with the last term in the right-hand side of (55). First, suppose that $z \in [m^k, h^k]$. Write

$$\mathtt{P}_\omega[\xi_s^z = m^k] \ge \mathtt{P}_\omega[\tau_{m^k}(\xi^z) \le s/2] \inf_{s/2\le u\le s} \mathtt{P}_\omega[\xi_u^{m^k} = m^k],$$

and since $t/2 \le s \le t$, using Lemma 3.1 and (12) for the first term in the right-hand side of the above display, and Lemma 3.5 for the second one, it is not difficult to get that

$$\ln \inf_{t/2\le s\le t} \mathtt{P}_\omega[\xi_s^z = m^k] = o(\ln t). \tag{62}$$

Using this fact together with (59), (61) and (13), we obtain from (55) that, if $z \in [m^k, h^k]$, then for (δ, A, L, t)-good environments from Γ_t

$$\begin{aligned}\ln \mathtt{P}_\omega[\xi_t^v = z] \quad \geq \quad & \max\{-(W(h^+) - W(h^-))^+; \ln t + W(m^0) - W(h^0)\} \\ & + \sum_{i=1}^{k-1} (\ln t + W(m^i) - W(h^i)) + W(m^k) - W(z) + o(\ln t). \qquad (63)\end{aligned}$$

Consider now the case $z \in [h^k, m^{k+1}]$. The Markov property implies that

$$\begin{aligned}\mathtt{P}_\omega[\xi_s^z = m^k] \quad \geq \quad & \mathtt{P}_\omega[\tau_{m^k}(\xi^z) < \tau_{m^{k+1}}(\xi^z); \tau_{m^k}(\xi^z) \leq s/2] \inf_{\frac{s}{2} \leq u \leq s} \mathtt{P}_\omega[\xi_u^{m^k} = m^k] \\ & + \mathtt{P}_\omega[\tau_{m^{k+1}}(\xi^z) < \tau_{m^k}(\xi^z); \tau_{m^{k+1}}(\xi^z) \leq s/3] \\ & \times \mathtt{P}_\omega[\tau_{m^k}(\xi^{m^{k+1}}) \leq s/3] \inf_{\frac{s}{3} \leq u \leq s} \mathtt{P}_\omega[\xi_u^{m^k} = m^k]. \qquad (64)\end{aligned}$$

Now, by (12) and Lemma 3.1 (note that if $z \in [h^k, m^{k+1}]$ then $h^{-,z} = h^k$), we have, as $s \geq t/2$,

$$\ln \mathtt{P}_\omega[\tau_{m^k}(\xi^z) < \tau_{m^{k+1}}(\xi^z); \tau_{m^k}(\xi^z) \leq s/2] = V(h^{+,z}) - V(h^{-,z}) + o(\ln t)$$

and, clearly,

$$\ln \mathtt{P}_\omega[\tau_{m^{k+1}}(\xi^z) < \tau_{m^k}(\xi^z); \tau_{m^{k+1}}(\xi^z) \leq s/3] = o(\ln t),$$

for (δ, A, L, t)-good environments from Γ_t. By means of Lemma 3.6, the probability $\mathtt{P}_\omega[\tau_{m^k}(\xi^{m^{k+1}}) \leq s/2]$ can be bounded from below quite analogously to (59), so, taking (62) into account, we obtain from (64) that

$$\begin{aligned}& \ln \inf_{t/2 \leq s \leq t} \mathtt{P}_\omega[\xi_s^z = m^k] \\ & \qquad \geq \max\{V(h^{+,z}) - V(h^{-,z}); \ln t - V(h^{-,z}) + V(m^{k+1})\} + o(\ln t).\end{aligned}$$

Finally, using the last formula, (59), (61) and (13), one gets from (55) that, for the case $z \in [h^k, m^{k+1}]$,

$$\begin{aligned}\ln \mathtt{P}_\omega[\xi_t^v = z] \quad \geq \quad & \max\{-(W(h^+) - W(h^-))^+; \ln t + W(m^0) - W(h^0)\} \\ & + \sum_{i=1}^{k-1} (\ln t + W(m^i) - W(h^i)) + W(m^k) - W(z) \\ & + \max\{-(W(h^{-,z}) - W(h^{+,z}))^+; \\ & \qquad \ln t + W(m^{k+1}) - W(h^{-,z})\} + o(\ln t), \qquad (65)\end{aligned}$$

for (δ, A, L, t)-good environments from Γ_t. Combining now (52), (54), (63), (65), (46), (37), we get

$$\ln \mathtt{P}_\omega[\xi_t^v = z] = \alpha(v, z) + o(\ln t)$$

as $t \to \infty$, uniformly in v, z, for (δ, A, L, t)-good $\omega \in \Gamma_t$. This completes the proof of Theorem 2.1. ■

5 Proof of Theorems 2.3 and 2.4

In this section we give a proof of the results concerning the "moderate deviations" and the nonexistence of the moments of the time of the first return to 0.

