A Characterization of Bound States of Nonlinear Wave and Schrödinger Equations *†

Randall Pyke

May 31, 2004

Abstract

We prove for a class of nonlinear wave and Schrödinger equations that bound state solutions are almost periodic in time. As an application we show how to use this result to establish necessary conditions for the existence of bound states.

1 Introduction

In this article we study solutions of nonlinear wave (NLW) and nonlinear Schrödinger (NLS) equations on space-time $\mathbb{R}^{n+1}$;

$$\partial_t^2 u - \Delta u + f(u) = 0 \qquad (NLW)$$

$$-i\partial_t u - \Delta u + f(u) = 0 \qquad (NLS)$$

Here $u : \mathbb{R}^n_x \times \mathbb{R}_t \to \mathbb{C}$ and $f : \mathbb{C} \to \mathbb{C}$ is a nonlinear function. In the absence of the nonlinearity f, these equations are purely dispersive and all finite energy solutions spread out and decay in amplitude. More precisely, they decay in a local (spatial)

*Department of Mathematics and Statistics, University College of the Fraser Valley, Abbotsford, British Columbia, Canada. Email: randall.pyke@ucfv.ca and pyke@math.toronto.edu

†Research supported by NSERC Research Grant #216981-99

norm to the zero solution. The nonlinearity, however, can act against dispersion with the result that some solutions may remain localized in space.

The most natural, and simplest, space-time classification for solutions of these equations are bound states and scattering states. Bound states are solutions that are localized in space uniformly in time, while scattering states are solutions that decay locally in space with respect to time. Bound states represent particle-like solutions (such as solitons and solitary waves) while scattering states represent dispersive waves (radiation travelling to infinity). Not every solution is a bound state or a scattering state, but these types of solutions can be used to describe the large-time behavior of a general solution (scattering theory).

Linear equations of the form $\partial_t^2 u + Hu = 0$ or $-i\partial_t u + Hu = 0$, where H is a self-adjoint operator, may possess bound states. Under certain assumptions on H every solution can be expressed as a superposition of bound states and scattering states (see for example [RS-III]). For integrable nonlinear equations (e.g., the sine-Gordon NLW $\partial_t^2 u - \Delta u + \sin u = 0$ or the cubic NLS $-i\partial_t u - \Delta u - 2|u|^2 u = 0$, both in dimension $n = 1$), the same can be done at least on a formal level. In the large-time limit these modes - the bound states and scattering states - decouple due to dispersion and consequently the solution converges in a local (spatial) norm to the bound states while radiating energy to infinity. For linear equations this decoupling of modes is a conclusion of the so-called RAGE Theorem [R],[AG], [E]; see also [GI], [RS-III], while for nonlinear integrable equations one can use the method of inverse scattering to establish this [FT],[NMPZ] (see also [CVZ] and [DZ] which makes this rigorous in certain cases).

This scenario is becoming a paradigm in the scattering theory of general nonlinear dispersive wave equations. That is, one expects that a general solution will converge locally to bound states as $t \to \infty$ while radiating energy, the convergence being driven by dispersion. Rigorous results corroborating this view include [SW1] and [BP] where it is proven that solutions of NLS initially near a soliton converge (locally) to a soliton. When the equation possesses many bound states (modulo symmetries) the dynamics among them as the solution disperses is a fascinating problem ([TY], [SW2]).

We see that bound states play an important role in describing the large-time behavior of solutions of nonlinear wave and Schrödinger equations. By definition, bound states share the property of being localized in space, uniformly in time. In this article we show that bound states also share a compactness property in time, namely, that they are almost periodic in time. This result extends part of the RAGE Theorem to nonlinear equations. As an application of this temporal property, we establish a necessary condition for the existence of bound state solutions. Proving existence or nonexistence of bound states for linear equations amounts to determining the spectrum of the linear operator H (this is part of the RAGE Theorem). For integrable nonlinear equations one can use the formalism of inverse scattering to address this question (see for example [NMPZ]). For general nonlinear equations with 'repulsive' nonlinearities one can prove apriori dispersive-like decay estimates (perhaps only for small solutions), which of course rule out the possibility of there being bound states (see [S] or [Ca] for a discussion and references). In this article we present new nonexistence results using virial relations and the almost periodicity of (the candidate) bound state solutions.

