RENORMALIZATION GROUP AND FIELD THEORETIC TECHNIQUES FOR THE ANALYSIS OF THE LINDSTEDT SERIES

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Abstract. The Lindstedt series were introduced in the XIX century in Astronomy to study perturbatively quasi-periodic motions in Celestial Mechanics. In Mathematical Physics, after getting the attention of Poincaré, who studied them widely by pursuing to all orders the analysis of Lindstedt and Newcomb, their use was somehow superseded by other methods usually referred to as KAM theory. Only recently, after Eliasson's work, they have been reconsidered as a tool to prove KAM-type results, in a spirit close to that of the Renormalization Group in quantum field theory. Following this new approach we discuss here the use of the Lindstedt series in the context of some model problems, like the standard map and natural generalizations, with particular attention to the properties of analyticity in the perturbative parameter.

1. Introduction

Many planetary motions are approximately periodic. Outstanding examples are given by the spin-orbit resonances between the revolution and the rotation satellite periods (see [43]) and by Lagrange's equilateral solutions for the three body problem, as in the case of Sun, Jupiter and the Trojan group (see [56]). From a mathematical point of view the existence of periodic solutions in Celestial Mechanics problems was proved rather soon; we can refer to the classical works by Poincaré [53] and Birkhoff [16].

Renewed interest arose about quasi-periodic motions since the originary formulation of the KAM theorem; see for instance [51] for a review. The perturbative series of the quasi-periodic solutions were well known in Astronomy: they were introduced and studied to first orders, independently, by Lindstedt [47] and Newcomb [52], and have become known as *Lindstedt series*. Then Poincaré showed that the series were well defined to all orders (see for instance [53]), but only with the work of Kolmogorov [44] the existence of quasi-periodic motions, hence the convergence of the series, was proved. Soon afterwards new proofs of Kolmogorov's theorem were given by Arnol'd [1] and by Moser [50], who treated the nonanalytic case. Successively a lot of works were devoted to such a field, and gave rise to what has since then referred to as *KAM theory*: all such works obtained the convergence of the Lindstedt series not by directly analyzing the series itself, but as a byproduct of the proof of existence of quasi-periodic solutions.

Recently[1] a new proof of Kolmogorov's theorem has been given by Eliasson by studying directly the perturbative series, hence without using the iterative rapidly convergent procedure typical of the standard versions of KAM techniques. Note that the approach followed Eliasson is quite natural, mostly from a physical point of view (and in fact it was the first to be introduced): if looking for quasi-periodic solutions the first attempt one can think about is to write formally the solution as a quasi-periodic function, with coefficients to be determined, and insert it into the equations of motion, to check if there is some choice of the coefficients for which the equations of motions are satisfied.

[1] Eliasson's results were announced in 1986 [26], and presented to full extent in 1988 in a Report of the Mathematics Department of the University of Stockholm; the latter was published only in 1996 [27].

Here we follow Eliasson's approach in the Renormalization Group interpretation given by [29] and developed in a series of subsequent papers; see [38, 39] and the review [40]. We do not insist on the wide field of applicability of the method, by referring to the literature for a detailed description of all the results that we shall not even mention in this paper (see [32, 30, 36, 33, 34, 35, 17, 18, 19, 3]; a review can be found in [37]). Rather we prefer to discuss some special dynamical systems, the *standard map* and other related models, which capture most of the general interesting features, and to describe in some details a series of original results which we have recently obtained with such techniques (see [7, 8, 9, 10, 11]).

The standard map is given by

$$T_\varepsilon : \begin{cases} x' = x + y + \varepsilon \sin x, \\ y' = y + \varepsilon \sin x. \end{cases}$$

and was introduced by Greene [41] and Chirikov [24] as a paradigmatical example of dynamical system: it is simple enough to keep separate the non-trivial dynamical features from the technical intricacies of other models of higher dimension.

The standard map has also more direct reasons to be interesting. In fact, it can be generated *formally* by the hamiltonian

$$\mathcal{H} = \frac{1}{2} y^2 + \varepsilon \cos x \sum_{k \in \mathbb{Z}} e^{2\pi i k t},$$

which is of course very singular, since the sum on the right gives a sum of δ functions at integer values of t (hence the name of "kicked rotator" which is given sometimes to the model). By truncating the sum (that is, by keeping only the terms with a slowly varying angular variable x), we obtain hamiltonians of the forced pendulum type, which are remarkably similar to those used as simplified models of the spin-orbit interaction in Celestial Mechanics . See [46] for the derivation of the hamiltonian $\mathcal{H}$ and [43] for the spin-orbit interaction (see also [21] for the so-called *spin-orbit model*).

The paper is organized as follows. In Section 2 we shall define the standard map and natural extensions of it, which we shall call *generalized standard maps*. We shall introduce also some simplified models, the *semistandard map* and, in the same spirit, the *generalized semistandard maps*. In Section 3 we shall state our main mathematical results about the analyticity properties of both periodic and quasi-periodic solutions. In particular we shall see that it is possible to write the solutions as functions of suitable variables in terms of which the dynamics is a trivial rotation; we shall call *conjugating functions* the functions which carry out such a task. Next, in Section 4 we shall develop the mathematical tools which shall be used in order to obtain the results described in the previous section. The method we shall describe uses techniques typical of quantum field theory (as first noted in [28] and [31]; see also [33]) and it is based on a diagrammatic representation of the Lindstedt series in terms of tree graphs (or trees *tout court* in the following). In Section 5 we shall give some hints about the proofs of the results discussed in Section 3, by referring to the original papers for more details, while in Section 6 we shall discuss explicitly the case of the semistandard map with the aim of giving some ideas of the methods in a particularly simple example. In Section 7 we shall brifely review some numerical results about the analyticity domains of the conjugating functions. Finally in Section 8 we shall describe some open problems, and we shall make some conclusive remarks.

2. The models

The standard map (SM) is a discrete one-dimensional dynamical system generated by the iteration of the symplectic map of the cylinder to itself, $T_\varepsilon \colon \mathbb{T}\times\mathbb{R} \mapsto \mathbb{T}\times\mathbb{R}$, given by

$$T_\varepsilon \colon \begin{cases} x' = x + y + \varepsilon \sin x, \\ y' = y + \varepsilon \sin x. \end{cases} \tag{2.1}$$

We can look for solutions of the form

$$\begin{cases} x = \alpha + u(\alpha,\varepsilon,\omega), \\ y = 2\pi\omega + v(\alpha,\varepsilon,\omega), \end{cases} \tag{2.2}$$

such that the dynamics in the α variable is a trivial rotation

$$\alpha' = \alpha + 2\pi\omega, \tag{2.3}$$

where $\omega \in [0,1]$ is called the *rotation number*.

One immediately checks that the function $v(\alpha,\varepsilon,\omega)$ is related to the function $u(\alpha,\varepsilon,\omega)$ by

$$v(\alpha,\varepsilon,\omega) = u(\alpha,\varepsilon,\omega) - u(\alpha - 2\pi\omega,\varepsilon,\omega), \tag{2.4}$$

while $u(\alpha,\varepsilon,\omega)$ is a solution of the functional equation

$$\begin{aligned} &(D_\omega u)(\alpha,\varepsilon,\omega) = \\ &\qquad u(\alpha+2\pi\omega,\varepsilon,\omega) + u(\alpha-2\pi\omega,\varepsilon,\omega) - 2u(\alpha,\varepsilon,\omega) = \varepsilon \sin\left(\alpha + u(\alpha,\varepsilon,\omega)\right), \end{aligned} \tag{2.5}$$

which is usually referred to as the *homological equation* (for the standard map).

We shall call $u = u(\alpha,\varepsilon,\omega)$ the *conjugating function*: it admits a formal expansion – the *Lindstedt series* – of the form

$$u(\alpha,\varepsilon,\omega) = \sum_{\nu\in\mathbb{Z}} u_\nu(\varepsilon,\omega)\, e^{i\nu\alpha} = \sum_{k\geq 1} u^{(k)}(\alpha,\omega)\,\varepsilon^k = \sum_{k\geq 1}\sum_{\nu\in\mathbb{Z}} u_\nu^{(k)}(\omega)\, e^{i\nu\alpha}\,\varepsilon^k. \tag{2.6}$$

For the standard map, it is easy to check that at order k in ε, the Fourier expansion in α contains only frequencies in the range $|\nu| \leq k$.

The *radius of convergence* of the Lindstedt series is defined as

$$\rho(\omega) = \inf_{\alpha\in\mathbb{T}} \left(\limsup_{k\to\infty} \left| u^{(k)}(\alpha,\omega) \right|^{1/k} \right)^{-1}. \tag{2.7}$$

For any $\omega \in [0,1]$ let us write $\omega = [0, a_1, a_2, a_3, \ldots]$, where $\{a_n\}$ are the *partial quotients* of ω and call $\{\omega_n\} = \{p_n/q_n\}$ the sequence of *convergents* of ω [54].