Proof of Theorem 2.3. First, let us describe informally the main idea of the proof. If $m_e^{1,0}$ and $m_e^{2,0}$ are two neighboring e-stable points and $h_e^{1,0}$ is the highest pass between them (recall the notation from the beginning of Section 2.1), then define

$$\begin{aligned} \beta_1 &= \mathbb{E}(m_e^{2,0} - m_e^{1,0}); &(66)\\ \beta_2 &= \mathbb{E}(W(h_e^{1,0}) - W(m_e^{1,0})) - 1 &(67) \end{aligned}$$

(recall that $W(\cdot)$ is the Brownian motion with the diffusion constant σ). Fix t; by the scaling property of the Brownian motion, the mean distance between two neighboring t-stable points is equal to $\beta_1 \ln^2 t$ (in the typical case, i.e., when the t-stable points under consideration are not those which are neighbours of 0), so there are roughly $\beta_1^{-1} z \ln^{-2} t$ points between 0 and z which are t-stable. By Lemmas 3.4 and 3.6, to go from $m_t^{i,0}$ to $m_t^{i+1,0}$, we pay essentially $\exp\{-W(h_t^{i,0}) + W(m_t^{i,0}) + \ln t\}$. From (67) and the scaling property, we have $\mathbb{E}(-W(h_t^{i,0})+W(m_t^{i,0})+\ln t) = -\beta_2 \ln t$, so the total cost is approximately $\exp\{-\beta_1^{-1}\beta_2 z \ln^{-1} t\}$ by the law of large numbers for independent, identically distributed random variables. In the following, we make the above argument rigorous, but this requires some care. We begin by providing an

Upper bound for $\mathtt{P}_\omega[\xi_t^0 \geq z]$: Recall from Section 2 that $m_t^{-,z}$ and $m_t^{+,z}$ are the two t-stable points which surround z. Let $N_\omega(t,z)$ be such that $m_t^{N_\omega(t,z),0} = m_t^{-,z}$. To get from 0 to z, one has to pass through $m_t^{N_\omega(t,z),0}$, so we have

$$\begin{aligned} \mathtt{P}_\omega[\xi_t^0 \geq z] &\leq \mathtt{P}_\omega[\tau_{m_t^{N_\omega(t,z),0}}(\xi^0) \leq t] \\ &\leq \mathtt{P}_\omega[\tau_{m_t^{1,0}}(\xi^0) \leq t] \prod_{i=1}^{N_\omega(t,z)-1} \mathtt{P}_\omega[\tau_{m_t^{i+1,0}}(\xi^{m_t^{i,0}}) \leq t] \\ &\leq \prod_{i=1}^{N_\omega(t,z)-1} \mathtt{P}_\omega[\tau_{h_t^{i,0}}(\xi^{m_t^{i,0}}) \leq t]. &(68) \end{aligned}$$

Fix arbitrary $\varepsilon' > 0$. Using (13) and Lemma 3.4, one gets from (68) that for all t large enough, on Γ_t

$$\ln \mathtt{P}_\omega[\xi_t^0 \geq z] \leq -\sum_{i=1}^{N_\omega(t,z)-1} (W(h_t^{i,0}) - W(m_t^{i,0}) - (1+\varepsilon')\ln t). \qquad (69)$$

Fix $\delta > 0$ and divide the time interval $[e, +\infty)$ into a countable collection of intervals $I_n := [e^{(1+\delta)^n}, e^{(1+\delta)^{n+1}})$, $n = 0, 1, 2, \ldots$. As $\mathcal{S}_t \subset \mathcal{S}_{t'}$ for $t' \leq t$, we have

$$N_\omega(e^{(1+\delta)^{n+1}}, z) \leq N_\omega(t,z) \leq N_\omega(e^{(1+\delta)^n}, z) \qquad (70)$$

for any z, ω and all $t \in I_n$. Define also

$$r^{(n)} = \inf \bigcup_{t \in I_n} R_t(\varphi, M) = \inf_{t \in I_n} (\ln^2 t \times \ln\ln\ln t \times \varphi(t)) = (1+\delta)^{2n} \varphi(e^{(1+\delta)^n}) \ln(n \ln(1+\delta)).$$

For any $\varepsilon > 0$, denote by B_n^ε the event

$$B_n^\varepsilon := \{N_\omega(t,z) \geq (1-\varepsilon)\beta_1^{-1} z (1+\delta)^{-2n}, \text{ for all } t \in I_n \text{ and } z \geq r^{(n)}\}.$$

For all $i \in \mathbb{Z}$, define the random variable $\zeta_i(t) := m_t^{i+1,0} - m_t^{i,0}$. Note that, by the Brownian scaling and Definition 2.1 it holds that

$$(\zeta_i(e^s); i \in \mathbb{Z}) \stackrel{\text{law}}{=} (s^2 \zeta_i(e); i \in \mathbb{Z}) \ . \tag{71}$$

where the symbol "$\stackrel{\text{law}}{=}$" stands for the equality in law. Abbreviate $g_0 = g_0(z) = (1-\varepsilon)\beta_1^{-1} z (1+\delta)^{-2n}$, so $z/g_0 = (1+\delta)^{2n}(1-\varepsilon)^{-1}\beta_1$. Using (70) and (71), we get

$$\begin{aligned} \mathbb{P}[B_n^\varepsilon] &\geq \mathbb{P}[N_\omega(e^{(1+\delta)^{n+1}}, z) \geq g_0(z), \text{ for all } z \geq r^{(n)}] \\ &= \mathbb{P}\Big[\sum_{i=1}^{g_0} \zeta_i(e^{(1+\delta)^{n+1}}) < z, \text{ for all } z \geq r^{(n)}\Big] \\ &= \mathbb{P}\Big[\sum_{i=1}^{g_0(z)} (1+\delta)^{2(n+1)} \zeta_i(e) < z, \text{ for all } z \geq r^{(n)}\Big] \\ &= \mathbb{P}\Big[\frac{1}{g_0} \sum_{i=1}^{g_0} \zeta_i(e) < (1-\varepsilon)^{-1}(1+\delta)^{-2}\beta_1 \ , \\ &\qquad \text{for all } g_0 \geq (1-\varepsilon)\beta_1^{-1} \varphi(e^{(1+\delta)^n}) \ln(n \ln(1+\delta))\Big]. \end{aligned} \tag{72}$$

Now, define $\eta_i(t) = W(h_t^{i,0}) - W(m_t^{i,0}) - \ln t$; by Definition 2.1 and the Brownian scaling,

$$(\eta_i(e^s); i \in \mathbb{Z}) \stackrel{\text{law}}{=} (s\eta_i(e); i \in \mathbb{Z}) \ . \tag{73}$$