2 Statement of Results

To state our main result we introduce some notation and definitions. Both NLW and NLS can be written as evolution equations;

$$\partial_t \varphi = A\varphi + G(\varphi), \tag{2.1}$$

where in the case of NLW, $A = \begin{bmatrix} 0 & 1 \\ \Delta & 0 \end{bmatrix}$, $G(\varphi) = \begin{bmatrix} 0 \\ -f(u) \end{bmatrix}$, $\varphi = (u, \partial_t u)$, and where in the case of NLS, $A = i\Delta$, $G(\varphi) = -if(\varphi)$, $\varphi = u$.

The spaces of functions we will work with are the following. In the case of NLW, we let $X = H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ and in the case of NLS we let $X = H^1(\mathbb{R}^n)$. For $\mu > 0$, we let $X_\mu = H^{1+\mu}(\mathbb{R}^n) \times H^\mu(\mathbb{R}^n)$ or $X_\mu = H^{1+\mu}(\mathbb{R}^n)$ in the case of NLW or NLS respectively, where $H^\mu(\mathbb{R}^n)$ denotes the $L^2(\mathbb{R}^n)$ Sobolev space of order μ; $H^\mu(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) : \langle k\rangle^\mu \hat{u} \in L^2(\mathbb{R}^n)\}$, where $\hat{u}$ denotes the Fourier transform of u and $\langle k\rangle \equiv (1 + |k|^2)^{1/2}$. We denote the associated norms on these spaces by $\|\cdot\|_X^2 = \|\cdot\|_{H^1(\mathbb{R}^n)}^2 + \|\cdot\|_{L^2(\mathbb{R}^n)}^2$ or $\|\cdot\|_X^2 = \|\cdot\|_{H^1(\mathbb{R}^n)}^2$, and $\|\cdot\|_{X_\mu}^2 = \|\cdot\|_{H^{1+\mu}(\mathbb{R}^n)}^2 + \|\cdot\|_{H^\mu(\mathbb{R}^n)}^2$ or $\|\cdot\|_{X_\mu}^2 = \|\cdot\|_{H^{1+\mu}(\mathbb{R}^n)}^2$ in the case of NLW or NLS respectively.

We consider a solution $\varphi(x,t)$ of NLW or NLS as a map $\varphi:\ \mathbb{R}\to X$ which we will denote by $\varphi(t)$. For $\varphi\in X$, $\mathcal{O}(\varphi)$ will denote the orbit of φ; $\mathcal{O}(\varphi)=\{\varphi(t)\ \mid\ t\in\mathbb{R},\ \varphi(0)=\varphi\}$, and $\mathcal{AP}(X)$ will denote the Banach space of almost periodic functions from $\mathbb{R}$ to X with norm $\|\varphi\|_{\mathcal{AP}(X)}=\sup_{t\in\mathbb{R}}\|\varphi(t)\|_X$ (see Definition 3.1 below).

Definition 2.1 *A function $\varphi:\mathbb{R}\to X$ is a* bound state *if for any $\varepsilon>0$ there exists a ball $B_{R_\varepsilon}\subset\mathbb{R}^n$ of radius R_ε centered at 0 such that*

$$\sup_t \|\overline{\chi}_{R_\varepsilon}\varphi(t)\|_X<\varepsilon \tag{2.2}$$

where $\overline{\chi}_{R_\varepsilon}$ is the characteristic function of $\mathbb{R}^n\setminus B_{R_\varepsilon}$.

Remark: Bound states as defined here are stationary. Both NLW and NLS possess a group of symmetry transformations which map one solution into another (e.g., the Lorentz group for NLW and the Galilean group for NLS). These symmetry transformation can transform a bound state into a 'travelling' bound state. Thus, some solutions may not be bound states in the sense of the Definition 2.1, but can be transformed into bound states by an appropriate symmetry transformation.