If $\omega \in \mathbb{Q}\cap[0,1]$, *i.e.* $\omega = p/q$, with $p \leq q$ and $\gcd(p,q) = 1$, then there exists $N = N(\omega)$ such that $\omega = [0, a_1, a_2, a_3, \ldots, a_N]$, *i.e.* such that $a_{N+1} = \infty$: [2] in such a case the sequence of convergents is finite and the last one is given by $p_N/q_N = p/q$. For such ω define

$$B_1(\omega) = \sum_{n=0}^{N-1} \frac{\log q_{n+1}}{q_n}. \tag{2.8}$$

For any $\omega \in [0,1]\cap\mathbb{R}\setminus\mathbb{Q}$ define

$$B_1(\omega) = \sum_{n=0}^{\infty} \frac{\log q_{n+1}}{q_n}, \tag{2.9}$$

and define ω a *Bryuno number* if $B_1(\omega) < \infty$. With a slight abuse of notation we shall call $B_1(\omega)$ the *Bryuno function*; by analogy we shall define (2.8) the *truncated Bryuno function* of the

[2] There is an intrinsic ambiguity for rational numbers as $[\ldots, a_N]$ and $[\ldots, a_N - 1, 1]$ define the same number. In the following we shall be interested essentially in rotation numbers obtained as truncations of the continued fraction representation of an irrational one, so that the ambiguity is automatically solved.

rational number ω. [3] Notice that the Bryuno function is usually defined [60] as a solution of the functional equation

$$\begin{cases} B(\omega+1) = B(\omega), \\ B(\omega) = -\log\omega + \omega B(1/\omega), \quad \text{if } \omega \in \mathbb{R}\setminus\mathbb{Q}. \end{cases} \tag{2.10}$$

It is well known that $B(\omega)$ differs from $B_1(\omega)$ by an essentially bounded function. It is also obvious that if ω is a Bryuno number and $\{\omega_N\}$ are the convergents of ω, then

$$\lim_{N\to\infty} B_1(\omega_N) = B_1(\omega). \tag{2.11}$$

By inserting (2.6) into the homological equation (2.5), and equating the Taylor-Fourier coefficients we obtain the recursion relation for the coefficients of the Lindstedt series

$$u_\nu^{(k)}(\omega) = g(\omega\nu) \sum_{m\ge0} \frac{1}{m!} \sum_{\substack{k_1,\ldots,k_m\in\mathbb{N} \\ k_1+\ldots+k_m=k-1}} \sum_{\substack{\nu_0,\nu_1,\ldots,\nu_m\in\mathbb{Z} \\ \nu_0+\nu_1+\ldots+\nu_m=\nu}} f_{\nu_0}\,(i\nu_0)^m \prod_{j=1}^m u_{\nu_j}^{(k_j)}, \tag{2.12}$$

with $f_{\nu_0} = -i\nu_0\delta_{|\nu_0|,1}/2$ and

$$g(\omega\nu) = \frac{1}{\gamma(\omega\nu)}, \qquad \gamma(\omega\nu) = 2\left[\cos(2\pi\omega\nu) - 1\right]. \tag{2.13}$$

Note how the denominators $\gamma(\omega\nu)$ become *arbitrarily small* for ω irrational, making their reciprocal *arbitrarily large*, while for $\omega = p/q$ they are zero if ν is a multiple of q, so that the Lindstedt series is not well defined. This is a manifestation of the famous *small divisors problem*.

If ω is a Bryuno number then there exists a solution of the form (2.2), (2.3), with u, v analytic in α, ε, for ε small enough, and 2π-periodic in α. A more formal statement will be given in Section 3. Letting α vary in $[0, 2\pi]$, (2.2) describes an invariant curve (*KAM curve*), which is densely covered by the quasi-periodic solution; we shall denote with $\mathcal{C}_\varepsilon(\omega)$ such a curve.

If $\omega = p/q$ is rational then the homological equation (2.5) admits no solution: in fact, the Lindstedt series becomes undefined because of the exploding small denominators. However we shall see in Section 3 that if we let α vary only on a discrete set – actually if we suitably choose α_0 and let $\alpha \in \{\alpha_0 + 2n\pi/q\}$, $n \in \mathbb{Z}$ –, then the homological equation obviously "closes", and $(x_0, y_0) = (\alpha_0 + u(\alpha_0, \varepsilon, \omega), 2\pi\omega + v(\alpha_0, \varepsilon, \omega))$ becomes the initial datum of a periodic solution with period $2\pi p$, *i.e.* such that

$$u(\alpha_0 + 2\pi p) = u(\alpha_0). \tag{2.14}$$

This means that, after q iterates of the dynamics, the variable α has been shifted by $2\pi p$, so that the variables (x, y) have came back to their original values (x_0, y_0), up to a shift by $2\pi p$ in the x direction.

We shall also consider *complex rotation numbers* of the form

$$\omega = \frac{p}{q} + i\eta, \qquad \eta \to 0, \tag{2.15}$$

with $p, q \in \mathbb{Z}$, $\gcd(p,q) = 1$ and $\eta \in \mathbb{R}$, in the limit $\eta \to 0$, or, more generally, rotation numbers tending to p/q along any path in the complex ω-plane non-tangential to the real axis.

Note that if the rotation number is complex then no small denominators appear, and the homological equation always has an analytic solution for small ε. Adding therefore an imaginary part to ω has the meaning of "regularizing" the Lindstedt series making quite tamer its singularities. The existence or not of a KAM invariant curve with given (real) rotation number ω_0 can therefore be thought as the existence (and analyticity) or not of the limit of u as the parameter ω tends

[3] For a rational number ω, admitting two equivalent representations $[\ldots, a_N]$ and $[\ldots, a_N - 1, 1]$, the value of the truncated Bryuno function depends on the particular representation one chooses. However the two values differ at most by a universal constant C – one can take $C = 1 + 2\log 2$; so the difference is really irrelevant for our purposes.

to ω_0. One may therefore ask what happens if ω tends, always from the complex plane, to a *rational* value p/q, as in (2.15), *i.e.* "how" the series diverge. Hence we shall be interested to the exact (asymptotic) dependence of $\rho(\omega)$, in the limit $\eta \to 0$; we shall also consider the limit of the conjugating function u when ε is suitably rescaled in such a way to keep the radius of convergence bounded, so that only the "leading singularities" of u play a role.

We can also consider more general symplectic maps of the form

$$T_{\varepsilon,f}: \begin{cases} x' = x + y + \varepsilon f(x), \\ y' = y + \varepsilon f(x), \end{cases} \tag{2.16}$$

where the *perturbation*

$$f(x) = \sum_{\nu \in \mathbb{Z}} e^{i\nu x} f_\nu \tag{2.17}$$

is a 2π-periodic function, analytic in a strip $\mathcal{D} = \{|\operatorname{Im} x| < \xi\}$, so that the Fourier coefficients decay exponentially:

$$\forall \xi' < \xi, \; \exists F : \; |f_\nu| < F e^{-\xi|\nu|}. \tag{2.18}$$

If $\mathcal{B}_\rho$ denotes the Banach space $\mathcal{B}_\rho = \{g | \; \|g\|_\rho < \infty\}$, where

$$\|g\|_\rho = \sum_{\nu \in \mathbb{Z}} |g_\nu| \, e^{\rho|\nu|}, \tag{2.19}$$

then $f \in \mathcal{B}_{\xi'}$ $\forall \xi' < \xi$.

We assume that f has zero average, *i.e.* that $f_0 = 0$; we also assume f to be real, so that $f_\nu^* = f_{-\nu}$. If f is a trigonometric polynomial of order N then in (2.17) the sum is over ν such that $|\nu| \le N$.

Denote by $\mathcal{S}_1$ the set

$$\mathcal{S}_1 = \{f \in \mathcal{B}_{\xi'} \; \forall \xi' < \xi \,|\, f_0 = 0, \; f_\nu^* = f_{-\nu}, \; \|f\|_0 = 1\}. \tag{2.20}$$

We shall refer to the systems (2.16), (2.17) as *generalized standard maps* (GSM).

Also for the system (2.16) we can look for solutions of the form (2.2), (2.3): the only difference is that now the homological equation (2.5) has te be replaced with

$$\left(D_\omega u\right)(\alpha, \varepsilon, \omega) = \varepsilon f\left(\alpha + u(\alpha, \varepsilon, \omega)\right), \tag{2.21}$$

with the same meaning for the operator D_ω.

We shall consider also maps of the kind (2.16), with $(x, y) \in \mathbb{C}/2\pi\mathbb{Z} \times \mathbb{C}$, and $f(x)$ a 2π-periodic function of $\operatorname{Re} x$, analytic in $\mathcal{D} = \{\operatorname{Im} x > -\xi\}$, $\xi > 0$, of the form

$$f(x) = \sum_{\nu \ge 1} e^{i\nu x} f_\nu, \tag{2.22}$$

where the coefficients f_ν still satisfy (2.18).

We can look for solutions of the form

$$\begin{cases} x = \alpha + u_0(\alpha, \varepsilon, \omega), \\ y = 2\pi\omega + v_0(\alpha, \varepsilon, \omega), \end{cases} \tag{2.23}$$

such that the dynamics in the α variable is again a trivial rotation (2.3).

The function $v_0(\alpha, \varepsilon, \omega)$ is still related to the function $u_0(\alpha, \varepsilon, \omega)$ by

$$v_0(\alpha, \varepsilon, \omega) = u_0(\alpha, \varepsilon, \omega) - u_0(\alpha - 2\pi\omega, \varepsilon, \omega), \tag{2.24}$$

so that one obtains the only equation

$$\left(D_\omega u_0\right)(\alpha, \varepsilon, \omega) = \varepsilon f\left(\alpha + u_0(\alpha, \varepsilon, \omega)\right), \tag{2.25}$$

Table 1: The non-linear term for the four models.

Model	$f(x)$
SM	$\sin x$
SSM	$\dfrac{e^{ix}}{2}$
GSM	$\displaystyle\sum_{\nu\in\mathbb{Z}\backslash\{0\}} e^{i\nu x} f_\nu, \quad f_\nu^* = f_{-\nu}$
GSSM	$\displaystyle\sum_{\nu\geq 1} e^{i\nu x} f_\nu$

for the *conjugating function* u_0.

The *semistandard map* (SSM) is obtained as a particular case of (2.16) by setting

$$f(x) = \frac{e^{ix}}{2}; \tag{2.26}$$

in such a case the parameter ε is not really necessary (as it is well known), but we prefer to introduce it for purposes of connection with the generalized standard maps. By analogy we shall call *generalized semistandard maps* (GSSM) the systems (2.16), (2.22). Note that in the case of the SSM the conjugating function still admits a formal expansion of the form (2.6), with the extra constraint $\nu = k$ in the last formula (which is evidence of the fact that it depends in fact on the only variable $z = e^{i\alpha}\varepsilon$).

Summarizing we have the four models listed in table 1, all deductible from (2.16) according to the choice of the non-linear term $f(x)$.

3. Mathematical results

We shall now state our main mathematical results. Theorems 1, 2 and 3 deal with real rotation numbers: in theorem 1 we find the optimal dependence on the rotation number of the radius of convergence of the Lindstedt series; in theorem 2 a partial, similar result is obtained for the series expansion in ε for the periodic orbits; in theorem 3 we show how an invariant curve with given rotation number ω is approximated – in an analytical sense described below – by the periodic orbits of (rational) rotation numbers obtained as approximants of ω.

Theorems 4 and 5 deal with the behaviour of the conjugating function u as the rotation number approaches a rational value p/q *from the complex plane*. We derive a scaling law for the radius of convergence, and show that, if the perturbative parameter ε is rescaled in such a way as to keep the radius of convergence bounded, then the conjugating function u has a limit when the (complex) rotation number tends non-tangentially to p/q, and the limit function satisfies a differential equation which is explicitly given. In the case of the standard map, such differential equation is basically the pendulum equation and can be solved exactly in terms of elliptic functions, whose singularities can be explicitly computed. In the case of generalized standard maps, the solutions are hyperelliptic functions and the explicit computation of their singularities seems difficult.