Again, as $\mathcal{S}_t \subset \mathcal{S}_{t'}$ for $t' \leq t$, we get from (69) and (70) that on Γ_t

$$\ln \mathsf{P}_\omega[\xi_t^0 \geq z] \leq - \sum_{i=1}^{N_\omega(e^{(1+\delta)^{n+1}}, z)-1} (\eta_i(e^{(1+\delta)^{n+1}}) - \varepsilon'(1+\delta)^{n+1}) \tag{74}$$

for all $t \in I_n$, where n is large enough. Define the sequence of events

$$D_n^\varepsilon \quad = \quad \Big\{\frac{1}{g} \sum_{i=1}^{g-1} \eta_i(e^{(1+\delta)^{n+1}}) \geq \beta_2(1-\varepsilon)(1+\delta)^{n+1}$$

$$\text{for all } g \geq (1-\varepsilon)\beta_1^{-1}\varphi(e^{(1+\delta)^n})\ln(n\ln(1+\delta))\Big\} \; .$$

Note that, on $D_n^\varepsilon \bigcap B_n^\varepsilon$, it follows from (74) that

$$\begin{aligned} \ln \mathsf{P}_\omega[\xi_t^0 \geq z] &\leq -\sum_{i=1}^{g_0-1}(\eta_i(e^{(1+\delta)^{n+1}}) - \varepsilon'(1+\delta)^{n+1}) \\ &\leq -\beta_1^{-1}[\beta_2(1-\varepsilon)-\varepsilon']z\ln^{-1}t \end{aligned} \tag{75}$$

for all $t \in I_n, z \in R_t(\varphi, M)$.

Observe now that for $i \neq 0$, $\beta_1 = \mathbb{E}\zeta_i(e)$, $\beta_2 = \mathbb{E}\eta_i(e)$, that both sequences $(\zeta_i(e))_{i\in\mathbb{Z}^*}$ and $(\eta_i(e))_{i\in\mathbb{Z}^*}$ are i.i.d., and also that the random variables $\zeta_i(e), \eta_i(e)$ have finite exponential moments (this last observation may not seem obvious; see the discussion in Section 6). It follows from Cramér theorem that, for $(1-\varepsilon)^{-1}(1+\delta)^2 > 1$, there exist finite constants $K_1, K_2, \dots > 0$ such that

$$\mathbb{P}\left[\frac{1}{g_0}\sum_{i=1}^{g_0}\zeta_i(e) \geq (1-\varepsilon)^{-1}(1+\delta)^{-2}\beta_1\right] \leq \exp\{-K_1 g_0\} \; ,$$

for all $g_0 \geq 1$, and, in view of (72),

$$\mathbb{P}[(B_n^\varepsilon)^c] \leq K_3\exp\{-K_2\varphi(e^{(1+\delta)^n})\ln n\} \; ,$$

and similarly,

$$\mathbb{P}[(D_n^\varepsilon)^c] \leq K_4\exp\{-K_5\varphi(e^{(1+\delta)^n})\ln n\}$$

for all n. Therefore, by the Borel-Cantelli lemma and since $\varphi \to \infty$, all but a finite number of events $D_n^\varepsilon \bigcap B_n^\varepsilon$ occur, implying (75). Since ε, δ are arbitrary, we conclude that

$$\limsup_{t\to\infty}\sup_{z\in R_t(\varphi,M)}\frac{\ln \mathsf{P}_\omega[\xi_t^0 \geq z]}{\beta_1^{-1}\beta_2 z\ln^{-1}t} \leq -1 \qquad \mathbb{P}\text{-a.s.} \tag{76}$$

Lower bound. Fix an arbitrary $\varepsilon_0 > 0$. From now on we work with the set of all $t^{1-\varepsilon_0}$-stable points $\{\dots, m_{t^{1-\varepsilon_0}}^{-1,0}, m_{t^{1-\varepsilon_0}}^{0,0}, m_{t^{1-\varepsilon_0}}^{1,0}, \dots\}$ and the respective mountain passes; for the sake of brevity in the sequel we suppress the subscript "$t^{1-\varepsilon_0}$" and the superscript "0" in m-s and h-s. Define

$$N_\omega^{\varepsilon_0}(t,z) = \min\Big\{i : m^i > m^{+,z}, \min\{W(h^{i-1}); W(h^i)\} - W(m^i) \geq (1+\varepsilon_0)\ln t\Big\}.$$

Define also

$$\Lambda_t^{\varepsilon_0} = \{\omega : m^{N_\omega^{\varepsilon_0}(t,z)} \leq \ln^{3+M} t, N_\omega^{\varepsilon_0}(t,z) \leq \ln^{1+M} t, \text{ for all } z \in R_t(\varphi, M)\};$$

from the fact that the random variable $\zeta_1(e) = m_e^{2,0} - m_e^{1,0}$ has exponential tail, it is not hard to get that $\omega \in \Lambda_t^{\varepsilon_0}$ for all t large enough, $\mathbb{P}$-a.s. Abbreviate $\tilde{m} = m^{N_\omega^{\varepsilon_0}(t,z)}$. Write

$$\begin{aligned}
\mathtt{P}_\omega[\xi_t^0 \geq z] &\geq \mathtt{P}_\omega[\tau_{\tilde{m}}(\xi^0) \leq t]\mathtt{P}_\omega[\tau_z(\xi^{\tilde{m}}) > t] \\
&\geq \mathtt{P}_\omega\Big[\tau_{m^1}(\xi^0) \leq \frac{t}{N_\omega^{\varepsilon_0}(t,z)}\Big]\mathtt{P}_\omega[\tau_z(\xi^{\tilde{m}}) > t] \\
&\quad \times \prod_{i=2}^{N_\omega^{\varepsilon_0}(t,z)} \mathtt{P}_\omega\Big[\tau_{m^i}(\xi^{m^{i-1}}) \leq \frac{t}{N_\omega^{\varepsilon_0}(t,z)}\Big]. \qquad (77)
\end{aligned}$$