If $\varphi(t)$ is a globally defined solution of NLW or NLS with $\varphi(0)=\varphi$, we let $S:\mathbb{R}\times\mathcal{O}(\varphi)\to\mathcal{O}(\varphi)$, $S(t)\ :\ \varphi_1\mapsto\varphi_1(t)\equiv S(t)\varphi_1$, denote the flow of the dynamical system (2.1) along $\mathcal{O}(\varphi)$. Note that $\{S(t)\}_{t\in\mathbb{R}}$ is a semi-group: $S(t_1+t_2)=S(t_1)S(t_2)$.

Our main result is the following.

Theorem 2.2 *Let $\varphi\in C(\mathbb{R};X)$ be a bound state solution of NLW or NLS such that $\sup_t\|\varphi(t)\|_{X_\mu}<\infty$ for some $\mu>0$. If the set $\{S(t)\}_{t\in\mathbb{R}}$ is equicontinuous on $\mathcal{O}(\varphi)$, then φ is an almost periodic function from $\mathbb{R}$ to X.*

The regularity condition $\sup_t\|\varphi(t)\|_{X_\mu}<\infty$, along with the hypothesis of localization (i.e., that $\varphi(t)$ is a bound state), assures that $\mathcal{O}(\varphi)$ is relatively compact in X (see Proposition 3.3). This compactness, combined with the equicontinuity of $\{S(t)\}_{t\in\mathbb{R}}$, guarantees the almost periodicity of φ (as we will show). Recall that $\{S(t)\}_{t\in\mathbb{R}}$ is equicontinuous if for any $\varepsilon>0$ there is a δ such that if $\varphi_1,\varphi_2\in$

$\mathcal{O}(\varphi)$ with $\|\varphi_1 - \varphi_2\|_X < \delta$, then $\sup_{t\in\mathbb{R}} \|S(t)\varphi_1 - S(t)\varphi_2\|_X < \varepsilon$. In the linear case the flow is equicontinuous on any trajectory if the family $\{S(t)\}_{t\in\mathbb{R}}$ is bounded; $\|S(t)\varphi_1 - S(t)\varphi_2\|_X \le (\sup_t \|S(t)\|_{X\to X})(\|\varphi_1 - \varphi_2\|_X)$. In particular, this is true for the free linear wave and Schrödinger equations (i.e., $f(u) \equiv 0$), and for general linear equations under mild assumptions on the operator H. In the nonlinear case equicontinuity will have to be verified for each equation. We remark that without the hypothesis of equicontinuity, the orbit $\mathcal{O}(\varphi)$, although relatively compact, may only be recurrent and not almost periodic (see for example Ch V.8 in [NS]).

Example

Consider the following assumptions on the nonlinearity f;

$$F(u) \ge -c|u|^2 - c|u|^{q+1} \quad \text{for} \quad q < 1 + \frac{4}{n} \quad \text{in the case of NLW,}$$

$$F(u) \ge -c|u|^2 \quad \text{in the case of NLS,}$$

where $F' = f$. Furthermore, suppose $|f'(u)| \le c(1 + |u|^{p-1})$ where $1 < p < 1 + \frac{4}{n-2}$ ($1 < p < \infty$ if $n = 1, 2$). Then if $\varphi_o \in X_1$, there exists a global solution $\varphi \in C(\mathbb{R}; X_1)$ with $\varphi(0) = \varphi_o$ (see [S] Chapter 3). Applying Theorem 2.2, if the solution $\varphi(t)$ is a bound state such that $\sup_t \|\varphi(t)\|_{X_\mu} < \infty$, $\mu > 0$, and if $\{S(t)\}_{t\in\mathbb{R}}$ is equicontinuous on $\mathcal{O}(\varphi_o)$, then $\varphi(t)$ is an almost periodic function from $\mathbb{R}$ to X.

Nonexistence of bound states

Here we combine the use of virial relations, as discussed in [P], with the almost periodicity of bound states (Theorem 2.2) to prove nonexistence of bound states for certain equations.