Theorem 1 [9]**.** Consider the standard map. *Let $\omega \in (0,1)$ be a Bryuno number. Then there exists $\rho(\omega) > 0$ such that there exists a solution of the form* (2.2), (2.3)*, with $u(\alpha, \varepsilon, \omega)$ analytic in ε for $|\varepsilon| < \rho(\omega)$. There exists two positive constants C and β such that*

$$|\log \rho(\omega) + \beta B_1(\omega)| < C, \tag{3.1}$$

uniformly in ω, if $B_1(\omega)$ is the Bryuno function (2.8)*. One must choose $\beta = 2$.*

We actually prove only the lower bound in [9], that is we prove the "existence" part implied by (3.1). The "non-existence" part – the upper bound on $\rho(\omega)$ – is proved by Davie [25], by comparison with the semistandard map case.

Note in fact that an identical result is known for the semistandard map, and was given by Davie [25]. In Section 6 we shall give a proof of the lower bound in Theorem 1, with our techniques, in the case of the semistandard map: as we shall see, some of the hardest difficulties of the proof in the standard map case are eliminated, while the essence of the Renormalization Group-like multiscale expansion is retained.

In the case of the GSM analogous lower bounds on the radius of convergence can be obtained, but with a value of β which in general is certainly not optimal; upper bounds also can be derived in the case of perturbations which are trigonometric polynomial by comparison with the GSSM (by an argument by Davie [25]). What is important to stress is that in general is *not* possible to have the same β for both the lower and upper bounds.

Theorem 2 [11]**.** Consider the standard map. *Let $\omega = p/q$ be a rational number in $[0,1]$, with* $\gcd(p,q) = 1$*. Then there exists $\rho(\omega) > 0$ and two $2\pi p$-periodic solutions which are are analytic in ε for $|\varepsilon| < \rho(\omega)$. There exist two positive constants C and β such that*

$$\log \rho(\omega) + \beta B_1(\omega) > -C, \tag{3.2}$$

uniformly in ω, if $B_1(\omega)$ is the truncated Bryuno function (2.9)*. One can choose $\beta = 2$.*

The two periodic orbits are one stable and one unstable for small ε, as it is straightforward to check.

By choosing α_0 in the set $\mathcal{A}(\omega)$ given by

$$\mathcal{A}(\omega) = \left\{ \frac{\pi k}{q} \;:\; k = 0, 1, 2, \ldots, 2q-1 \right\}, \tag{3.3}$$

we shall see in Section 4 that the trajectory (2.3) generated from such α_0 corresponds to a periodic solution of the equation of motion of the form (2.2), with initial datum $(\alpha_0 + u(\alpha_0, \varepsilon, \omega), 2\pi\omega + u(\alpha_0, \varepsilon, \omega) - u(\alpha_0 - 2\pi\omega, \varepsilon, \omega))$, where

$$u(\alpha_0, \varepsilon, \omega) = \bar{u}(\alpha_0, \varepsilon, \omega) \equiv \sum_{\nu \in \mathbb{Z} \setminus q\mathbb{Z}} e^{i\nu\alpha_0} u_\nu(\varepsilon, \omega) \tag{3.4}$$

can be interpolated by a continuous (analytic) function

$$u(\alpha, \varepsilon, \omega) = \bar{u}(\alpha, \varepsilon, \omega) = \sum_{\nu \in \mathbb{Z} \setminus q\mathbb{Z}} e^{i\nu\alpha} u_\nu(\varepsilon, \omega) = \sum_{k=1}^{\infty} \varepsilon^k \sum_{\nu \in \mathbb{Z} \setminus q\mathbb{Z}} e^{i\nu\alpha} u_\nu^{(k)}(\omega); \tag{3.5}$$

for $\alpha \neq \alpha_0$ the function (3.5) does not describe anymore a periodic solution of the equation of motion: it is simply a 2π-periodic function which is equal to the solution only when $\alpha = \alpha_0$, with $\alpha_0 \in \mathcal{A}(\omega)$.

We can rewrite (3.5) as

$$u(\alpha, \varepsilon, \omega) = \bar{u}(\alpha, \varepsilon, \omega) = \sum_{\nu \in \mathbb{Z}} e^{i\nu\alpha} u_\nu(\varepsilon, \omega) = \sum_{k=1}^{\infty} \varepsilon^k \sum_{\nu \in \mathbb{Z}} e^{i\nu\alpha} u_\nu^{(k)}(\omega), \tag{3.6}$$

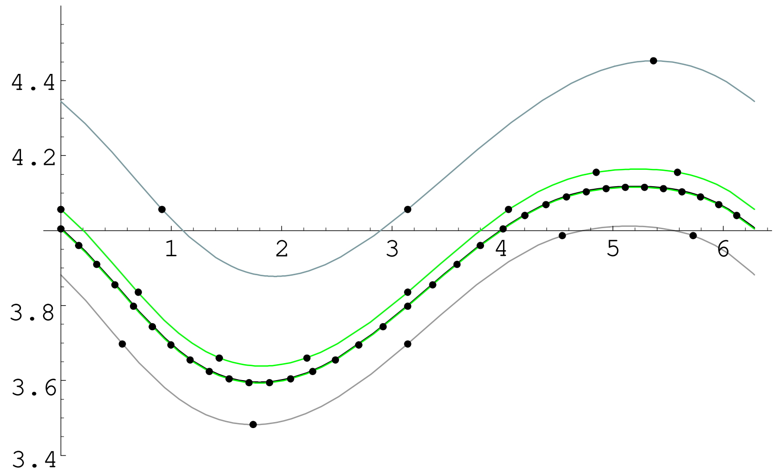

Figure 1: The invariant curve with rotation number $\gamma = [0, 1^\infty]$ (golden mean), and the interpolation curves of the stable periodic orbits corresponding to the convergents 2/3, 3/5, 5/8, 21/34 and 34/55 for $\varepsilon = 0.5$ (one has $\rho(\gamma) \approx 0.9716$): the last two are apparently overlapping the invariant curve, even if a better resolution shows that there is a little gap between them (see Figure 2). The interpolation curves can be easily recognized by the number of points of the corresponding periodic orbits; the points of the periodic orbit with rotation number 34/55 are not shown.

if we set $u_\nu^{(k)}(\omega) = 0$ for ν such that $\nu \in q\mathbb{Z}$. We shall call (3.6) the *interpolating function* (for the periodic orbit with rotation number ω).

Theorem 3 [11]**.** Consider the standard map. *Let ω be a Bryuno number; if $\{\omega_N\}$ are the convergents of ω, denote by $u_N = u(\alpha, \varepsilon, \omega_N)$ the functions interpolating the periodic solutions with rotation number ω_N as given by* (3.6)*, and by $u = u(\alpha, \varepsilon, \omega)$ the quasi-periodic solution with rotation number ω. Then there exist two positive constants ρ_0 and β, such that the sequence $\{u_N\}$ converges to the function u, uniformly for $|\varepsilon| < \rho_0 e^{-\beta B_1(\omega)}$. One can choose $\beta = 2$.*

The situation described by Theorem 3 is exemplified by the Figure 1, where the invariant curve u correponding to the golden mean $\omega = \gamma = (\sqrt{5} - 1)/2$ and the curves parameterized by the functions u_N (*interpolating curves*) corresponding to the (stable) periodic solutions with rotation numbers given by some of its approximants are shown.

Note that the interpolating curves are more and more close to the invariant curve; for instance in Figure 1 the interpolating curves of the periodic orbits with rotation numbers 21/34 and 34/55 cannot be distinguished from the invariant curve with $\omega = \gamma$. Therefore for N large enough one needs a strong resolution in order to distinguish them from the invariant curve (see Figure 2).

We turn now to considering complex rotation numbers.

Theorem 4 [7]**.** Consider the standard map. *Let $\omega = p/q + i\eta$, $p, q \in \mathbb{Z}$,* $\gcd(p, q) = 1$ *and $\eta \in \mathbb{R}$. Then the following results hold.*

(1) *For fixed $\eta \neq 0$ the function $u(\alpha, \varepsilon, \omega)$ is divisible by ε and jointly analytic in (α, ε) in the product of a strip around the real axis in the complex α-plane and a neighborhood $|\varepsilon| < \rho(\omega)$ of the origin in the complex ε-plane, with $\rho(\omega) = O(\eta^{2/q})$.*
(2) *The function $u'(\alpha, \varepsilon, \omega) \equiv u(\alpha, (2\pi\eta)^{2/q}\varepsilon, \omega)$ is well defined for $\eta \to 0$ and converges to a function $\bar{u} = \bar{u}_{p/q}(\alpha, \varepsilon)$, divisible by ε^q and analytic in ε^q in a neighborhood of the origin,*

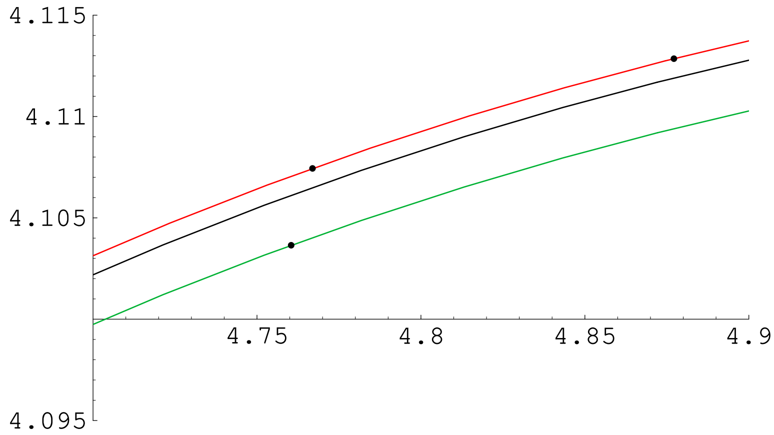

Figure 2: Enlargement for the invariant curve (middle) with rotation number $\gamma = [0, 1^\infty]$ (golden mean), and the interpolation curves of the periodic orbits corresponding to the convergents 21/34 (below) and 34/55 (above).

which is $2\pi/q$-periodic and solves the differential equation

$$\frac{\mathrm{d}^2 \bar{u}}{\mathrm{d}\alpha^2} = C_{p/q} \varepsilon^q \sin\left(q(\alpha + \bar{u})\right), \tag{3.7}$$

with boundary conditions $\bar{u}(0) = \bar{u}(2\pi) = 0$, for some nonvanishing explicitly computable constant $C_{p/q}$.

