Wegner Estimate for some Indefinite Anderson-type Schrödinger Operators

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Abstract

We study Schrödinger operators with a random potential of Anderson-type. The single site potentials are allowed to change sign. For a certain class of them we prove a Wegner estimate (Theorem 5). This is a key ingredient in an existence proof of pure point spectrum of the considered random Schrödinger operators. Our estimate is valid for all bounded energy intervals and all space dimensions.

Keywords: integrated density of states, random Schrödinger operators, Wegner estimate, multiscale analysis, localisation, indefinite single site potential

1 Introduction

The subject matter of this work are families of random Schrödinger operators $\{H_\omega\}_{\omega\in\Omega}$. They have been introduced as quantum mechanical models for disordered media in solid state physics. The random Schrödinger operator we consider is of Anderson- or alloy-type

$$H_\omega = -\Delta + V_0 + V_\omega,\ \omega \in \Omega \tag{1}$$

where the negative Laplace operator $-\Delta$ corresponds to the kinetic energy, V_0 is a bounded $\mathbb{Z}^d$-periodic potential and V_ω is the random potential given by the stochastic process

$$V_\omega(x) = \sum_{k\in\mathbb{Z}^d} \omega_k\, u(x-k)\ . \tag{2}$$

The function $u\colon \mathbb{R}^d \to \mathbb{R}$ is called *single site potential* and represents the contribution to V_ω due to a single nucleus or ion situated at a lattice point $k \in \mathbb{Z}^d$. We assume that u is compactly supported and bounded. The ω_k are real-valued random *coupling constants*. I.e. while we fix the shape of the single site potential at each $k \in \mathbb{Z}^d$, its strength is allowed to vary randomly. We assume that the random variables $\omega_k, k \in \mathbb{Z}^d$ are independent and identically distributed (iid) and their distribution measure μ has a density $f \in L^\infty(\mathbb{R})$ with support in $[0, \omega_+], \omega_+ > 0$. We consider the coupling constants as components of a random vector $\omega := \{\omega_k\}_{k\in\mathbb{Z}^d} \in \Omega := \bigotimes_{k\in\mathbb{Z}^d}[0, \omega_+]$. The probability space Ω is equipped with the product measure $\mathbb{P} := \bigotimes_{k\in\mathbb{Z}^d} \mu$.

In the remainder of this section we outline briefly some spectral properties of H_ω which have been studied in the literature. Section 2 contains our main result (Theorem 5) and relates it to previous Wegner estimates for indefinite models. Sections 3 to 5 contain the proof of the main theorem and the last two sections are devoted to the discussion of the results and the application to localisation.

Under our assumptions the family $H_\omega, \omega \in \Omega$ fits into the general theory of ergodic random Schrödinger operators [Kir89, CL90, PF92]. We infer two central results from this theory.

Theorem 1
The spectrum of the family $H_\omega, \omega \in \Omega$ is non-random in the following sense. There exists a subset Σ of the real line and an $\Omega' \subset \Omega$, $\mathbb{P}(\Omega') = 1$ such that for all $\omega \in \Omega'$ one has

$$\sigma(H_\omega) = \Sigma\ . \tag{3}$$

The analogous statement holds true for the essential, discrete, continuous, absolutely continuous, singular continuous, and pure point part of the spectrum. Note that the pure point spectrum σ_{pp} is the closure of the set of eigenvalues of H_ω.

We denote by Λ_l the cube of side length l and centre at 0 and with H_ω^l the restriction of H_ω to Λ_l with Dirichlet boundary conditions (b.c.). Note that H_ω^l has purely discrete spectrum so we can enumerate its eigenvalues $\lambda_i(H_\omega^l)$ in non-decreasing order and counting multiplicities.

Theorem 2
There exists a self-averaging integrated density of states associated with the family $H_\omega, \omega \in \Omega$. This means that the normalised eigenvalue counting functions

$$N_\omega^l(E) = l^{-d}\#\{i|\ \lambda_i(H_\omega^l) < E\} \tag{4}$$

of H_ω^l converge for almost all ω to a limit

$$N(E) := \lim_{l\to\infty} N_\omega^l(E) \tag{5}$$

which is ω-independent.

We call $N(\cdot)$ the integrated density of states (IDS) of H_ω and $N_\omega^l(\cdot)$ the (finite volume) IDS of H_ω^l.

Remark 3
While the two above Theorems 1 and 2 follow from the general theory, one can ask for more specific spectral properties of concrete models $H_\omega, \omega \in \Omega$, e.g.

1. Which spectral types can occur in $\sigma(H_\omega)$? Can different spectral types coexist, and if yes, can the supports of the corresponding spectral measures overlap?

2. Can something be said about the regularity of the IDS N as a function of the energy E? Is it Hölder continuous; or its ω-averaged finite box analoga $\mathbb{E}[N_\bullet^l]$?

Actually both questions are related to the transport properties of the medium modeled by H_ω. A perfect crystal is described by a Schrödinger operator with periodic potential. It has purely absolutely continuous spectrum, which reflects its good electric transport properties. In contrast to this, it has been proven that random perturbations of this regular structure give rise to energy intervals with pure point spectrum. This corresponds to the less effective transport properties of random media. The existence of pure point spectrum in this context is called *localisation.*

A powerful tool for proving localisation is the so-called *multiscale analysis* (MSA), an induction argument over increasing length scales $l_k, k \in \mathbb{N}$. This technique was first applied by Fröhlich and Spencer [FS83] to the discretisation of the Schrödinger operator (1) and underwent since then a number of strengthenings [MS85, FMSS85], simplifications [vDK89] and adaptations to the continuous model on $L^2(\mathbb{R}^d)$, which we are considering [MH84, Klo95a, CH94]. More recently it was used also for Hamiltonians governing the motion of classical waves [FK96, FK97, Sto98, CHT99].

At the same time extensive research has been done to identify physical situations where one can prove the key ingredients needed to start and carry trough the MSA [Klo95b, BCH97, KSS98b, KSS98a, Ves98, FLM98].