For all $i = 1, \dots, N_\omega^{\varepsilon_0}(t,z)$ we have

$$\begin{aligned}
&\mathtt{P}_\omega\Big[\tau_{m^i}(\xi^{m^{i-1}}) \leq \frac{t}{N_\omega^{\varepsilon_0}(t,z)}\Big] \\
&\quad \geq \mathtt{P}_\omega\Big[\tau_{h^{i-1}}(\xi^{m^{i-1}}) \leq \frac{t}{2N_\omega^{\varepsilon_0}(t,z)}\Big]\mathtt{P}_\omega\Big[\tau_{m^i}(\xi^{h^{i-1}}) \leq \frac{t}{2N_\omega^{\varepsilon_0}(t,z)}\Big]. \qquad (78)
\end{aligned}$$

Now, analogously to (39), we obtain

$$\begin{aligned}
\mathtt{P}_\omega\Big[\tau_{m^i}(\xi^{h^{i-1}}) \leq \frac{t}{2N_\omega^{\varepsilon_0}(t,z)}\Big] &\geq \mathtt{P}_\omega[\tau_{m^i}(\xi^{h^{i-1}}) < \tau_{m^{i-1}}(\xi^{h^{i-1}})] \\
&\quad - \mathtt{P}_\omega\Big[\tau_{\{m^{i-1},m^i\}}(\xi^{h^{i-1}}) > \frac{t}{2N_\omega^{\varepsilon_0}(t,z)}\Big]. \qquad (79)
\end{aligned}$$

Note that, with Δ_1, γ_1 from Lemma 3.1, we have $\Delta_1 \leq \tilde{m}$, and, by Condition B, $\gamma_1 \leq K\tilde{m}$ on $\Gamma_t \cap \Lambda_t^{\varepsilon_0}$ for some $K > 0$. Note also that, as m^{i-1}, m^i are $t^{1-\varepsilon_0}$-stable points, it holds that

$$\mathfrak{E}[m^{i-1}, h^{i-1}] \vee \mathfrak{E}[h^{i-1}, m^i] \leq (1-\varepsilon_0)\ln t. \qquad (80)$$

So, by Lemma 3.1 and (12), we get from (79) that on $\Lambda_t^{\varepsilon_0} \cap \Gamma_t$ it holds that

$$\begin{aligned}
&\mathtt{P}_\omega\Big[\tau_{m^i}(\xi^{h^{i-1}}) \leq \frac{t}{2N_\omega^{\varepsilon_0}(t,z)}\Big] \\
&\quad \geq \frac{\tilde{m}^{-1}}{\ln^{2K_0} t} - 2\exp\Big\{-t^{\frac{\varepsilon_0}{2}}\Big(\frac{\tilde{m}^{-1}}{\ln^{2K_0} t} - (K\tilde{m})^{1/2}\exp\Big\{-\frac{t^{\frac{\varepsilon_0}{2}}\lambda_i^*}{4N_\omega^{\varepsilon_0}(t,z)}\Big\}\Big)\Big\}, \qquad (81)
\end{aligned}$$

where $\lambda_i^* := \min\{\lambda[m^{i-1}, h^{i-1}]e^{-\mathfrak{E}[m^{i-1},h^{i-1}]}, \lambda[h^{i-1}, m^i]e^{-\mathfrak{E}[h^{i-1},m^i]}\}$. Thus, from Proposition 3.1 and (81) it follows that

$$\frac{1}{z\ln^{-1} t} \inf_{\omega \in \Lambda_t^{\varepsilon_0} \cap \Gamma_t} \sum_{i=1}^{N_\omega^{\varepsilon_0}(t,z)} \ln \mathtt{P}_\omega\Big[\tau_{m^i}(\xi^{h^{i-1}}) \leq \frac{t}{2N_\omega^{\varepsilon_0}(t,z)}\Big] \longrightarrow 0 \qquad (82)$$

as $t \to \infty$, uniformly in $z \in R_t(\varphi, M)$.

To get a lower bound for the first term in the right-hand side of (78), we are going to consider two cases.

Case 1: $\min\{W(h^{i-2}); W(h^{i-1})\} - W(m^{i-1}) \geq (1 - \frac{\varepsilon_0}{3}) \ln t$.

Suppose that t is so large that $t^{1-\frac{\varepsilon_0}{2}} \leq t/(2N_\omega^{\varepsilon_0}(t,z))$ for all $z \in R_t(\varphi, M)$ on $\Lambda_t^{\varepsilon_0}$. This means that

$$\mathtt{P}_\omega\Big[\tau_{h^{i-1}}(\xi^{m^{i-1}}) \leq \frac{t}{2N_\omega^{\varepsilon_0}(t,z)}\Big] \geq \mathtt{P}_\omega\Big[\tau_{h^{i-1}}(\xi^{m^{i-1}}) \leq t^{1-\frac{\varepsilon_0}{2}}\Big]. \tag{83}$$

We are going to use Lemma 3.6 to bound the right-hand side of the above display from below. By (80), one gets that (29) holds for all t large enough on $\Lambda_t^{\varepsilon_0}$. Again by (80), we get that the quantity defined by (25) is bounded from below by $const \times \ln^{-1} \ln t \times \ln^{-(3+M)} t$ on $\Lambda_t^{\varepsilon_0} \cap \Gamma_t$. Applying (30), we obtain from (83) that

$$\ln \mathtt{P}_\omega\Big[\tau_{h^{i-1}}(\xi^{m^{i-1}}) \leq \frac{t}{2N_\omega^{\varepsilon_0}(t,z)}\Big] \geq -\Big(V(h^{i-1}) - V(m^{i-1}) - \Big(1 - \frac{\varepsilon_0}{2}\Big) \ln t\Big) + o(\ln t) \tag{84}$$

on $\Lambda_t^{\varepsilon_0} \cap \Gamma_t$.