As the differential equation (3.7) has an explicit solution, theorem 4 has some interesting consequences. We start noting that (3.7) has the form

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = \lambda \sin x, \qquad x(0) = 0, \quad x(2\pi) = 2\pi, \tag{3.8}$$

by letting

$$x(t) = q\left(\alpha + \bar{u}_{p/q}(\alpha, \varepsilon)\right), \qquad t = q\alpha, \qquad \lambda = \frac{C_{p/q}}{q} \varepsilon^q, \tag{3.9}$$

in (3.7). Trivial qualitative analysis shows that for all real values of λ, (3.8) has a unique solution with the given boundary conditions. Its solution can actually be written in terns of Jacobian elliptic functions as

$$x(t) = \pi + 2 \text{ am}\left(\frac{K(k)}{\pi}(t - \pi), k\right), \tag{3.10}$$

where

$$K(k) = \int_0^{\pi/2} \mathrm{d}\phi \frac{1}{\sqrt{1 - k^2 \sin^2 \phi}}; \tag{3.11}$$

see [58] for the solution of (3.8), and [59, 23] for the relevant properties and formulae satisfied by the elliptic functions and integrals we use.

To compute the modulus k in terms of λ, we must solve the equation

$$kK(k) = \pi\sqrt{\lambda}, \tag{3.12}$$

as a simple calculation based on the properties of the elliptic functions could show (see [59]).

The solution (3.10) gives us immediately the singularities of $x(t)$ on the complex t plane, and therefore of $\bar{u}_{p/q}(\alpha, \varepsilon)$ on the complex α plane. In fact, $\mathrm{am}(s, k)$ has branch points of infinite order at $s = 2nK + (2m+1)iK'$, with $(m, n) \in \mathbb{Z}^2$ and $K' = K(k')$, where k' is the complementary modulus satisfying $k^2 + k'^2 = 1$. Therefore the singularities closest to the real axis are $\pm i\pi K'/K +$

$(2n+1)\pi$. Using the modular variables $\tau = iK'/K$, $q = e^{i\tau}$, the singularities are at $\alpha = \pm\pi\tau + (2n+1)\pi$, or, in the complex plane of the variable $z = e^{i\alpha}$, at $-q$ and $-1/q$ (note that $\operatorname{Im}\tau > 0$, $|q| < 1$). Therefore $\bar{u}_{p/q}(\alpha, \varepsilon)$ has, in the complex α-plane, $2q$ branch points along the boundary of the strip of width $\pi\tau/q$, uniformly spaced.

The analytic structure in ε seems to be harder to derive exactly. The Jacobian amplitude $\mathrm{am}(s,k)$ can be Fourier expanded giving

$$x(t) = t + 4\sum_{n=1}^{\infty} \frac{(-1)^n q^n}{n(1+q^{2n})} \sin nt, \tag{3.13}$$

in terms of the modular variable q. The series converges in a strip around the real t-axis if $|q| < 1$, and depends analytically on q inside the unit circle. As q always lies inside the unit circle, and all the dependence on λ is through q, the singularities in λ are given by the singularities of q as a function of λ. To determine them exactly, one should solve exactly (3.12), which seems to be impossible. It is though possible to write the inverse function $\lambda(q)$ explicitly; it is in fact given by

$$\lambda = \frac{1}{4}\vartheta_2^4(0,q) = 4q\left(\sum_{n=0}^{\infty} q^{n^2+n}\right)^4, \tag{3.14}$$

where the notations of [59] are used for ϑ-functions. Note also that λ, as a function of τ, is a modular form with respect to the principal congruence subgroup $\Gamma(2)$ of the modular group.

The singularities of $q(\lambda)$ are therefore located as critical points of its inverse function (see [13] for further details and calculations). Numerical calculations in [13] give a critical value of q equal to about 0.3281 and so a value of λ_c for the singularity nearest to the origin of abount $\pm 0.8275i$. Note that the position of the singularities in λ does not therefore depend on t (except of course for the cases $\alpha = 0$, $\alpha = \pi$). For $\bar{u}_{p/q}(\alpha, \varepsilon)$ this implies $2q$ singularities along a circle centered at the origin in the complex ε-plane.

It we *assume* that, when ω is a Bryuno number very "close" (in a sense to be specified in a while) to a rational value p/q (*i.e.* with $\eta = \omega - p/q$ very small), not only the function $u(\alpha, \varepsilon, \omega)$ is close to $\bar{u}_{p/q}(\alpha, (2\pi\eta)^{-2/q}\varepsilon)$, but also the singularities of the two functions in the complex variable ε are close, then we can infer the location of the singularities of the first from the location of the singularities of the second. Note that there is no mathematical argument for such an assumption: the funtion $\bar{u}$ has been obtained by rescaling u when the rational rotation number is approximated from the complex ω-plane, and, at least in principle, the corrections which disappear when η is set equal to zero could contribute with other singularities than those of the limit function. Furthermore it is obvious that the above statement is certainly false if we do not specify better what we mean by "close": in fact the limit for ω tending to p/q along a sequence of Bryuno numbers will depend on the particular sequence we choose, so that, for such an assumption to be hopefully verified, we have to take the limit superior rather than the limit (contrary to the case of imaginary η). Indeed the numerical experiments of [13], for complex ω, and of [4], for real ω, suggest just that the above assumption is correct. Then, by locating the singularities closest to the origin in the complex ε-plane for the rescaled function and "rescaling back" ε to its value by multiplying by $(2\pi\eta)^{-2/q}$, we obtain an estimate for the radius of convergence

$$\rho(\omega) \approx \left((2\pi\eta)^2 q C_{p/q}^{-1} |\lambda_c|\right)^{1/q}, \tag{3.15}$$

where $\eta = \omega - p/q$ is very small. The agreement with the numerically observed values is remarkable; see [4].

Let us now consider the case of the generalized standard map. Let $\omega \to p/q$, where $\gcd(p,q) = 1$, in the complex plane. We then consider the q sequences $\tilde{I}_c(f) = \{f_{lq+c}\}_{l\in\mathbb{Z}_+}$, $c = 1, \ldots, q$ (recall that f_0 is assumed to vanish). For each sequence $\tilde{I}_c(f)$, $c = 1, \ldots, q$, let

$$I_c(f) = \{f_\nu \in \tilde{I}_c(f) | f_\nu \neq 0\} \tag{3.16}$$

be the set of nonzero values of the sequence $\tilde{I}_c(f)$.

We define the following sets of integers:

$$A_{p/q}(f) = \{c \in \{1, \ldots, q\} | I_c(f) \neq \emptyset\}. \tag{3.17}$$

and

$$B_c(f) = \{l \in \mathbb{Z}_+ | f_{lq+c} \neq 0\}. \tag{3.18}$$

Of course $A_{p/q}(f) = \{c_1, \ldots, c_M\}$, where $1 \le c_1 < \cdots < c_M \le q$, and $M \le q$. Note that $A_{p/q}(f)$ is the set of equivalence classes modulo q of frequencies actually appearing in the Fourier expansion of the perturbation,

If $q \notin A_{p/q}(f)$ and $|I_c(f)| = 1$ $\forall c \in A_{p/q}(f)$, define

$$\mathcal{A}(f) = \{\nu \in \mathbb{N} | f_\nu \neq 0\}; \tag{3.19}$$

then $\mathcal{A}(f) = \{\nu_1, \ldots, \nu_M\}$, with $1 \le M \le q-1$ and $\nu_i - \nu_j \notin q\mathbb{Z}$ $\forall i \neq j$. This can be easily proved by reasoning as follows. If $q \notin A_{p/q}(f)$ then $f_\nu = 0$ for all $\nu \in q\mathbb{Z}$, so that $I_q(f) = \emptyset$: then $M = |A_{p/q}(f)| \le q-1$. As $|I_c(f)| = 1$ $\forall c \in A_{p/q}(f)$, for any $c \in A_{p/q}(f)$ there is only one $\nu \in \mathbb{N}$ of the form $\nu = c + lq$, with $l \in \mathbb{Z}_+$, such that $f_\nu \neq 0$. So the number of Fourier labels ν's for which $f_\nu \neq 0$ is given by $M \le q-1$; of course the bound $M \ge 1$ is obvious. Then the assertion follows.

If $q \notin A_{p/q}(f)$ and $|I_c(f)| = 1$ $\forall c \in A_{p/q}(f)$, we define $2M$ integers $r_1, \ldots, r_M$, $r'_1, \ldots, r'_M$, with $r_i, r'_i \ge 0$, and an integer $R > 0$ as those integers which satisfy the following conditions:

$$(r_1 - r'_1)\,\nu_1 + \cdots + (r_M - r'_M)\,\nu_M = Rq, \tag{3.20a}$$

$$r_1 + r'_1 + \cdots + r_M + r'_M = r_0 \ge 2, \tag{3.20b}$$

$$r_0 \text{ is minimal,} \tag{3.20c}$$

where $\{\nu_1, \ldots, \nu_M\} = \mathcal{A}(f)$.

We can easily prove that the Diophantine problem (3.20) has always a finite, nonzero number of solutions and $r_0 \le q$. Solutions to (1) and (2) exist: one is given for example by $R = \nu_i$ for some i and $r_i = q$, while $r_j = 0$ $\forall j \neq i$ and $r'_j = 0$ $\forall j$. Note that in this case (2) yields $r_0 = q$. As the set of solutions to (1) and (2) is not empty and as each solution has associated a positive value of r_0, there must be at least one such that r_0 is minimal. On the other hand, a minimal solution has $r_0 \le q$ because of the above observation, so there can be only a finite number of them.

Define $r^*(f) = r^*$ as

$$r^* = \begin{cases} 1 & \text{if } q \in A_{p/q}(f), \\ 2 & \text{if } q \notin A_{p/q}(f) \text{and } \exists \bar{c} \in A_{p/q}(f) \text{such that } |I_{\bar{c}}(f)| \ge 2, \\ r_0 & \text{otherwise} \end{cases} \tag{3.21}$$

where r_0 is defined as in (3.20). Note that $1 \le r^* \le q$.