Remark 4 (MSA Hypotheses)
Let us fix some notation. For points $x \in \mathbb{R}^d$ in the configuration space let $\|x\|_\infty := \sup\{|x_i|, i = 1, \dots, d\}$ denote the sup-norm. Let $\delta > 0$ be a small constant independent of the length scale l_k and $\phi_k(x) \in C^2$ a function which is identically equal to 0 for x with $\|x\|_\infty > l_k - \delta$ and identically equal to one for x with $\|x\|_\infty < l_k - 2\delta$. The commutator $W(\phi_k) := [-\Delta, \phi_k] := -(\Delta\phi_k) - 2(\nabla\phi_k)\nabla$ is a local operator acting on functions which live on a ring of width δ near the boundary of $\Lambda_k := \Lambda_{l_k}$. We say that a pair $(\omega, \Lambda_k) \in \Omega \times \mathcal{B}(\mathbb{R}^d)$ is *m-regular* for

a given energy E, if

$$\sup_{\epsilon \neq 0} \|W(\phi_k)(H^l_\omega - E + \epsilon i)^{-1} \chi_{l_k/3}\|_{\mathcal{L}(L^2)} \leq e^{-ml_k} . \tag{6}$$

Here $\chi_{l_k/3}$ is the characteristic function of $\Lambda_{l_k/3} := \{y | \, \|y\|_\infty \leq l_k/6\}$. Thus the distance of the supports of $\nabla\phi_k$ and $\chi_{l_k/3}$ is at least $l_k/3 - 2\delta \geq l_k/4$.

We formulate two key hypotheses for the MSA associated to energies E in the interval $I \subset \mathbb{R}$ in which we want to prove the existence of pure point spectrum.

(H1) $\Leftrightarrow$ There exist constants $Q_1, m \in]0, \infty[, q > 0$ such that

$$\mathbb{P}\{\omega | \, (\omega, \Lambda_{Q_1}) \text{ is } m\text{-regular}\} \geq 1 - Q_1^q. \tag{7}$$

(H2) $\Leftrightarrow$ There exist constants $Q_2, \eta_0, \in]0, \infty[$ such that

$$\mathbb{P}\{\omega | \, d(\sigma(H_\omega|_\Lambda), E) \leq \eta\} \leq C_W \, \eta |\Lambda| \tag{8}$$

for all boxes Λ with side length larger than Q_2 and all $\eta \leq \eta_0$.

The first hypothesis (H1) is commonly called *initial scale estimate*. It provides the induction anchor for the MSA. Most papers deduce (H1) from the asymptotic behaviour of the IDS at so-called spectral *fluctuation boundaries*. This asymptotics reflect the fact that "electron levels" are very sparse near such edges of the spectrum. The existence of these "tails" has been first observed and proven by physical arguments by Lifshitz [Lif65].

The estimate (H2) is associated with a paper of Wegner [Weg81] where he — like Fröhlich and Spencer [FS83] — considers Schrödinger operators on $l^2(\mathbb{Z}^d)$. Wegner's estimate is needed to draw the induction conclusion on each length scale $l_k, k \in \mathbb{N}$ of the MSA. This is the reason why — in contrast to (H1) — it has to be valid for arbitrarily large scales $l \geq Q_2$ [1].

Once the MSA has been accomplished, one proceeds to prove localisation using the spectral averaging technique and expansion in generalised eigenfunctions, cf. Sections 7 and 8 in [KSS98b].

Actually for the MSA a variety of weaker bounds than (8) is sufficient. It is enough to know

(H2') $\Leftrightarrow$ There exist constants $Q_2, \eta_0, a, b \in]0, \infty[$ such that

$$\mathbb{P}\{\omega | \, d(\sigma(H_\omega|_\Lambda), E) \leq \eta\} \leq C_W \, \eta^a |\Lambda|^b \tag{9}$$

for all boxes Λ with side length larger than Q_2 and all $\eta \leq \eta_0$. Inequality (9) is implied by the Hölder continuity of the averaged finite volume IDS, mentioned in Remark 3:

$$\mathbb{E}\{N^l_\bullet(E + \eta) - N^l_\bullet(E - \eta)\} \leq C_W \, \eta^a |\Lambda_l|^{b-1} . \tag{10}$$

[1] Each hypothesis in Remark 4 has its own initial scale: Q_1 and Q_2. On the other hand the MSA itself needs a sufficiently large starting scale Q_0. For the whole argument to work out one has to make sure that Q_1 can be chosen at least as large as the maximum of Q_0 and Q_2.

2 Indefinite Wegner Estimates: previous and new results

In this section we will try to shed some light on the role played by monotonicity in the proof of the Wegner estimate. We discuss briefly earlier results on Wegner estimates in non-monotonous situations and state our main theorem.

For a qualitative discussion assume that V_ω depends on some component ξ_k of the "randomness" in a monotone increasing way on some subset $A_k \subset \mathbb{R}^d$ of configuration space. This is e.g. the case for the potential (2) if $\xi_k = \omega_k, k \in \mathbb{Z}^d$ and we consider the particular single site potential $u = \chi_{\Lambda_1}$. (In the sequel we will abbreviate the characteristic function χ_{Λ_1+k} of the unit cube $\Lambda_1 + k$ at site $k \in \mathbb{Z}^d$ by χ_k.) Because of the linear coupling $\omega_k\, u(\cdot - k)$ the function $\omega_k \mapsto V_\omega|_{A_k}, A_k = \Lambda_1 + k$ is then strictly increasing. Additionally we assume that

- the A_k form a partition of $\mathbb{R}^d$ (as it is the case with the cubes $\Lambda_1 + k$) and
- for each ξ_k its conditional distribution with respect all the remaining randomness is absolutely continuous. This is immediate for $\xi_k = \omega_k$ in (2) since these random variables are iid and have a density f.

It is obvious that the eigenvalues of H_ω^l are non-decreasing in ω_k. The Wegner estimate shows that this can be controlled in a quantitative way, so to rule out the possibility of clustering of eigenvalues (w.r.t the randomness ω) in a small energy interval $[e - \eta, E + \eta]$. [1] If we allow u to change sign the picture becomes much more complicated since $\omega_k \mapsto V_\omega$ is not monotonous any more and thus $\omega_k \mapsto \lambda_i(H_\omega^l)$ neither. This rouses the question in which situations an estimate like (H2) still can be proven. The solution to this problem can be understood as a search for hidden monotonous parameters in V_ω. Here "situations" means

1. For which energy intervals $[E_1, E_2] \subset \mathbb{R}$?
2. For which dimensions d of configuration space $\mathbb{R}^d$?
3. For what kind of indefinite single site potential u?