A virial relation is an integral identity involving the solution of a differential equation. The identity can be derived directly from the equation itself or, if the equation can be formulated as a variational problem, from the action functional associated to the equation. A well known example from physics relates the time-average kinetic and potential energies of an n-particle system under the influence of central forces (usually referred to as *the* virial theorem; see for example [LL]). In mathematics virial relations have been used extensively, for example, in deriving necessary conditions for the existence of solutions of differential equations beginning with the work of Pohozaev [Po].

Let's consider NLW, and suppose $\varphi(t)$ is a bound state solution. Assuming the hypotheses of Theorem 2.2 are satisfied, $\varphi(t)$ is almost periodic in time. As described in [P], by combining the dilation and gauge transformations, one obtains the following virial relation valid for any $c \in \mathbb{R}$;

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \int_{\mathbb{R}^n} \left\{ (\frac{1}{2} - c)(\partial_t \varphi)^2 + (c + \frac{2-n}{2n}) \mid \nabla\varphi \mid^2 \right\} = \lim_{T\to\infty} \frac{1}{T}\int_0^T \int_{\mathbb{R}^n} \{F(\varphi) - c\varphi f(\varphi)\} \tag{2.3}$$

We assume that $\varphi \in \mathcal{AP}_q$ for some $q \in [2,\infty]$ where $\mathcal{AP}_q \equiv \{\varphi \in \mathcal{AP}(H^1(\mathbb{R}^n) \cap L^q(\mathbb{R}^n))$ such that $\partial_t \varphi$ exists in the strong sense as a uniformly continuous map $\mathbb{R} \to L^2(\mathbb{R}^n)\}$. Here $\mathcal{AP}(\mathcal{B})$ denotes the set of almost periodic functions from $\mathbb{R}$ to the space $\mathcal{B}$ (see Definition 3.1 below). We also assume that $f \in C(\mathbb{R},\mathbb{R})$ with $f(0) = 0$ and that either $\mid f(z) \mid \leq c(\mid z \mid + \mid z \mid^{q/2})$, or f satisfies a Lipschitz condition at the origin.

If for some $c \in [\frac{n-2}{2n}, \frac{1}{2}]$, the inequality

$$F(z) - czf(z) \leq 0 \tag{2.4}$$

holds, then both sides of Equation (2.3) must be zero. In the case $c > \frac{n-2}{2n}$ this then implies that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \int_{\mathbb{R}^n} \mid \nabla\varphi \mid^2 \;=\; 0 \;=\; \lim_{T\to\infty} \frac{1}{T}\int_0^T \|\nabla\varphi\|_2^2. \tag{2.5}$$

Parseval's relation for Hilbert space valued almost periodic functions ([LZ] pp 31) states that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \|\nabla\varphi\|_2^2 = \sum_{k=1}^{\infty} \|a_k\|_2^2 \tag{2.6}$$

where $\sum_{k=1}^{\infty} a_k e^{i\lambda_k t}$ is the Fourier series associated to $\nabla\varphi(t)$. By the uniqueness of these series it follows from equations (2.5) and (2.6) that $\nabla\varphi(t) = 0$ (in $L^2(\mathbb{R}^n)$) for all t which implies that, since $\varphi(t) \in L^2(\mathbb{R}^n)$, $\varphi(t) = 0$ for all t. Thus, if f satisfies (2.4) with $c \in (\frac{n-2}{2n}, \frac{1}{2}]$, then this NLW has no (non-trivial) bound states.

Remark: The virial relation (2.3) contains no boundary terms (in space and in time). The occurrence of boundary terms themselves do not exclude the use of virial relations, but one then one has to control these terms. A virial relation for NLW or NLS will have no boundary terms if the solution has vanishing or periodic boundary conditions. In the spatial variable this is assured by assuming the solution is integrable in x. Only dispersive solutions will have vanishing boundary conditions in time, so to obtain virial relations without boundary terms for bound states one must assume some kind of compactness property in time of the solution. For example, the existence of the ergodic mean

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \varphi(t)\, dt$$

for $\varphi(t)$ and the other terms appearing in the action functional defining the equation, allows us to obtain virial relations without boundary terms in time. One important property of almost periodic functions is that they do have ergodic means.