Theorem 5 [8, 10]**.** Consider generalized standard maps. *Let f be any function in $\mathcal{S}_1$, see* (2.20)*. Consider the cone $\mathcal{C}_{p/q,\beta} = \{z \in \mathbb{C} :\ |\operatorname{Im} z| > 0,\ |\operatorname{Re} z - p/q| \le \beta |\operatorname{Im} z|,\ \beta \ge 0\}$; let $\omega \in \mathcal{C}_{p/q,\beta}$. Then the* rescaled conjugating function

$$u'_{p/q}(\alpha, \varepsilon, \omega) = u\big(\alpha, \varepsilon(\omega - p/q)^{2/r^*(f)}, \omega\big), \tag{3.22}$$

extends to a function continuous in ω in the closure of the cone $\mathcal{C}_{p/q,\beta}$ and analytic in ω in the interior of $\mathcal{C}_{p/q,\beta}$, for any $\beta \ge 0$, analytic in ε for $|\varepsilon| < a$ and analytic in α for $|\operatorname{Im} \alpha| < b$, with a *and b two positive constants. In particular, the following limit exists:*

$$\bar{u}_{p/q}(\alpha,\varepsilon) = \lim_{\omega\to p/q} u'_{p/q}(\alpha,\varepsilon,\omega), \tag{3.23}$$

it is independent from the non-tangential path chosen either in the complex upper half plane $\operatorname{Im}\omega > 0$ *or in the complex lower half plane* $\operatorname{Im}\omega < 0$*, and it is analytic for* $|\varepsilon| < \bar{\rho}_f(p/q)$*, for some* $\bar{\rho}_f(p/q) > 0$*. Defining*

$$R(p/q) = \inf_{f\in\mathcal{S}_1} \bar{\rho}_f(p/q), \qquad R_r(p/q) = \inf_{\substack{f\in\mathcal{S}_1\\ r^*(f)=r}} \bar{\rho}_f(p/q), \tag{3.24}$$

there exist finite strictly positive constants C_r, C_0, D_r, D_0*, depending only on* p/q*, such that one has*

$$C_0 \le R(p/q) \le D_0, \tag{3.25a}$$

$$C_r \le R_r(p/q) \le D_r, \tag{3.25b}$$

for all integer $r \le q$*. Generically one has* $r^*(f) = 1$*.*

Unfortunately, the differential equation satisfied by $\bar{u}_{p/q}(\alpha,\varepsilon)$ in this case (see [8] for details) does not lend itself to such a refined analysis as in the standard map case. In fact, even if $f(x)$ is a trigonometric polynomial, the solutions to this differential equation are given by the inversion of hyperelliptic integrals, and nothing quite as explicit and concrete as in the elliptic case seems to be known for them, even if lower bounds on the radius of convergence (in the sense of (3.25)) can be obtained by comparison with the generalized semistandard maps case [10].

Results similar to those of theorems 4 and 5 have been obtained for the semistandard map by [15] and for Siegel's problem [55] by [60] and by [15]. Note that in the semistandard map case the limit function is given explicitly in terms of elementary functions (indeed, a logarithm) and all its singularities are trivially computed. For the semistandard map, in [15] also the first few corrections beyond the limit $\bar{u}_{p/q}(\alpha,\varepsilon)$ are computed. We refer to [15] and [60] for further details on the analogous results for the semistandard map and for Siegel's problem.

4. Diagrammatic formalism

We can graphically represent (2.12) as in Figure 3. By iterating the graphical construction (which corresponds, analytically, to apply to each $u_{\nu_j}^{(k_j)}$ in (2.12) the same decomposition used for $u_\nu^{(k)}$ itself), we see that at the end the coefficient $u_\nu^{(k)}$ can be written as sums of contributions which are represented in terms of tree graphs (or simply *trees*).

A tree ϑ consists of a family of k *lines* arranged to connect a partially ordered set of points called *nodes*, with the lower nodes to the right; see Figure 4.

All the lines have two nodes at their extremes, except the highest which has only one node, the *last node* u_0 of the tree (which is the leftmost one); the other extreme r will be called the *root* of the tree and it will not be regarded as a node.

We denote by $\preceq$ the partial ordering relation between nodes: given two nodes u_1 and u_2, we say that $u_2 \preceq u_1$ if u_1 is along the path of lines connecting u_2 to the root r of the tree (they could coincide: we say that $u_2 \prec u_1$ if they do not).

Each line carries an arrow pointing from the node u to the right to the node u' to the left (*i.e.* directed toward the root): we say that the line exits from u and enters u', and we write $u'_0 = r$ even if, strictly speaking, r is not a node. For each node there are only one exiting line and $m_u \ge 0$ entering ones; as there is a one-to-one correspondence between nodes and lines, we can associate to each node u a line ℓ_u exiting from it. The line $\ell_0 = \ell_{u_0}$ connecting the node u_0 to the root r will be called the *root line*. Note that each line ℓ_u can be considered the root line of

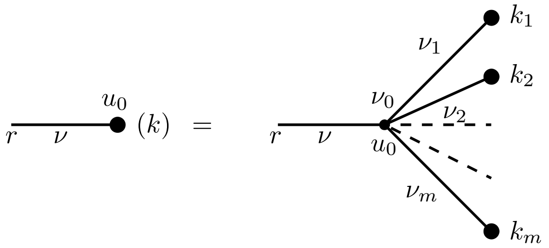

Figure 3: Graphical interpretation of (2.12). The graph element formed by a line with a label ν and a black circle at its right with a label (k) symbolizes the Taylor-Fourier coefficient $u_\nu^{(k)}$, while the node u_0 carrying a label ν_0 represents the quantity $f_{\nu_0}(i\nu_0)^m$; note that m is uniquely determined by the number of lines emerging from u_0. The sum over the labels (with the constraints as in (2.12)) is not explicitly shown.

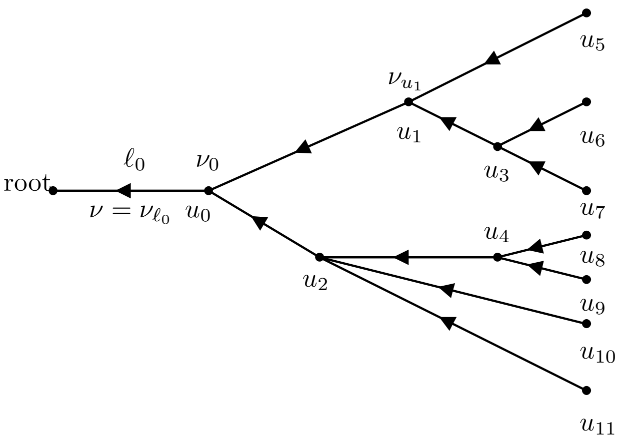

Figure 4: A tree ϑ with 12 nodes; one has $m_{u_0} = 2, m_{u_1} = 2, m_{u_2} = 3, m_{u_3} = 2, m_{u_4} = 2$, and m_u for all the other nodes u. The length of the lines should be the same but it is drawn of arbitrary size.

the subtree consisting of the nodes satisfying $w \preceq u$ and of the lines connecting them: u' will be the root of such subtree, which we denote by ϑ_u. The *order* k of the tree is defined as the number of nodes of the tree.

To each node $u \in \vartheta$ we associate a *mode label* $\nu_u \in \mathbb{Z} \setminus \{0\}$ (which is the Fourier label of the function $f(x)$ in (2.22)), and define the *momentum* flowing through the line ℓ_u as

$$\nu_{\ell_u} = \sum_{w \preceq u} \nu_w, \qquad \nu_w \in \mathbb{Z} \setminus \{0\}. \tag{4.1}$$

Let us denote by $\mathcal{T}^0_{\nu,k}$ the set of all trees of order k (*i.e.* with k nodes) and with momentum ν flowing through the root line, and by $V(\vartheta)$ and $\Lambda(\vartheta)$, respectively, the set of nodes and the set of lines of the tree ϑ.

To each node $u \in V(\vartheta)$ we associate a *node factor*

$$F_v = \frac{1}{2}\,\frac{\nu_u^{m_u+1}}{m_u!}, \tag{4.2}$$

while to each line $\ell \in \Lambda(\vartheta)$ we associate a *propagator*

$$G_\ell = g(\omega\nu_\ell), \tag{4.3}$$

Table 2: The choice of the mode labels for the four models.

Model	ν_u
SM	$\nu_u = \pm 1$
SSM	$\nu_u = 1$
GSM	$\nu_u \in \mathbb{Z} \setminus \{0\}$
GSSM	$\nu_u \in \mathbb{N}$

where $g(\omega\nu)$ is defined as in (2.13). Then, by defining the *value* of a tree ϑ as

$$\mathrm{Val}(\vartheta, \omega) = -i \Big(\prod_{u \in V(\vartheta)} F_v \Big) \Big(\prod_{\ell \in \Lambda(\vartheta)} G_\ell \Big), \tag{4.4}$$

we have formally

$$u_\nu^{(k)}(\omega) = \sum_{\vartheta \in \mathcal{T}^0_{k,\nu}} \mathrm{Val}(\vartheta, \omega), \tag{4.5}$$

as it is easy to verify by iterating (2.12) and using the graphical representation of Figure 3 (equivalently the proof can be carried out by induction).

Note that a group $\mathcal{G}$ of transformations acts on the trees, generated by the permutations of all the subtrees emerging from each node with at least one entering line. Two trees which can be transformed into each other by the action of the group $\mathcal{G}$ are called equivalent. The sum in (4.5) is meant as a sum over all the trees which are not equivalent; that this is the correct way to count the trees follows from the fact that it keeps trace of the combinatorial factors naturally arising from the Taylor expansion (2.12) when we iterate the graphical construction represented in Figure 3. The number of elements of $\mathcal{T}^0_{k,\nu}$ is bounded by $2^k \cdot 2^{2k}$ in the case of the SM and by 2^{2k} in the case of the SSM; note that for the SSM one has $\mathcal{T}_{k,\nu} = \emptyset$ for $\nu \neq k$.

According to the constraints imposed on the mode labels, one recovers therefore the four models introduced in table 1, as shown in table 2

Of course the formal solution with coefficients given by (4.5) is plagued by the *small divisors problem* (*i.e.* the presence of small denominators). First of all one has $\gamma(\omega\nu) = 0$ when $\omega\nu = 0$ mod 2π: for $\omega = p/q \in \mathbb{Q}$ this occurs for all ν which are multiple of q, while for $\omega \in \mathbb{R} \setminus \mathbb{Q}$ this occurs only for $\nu = 0$. Moreover, in the latter case, the quantity $\omega\nu$ can be in general arbitrarily close to zero, for ν large enough, so that, without imposing some condition on ω, there is no hope to control the propagators.