In [Klo95b] Klopp proves a Wegner estimate for the alloy-type model (1) at low energies. This means that the estimate (H2') is valid for $[E-\eta, E+\eta] \subset [E_{\text{inf}}, E_c]$ where E_{inf} is the infimum of the spectrum of H_ω^l and E_c an energy strictly above E_{inf}. The result is valid for arbitrary dimensions d, and for the single site potential u only some mild regularity and decay assumptions, but no sign-definiteness is required. Crucial is that the density f belongs to a nice class of functions which contains as a subset $C^1(\mathbb{R})$.

[1] A very instructive derivation of Wegner's estimate for V_ω a Gaussian random field is [FHLM97]. In this case there are no natural randomness components like the ω_k in (1) and one has to extract them ingeniously from the Gaussian field.

A technique developed by Buschmann and Stolz [BS98] for one-dimensional Schrödinger operators with Poissonian or random displacement potentials is another instance of an indefinite Wegner estimate. Stolz adopted it in [Sto99] to the alloy-type model (1) in one dimension with no sign restrictions on the single site potential u. In the proof analyticity assumes the role usually played by monotonicity. This again implies piecewise monotonicity.

We state now our main result. It is valid only for a quite restricted class of single site potentials u but in any dimension d and for all bounded energy intervals. So it may be used as an ingredient of a proof of localisation at internal spectral band edges of multidimensional random Schrödinger operators.

Theorem 5
Let $\Gamma \subset \mathbb{Z}^d$ be a finite set and $\alpha = \{\alpha_k, k \in \Gamma\}$ a vector of real numbers with the property $\alpha_0 = 1$ and $\alpha^* = \sum_{k \neq 0} |\alpha_k| < 1$. Let the single site potential be

$$u(x) = \sum_{k \in \Gamma} \alpha_k \chi_0(x-k) \tag{11}$$

and the single site density $f \in C_0^2(]0, \omega_+[)$. Then for the finite volume IDS $N_\omega^l(E)$ of H_ω^l we have

$$\mathbb{E}\left[N_\bullet^l(E_2) - N_\bullet^l(E_2)\right] \leq const\ |E_2 - E_1|\ . \tag{12}$$

The theorem is proven in Sections 3 to 5. We call α the *convolution vector.*

Corollary 6
Let $p = 2$ if $d \leq 3$, $p > 2$ if $d = 4$ and $p \geq d/2$ if $d \geq 5$. By the same proof Theorem 5 extends to the case where χ_0 is replaced by some real valued $w \in L^p(\mathbb{R}^d)$ with compact support and

$$w \geq \epsilon \chi_0 \tag{13}$$

for some $\epsilon > 0$.

3 From spectral projections to localised resolvents

In this section we reduce the bound of the expectation value of spectral projections to averaging of resolvents localised in space. We follow the arguments of [CH94, Section 4] which in turn are based on [KS87b]. Let $I :=]E_1, E_2[$ be an open energy interval, $P_\omega^l(I)$ the spectral projection of H_ω^l onto the interval I and let $\mathrm{Tr}(A)$ denote the trace of an operator A. For the finite volume IDS and any $\epsilon > 0$ we have

$$\mathbb{E}\left[N_\omega^l(E_2) - N_\omega^l(E_1 + \epsilon)\right] \leq \frac{1}{l^d} \mathbb{E}\left[\mathrm{Tr}\, P_\omega^l(I)\right]\ . \tag{14}$$

Let $\tilde{\Lambda} := \Lambda \cap \mathbb{Z}^d$ be the lattice points in Λ. As in [FHLM97, Section 3] we estimate

$$\mathbb{E}\left[\operatorname{Tr} P^l_\omega(I)\right] \leq e^{E_2} C_V \sum_{j \in \tilde{\Lambda}} \left\| \mathbb{E}\left[\chi_j P^l_\omega(I) \chi_j\right] \right\| \tag{15}$$

where the constant C_V is an uniform upper bound on $\operatorname{Tr}(\chi_j e^{-H^{\Lambda+j}_\omega} \chi_j)$, cf. proof of Theorem 76 in [RS78]. For the bound on the operator norm in (15) it is sufficient to consider $\mathbb{E}[\langle \phi, \chi_j P^l_\omega(I) \chi_j \phi \rangle]$ for all normalised $\phi \in L^2(\Lambda_l)$. Stone's formula for spectral projections [RS78, Wei80] and Lebesgue's theorem on dominated convergence give the following inequality

$$\mathbb{E}\left[\langle \phi, \chi_j P^l_\omega(I) \chi_j \phi \rangle\right] \leq \frac{1}{\pi} \lim_{\delta \to 0} \left| \int_I \mathbb{E}\left[\langle \phi, \chi_j (H^l_\omega - E + \delta i)^{-1} \chi_j \phi \rangle\right] dE \right| . \tag{16}$$

Up to now we did not consider how the potential V_ω depends on the random parameter ω. But to bound the expectation value on the rhs of (16) this will be crucial. Note that I is a subset of $\sigma(H_\omega)$, so there may be well eigenvalues of H^l_ω in this interval. This would give rise to a non-integrable singularity in the variable E, if there would not be any averaging over ω.

4 Transformation of variables

In this section we introduce a transformation of variables on the probability space Ω. It will enable us to use a spectral averaging result from [CHM96] to bound the expectation value on the rhs of (16). At the same time we have to keep control of the new probability density, which will loose its simple product structure by the transformation.