3 Proof of Theorem 2.2

We state the definition of almost periodic functions that we will use (see for example [LZ]).

Definition 3.1 *A strongly continuous function $\varphi:\ \mathbb{R} \to X$ is* almost periodic *if the set of translates of φ, $\mathcal{T}(\varphi) = \{\varphi^h(t) \equiv \varphi(t+h);\ h \in \mathbb{R}\}$, has compact closure in $C_b(\mathbb{R};\ X)$ where $C_b(\mathbb{R};\ X)$ denotes the metric space of bounded strongly continuous functions from $\mathbb{R}$ to X equipped with the supremum norm: $\|\varphi\|_{C_b} \equiv \sup_t \|\varphi(t)\|_X$. The set of almost periodic functions from $\mathbb{R}$ to X will be denoted by $\mathcal{AP}(X)$.*

Remark: Almost periodic functions admit an elegant representation, from which their many properties may be derived [Co],[LZ]. For example, we can associate a Fourier series to any $\varphi \in \mathcal{AP}(X)$;

$$\varphi(t)\ \sim\ \sum_{k\in\mathbb{N}} \phi_k e^{i\lambda_k t}, \quad \lambda_k \in \mathbb{R},\ \phi_k \in X \tag{3.7}$$

where $\phi_k = \lim_{T\to\infty} \frac{1}{T}\int_0^T e^{-i\lambda_k t}\varphi(t)\,dt$. The question of convergence of this Fourier series is just as delicate as for periodic functions. In general though, by re-summing the Fourier series of an almost periodic function one can find trigonometric polynomials that approximate the function arbitrarily well, uniformly in time. More precisely, for any $\varepsilon > 0$ there exists a trigonometric polynomial $p_\varepsilon(t) = a_0 + a_1 e^{i\lambda_{k_1} t} + \cdots + a_m e^{i\lambda_{k_m} t}$, $a_i \in X$, such that $\sup_{t\in\mathbb{R}} \|\varphi(t) - p_\varepsilon(t)\|_X < \varepsilon$.

Proposition 3.2 *If $\varphi \in \mathcal{AP}(X)$ then φ is a bound state.*

An important property of almost periodic functions is that the closure of their orbits in the phase space X are compact [LZ]. From this Proposition 3.2 follows. Conversely, to prove that a bound state is almost periodic we first show that the closure of its orbit is compact under a regularity assumption. For this we require a convenient characterization of compact subsets of the spaces X.

Proposition 3.3 *A bounded subset $K \subset X$ has compact closure in X if for any $\varepsilon > 0$ there exists an $R_\varepsilon > 0$ such that $\sup_{\varphi\in K} \|\overline{\chi}_{R_\varepsilon}\varphi\|_X < \varepsilon$ and if for some $\mu > 0$, $\sup_{\varphi\in K}\|\varphi\|_{X_\mu} < \infty$.*

Proof: We show that K is totally bounded in X. Let $\varepsilon > 0$ and find R_ε so that $\sup_{\varphi\in K} \|\overline{\chi}_{R_\varepsilon}\varphi\|_X < \varepsilon$. Let $R > R_\varepsilon$. For $\varphi \in K$ let $\tilde{\varphi} \equiv \chi_R\varphi$ and $\overline{\varphi} \equiv \overline{\chi}_R\varphi$ where χ_R is a smooth function such that $\chi_R \equiv 1$ on B_{R_ε} and $\chi_R \equiv 0$ on B_R^C, and $\overline{\chi}_R = 1 - \chi_R$. We set $\tilde{K} = \{\tilde{\varphi} \,|\, \varphi \in K\}$. For any subset $\Omega \subset \mathbb{R}^n$ let $X(\Omega)$ denote the corresponding Sobolev spaces defined on Ω (as defined above; so for example $X(\mathbb{R}^n) = X$). Since $\tilde{K}$ is bounded in $X_\mu(B_R)$, $\mu > 0$, and every $\tilde{\varphi} \in \tilde{K}$ is supported in B_R, by the Rellich-Kondrachov Theorem, $\tilde{K}$ has compact closure in $X(B_R)$. Therefore, there exists an ε-net for $\tilde{K}$ in $X(B_R)$, i.e., a finite set $\{\phi_1, \ldots, \phi_N\} \subset X(B_R)$ such that for any $\tilde{\varphi} \in \tilde{K}$, there is a $k \in \{1, \ldots, N\}$ such that $\|\tilde{\varphi} - \phi_k\|_{X(B_R)} = \|\tilde{\varphi} - \phi_k\|_X < \varepsilon$ (here we extend $\tilde{\varphi}$ and ϕ_k to all of $\mathbb{R}^n$ with $\tilde{\varphi} \equiv 0$ and $\phi_k \equiv 0$ on $\mathbb{R}^n \setminus B_R$). Since $\varphi = \tilde{\varphi} + \overline{\varphi}$, we have that $\|\varphi - \phi_k\|_X \le \|\tilde{\varphi} - \phi_k\|_X + \|\overline{\varphi}\|_X < 2\varepsilon$. Thus, $\{\phi_1, \ldots, \phi_N\}$ is a 2ε-net for K in X $\square$