So we have two problems: (i) to check that no division by zero occurs, so that the formal series expansion in ε

$$u(\alpha, \varepsilon, \omega) = \sum_{k=1}^{\infty} u^{(k)}(\alpha, \omega)\, \varepsilon^k \tag{4.6}$$

has well defined Taylor coefficients $u^{(k)}(\alpha, \omega)$ to all perturbative orders, and, if it is the case, (ii) to prove that the series (4.6) converges. Here we confine ourselves to the first problem, by deferring to Section 5 the discussion of the latter.

The formal solubility of the homological equation can be easily studied by using the graphical representation (4.5).

For irrational rotation numbers, in the case of the SSM and GSSM, there is nothing really to prove, as in such a case $\nu_\ell \geq 1$, as it follows from (4.1) and table 2, so that the quantity $\gamma(\omega\nu)$ is always nonvanishing.

For the SM the formal solubility can be easily proved by parity arguments. In fact we can write the homological equation (2.5) to order k as

$$\big(D_\omega u^{(k)}\big)(\alpha,\omega) = [S(\alpha,\varepsilon,\omega)]^{(k)}, \qquad S(\alpha,\varepsilon,\omega) = \varepsilon \sin\left(\alpha + u(\alpha,\varepsilon,\omega)\right), \tag{4.7}$$

where, given any function $F(\alpha,\varepsilon,\omega)$, we are denoting by $[F(\alpha,\varepsilon,\omega)]^{(k)}$ the coefficient of order k in its Taylor expansion in ε; note that $[S(\alpha,\varepsilon,\omega)]^{(k)}$ depends only on the coefficients $u^{(k')}(\alpha,\omega)$ with $k' < k$. Then, if we suppose inductively that the functions $u^{(k')}(\alpha,\omega)$ are odd in α for $k' < k$, we obtain that the function $[S(\alpha,\varepsilon,\omega)]^{(k)}$ has zero average, and also $u^{(k)}(\alpha,\omega)$ is odd (for $k=1$ one has $u^{(1)}(\alpha,\omega) = g(\omega)\sin\alpha$).

The same argument extends to any GSM with an odd function $f(x)$ in (2.16). For general zero average functions $f(x)$, the discussion is a little less trivial but it is well known and dates basically back to Poincaré [53]; a proof based on the tree expansion can be found in [8] in all details.

In the case of rational numbers $\omega = p/q$, with $\gcd(p,q) = 1$, for the SM one finds that it is possible to fix α to a value α_0 in the set $\mathcal{A}(\omega)$ in (3.3), such that no line ℓ with momentum $\nu_\ell \in q\mathbb{Z}$ appears in $\Lambda(\vartheta)$. The condition $\alpha_0 \in \mathcal{A}(\omega)$ follows from the analysis of the trees of order q. By considering (4.7) one sees that $[S(\alpha,\varepsilon,\omega)]^{(q)}$ contains only terms with Fourier labels $|\nu| \leq q$, and the terms with $|\nu| = q$ are given by

$$S_q^{(q)} e^{iq\alpha} + S_{-q}^{(q)} e^{-iq\alpha} = 2iS^{(q)} \sin q\alpha, \tag{4.8}$$

where the last equality follows again by parity considerations. As the Fourier components with $\nu \in q\mathbb{Z}$ in the left hand side of (4.7) are vanishing, the (formal) solubility of (4.7) requires $\alpha \in \mathcal{A}(\omega)$, so that one has $\sin q\alpha = 0$. Such a condition is sufficient to assure also the formal solubility to higher perturbative orders (again a parity property): for details we refer to [11].

Of course for SSM and GSSM there is no periodic solution; in fact it is straightforward to realize that the above cancellation mechanism does not apply, as all Fourier lables are strictly positive (see table 2).

To control the product of the propagators in (4.4) one needs a multiscale analysis wich can be pursued as follows. We say that a line ℓ has scale n if

$$\|\omega\nu_\ell\| = \min_{p\in\mathbb{Z}} |\omega\nu_\ell - p| \tag{4.9}$$

is approximately equal to $1/q_{n+1}$. A more formal statement can be obtained by introducing a C^∞ partition of unity in the following way. Let $\chi(x)$ a C^∞ non-increasing compact-support function defined on $\mathbb{R}^+$, such that

$$\chi(x) = \begin{cases} 1 & \text{for } x \leq 1, \\ 0 & \text{for } x \geq 2, \end{cases} \tag{4.10}$$

and define for each $n \in \mathbb{N}$

$$\begin{cases} \chi_0(x) = 1 - \chi(96 q_1 x), \\ \chi_n(x) = \chi(96 q_n x) - \chi(96 q_{n+1} x), \text{for } n \geq 1; \end{cases} \tag{4.11}$$

then for each line ℓ set

$$G_\ell = g(\omega\nu_\ell) = \sum_{n=0}^{\infty} g_n(\omega\nu_\ell) \equiv \sum_{n=0}^{\infty} G_\ell^{(n)}, \qquad G_\ell^{(n)} = \frac{\chi_n(\|\omega\nu_\ell\|)}{\gamma(\omega\nu_\ell)}, \tag{4.12}$$

and call $G_\ell^{(n)} = g_n(\omega\nu_\ell)$ the *propagator on scale* n.

Given a tree ϑ, we can associate to each line ℓ of ϑ a scale label n_ℓ, using the multiscale decomposition (4.12) and singling out the summands with $n = n_\ell$. We shall call n_ℓ the *scale label* of the line ℓ, and we shall say also that the line ℓ is on scale n_ℓ.

Note that if a line ℓ has momentum ν_ℓ and scale n_ℓ, then

$$\frac{1}{96 q_{n_\ell+1}} \le \|\omega\nu_\ell\| \le \frac{1}{48 q_{n_\ell}}, \tag{4.13}$$

provided that one has $\chi_{n_\ell}(||\omega\nu_\ell||) \neq 0$. Note also that given a line ℓ at most only two summands in (4.12) are really nonvanishing, so that in fact the series (4.12) is a finite sum.

Therefore (4.5) can be rewritten as

$$\begin{aligned} u_\nu^{(k)}(\omega) &= \sum_{\vartheta\in\mathcal{T}_{k,\nu}} \mathrm{Val}(\vartheta,\omega), \\ \mathrm{Val}(\vartheta,\omega) &= -i\Big(\prod_{u\in V(\vartheta)} F_v\Big)\Big(\prod_{\ell\in\Lambda(\vartheta)} G_\ell^{(n_\ell)}\Big), \end{aligned} \tag{4.14}$$

where $\mathcal{T}_{k,\nu}$ is the set of all trees of order k and with $\nu_{\ell_0} = \nu$ carrying also scale labels (in addition to the node labels); in the case of the SM the number of elements in $\mathcal{T}_{k,\nu}$ is bounded by $2^k 2^k 2^{2k}$, while in the case of the SSM the number of elements in $\mathcal{T}_{k,k}$ is bounded by $2^k 2^{2k}$.

Given a tree ϑ, a *cluster* T of ϑ on scale n is a maximal connected set of lines of lines on scale $\le n$ with at least one line on scale n. We shall say that such lines are internal to T, and write $\ell \in \Lambda(T)$. A node u is called *internal* to T, and we write $u \in V(T)$, if at least one of its entering lines or exiting line is in T. Each cluster has an arbitrary number $m_T \ge 0$ of entering lines but only one or zero exiting line; we shall call *external* to T the lines entering or exiting T (which are all on scale $> n$). We shall denote with n_T the scale of the cluster T, and with k_T the number of nodes in T.

Note that there is an inclusion relation between clusters: the innermost ones are those with highest scale, while the outermost ones are on the lowest scale. The aim of introducing the clusters is to characterize the lines of the trees on the basis of the sizes of the corresponding propagators: the lines which are contained inside the outermost clusters correspond to the propagators with the smallest small divisors, and so on.

If we confine ourselves to the SSM and to GSSM, the notion of cluster is sufficient to prove the lower bound in (3.1), as we shall see in Sections 5 and 6.

However in the case of the SM – and *a fortiori* of the GSM – additional problems arise, due to the fact that for the SM for each node $u \in V(\vartheta)$ one has $\nu_u = \pm 1$, whereas $\nu_u = 1$ for the SSM (see Section 5). To single out the cases which can really give problems, we have to introduce the notion of resonance.

A cluster T will be called a *resonance* with *resonance-scale* n if the following three properties are verified:

$$(1) \qquad m_T = 1, \tag{4.15a}$$

$$(2) \qquad \sum_{u\in V(T)} \nu_u = 0, \tag{4.15b}$$

$$(3) \qquad k_T < q_n, \tag{4.15c}$$

where n is minimum between the scales of the external lines of T. The condition (1) means that T has only one entering line, which, by the condition (2) must have the same momentum of the exiting one (so that the scales of the two lines can differ at most by one unit). We note also that the condition (3) could be replaced with that of requiring that there must be a sufficiently large

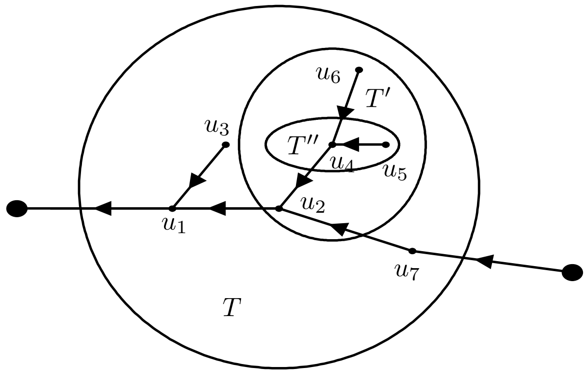

Figure 5: An example of three clusters symbolically delimited by circles, as visual aids, inside a tree (whose remaining lines and clusters are not drawn and are indicated by the bullets). Note that T is a resonance as defined in (4.13) only if $\nu_{u_1}+\nu_{u_2}+\nu_{u_3} = 0$ and $k_T = 3 < q_n$; analogous considerations hold for the other clusters.

difference between the resonant-scale n of T and the scale n_T of T as a cluster (for instance one can impose $n_T \geq n+5$ as in [37]): it is easy to realize that the two conditions are related. See Figure 5 for a graphical visualization of clusters and resonances.

Then, among the new trees which appear in the case of the SM (with respect to those of the SSM), there are some in which there can be accumulation of small divisors (as it will be discussed in Section 5), but, when the values of all trees in $\mathcal{T}_{k,\nu}$ are summed together, if we group the trees into suitable classes then some remarkable cancellation mechanisms intervene between them, and the overall contribution still admits a bound of the same kind of that of the SSM.