Let $A := \{a_{j,k}\}_{j,k \in \mathbb{Z}^d}$ be an infinite Toeplitz matrix with entries $a_{j,k} = \alpha_{j-k}$. It transforms the components of the random vector linearly: $\eta := A\,\omega \in \Omega$. Note that due to the assumptions on the vector α the matrix A is invertible and one can derive a bound on its inverse by a Neumann series. The same is true for its truncations to a finite cube $\tilde{\Lambda} \subset \mathbb{Z}^d$. The row-sum matrix norm we use is given by

$$\|A\| := \|A\|_1 := \sup_{j \in \mathbb{Z}^d} \sum_{k \in \mathbb{Z}^d} |a_{j,k}| . \tag{17}$$

Since A is a Toeplitz matrix generated by the vector α we can write it as

$$A =: \mathrm{Id} + S, \; \|S\| = \sum_{j \neq 0} |\alpha_j| = \alpha^* < 1 \tag{18}$$

by the definition of α^*. So A^{-1} exists and we have $\|A^{-1}\| \leq \frac{1}{1-\alpha^*}$. Now we introduce truncations A_Λ of the matrix A associated to a cube $\Lambda \subset \mathbb{R}^d$. Denote with $\Lambda^+ = \tilde{\Lambda} - \Gamma = \{\lambda - \gamma |\; \lambda \in \tilde{\Lambda}, \gamma \in \Gamma\}$ the set of all sites k whose associated potentials $u(\cdot - k)$ influence the potential V_ω within the cube Λ. The truncated matrix $A_\Lambda = \{\alpha_{j-k}\}_{j,k\in\Lambda^+}$ acts on $\omega_\Lambda = \{\omega_k, k \in \Lambda^+\}$ to give a vector $\eta_\Lambda = \{\eta_k, k \in \Lambda^+\}$

$$(\eta_\Lambda)_j = (A_\Lambda \omega_\Lambda)_j = \sum_{k\in\Lambda^+} \alpha_{j-k}(\omega_\Lambda)_k \ . \tag{19}$$

The decomposition and the bound in (18) remain true for the truncations A_Λ.

We write now the restricted Schrödinger operator H^l_ω in the new variables η. We drop for the remainder of this section the index in the truncated matrix and vectors A_Λ, respectively $\omega_\Lambda, \eta_\Lambda$, and write $\tilde{V}_\eta$ for $V_\omega = V_{A^{-1}\eta}$ and $\tilde{H}^l_\eta$ for $H^l_\omega = H^l_{A^{-1}\eta}$ for the operators living on $\Lambda = \Lambda_l$. For $x \in \Lambda$ we have

$$\begin{aligned} V_\omega(x) &= \sum_{k\in\Lambda^+} \omega_k \sum_{l\in\Gamma} \alpha_l \, \chi_{k+l}(x) \\ &= \sum_{j\in\tilde{\Lambda}} \chi_j(x) \sum_{k\in\Lambda^+} \alpha_{j-k}\omega_k \\ &= \sum_{j\in\tilde{\Lambda}} \eta_j \chi_j(x) \\ &= \tilde{V}_\eta(x) \ . \end{aligned} \tag{20}$$

Note that it does not matter how the transformation A acts on the components of ω which have index outside Λ^+ because this does not influence the potential on Λ.

The common density of the random variable $\eta_k, k \in \Lambda^+$ is given by

$$k(\eta) = |\det A^{-1}| \prod_{k\in\Lambda^+} f\left[(A^{-1}\eta)_k\right] \tag{21}$$

or more formally

$$k = |\det A^{-1}| \left(\bigotimes_{k\in\Lambda^+} f \right) \circ A^{-1} \ . \tag{22}$$

By the properties of f and A the density $k\colon \mathbb{R}^{|\Lambda^+|} \to \mathbb{R}$ is bounded and twice continuously differentiable. Its support is contained in

$$M := A\left( ]0, \omega_+[^{|\Lambda^+|} \right) \ . \tag{23}$$

This is the set we have as the integration domain if we want to take the expectation value w.r.t. the random variables $\eta_k, k \in \Lambda^+$.

5 Bounds on the density

The structure of the random operator $\tilde{H}^l_\eta$ in the new variables makes it easier to estimate the expectation value on the rhs of (16)

$$\mathbb{E}\left[\langle \phi, \chi_j (H^l_\omega - E + \delta i)^{-1} \chi_j \phi \rangle\right] \ . \tag{24}$$

We single out the lattice point $j \in \tilde{\Lambda}$ and consider the one-parameter family of operators

$$\eta_j \mapsto \tilde{H}^l_{\hat{\eta}}(\eta_j) \ , \tag{25}$$

where $\hat{\eta} = \{\eta_k, k \in \Lambda^+ \setminus \{j\}\}$ and $\tilde{H}^l_{\hat{\eta}}(\eta_j) = \tilde{H}^l_\eta$. Similarly we write $k_{\hat{\eta}}(\eta_j) = k(\eta)$ for the common density. Locally on the cube $\Lambda_1 + j$ the dependence (25) on the parameter η_j is strictly increasing. The price we had to pay is that the density of the random variable η_j depends on the other components $\hat{\eta}$.

Let us introduce some notation. For a positive $\delta > 0$ and a normalised vector $\phi \in L^2(\Lambda_l)$ set

$$s(\eta) := \langle \phi, \chi_j (\tilde{H}^l_\eta - E + \delta i)^{-1} \chi_j \phi \rangle \ . \tag{26}$$

The projection $\pi \colon \mathbb{R}^{|\Lambda^+|} \to \mathbb{R}^{|\Lambda^+|-1}$,

$$\pi\left[\{\eta_k, k \in \Lambda^+\}\right] = \{\eta_k, k \in \Lambda^+ \setminus \{j\}\} \tag{27}$$

maps M onto

$$\begin{aligned} \hat{M} &= \pi(M) \qquad (28) \\ &= \Bigg\{ x = \{x_k, k \in \Lambda^+ \setminus \{j\}\} \in \mathbb{R}^{|\Lambda^+|-1} \Bigg| \\ &\qquad \exists y \in \mathbb{R} : \eta = \left\{ \eta_k = \left\{ \begin{array}{c} x_k, k \neq j \\ y, k = j \end{array} \right\}, k \in \Lambda^+ \right\} \in M \Bigg\} \ . \end{aligned}$$

For a given $\hat{\eta} = \{\eta_k, k \in \Lambda^+ \setminus \{j\}\} \in \mathbb{R}^{|\Lambda^+|-1}$ we define

$$M_{\hat{\eta}} = \left\{ y \in \mathbb{R} \middle| \ \eta = \left\{ \begin{array}{c} \eta_k, k \in \Lambda^+ \setminus \{j\} \\ y, k = j \end{array} \right\} \in M \right\} \ . \tag{29}$$

which is the set of the jth components of the vectors in $\pi^{-1}(\hat{\eta}) \cap M$. Since M is convex, $\hat{M}$ and $M_{\hat{\eta}}$ are this too, moreover $M_{\hat{\eta}}$ is an interval. By Fubini's theorem we have for (24) the upper bound

$$\sup_{\delta > 0} \left| \int_M d\eta \, k(\eta) \, s(\eta) \right| \leq \int_{\hat{M}} d\hat{\eta} \sup_{\delta > 0} \left| \int_{M_{\hat{\eta}}} d\eta_j \, k(\eta) \, s(\eta) \right| \tag{30}$$

We bound the rhs of (30) by the following proposition which is [CHM96, Theorem 1.1] adopted to our model.