Case 2: $\min\{W(h^{i-2}); W(h^{i-1})\} - W(m^{i-1}) \in [(1 - \varepsilon_0) \ln t, (1 - \frac{\varepsilon_0}{3}) \ln t)$.

Analogously to (39), we write

$$\begin{aligned} \mathtt{P}_\omega\Big[\tau_{h^{i-1}}(\xi^{m^{i-1}}) \leq \frac{t}{2N_\omega^{\varepsilon_0}(t,z)}\Big] \;\; &\geq \;\; \mathtt{P}_\omega[\tau_{h^{i-1}}(\xi^{m^{i-1}}) < \tau_{h^{i-2}}(\xi^{m^{i-1}})] \\ &\qquad -\mathtt{P}_\omega\Big[\tau_{\{h^{i-2}, h^{i-1}\}}(\xi^{m^{i-1}}) > \frac{t}{2N_\omega^{\varepsilon_0}(t,z)}\Big] \end{aligned} \tag{85}$$

and then proceed as follows. Using the fact that $\mathfrak{I}(m^{i-1}) \geq (1 - \varepsilon_0) \ln t$, it is straightforward to get that on Γ_t

$$(V(h^{i-1}) - V(h^{i-2}))^+ \leq V(h^{i-1}) - V(m^{i-1}) - (1 - \varepsilon_0) \ln t + o(\ln t).$$

Thus, by (12), on $\Lambda_t^{\varepsilon_0} \cap \Gamma_t$

$$\begin{aligned} \ln \mathtt{P}_\omega[\tau_{h^{i-1}}(\xi^{m^{i-1}}) < \tau_{h^{i-2}}(\xi^{m^{i-1}})] \;\; &\geq \;\; -(V(h^{i-1}) - V(h^{i-2}))^+ + o(\ln t) \\ &\geq \;\; -V(h^{i-1}) + V(m^{i-1}) + (1 - \varepsilon_0) \ln t + o(\ln t). \end{aligned} \tag{86}$$

Applying now Lemma 3.2 to the second term in the right-hand side of (85) and taking (84) and (86) into account, we obtain that in both cases

$$\ln \mathtt{P}_\omega\Big[\tau_{h^{i-1}}(\xi^{m^{i-1}}) \leq \frac{t}{2N_\omega^{\varepsilon_0}(t,z)}\Big] \geq -V(h^{i-1}) + V(m^{i-1}) + (1 - \varepsilon_0) \ln t + o(\ln t) \tag{87}$$

on $\Lambda_t^{\varepsilon_0} \cap \Gamma_t$.

Now, let us consider the first two terms in (77). As, by definition, $\tilde{m}$ is a $t^{1+\varepsilon_0}$-stable point, and z is not in the $t^{1+\varepsilon_0}$-stable well of $\tilde{m}$, by Lemma 3.4 it is straightforward to get that on Γ_t

$$\mathtt{P}_\omega[\tau_z(\xi^{\tilde{m}}) > t] \geq \frac{1}{2} \tag{88}$$

for all t large enough and *all* z, ω. Then, we bound

$$\mathtt{P}_\omega\Big[\tau_{m^1}(\xi^0) \leq \frac{t}{2N_\omega^{\varepsilon_0}(t,z)}\Big] \geq \mathtt{P}_\omega[\tau_{m^1}(\xi^0) < \tau_{-1}(\xi^0)] - \mathtt{P}_\omega\Big[\tau_{\{-1,m^1\}}(\xi^0) > \frac{t}{2N_\omega^{\varepsilon_0}(t,z)}\Big].$$

The first term in the right-hand side of the above display is essentially $e^{-V(h^+)}$, and the second one can be treated by using Lemma 3.1; from the existence of exponential moments of $\hat{\zeta}_0(e)$ it is not difficult to get that

$$\frac{1}{z\ln^{-1} t} \inf_{\omega\in\Lambda_t^{\varepsilon_0}\cap\Gamma_t} \ln \mathtt{P}_\omega\Big[\tau_{m^1}(\xi^0) \leq \frac{t}{2N_\omega^{\varepsilon_0}(t,z)}\Big] \longrightarrow 0 \tag{89}$$

as $t \to \infty$, uniformly in $z \in R_t(\varphi, M)$.

Now, we insert (82) and (87) into (78), and then use (13), (78), (88), (89) to obtain from (77) that

$$\ln \mathtt{P}_\omega[\xi_t^0 \geq z] \geq -\sum_{i=2}^{N_\omega(t,z)} \big(W(h_t^{i,0}) - W(m_t^{i,0}) - (1-\varepsilon_0)\ln t\big) + o(\ln t) \tag{90}$$

for all t large enough on $\Lambda_t^{\varepsilon_0} \cap \Gamma_t$.