With the notion of resonance given above we are able to prove the lower bound in (3.1) with $\beta = 4$ (for an explicit derivation see for instance [37]). In order to obtain $\beta = 2$ a more careful analysis is needed: in particular the cancellation mechanisms have to be extended to a larger class of trees. We do not enter into the details here, by referring to [9] for a complete discussion, and we confine to note that a much more involved notion of resonance has to be introduced, where, among other properties, the condition $m_T = 1$ has to be dropped out (while the conditions (2) and (3) in (4.15) are retained).

So far we considered a very refined multiscale decomposition. In the case of periodic orbits, say with rotation number $\omega = p/q$, if we are not looking for optimal bounds, this is not really necessary.

Given a tree ϑ, we can associate to each line ℓ a scale label n_ℓ, setting $n_\ell = 0$ if its momentum ν_ℓ is a multiple of q, and $n_\ell = 1$ otherwise; note that with such a presciption the scale is a label which is uniquely determined by the momentum, and it has only the function of helping to visualize the size of the propagator.

Given a tree ϑ, a *cluster* T of ϑ is a maximal connected set of lines on scale $n = 1$; we shall say that such lines are *internal* to T, and write sometimes $\ell \in \Lambda(T)$; a node u is called internal to T, and we write $u \in V(T)$, if at least one of its entering or exiting lines belongs to $\Lambda(T)$. The lines outside the clusters are all on scale $n = 0$, and each cluster has an arbitrary number $m_T \geq 0$ of entering lines but only one exiting line.

A cluster T will be called a *resonance* if

$$\sum_{u \in V(T)} \nu_u = 0, \tag{4.16}$$

and, in such a case, the exiting line of the cluster T will be called a *resonant line*.

Note that in such a case no condition is imposed on the number of nodes in $V(T)$ (this could be done, but no real difference would follow in the analysis of resonances). On the contrary we did not impose also the condition $m_T = 1$ as in (4.15), as, if we avoid adding such a constraint – hence enlarging the classes of clusters which have to considered as resonances – we are able to deal with both the case of rational rotation numbers and the case of rotation numbers of the form (2.15)

For $\omega \in \mathbb{Q}$, if a line is on scale $n = 0$, then one has $\gamma(\omega\nu_\ell) = 0$, so that the corresponding propagator would be formally infinite. On the other hand we have seen that for such ω no line ℓ with $\nu_\ell \in q\mathbb{Z}$ (hence on scale 0) appear. Therefore for all trees $\vartheta \in \mathcal{T}_{k,\nu}$ and for all lines $\ell \in \Lambda(T)$, we can bound their propagators by

$$|g(\omega\nu_\ell)| < Cq^2, \tag{4.17}$$

for some constant C, as $\nu_\ell \notin q\mathbb{Z}$.

For $\omega = p/q + i\eta$, $\eta \in \mathbb{R}$, one has

$$|g(\omega\nu_\ell)| < \begin{cases} C|\nu\eta|^{-2}, & \text{for } \nu \in q\mathbb{Z} \setminus \{0\}, \\ Cq^2, & \text{otherwise}, \end{cases} \tag{4.18}$$

for some other constant C, so that the lines on scale $n = 1$ are bounded as in (4.17), while the lines on scale $n = 0$ can be present (recall that we are not fixing α), and obviously have divergent propagators for $\eta \to 0$.

5. About the proofs

The aim of this Section is to give an idea of the proof of the theorems stated in Section 3, without giving all the details, for which we refer to the original papers. As we anticipated in the previous Section, we are left with the problem of proving the convergence of the envisaged perturbative expansions (Lindstedt series).

First of all note that, if $\{q_n\}$ are the denominators of the convergents of ω, then [54]

$$\frac{1}{2q_{n+1}} < \|\omega q_n\| < \frac{1}{q_n}, \tag{5.1a}$$

$$\|\omega\nu\| > \|\omega q_n\| \qquad \forall|\nu| < q_{n+1},\ |\nu| \neq q_n, \tag{5.1b}$$

a property which will play a crucial role in the following.

Roughly speaking, in the case of the SSM, the idea behind the proof is the following. Even if the quantities $\|\omega\nu_\ell\|$ can become very small for ν large enough, they cannot "accumulate" too much. In fact once a line ℓ on scale n (*i.e.* with momentum ν_ℓ such that $\|\omega\nu_\ell\|$ is of order $1/q_{n+1}$) has been obtained, in order to have again a line ℓ_1 on the same scale along the path connecting ℓ to the root, one needs many nodes between the two lines, [4] as each node contributes a mode label 1 to the momentum ν_{ℓ_1} (see (4.1)) and to have $\|\omega(\nu_{\ell'} - \nu_\ell)\| \le \|\omega\nu_{\ell'}\| + \|\omega\nu_\ell\| = O(1/q_{n+1})$ requires $\nu_\ell - \nu_{\ell'}$ to be large enough, as it follows from (5.1). For a formal discussion see Section 6.

The above argument can be easily extended to the GSSM, by using the fact that, if on one hand the mode labels can assume any value in $\mathbb{N}$, on the other hand each mode label ν is associated to a Fourier coefficient f_ν satisfying (2.18). As a consequence there can be also only a few nodes between the lines ℓ and ℓ_1 but we can use the exponential decay of the factors f_ν associated to such nodes (as their Fourier labels have to be large if $\nu_\ell - \nu_{\ell'}$ is large and the number of the nodes is small). For further details we refer to [40], where such an idea is developed for continuous time

[4] More precisely, if we denote by v and v_1 the nodes such that $\ell = \ell_v$ and $\ell_1 = \ell_{v_1}$, there must be many nodes preceding v_1 but following v.

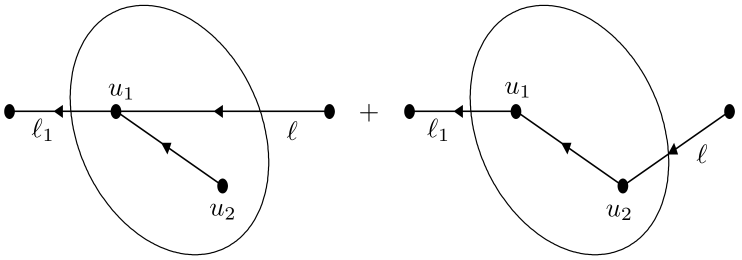

Figure 6: The simplest example of cancellation between resonances. The circles represent resonances, so that $\nu_{u_1} + \nu_{u_2} = 0$. The parts of the three outside the resonance are represented as bullets. Each tree represents two contributions: $\nu_{u_1} = -\nu_{u_2} = 1$ and $\nu_{u_1} = -\nu_{u_2} = -1$.

Hamiltonian systems (the adaptation to the GSSM being trivial), and to [8], where the case of comlex rotation numbers was considered.

On the contrary in the case of the SM, once a line ℓ on a large scale n has been obtained, another line ℓ_1 on the same scale can be easily obtained once more along the path connectig ℓ to the root, without going much further along the tree, as the mode labels now can also be negative (see table 2): for instance it is enough to have two nodes u_1 and u_2 with $\nu_{u_1} = -\nu_{u_2} = 1$ in order to have $\nu_{\ell_1} = \nu_\ell + \nu_{u_1} + \nu_{u_2} = \nu_\ell$, so that the line ℓ_1 has the same scale of the line ℓ.

The key remark is that, when summing the tree values over all possible trees as in (4.14), all terms containing the same resonance cancel almost exactly: more precisely, for any resonance T, the values of the trees containing that resonance cancel to order 2 in $\|\omega\nu\|$, if ν is the momentum of the line entering T. The criterion to single out the trees between which the cancellation operates is the following: given a tree ϑ with a resonance T, consider the class $\mathcal{F}_T(\vartheta)$ of all trees obtained by detaching the line entering T and reattaching it to all the nodes inside T, and for each of such trees consider also the tree obtained by reverting the sign of the mode labels of the nodes contained in T (*i.e.* by replacing each $\nu_u = 1$ with $\nu_u = -1$ and *vice versa*); see Figure 6 for an example.

Then it is easy to realize that the sum of all the so obtained trees cancel to zero and first order. For instance, with reference to Figure 6, by denoting with ϑ_1 and ϑ_2 the two trees obtained by fixing the values of ν_{u_1} and ν_{u_2}, we observe that their values contain a common factor $A = A(\vartheta_1) = A(\vartheta_2)$ (arising from the part of the trees outside the resonance) times

$$\left(\frac{1}{2} \frac{(\nu_{u_1})^3}{2!} \right) \frac{1}{\gamma(\omega\nu_{u_2})} \left(\frac{1}{2} \frac{\nu_{u_1}}{1!} \right) \tag{5.2}$$

for the first one, and times

$$\left(\frac{1}{2} \frac{(\nu_{u_1})^2}{1!} \right) \frac{1}{\gamma(\omega\nu_{u_2})} \left(\frac{1}{2} \frac{(\nu_{u_1})^2}{1!} \right) \tag{5.3}$$

for the second one. Moreover there are two non-equivalent trees of the form ϑ_1 (obtained by the action of the group $\mathcal{G}$ corresponding to the node u_1), whereas there is only one of the form ϑ_2, so that, if we sum all the non-equivalent trees, we obtain from ϑ_1 and ϑ_2 a contribution

$$A \left(\frac{1}{2} (\nu_{u_1})^2 \frac{1}{2} \nu_{u_2} \right) \left(\frac{\nu_{u_1}}{\gamma(\omega\nu_{u_2})} + \frac{\nu_{u_2}}{\gamma(\omega\nu_{u_2} + \omega\nu_\ell)} \right), \tag{5.4}$$

where the quantity in the last parentheses is vanishing for $\omega\nu_\ell = 0$ (as it follows from the property (2) in (4.15)); a second order zero is immediately obtained from the parity of the function $\gamma(\omega\nu)$ (see (2.13)).

But a second order cancellation produces a factor proportional to $\|\omega\nu_\ell\|^2$ which is exactly of the same size of the propagator of the line ℓ_1 (recall that $\nu_{\ell_1} = \nu_\ell$). In other words, given a line ℓ on a very large scale n, it is possible to create another line ℓ_1 with the same scale adding only a few nodes (for instance 2 in the above example), but in such a way a resonance arises, and the cancellation mechanism described above produces a gain factor which compensates the propagator of the newly added line ℓ_1: this means that also in such a case there cannot be any accumulation of small divisors. Again the argument can be extended to the GSM. For more details we refer to the original papers (see in particular [29, 38, 9, 8]).