Proposition 7
Let $\tilde{H}^l_\eta$, $s(\eta)$, $M_{\hat\eta}$ and $k_{\hat\eta}(\eta_j)$ be defined as above. Then we have

$$\sup_{\delta>0} \left| \int_{M_{\hat\eta}} d\eta_j \, k(\eta)\, s(\eta) \right| \le \| \, k_{\hat\eta}(\cdot) + |k'_{\hat\eta}(\cdot)| + 2|k''_{\hat\eta}(\cdot)| \, \|_{L^1(M_{\hat\eta})} \, . \tag{31}$$

Using Fubini's theorem in the reverse direction and transforming back to the ω variables we estimate (24) by

$$\begin{aligned} & \int_{\hat M} d\hat\eta \int_{M_{\hat\eta}} d\eta_j \, \left[k_{\hat\eta}(\eta_j) + |k'_{\hat\eta}(\eta_j)| + 2|k''_{\hat\eta}(\eta_j)| \right] \\ = & \int_M d\eta \, \left[k_{\hat\eta}(\eta_j) + |k'_{\hat\eta}(\eta_j)| + 2|k''_{\hat\eta}(\eta_j)| \right] \\ = & \int_{[0,\omega_+]^{|\Lambda^+|}} d\omega \, \left\{ F(\omega) + |k'_{\hat{A\omega}}[(A\omega)_j]| + 2|k''_{\hat{A\omega}}[(A\omega)_j]| \, \right\} \end{aligned} \tag{32}$$

where $\hat{A\omega} := \pi(A\omega)$ and $F(\omega) = \prod_{k\in\Lambda^+} f(\omega_k)$.

The first term gives $\int d\omega \, F(\omega) = 1$, for the others we have to calculate the derivatives of k w.r.t. η_j. Let $B = \{b_{i,j}\}_{i,j\in\Lambda^+}$ denote the inverse of $A = A_\Lambda$. With this notation we have for the derivatives

$$k'_{\hat{A\omega}}[(A\omega)_j] = \sum_{k\in\Lambda^+} f'(\omega_k) \; b_{k,j} \prod_{\substack{i\in\Lambda^+ \\ i\neq k}} f(\omega_i) \tag{33}$$

$$k''_{\hat{A\omega}}[(A\omega)_j] = \sum_{\substack{k,l\in\Lambda^+ \\ k\neq l}} f'(\omega_k) f'(\omega_l) \, b_{k,j} b_{l,j} \prod_{\substack{i\in\Lambda^+ \\ i\neq l,k}}^{L} f(\omega_i) + \sum_{k\in\Lambda^+} f''(\omega_k) \, b^2_{k,j} \prod_{\substack{i\in\Lambda^+ \\ i\neq k}}^{L} f(\omega_i) \tag{34}$$

and for the corresponding integrals

$$\int_{[0,\omega_+]^{|\Lambda^+|}} d\omega \, |k'_{\hat{A\omega}}[(A\omega)_j]| \;\; \le \;\; \|f'\|_{L^1} \sum_{k\in\Lambda^+} |b_{k,j}| \tag{35}$$

$$\int_{[0,\omega_+]^{|\Lambda^+|}} d\omega \, |k''_{\hat{A\omega}}[(A\omega)_j]| \;\; \le \;\; \|f'\|^2_{L^1} \sum_{\substack{k,l\in\Lambda^+ \\ k\neq l}} |b_{k,j} b_{l,j}| + \|f''\|_{L^1} \sum_{k\in\Lambda^+} b^2_{k,j} \, . \tag{36}$$

This gives for the expectation (24) the estimate

$$\begin{aligned} & \mathbb{E}\left[\langle \phi, \chi_j (H^l_\omega - E + \delta i)^{-1} \chi_j \phi \rangle \right] \\ & \quad \le 1 + \|f'\|_{L^1} \sum_{k\in\Lambda^+} |b_{k,j}| + 2\|f'\|^2_{L^1} \sum_{\substack{k,l\in\Lambda^+ \\ k\neq l}} |b_{k,j} b_{l,j}| + 2\|f''\|_{L^1} \sum_{k\in\Lambda^+} b^2_{k,j} \, . \end{aligned} \tag{37}$$

By the bound (18) on the inverse of A we know

$$\sum_{k\in\Lambda^+} |b_{k,j}| \;\leq\; \|B\| \qquad\qquad \leq \frac{1}{1-\alpha^*} \tag{38}$$

$$\sum_{\substack{k,l\in\Lambda^+\\ k\neq l}} |b_{k,j} b_{l,j}| \;\leq\; \sum_{k\in\Lambda^+} |b_{k,j}| \sum_{l\in\Lambda^+} |b_{l,j}| \qquad \leq \left(\frac{1}{1-\alpha^*}\right)^2 \tag{39}$$

$$\sum_{k\in\Lambda^+} b_{k,j}^2 \;\leq\; \left(\sum_{k\in\Lambda^+} |b_{k,j}|\right)^2 \qquad \leq \left(\frac{1}{1-\alpha^*}\right)^2 . \tag{40}$$

So (24) is bounded by

$$C_\alpha := 1 + \|f'\|_{L^1} \frac{1}{1-\alpha^*} + 2\left(\|f'\|_{L^1}^2 + \|f''\|_{L^1}\right)\left(\frac{1}{1-\alpha^*}\right)^2 \tag{41}$$

which is independent of I, δ, Λ_l and j. The average trace in (15) is thus bounded by

$$\frac{e^{E_2} C_V C_\alpha}{\pi} \, |I| \, |\Lambda| \; . \tag{42}$$

Note that $|\Lambda| = |\Lambda_l| = l^d$. Thus we proved that the averaged finite volume IDS is Lipschitz continuous

$$\mathbb{E}\left[N_\bullet^l(E_2) - N_\bullet^l(E_1+\epsilon)\right] \leq C\,|E_2 - E_1|, \;\forall\; \epsilon > 0 \tag{43}$$

with $C := \frac{e^{E_2} C_V C_\alpha}{\pi}$. By the Čebyšev inequality the Wegner estimate (H2) now follows.