Note that the lower bound (90), to which we arrived, is essentially the same as the upper bound (69). Analogously to what was done to derive (76) from (69), one can obtain that

$$\liminf_{t\to\infty} \inf_{z\in R_t(\varphi,M)} \frac{\ln \mathtt{P}_\omega[\xi_t^0 \geq z]}{\beta_1^{-1}\beta_2 z \ln^{-1} t} \geq -1 \qquad \mathbb{P}\text{-a.s.}; \tag{91}$$

details are omitted. Combining (76) and (91), we get that

$$\sup_{z\in R_t(\varphi,M)} \Big|\frac{\ln \mathtt{P}_\omega[\xi_t^0 \geq z]}{\beta_1^{-1}\beta_2 z \ln^{-1} t} + 1\Big| \longrightarrow 0 \qquad \mathbb{P}\text{-a.s.}$$

In Section 6, we compute the value of the constants β_1 and β_2, and together with the previous limit, this proves (10):

Proposition 5.1 (to be proved in Section 6) *For the constants β_1, β_2 defined in (66) and (67), we have*

$$\beta_1 = 2\sigma^{-2}, \qquad \beta_2 = 1.$$

As for the proof of the corresponding statement for $\max_{s\leq t}\xi_s^0$, note that, analogously to (68)

$$\mathtt{P}_\omega[\max_{s\leq t}\xi_s^0 \geq z] \leq \prod_{i=1}^{N_\omega(t,z)-1} \mathtt{P}_\omega[\tau_{h_t^{i,0}}(\xi^{m_t^{i,0}}) \leq t],$$

so the derivation of the upper bound for $\max_{s\leq t}\xi_s^0$ goes through with practically no changes. Since, on the other hand,

$$\mathtt{P}_\omega[\max_{s\leq t}\xi_s^0 \geq z] \geq \mathtt{P}_\omega[\xi_t^0 \geq z],$$

the lower bound is straightforward. ■

Proof of Theorem 2.4. First, let us introduce some notations. For all $n \geq 1$, we define

$$v_n = \min\{x \in \mathbb{R}_+ : W(x) \leq -2n\},$$

and let u_n be such that

$$W(u_n) = \max_{z\in[0,v_n]} W(z).$$

Then, define $U_n = W(u_n)/n$. Using a well-known formula about the hitting probabilities of the Brownian motion, we write for any $b \geq 0$

$$\begin{aligned} \mathbb{P}[U_n \leq b] &= \mathbb{P}[\text{Brownian motion hits } (-2n) \text{ before hitting } bn] \\ &= \frac{b}{b+2}, \end{aligned} \tag{92}$$

so the distribution of the random variable U_n does not depend on n.

Now, for any $t < e^n$, we have

$$\mathtt{P}_\omega[\hat{\tau} > t] \geq \tilde{K}\mathtt{P}_\omega[\tau_{v_n}(\xi^1) < \tau_0(\xi^1)]\mathtt{P}_\omega[\tau_{u_n}(\xi^{v_n}) \geq t], \tag{93}$$

where $\tilde{K} = \tilde{K}(\omega) = \mathtt{P}_\omega[\tau_1(\xi^0) < \tau_{-1}(\xi^0)]$. It is elementary to get that for $\mathbb{P}$-almost all environments it holds that $v_n \leq n^3$ for all n large enough. By definition of u_n, v_n, it is true that $W(u_n) - W(v_n) \geq 2n$; from (2) we get that for $\mathbb{P}$-almost all environments

$$\max_{z\in[0,v_n]} V(z) - V(v_n) \geq \frac{3n}{2}$$

for all n large enough. This shows that, by Lemma 3.4, the last term in (93) is greater than $1/2$ for all n large enough. Applying (12) to the first term, we get from (93) that for all n large enough

$$\mathtt{P}_\omega[\hat{\tau} > t] \geq \frac{\tilde{K}e^{-nU_n}}{2n^3} \tag{94}$$

for all $t \leq e^n$. Now, integrating by parts, we get from (94) that

$$\mathtt{E}_\omega\hat{\tau}^a = \int_0^{+\infty} x^a d\mathtt{P}_\omega[\hat{\tau} \leq x]$$

$$
\begin{aligned}
&= a\int_0^{+\infty} x^{a-1}\mathtt{P}_\omega[\hat\tau > x]dx \\
&\geq a\int_0^{e^n} x^{a-1}\frac{\tilde K e^{-nU_n}}{2n^3}dx \\
&= \frac{\tilde K e^{n(a-U_n)}}{2n^3},
\end{aligned} \tag{95}
$$

for all n large enough.

As U_n, $n = 1, 2, \dots$, is an ergodic sequence and by (92), there exist an increasing sequence n_i such that $U_{n_i} \leq a/2$, $i = 1, 2, \dots$. Thus, from (95) we get that $\mathtt{E}_\omega\hat\tau^a \geq \tilde K n_i^{-3} e^{an_i/2}/2$ for all i, and so $\mathtt{E}_\omega\hat\tau^a = \infty$ $\mathbb{P}$-a.s. ∎

6 Values of β_1, β_2, and proof of Theorem 2.2

In this section we deal with the results which require explicit calculations of the laws of some functionals of the Brownian motion. Namely, here we calculate the constants β_1, β_2 from (66), (67), in the proof of Theorem 2.3, and prove Theorem 2.2. For that, we need first to recall some known facts. Let $B(t)$, $t \geq 0$, be the standard Brownian motion starting from 0. We define a stopping time T by

$$T = \inf\{t > 0 : B(t) - \inf_{0\leq s\leq t} B(s) = 1\}$$

and two random variables R_1, R_2 by

$$R_1 = -\inf_{0\leq t\leq T} B(t)\ , \qquad R_2 = \sup_{0\leq t\leq T'} B(t)\ ,$$

where $T' = \inf\{t : B(t) = -R_1\}$.

Lemma 6.1 *For the random variables T, R_1, R_2 it holds that*

(i) $R_1 \geq 0$, $0 \leq R_2 \leq 1$ *a.s., and the joint density $f(x, y)$ of the pair (R_2, R_1) is given by*

$$f(x,y) = \begin{cases} \dfrac{y}{(x+y)^2}, & \text{if } x + y \leq 1, \\ (1-x)\exp\{-(y-1+x)\}, & \text{if } x+y \geq 1 \end{cases}$$

for $0 \leq x \leq 1, y \geq 0$;

(ii) R_1 *has the exponential distribution with mean* 1;

(iii) $\mathbb{E}T = 1$.