A necessary and sufficient condition that K has compact closure in X is that K be uniformly localized in space and momentum, i.e., that for any $\varepsilon > 0$ there is an R_ε such that $\sup_{\varphi\in K} \|\overline{\chi}_{R_\varepsilon}\varphi\|_X < \varepsilon$ and $\sup_{\hat{\varphi}\in K} \|\overline{\chi}_{R_\varepsilon}\hat{\varphi}\|_X < \varepsilon$ (see for example [RS-IV] or [GI]). The condition $\sup_{\varphi\in K} \|\varphi\|_{X_\mu} < \infty$ for some $\mu > 0$ guarantees uniform localization in momentum, but it is not a necessary condition so this hypothesis could be weakened (and hence the regularity hypothesis in Theorem 2.2 could be weakened).

The idea of the proof of Theorem 2.2 is to go from compactness of $\mathcal{O}(\varphi)$ in X to compactness of $\mathcal{T}(\varphi)$ in $C_b(\mathbb{R}; X)$ via the equicontinuity of $\{S(t)\}_{t\in\mathbb{R}}$.

Proof of Theorem 2.2: Let $\{\varphi^{h_j}\} \equiv \{\varphi(t + h_j)\}$ be any sequence from $\mathcal{T}(\varphi)$. By Proposition 3.3, $\overline{\mathcal{O}(\varphi)}$ is compact in X so there exists a subsequence $\{h_j'\}$ of $\{h_j\}$ such that $\{\varphi(h_j')\}$ is a Cauchy sequence in X. Now consider the subsequence $\{\varphi^{h_j'}\}$ of $\{\varphi^{h_j}\}$. Let $\varepsilon > 0$. Since we are assuming that the flow $S(t)$ on $\overline{\mathcal{O}(\varphi)}$ is equicontinuous, there is a $\delta > 0$ such that $\|S(t)\varphi_1 - S(t)\varphi_2\|_X < \varepsilon$ for all t if $\varphi_1, \varphi_2 \in \overline{\mathcal{O}(\varphi)}$ and $\|\varphi_1 - \varphi_2\|_X < \delta$. Because $\{\varphi(h_j')\}$ is a Cauchy sequence, we can find an N_δ such that $\|\varphi(h_k') - \varphi(h_l')\|_X < \delta$ if $k, l > N_\delta$. Therefore, $\|\varphi(t + h_k') - \varphi(t + h_l')\|_X = \|S(t)\varphi(h_k') - S(t)\varphi(h_l')\|_X < \varepsilon$ for all t if $k, l > N_\delta$ which implies that $\{\varphi^{h_j'}\}$ is a Cauchy sequence in $C_b(\mathbb{R};\ X)$. Thus, $\overline{\mathcal{T}(\varphi)}$ is sequentially compact, and hence compact, in $C_b(\mathbb{R}; X)$ $\square$

References

[AG] Amrein, W.O. and Georgescu, V. *On the Characterizaation of Bound States and Scattering States in Quantum Mechanics.* Helvetica Physica Acta, Vol. 46, 635- 658 (1973).