6 Discussion

In this section we consider some generalisations of Theorem 5 and discuss applications.

Remark 8 (Existence of the density of states)
Note that the estimate (43) on the finite volume IDS for a bounded energy interval is independent of $|\Lambda_l|$. So one can take the limit $l \to \infty$ and obtain the Lipschitz continuity of the (infinite volume) IDS

$$N(E_2) - N(E_1) \leq \text{const}\; |E_2 - E_1| \; . \tag{44}$$

By Rademacher's theorem it follows that the derivative $dN(E)$ exists for almost all E and is bounded by $\frac{C_V C_\alpha}{\pi} e^E$. $dN(E)$ is called the density of states.

Remark 9 (Discrete model)
One could also consider the discrete Schrödinger operator

$$h_\omega = -\Delta_{\rm disc} + V_\omega \text{ on } l^2(\mathbb{Z}^d) \tag{45}$$

where $[-\Delta_{\rm disc}\phi](i) = \sum_{|i-n|=1}(\phi(i) - \phi(n))$ and in the definition of the multiplication operator V_ω the characteristic function of the unit cube $\chi_0(x)$ is replaced by the Kronecker symbol $\delta_0(i)$. An additional $\mathbb{Z}^d$-periodic potential V_0 would just shift the Hamiltonian h_ω by a constant.

Our proof works for this model, too, since the results in [CH94, Section 4] and [CHM96, Theorem 1.1] are formulated for abstract one-parameter families of operators. In Sections 3 to 5 of this paper we would just have to change the notations.

Remark 10 (Correlated potentials)
We can regard Theorems 5 as a result about alloy-type Schrödinger operators with non-negative single site potential u but negatively correlated coupling constants. I.e. we consider $\tilde{H}_\eta$ as the original operator.

Correlated random potentials are the subject of some papers [vDK91, KSS98a] on Anderson-type models. However there the long range correlations are studied, and the way the MSA has to be adapted to yield localisation in this case.

Remark 11 (More general convolution vectors α)
The condition $\alpha^* < 1$ in Theorem 5 can be relaxed as can be seen from the following example.

Example 12
Let $e = (1, 0, \ldots, 0) \in \mathbb{Z}^d$ and $u = \chi_0 - \chi_e$ be the single site potential of H_ω (1). This corresponds to $\alpha_0 = 1, \alpha_e = -1$ and $\alpha_k = 0$ otherwise. The truncations A_Λ have inverses $B_\Lambda := \{b_{j,k}\}_{j,k\in\Lambda^+}$ with entries $b_{j,k} = 1$ for $j, k \in \Lambda^+$, $k_1 \le j_1$ and $k_i = j_i$ for $i = 2, \ldots, d$ and $b_{j,k} = 0$ otherwise. Here for a $i = 1, \ldots, d$ the numbers k_i and j_i denote the i-th components of the vectors $k, j \in \mathbb{Z}^d$. In this case the B_Λ are not uniformly bounded in Λ. However, the terms in (37) depending on B can be estimated by putting all $b_{k,j} = 1$:

$$\sum_{k\in\Lambda^+} b_{k,j}^2, \sum_{k\in\Lambda^+} |b_{k,j}| \le |\Lambda_l^+| \tag{46}$$

$$\sum_{\substack{k,l\in\Lambda^+\\ k\neq l}} |b_{k,j} b_{l,j}| \le |\Lambda_l^+|^2 \ . \tag{47}$$

Since $|\Lambda_l^+| \le (l+g)^d \le C_\Gamma \, l^d$ where $g = \operatorname{diam}\Gamma$ we obtain the estimate

$$\mathbb{E}\left[N_\bullet^l(E_2) - N_\bullet^l(E_1)\right] \le \tilde{C}\, |E_2 - E_1| |\Lambda_l|^2 \tag{48}$$

where $\tilde{C} = \frac{e^{E_2} C_V C_\Gamma^2}{\pi} \left(1 + \|f'\|_{L^1} + 2\|f'\|_{L^1}^2 + 2\|f''\|_{L^1}\right)$ The same Wegner estimate extends to similar single site potentials, e.g.

$$u = \chi_{(0,\ldots,0)} + \chi_{(1,1,0,\ldots,0)} - \chi_{(1,0,\ldots,0)} - \chi_{(0,1,0,\ldots,0)} \,. \tag{49}$$

Note that while we have proven (10) with the exponent $b = 3$ we cannot deduce the Lipschitz continuity of the IDS because of the divergent term $|\Lambda_l|^2$. This example illustrates that (H2') can be proven for A (respectively α) with inverses whose norms grow at most polynomially in $|\Lambda|$. It would be desirable to get a nice description of this class of Toeplitz matrices in terms of the convolution vector α.

Remark 13 (Other single site densities)
The set of admissible single site densities $f \in C_0^2(]0, \omega_+[)$ is quite restrictive. It can be replaced by the Sobolev space $W_0^{2,1}(\mathbb{R})$ since the existence of the derivatives is only needed for partial integration. Here the index 2 stands for second order derivatives and the index 1 for the exponent of the integration.

There is a completely different f we can cope with, namely the uniform density, as the following example shows [Ves99].

Example 14 (Uniform density)
Consider again the single site potential $u = \chi_0 - \chi_e$ but now with the uniform density $f(x) = \frac{1}{\omega_+}\chi_{[0,\omega_+]}$ for the coupling constants $\omega_k, k \in \mathbb{Z}^d$. The reasoning of Sections 3 and 4 remain valid for this case, too. Section 5 has to be replaced by explicit estimates on the volume of the integration domain M and the common density k of the transformed variables using $|\operatorname{supp} f| \, \|f\|_\infty = 1$ to get

$$\mathbb{E}\left[N_\bullet^l(E_2) - N_\bullet^l(E_1)\right] \leq \text{ const } |E_2 - E_1||\Lambda_l| \,. \tag{50}$$

Unfortunately we cannot deal with the superposition of the uniform and C_0^2-densities due to the transformation A^{-1} which appears in the common density k.