Proof. The part (i) is Corollary 2.12 from [11]. As for the part (ii), one can deduce it directly from (i) (but it is quite simple to prove it directly; see e.g. the proof of Lemma 2.5.18 in [22]). To prove the part (iii), note that, by Levy's Theorem (cf. [16], Chapter VI, Theorem 2.3) the process $B(t) - \inf_{s\le t} B(s)$ is the reflected Brownian motion, and so

$$T \overset{\text{law}}{=} \inf\{t > 0 : |B(t)| = 1\}.$$

Applying Proposition 3.7 of Chapter II of [16], we conclude the proof. ■

Proof of Proposition 5.1. To evaluate β_1, β_2, we need to introduce some notations. Let

$$\begin{aligned} v_1 &= \inf\{t > m_e^{1,0} : W(t) - W(m_e^{1,0}) = 1\}, \\ v_2 &= \inf\{t > h_e^{1,0} : W(h_e^{1,0}) - W(t) = 1\}, \\ v_3 &= \inf\{t > m_e^{2,0} : W(t) - W(m_e^{2,0}) = 1\}. \end{aligned}$$

As $v_1 - m_e^{1,0} \overset{\text{law}}{=} v_3 - m_e^{2,0}$, we get that $\beta_1 = \mathbb{E}(v_3 - v_1)$. Using that $v_2 - h_e^{1,0} \overset{\text{law}}{=} v_3 - m_e^{2,0}$ and $m_e^{2,0} - v_2 \overset{\text{law}}{=} h_e^{1,0} - v_1$, we obtain that $\beta_1 = 2\mathbb{E}(v_3 - v_2)$. Now, using the fact that $W'(t) := W(\sigma^{-2}t)$ is the standard Brownian motion we see that $v_3 - v_2 \overset{\text{law}}{=} \sigma^{-2}T$, and from Lemma 6.1 (iii), we derive that $\beta_1 = 2\sigma^{-2}$.

As for the value of β_2, note that $\beta_2 = \mathbb{E}[W(h_e^{1,0}) - W(v_1)]$, and that $W(h_e^{1,0}) - W(v_1) \overset{\text{law}}{=} R_1$ (as the process $W''(t) = -(W(\sigma^{-2}t + v_1) - W(v_1))$ is the standard Brownian motion). Thus, Lemma 6.1 (ii) implies that $\beta_2 = 1$. ■

Proof of Theorem 2.2. Let us consider a set of 14 random variables $T_0^\pm$, $T_1^\pm$, $T_2^\pm$, $H_0^\pm$, $H_1^\pm$, $M_0^\pm$, $M_1^\pm$, defined in the following way (see Figure 3):

$$\begin{aligned} T_0^+ &= \inf\{t > 0 : W(t) - \inf_{0\le s\le t} W(s) = 1\}, \\ T_0^- &= \sup\{t < 0 : W(t) - \inf_{t\le s\le 0} W(s) = 1\}, \\ T_1^+ &= \inf\{t > T_0^+ : \sup_{T_0^+\le s\le t} W(s) - W(t) = 1\}, \\ T_1^- &= \sup\{t < T_0^- : \sup_{t\le s\le T_0^-} W(s) - W(t) = 1\}, \\ T_2^+ &= \inf\{t > T_1^+ : W(t) - \inf_{T_1^+\le s\le t} W(s) = 1\}, \\ T_2^- &= \sup\{t < T_1^- : W(t) - \inf_{t\le s\le T_1^-} W(s) = 1\}, \\ M_0^+ &= \inf\{t \in [0, T_0^+] : W(t) = \inf_{0\le s\le T_0^+} W(s)\}, \\ M_0^- &= \sup\{t \in [T_0^-, 0] : W(t) = \inf_{T_0^-\le s\le 0} W(s)\}, \\ M_1^+ &= \inf\{t \in [T_1^+, T_2^+] : W(t) = \inf_{T_1^+\le s\le T_2^+} W(s)\}, \end{aligned}$$