[BP] Buslaev, V.S. and Perel'man, G.S. *Scattering for the Nonlinear Schrodinger Equation: States Close to a Soliton.* St. Petersburg Math. J., vol. 4, no. 6, 1111-1142 (1993).

[Ca] Cazanave, T. *An Introduction to Nonlinear Schrödinger Equations.* Instituto de Matemática UFRJ, Rio de Janeiro, RJ, 1996.

[Co] Corduneanu, C. *Almost Periodic Functions.* Chelsea (1989).

[CVZ] Cheng, P-J., Venakides, S. and Zhou, X. *Long-Time Asymptotics for the Pure Radiation Solution of the sine-Gordon Equation.* Comm. P.D.E., **24** (7&8), 1195-1262 (1999).

[DZ] Deift, P. and Zhou, X. *Perturbation Theory for Infinite-Dimensional Integrable Systems on the Line. A Case Study.* Acta Math, vol. 188, pp. 163-262 (2002).

[E] Enss, V. *Asymptotic Completeness for Quantum-Mechanical Potential Scattering, I. Short-Range Potentials.* Comm. Math. Phys., **61**, 285-291 (1978).

[FT] Faddeev, L. and Takhtajan, L. *Hamiltonian Methods in the Theory of Solitons.* Springer-Verlag, New York (1987).

[GI] Georgescu, V., Iftimivici, A., *Riesz-Kolmogorov Compacity Criterion, Ruelle Theorem and Lorentz Convergence on Locally Compact Albelian Groups.* Preprint (2000). `http://rene.ma.utexas.edu/mp_arc-bin/mpa?yn=00-520`.

[LL] Landau, L.D., Lifshitz, E.M. *Mechanics.* Pergamon Press, New York (1976).

[LZ] Levitan, B.Z. and Zhikov, V.V. *Almost Periodic Functions and Differential Equations*. Cambridge University Press (1982).

[NMPZ] Novikov, S., Manakov, S.V., Pitaevskii, L.P., Zakharov, V.E. *Theory of Solitons: The Inverse Scattering Method*. Consultants Bureau, New York (1984).

[NS] Nemytskii, V.V. and Stepanov, V.V. *Qualitative Theory of Differential Equations*. Dover (1989).

[P] Pyke, R. *Virial Relations for Nonlinear Wave Equations and Nonexistence of Almost Periodic Solutions*. Review of Mathematical Physics, Vol. 8, No. 7, 1001-1039 (1996).

[Po] Pohozaev, S.I. *Eigenfunctions of the Equation* $\Delta u + \lambda f(u) = 0$. Sov. Math. Doklady, **5**, 1408-1411 (1965).

[R] Ruelle, D. *A Remark on Bound States in Potential Scattering Theory*. Nuovo Cimento, **61A**, 655-662 (1969).

[RS-III] Reed, M. and Simon, B. *Methods of Modern Mathematical Physics III. Scattering Theory*. Academic Press, New York (1979).

[RS-IV] Reed, M. and Simon, B. *Methods of Modern Mathematical Physics IV. Analysis of Operators*. Academic Press, New York (1978).

[S] Strauss, W.A. *Nonlinear Wave Equations*. Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics, number 73. American Mathematical Society (1989).

[SW1] Soffer, A. and Weinstein, M.I. *Multichannel Nonlinear Scattering for Nonintegrable Equations*. Comm. Math. Phys., **133**, 119-146 (1990) and J. Diff. Eq. **98**, 376-390 (1992).

[SW2] Soffer, A. and Weinstein, M.I. *Selection of the Ground State for Nonlinear Schrödinger Equations*. Preprint (2003).
`http://www.ma.utexas.edu/mp_arc-bin/mpa?yn=03-398`

[TY] Tsai, T.-P., Yau, H.-T. *Asymptotic Dynamics of Nonlinear Schrodinger Equations: Resonance Dominating and Radiation Dominating Solutions*. Preprint, 2000. http://www.ma.utexas.edu/mp_arc-bin/mpa?yn=00-413