7 Localisation

An important application of Wegner's estimate is the proof of localisation. This raises the question whether there is a class of single site potentials u for which Theorem 5 is valid and additionally an initial scale estimate (H1) can be proven. As mentioned before, for non-negative u, Lifshitz tails can be used to deduce (H1) near the infimum of the spectrum of H_ω. Now, for u with changing sign there are no results on Lifshitz asymptotics. The proofs are not stable under a (even small) negative perturbation of u. However, while the standard deduction of the asymptotic behaviour is based on a sequence of inequalities for the first

Neumann eigenvalue of H_ω^l on arbitrarily large cubes Λ_l, (H1) is implied by this inequality on a sufficiently large, but fixed Λ_{Q_1}.

We will show that the basic estimate on the first Neumann eigenvalue on a fixed scale Λ_{Q_1} is stable under a negative perturbation of u as long as it is coupled with a small parameter ϵ. The dependence (see also Remark 20) $\epsilon = \epsilon(Q_1) \to 0$ for $Q_1 \to \infty$ explains why our estimate is no good for proving Lifshitz tails.

Notation 15 (Small negative perturbation uf u)
We decompose $u = u_+ - \epsilon u_-$ into a non-negative u_+ and a non-positive part $-\epsilon u_-$, with $\|u_-\|_\infty \leq 1, \epsilon \in [0,1]$ and $\operatorname{supp} u \subset \Lambda_g, g > 0$. We set $N = \| \sum_{k\in\mathbb{Z}^d} u_-(\cdot - k)\|_\infty$.

The following arguments are adaptations of inequality (2) and Proposition 3 in [KS86] to u with changing sign. The restriction of H_ω and $H_0 = -\Delta + V_0$ to Λ_l with Neumann b.c. will be denoted by $H_\omega^{l,N}$ and $H_0^{l,N}$ respectively. $N_\omega^{l,N}$ is the IDS of $H_\omega^{l,N}$. Assume that V_0 is symmetric under the reflection along the coordinate axes. Let ϕ be the ground state of $H_0^{1,N}$ and Φ its periodic extension to $\mathbb{R}^d$. Then for $l \in \mathbb{N}$, $\psi_l = |\Lambda_l|^{-1/2}\, \Phi\, \chi_{\Lambda_l}$ is the ground state of both $H_\omega^{l,N}$ and $H_\omega^{l,\mathrm{per}}$, where "per" stands for periodic b.c. So we have

$$\inf \sigma(H_0) = \lambda_1\left(H_0^{1,\mathrm{per}}\right) = \lambda_1\left(H_0^{l,\mathrm{per}}\right) = \lambda_1\left(H_0^{l,N}\right) \ . \tag{51}$$

By adding a constant we get

$$\inf \sigma(H_0) = 0 \ . \tag{52}$$

Set $m_1 = \int dx\, u(x)\Phi^2(x)$ and assume that ϵ is so small that $m_1 > 0$. For a given energy $E \in]0,1[$ and a parameter $\beta > 0$ choose the length scale $l := [(\beta E)^{-1/2}]_{\text{Gauß}}$.

By Dirichlet-Neumann bracketing and Čebyšev's inequality we know

$$\begin{aligned}\mathbb{P}\{\sigma(H_\omega^l) \cap]-\infty, E[\neq \emptyset\} &\leq \mathbb{P}\{\sigma(H_\omega^{l,N}) \cap]-\infty, E[\neq \emptyset\} \\ &\leq l^d \int_\Omega N_\omega^{l,N}(E)\, d\mathbb{P}(\omega) \\ &\leq l^d \sup_\omega N_\omega^{l,N}(E)\ \mathbb{P}\{\omega |\, \lambda_1(H_\omega^{l,N}) < E\}\,. \end{aligned} \tag{53}$$

In the last inequality we used that ω with $\lambda_1(H_\omega^{l,N}) \geq E$ do not contribute to the integral. Since the random variables $\omega_k, k \in \mathbb{Z}^d$ are bounded we can give uniform upper V^+ and lower V^- comparison potentials for V_ω. At this point we slightly change our notation. Let $\omega_- = \inf \operatorname{supp} f$. Since $f \in C_0^2(]0, \omega_+[)$, ω_- has to be strictly larger than zero. By replacing the random variables ω_k by $\tilde{\omega}_k = \omega_k - \omega_-$ we achieve $\inf \operatorname{supp} \tilde{f} = 0$, where $\tilde{f}$ is the density of the transformed variables $\tilde{\omega}_k$.

This change can be compensated by adding a periodic potential to V_0:

$$V_0 + V_\omega = \left[V_0 + \omega_- \sum_{k \in \mathbb{Z}d} u(\cdot - k) \right] + \sum_{k \in \mathbb{Z}d} (\omega_k - \omega_-)\, u(\cdot - k) \tag{54}$$

$$=: \tilde{V}_0 + V_{\tilde{\omega}}\,. \tag{55}$$

In the following we omit the tildes and assume $\omega_- = 0$. By this convention the bounds on V_ω read:

$$V^-(x) = -\sum_{k \in \mathbb{Z}^d} \omega_+ \, \epsilon u_-(x-k) \qquad \leq V_\omega \tag{56}$$

$$V^+(x) = \sum_{k \in \mathbb{Z}^d} \omega_+ \, u_+(x-k) \qquad \geq V_\omega \tag{57}$$

Let $N_-^{l,N}$ (resp. $N_0^{l,N}$) denote the IDS of $H_0 + V_-$ (resp. $-\Delta$) on Λ_l with Neumann b.c. By the Min-Max principle for eigenvalues we know

$$N_0^{l,N}(E - \inf V_0 + \epsilon\omega_+) \geq N_-^{l,N}(E) \geq N_\omega^{l,N}(E) \;\forall\;\; \omega \in \Omega\,.$$

For the IDS of the pure kinetic energy one knows $N_0^{l,N}(E) \leq \text{const}\; l^d$, with an uniform constant for bounded energy intervals. We will derive an upper bound on $\mathbb{P}\{\omega | \lambda_1(H_\omega^{l,N}) < E\}$ which is exponentially small in $|\Lambda_l|$. It will dominate the two polynomially growing factors of l^d each. The exponential bound follows from the combination of a Large Deviations result and the fact that $\lambda_1(H_\omega^{l,N})$ can attain a small value only for very rare configurations of ω.