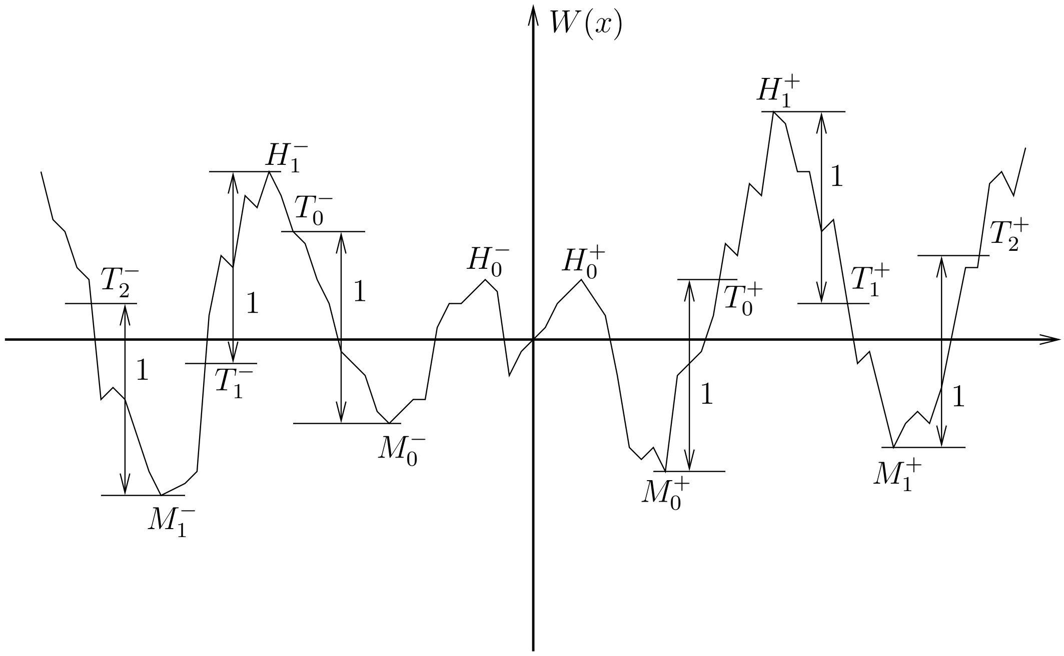

Figure 3: On the definition of auxiliary random variables

$$
\begin{aligned}
M_1^- &= \sup\{t \in [T_2^-, T_1^-] : W(t) = \inf_{T_2^- \le s \le T_1^-} W(s)\},\\
H_0^+ &= \inf\{t \in [0, M_0^+] : W(t) = \sup_{0 \le s \le M_0^+} W(s)\},\\
H_0^- &= \sup\{t \in [M_0^-, 0] : W(t) = \sup_{M_0^- \le s \le 0} W(s)\},\\
H_1^+ &= \inf\{t \in [T_0^+, T_1^+] : W(t) = \sup_{T_0^+ \le s \le T_1^+} W(s)\},\\
H_1^- &= \sup\{t \in [T_1^-, T_0^-] : W(t) = \sup_{T_1^- \le s \le T_0^-} W(s)\}.
\end{aligned}
$$

Now, we need to compute the law of the random variable $\hat{\alpha}_e$ defined by (8). Clearly, $m_e^+ \in \{M_0^+, M_1^+\}$, $m_e^- \in \{M_0^-, M_1^-\}$, and $\{m_e^+, m_e^-\} \cap \{M_0^+, M_0^-\} \neq \emptyset$, and also $h_e^+ \in \{H_0^+, H_1^+\}$, $h_e^- \in \{H_0^-, H_1^-\}$, and $\{h_e^+, h_e^-\} \cap \{H_0^+, H_0^-\} \neq \emptyset$. Introduce a partition A_1, A_2, A_3 of the sample space, with

$$
\begin{aligned}
A_1 &= \{m_e^- = M_0^-, m_e^+ = M_0^+\}\\
&= \{m_e^- = M_0^-, m_e^+ = M_0^+, h_e^- = H_0^-, h_e^+ = H_0^+\}\\
&= \Big\{\max\{W(H_0^-); W(H_0^+)\} - \max\{W(M_0^-); W(M_0^+)\} \ge 1\Big\},\\
A_2 &= \{m_e^- = M_0^-, m_e^+ = M_1^+\}
\end{aligned}
$$

$$
\begin{aligned}
&= \{m_e^- = M_0^-, m_e^+ = M_1^+, h_e^- = H_0^-, h_e^+ = H_1^+\} \\
&= \Big\{ \max\{W(H_0^-); W(H_0^+)\} - W(M_0^+) < 1, W(M_0^-) < W(M_0^+) \Big\}, \\
A_3 &= \{m_e^- = M_1^-, m_e^+ = M_0^+\} \\
&= \{m_e^- = M_1^-, m_e^+ = M_0^+, h_e^- = H_1^-, h_e^+ = H_0^+\} \\
&= \Big\{ \max\{W(H_0^-); W(H_0^+)\} - W(M_0^-) < 1, W(M_0^-) > W(M_0^+) \Big\}.
\end{aligned}
$$

On A_1, we may rewrite (8) as

$$
\hat{\alpha}_e = \min\{2(W(H_0^+) - W(H_0^-))^+ - W(M_0^+); 2(W(H_0^-) - W(H_0^+))^+ - W(M_0^-)\}. \tag{96}
$$

Using the fact that on A_2 it holds that $W(H_1^+) \geq \max\{W(H_0^+), W(H_0^-)\} \geq W(H_0^-)$, we get that

$$
\begin{aligned}
\hat{\alpha}_e &= \min\{2(W(H_1^+) - W(H_0^-))^+ - W(M_1^+); 2(W(H_0^-) - W(H_1^+))^+ - W(M_0^-)\} \\
&= \min\{2W(H_1^+) - 2W(H_0^-) - W(M_1^+); -W(M_0^-)\} \qquad (97)
\end{aligned}
$$

on A_2, and, analogously,

$$
\hat{\alpha}_e = \min\{2W(H_1^-) - 2W(H_0^+) - W(M_1^-); -W(M_0^+)\} \tag{98}
$$

on A_3.

Thus, it is a fact that the law of $\hat{\alpha}_e$ depends only on the law of the random vector $(W(H_0^\pm),\ W(H_1^\pm), W(M_0^\pm), W(M_1^\pm))$. At this point it is important to note that, since $t \mapsto W(\sigma^{-2}t)$ is a standard Brownian motion, from the definitions it follows that the law of this random vector is the same for all $\sigma \in (0, +\infty)$. Hence, it suffices to prove the result in the case $\sigma = 1$. In this case, with the help of Lemma 6.1 (i)–(ii), the joint distribution of $(W(H_0^\pm), W(H_1^\pm), W(M_0^\pm), W(M_1^\pm))$ can be described as follows:

- the two random vectors $(W(H_0^+), W(H_1^+), W(M_0^+), W(M_1^+))$ and $(W(H_0^-), W(H_1^-), W(M_0^-), W(M_1^-))$ are independent and identically distributed;
- the joint distribution of $(W(H_0^+), -W(M_0^+))$ is that given in part (i) of Lemma 6.1;
- the random variables $W(H_1^+) - (W(M_0^+) + 1)$ and $(-W(M_1^+) + W(H_1^+) - 1)$ are independent of the pair $(W(H_0^+), -W(M_0^+))$, and have exponential distribution with mean 1.

Using this together with (96)–(98), we get (after some tedious but elementary computations) the formula (9). Furthermore, it is easy to check that the Laplace transform of the function p defined in the theorem coincides with formula (9), and therefore, p is the density of $\hat{\alpha}_e$. ■

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