Proposition 16
There exist $\beta_0, Q_1 < \infty$, such that for $l \geq Q_1, \beta \geq \beta_0$ and $\epsilon \leq \frac{E}{4\omega_+ N}$ we have the estimate:

$$\lambda_1\left(H_\omega^l\right) < E \Longrightarrow \#\left\{ k \in \Lambda_l \,\middle|\, \omega_k < \frac{4E}{m_1} \right\} > \frac{l^d}{2}\,.$$

The proof is the same as of [KS87a, Proposition 3]. One has just to control the contributions from u_- and is not allowed to replace u by $u\chi_0$ as done there.

Now by Large Deviations we know

$$\mathbb{P}\left\{ \#\left\{k \in \Lambda_l \,|\, \omega_k < 4E/m_1 \right\} > \frac{l^d}{2} \right\} \leq e^{-c|\Lambda_l|} \tag{58}$$

where we choose E sufficiently small so that $\mathbb{E}(\omega_0) > \frac{4E}{m_1}$. $c > 0$ is a constant independent of l. Combining (53), Proposition 16 and (58) we arrive at the bound

$$\mathbb{P}\{\sigma(H_\omega^l) \cap]-\infty, E[\neq \emptyset\} \leq l^{2d} e^{-cl^d}\,. \tag{59}$$

Now for any $n, q \in \mathbb{N}$ we can achieve by choosing l sufficiently large

$$\mathbb{P}\{\sigma(H_\omega^l) \cap]-\infty, l^{-1/n}[\neq \emptyset\} \leq l^{-q}. \tag{60}$$

Note that due to the condition in Proposition 16 this inequality is applicable for single site potentials $u = u_+ - \epsilon u_-$ with $\epsilon \leq (4\omega_+ N\beta l^2)^{-1}$. Applying the Combes-Thomas argument (cf. [Klo95a, BCH97]) the initial scale estimate (H1) follows. We still have to settle the question whether there is spectrum of H_ω in the interval $]-\infty, l^{-1/n}[$ at all. This is true since $0 \in \operatorname{supp} f$ and therefore by [KSS98b, equation(1.1)] $\sigma(H_\omega) \supset \sigma(H_0) \ni 0$.

Theorem 17
Let H_ω, u and α be as in Theorem 5 and $\omega_- = 0$. Then there exist $\epsilon > 0$ and $E^* > 0 = \inf \sigma(H_0)$ such that if $\min_{i=1}^d \alpha_i \geq -\epsilon$ then

$$\emptyset \neq \sigma(H_\omega) \cap]-\infty, E^*[\subset \sigma_{pp}(H_\omega). \tag{61}$$

We turn our attention now to localisation away from the infimum of $\sigma(H_\omega)$. The spectrum of the periodic Schrödinger operator H_0 consists of intervals called spectral bands. If there are gaps belonging to the resolvent set between them, there exist internal spectral (band) edges. If the perturbation V_ω is small, the spectral gaps will be preserved, although (possibly) with shifted spectral edges. It is natural to ask whether the localisation results for energies near $\inf \sigma(H_\omega)$ can be extended to small neighbourhoods of internal edges. It turns out that the study of Lifshitz tails in this energy regime is quite involved [Klo99].

As a substitute for the Lifshitz asymptotic a special disorder regime has been assumed in several papers [BCH97, KSS98b]. In this case the density f of the coupling constants is required to satisfy

$$\int_{\omega_-}^{\omega_-+\delta} f(x)\, dx \leq \delta^\tau \text{ or } \int_{\omega_+-\delta}^{\omega_+} f(x) \leq \delta^\tau \text{ for some } \tau > d/2 \text{ and small } \delta. \tag{62}$$

The first condition is needed when considering lower band edges, the second for upper band edges. In this case we can prove using the arguments of [KSS98b, pp.10-11]:

Proposition 18
Let $H_\omega, u, \alpha, \omega_+$ be as in Theorem 17, the density f as in (62) and E a lower spectral band edge. Let $p \in]0, 2\tau - d[$ and $\xi \in]0, 1 - \frac{d}{2\tau}[$. Then there exists a Q_1 such that for all $l \geq Q_1$ and $\epsilon \leq \frac{l^{p-2}}{\omega_+ N}$ we have

$$\min_{i=1}^d \alpha_i \geq -\epsilon \Longrightarrow \mathbb{P}\{\sigma(H_\omega^l) \cap [E - l^{p-2}, E + l^{p-2}] \neq \emptyset\} \leq l^{-\xi}. \tag{63}$$

Now one proceeds as in [KSS98b] using Combes-Thomas arguments to prove an initial scale estimate and thereby localisation. Note that in the mentioned paper the bound (H1) is slightly more flexible, i.e. weaker, then the one we presented in Section 1.

Theorem 19
Let $H_\omega, u, \alpha, \omega_+$ be as in Theorem 17, the density f as in (62) and E a spectral band edge. There exist $r, \epsilon > 0$ such that if $\min_{i=1}^d \alpha_i \geq -\epsilon$ the spectrum of H_ω in the interval $[E-r, e+r]$ is pure point.

Remark 20
For the localisation Theorems 17 and 19 we had to choose ϵ small depending on the initial scale Q_1. It might seem irritating that Q_1 in turn depends on V_ω, i.e. implicitly on ϵ. However an admissible initial scale Q_1 for $\epsilon = 1$, i.e. the potential $V_\omega^{\epsilon=1}(x) = \sum_{k\in\mathbb{Z}^d} \omega_k\,(u_+ - u_-)(x-k)$ is admissible for the potentials for other values of $\epsilon \in [0,1]$, also. In this sense the dependence of Q_1 on ϵ is uniform for $\epsilon \in [0,1]$. This means that choosing ϵ closer to 0 does not change the initial scale Q_1.

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