The just described cancellation mechanism is enough to prove the convergence of the perturbative series (hence the existence of the corresponding invariant curve), but produces (at most) a value $\beta = 4$ for the lower bound in (3.1); we refer to [37] for a detailed proof. To obtain the optimal value $\beta = 2$ requires an enlargement of the classes of trees whose values have to cancel each other to second order, as we already observed in Section 4; we refer to [9] for an exaustive discussion. Note however that an idea why it is necessary to consider as resonances also clusters T with $m_T > 1$ can given – in a simpler case – by studying the Lindstedt series for complex rotation numbers (see below).

For the periodic orbits there is no small divisors problem (once one has realized that no division by zero occurs after fixing suitably the value of α). Therefore the proof of convergence of the Lindstedt series becomes trivial if one is not interested in obtaining optimal bounds. Nevertheless, in order to prove, the bound (3.2) some work is needed. In fact to prove the lower bound in (3.2) with $\beta = 2$ has the same difficulty as to obtain the analogous bound in (3.1): in fact the proof is essentially the same, as noticed in [11].

To prove that the interpolating functions, corresponding to the periodic orbits with rotation numbers given by the approximants of a Bryuno number ω, converge to the invariant curve $\mathcal{C}_{\varepsilon,\omega}$, one has simply to compare the respective Lindstedt series: this is what is done in [11], to which we refer for details.

Finally we can consider the case of rotation numbers of the form (2.15).[5] Let us start again by the SSM. The expansion (4.5) and bound (4.18) show that the coefficients $u_\nu^{(k)}(\omega)$ of the conjugating function are well defined for $\eta \neq 0$. When $\eta \to 0$ only the propagators G_ℓ correponding to the lines ℓ with momentum $\nu_\ell \in q\mathbb{Z}$ (*i.e.* with scale $n_\ell = 0$) can diverge: so we have the problem of counting how many such propagators can arise in a given tree ϑ.

It is not difficult to realize that the number of lines with scales $n = 0$ is bounded by $\lfloor k/q \rfloor$ (*i.e.* the highest integer smaller than k/q). To create a line ℓ with scale $n_\ell = 0$ one needs at least q nodes (recall that each node u carries a mode label $\nu_u = 1$). Moreover once a line ℓ with $n_\ell = 0$ has been obtained, in order to create another line ℓ_1 with $n_{\ell_1} = 0$, one needs at least other q nodes whose mode labels add with ν_ℓ contributing to ν_{ℓ_1}. In conclusion, if k the number of nodes in ϑ, one needs q nodes for each line on scale $n = 0$, so that the number of lines on scale $n = 0$ cannot be greater than $\lfloor k/q \rfloor$.

In the case of the SM there is the additional problem of the resonances, in the sense of the definition (4.16): but each time a resonance is formed, the cancellation mechanism described above (which trivially extends to the case in which there are more entering lines, provided that one shifts at least two lines; see [8] for details) assures that the newly created line on scale $n = 0$ contributes a small denominator which is in fact compensated. This means that once more that, in order to have a propagator of order $O(\eta^{-2})$ *which is not compensated*, one needs a cluster T of lines on

[5] For simplicity we shall not explicitly consider the case of more general complex rotation numbers tending to a rational value along a curve in the complex plane non-tangential to the real axis, but this could be easily done, as discussed in [7, 8].

scale $n = 1$ such that $\nu_T \neq 0$ (*i.e.* a cluster *which is not a resonance*), so that the above argument still applies.

By resuming, the value of a tree of order k can be at most of order $O(\eta^{-2\lfloor k/q \rfloor})$, so that a rescaling $\varepsilon \to (2\pi\eta)^{2/q}\varepsilon$ of the perturbative parameter assures that the rescaled function $u'(\alpha,\varepsilon,\omega) = u(\alpha,(2\pi\eta)^{2/q}\varepsilon,\omega)$ admits a finite limit $\bar{u} = \bar{u}_{p/q}(\alpha,\varepsilon)$ for $\eta \to 0$. In fact such a limit is nonvanishing and solves the differential eqaution (3.7) with vanishing boundary conditions: the latter statement can be proved by showing that the tree expansions of the limit function and of the formal solution of the differential equation coincide.

When passing to generalized semistandard and standard maps, the situation becomes a little more involved, as it is no more true that the nodes $u \in V(\vartheta)$ can contribute only a mode label ν_u with $|\nu_u| = 1$. Then the above argument can be modified as follows. To create a line ℓ with $n_\ell = 0$ one needs at least r nodes $u_1,\ldots,u_r$ with mode labels $\nu_{u_1},\ldots,\nu_{u_r}$ such that $\nu_{u_1} + \ldots + \nu_{u_r} = q$. Of course the value of r depends on f and q: for fixed q it is a value $r = r^*(f)$ depending on f. For instance if $q = 3$ and $f(x) = a\sin x + b\sin 2x$, with a, b nonvanishing constants, one has $r = 2$ and (say) $\nu_{u_1} = 1$, $\nu_{u_2} = 2$; the case of the SM is easily recoved as one has $r = q$ and $\nu_{u_j} = 1$ for all $j = 1,\ldots,q$ in such a case. Coming back to the general case, the rescaling implying the finiteness of the limit function is $\varepsilon \to (2\pi\eta)^{2/r^*(f)}$; to see that in general the limit function has in fact (*i.e.* in the sense of the definition (3.24)) a finite radius of convergence one can reason by comparison with the case of generalized semistandard maps in the spirit of [25]; see [10].

Note that generically an analytic function $f(x)$ has all the harmonics, so that one can choose $r = 1$ and (say) $\nu_{u_1} = q$: this implies the last statement of Theorem 5 about the genericity of the value of $r^*(f)$. In general the search of the right value r_0 of r leads to the Diophantine problem (3.20), as one can easily show (again we refer to [8] for details).

6. Lower bound for the semistandard map

In this section we want to prove the bound

$$\log\rho_0(\omega) + 2B_1(\omega) > -C, \tag{6.1}$$

where

$$\rho_0(\omega) = \left(\limsup_{k\to\infty}\left|u_k^{(k)}(\omega)\right|^{1/k}\right)^{-1}. \tag{6.2}$$

is the radius of convergence of the conjugating function

$$u_0(\alpha,\varepsilon,\omega) = \tilde{u}(e^{i\alpha}\varepsilon,\omega) = \sum_{k=1}^{\infty} u_k^{(k)}(\omega)\left(e^{i\alpha}\varepsilon\right)^k \tag{6.3}$$

for the SSM. This result was proved in [25]: here we give a proof based on the techniques described in Section 4. Such a proof is very simple, whereas to extend it to the case of the SM in order to obtain the lower bound in (3.1) requires a deep analysis of the cancellations between resonances, and it can be found in [9].

Let us denote by $N_n(\vartheta)$ the number of lines $\ell \in \Lambda(\vartheta)$ with scale $n_\ell = n$. Note that, given a tree ϑ of order $k < q_n$, the properties (5.1) imply that one has $N_n(\vartheta) = 0$.

A basic result which will be used in the proof is the following Davie's lemma (for the proof see [25] or [9]): *given $\nu \in \mathbb{Z}$ such that $\|\omega\nu\| \leq 1/4q_n$, then (1) either $\nu = 0$ or $|\nu| \geq q_n$, and (2) either $|\nu| \geq q_{n+1}/4$ or $\nu \in q_n\mathbb{Z}$.*

Then we want to prove that, in the case of the SSM, one has

$$N_n(\vartheta) \leq \frac{k}{q_n} + \frac{8k}{q_{n+1}}, \tag{6.4}$$

which immediately implies the bound (6.1). In fact (6.4), inserted into (4.14), gives

$$\begin{aligned} |\mathrm{Val}(\vartheta)| &\leq \left(\frac{1}{2}\right)^k \prod_{n=0}^{\infty} (Cq_{n+1})^{2N_n(\vartheta)} \\ &\leq \left(\frac{C}{2}\right)^k \exp\left[2k\sum_{n=0}^{\infty}\left(\frac{\log q_{n+1}}{q_n} + \frac{8\log q_{n+1}}{q_{n+1}}\right)\right], \end{aligned} \tag{6.5}$$

where, in the first line, $1/2$ is a bound on the node factor F_v (see (4.2)), while Cq_{n+1}^2 is a bound on the propagator of a line on scale n, with C a constant (see (4.3), (2.13) and (4.13)).

Then the bound (6.2) and the (trivial) bound

$$\sum_{n=1}^{\infty} \frac{\log q_n}{q_n} < D \tag{6.6}$$

holding for any irrational ω with convergents $\{p_n/q_n\}$ and for some (universal) constant D, give[6]

$$\left|u_k^{(k)}(\omega)\right| \leq \sum_{\vartheta\in\mathcal{T}_{k,k}} |\mathrm{Val}(\vartheta)| \leq 2^{3k}\left(\frac{Ce^{16D}}{2}\right)^k e^{2B_1(\omega)}, \tag{6.7}$$

from which the bound (6.1) follows for the radius of convergence.

So it remains to check the bound (6.4). One can prove inductively on the order k the following bounds:

$$\begin{cases} N_n(\vartheta) = 0, & \text{if } k < q_n, \\ N_n(\vartheta) \leq \dfrac{2k}{q_n} - 1, & \text{if } k \geq q_n, \end{cases} \tag{6.8}$$

for any $n \geq 0$, and:

$$\begin{cases} N_n(\vartheta) = 0, & \text{if } k < q_n, \\ N_n(\vartheta) \leq \dfrac{k}{q_n}, & \text{if } q_n \leq k < \dfrac{q_{n+1}}{4}, \\ N_n(\vartheta) \leq \dfrac{k}{q_n} + \dfrac{8k}{q_{n+1}} - 1, & \text{if } k \geq \dfrac{q_{n+1}}{4}, \end{cases} \tag{6.9}$$

for $q_{n+1} > 4q_n$, where k is the order of the tree ϑ. Then (6.8) and (6.9) immediately imply (6.4).

We give explicitly the proof of (6.9), as the proof of (6.8) is easier, and we leave to the reader.

[1] If the root line ℓ of ϑ has scale $\neq n$, let us denote with $\ell_1, \ldots, \ell_m$ the lines entering the last node u_0 of ϑ and with $\vartheta_1, \ldots, \vartheta_m$ the subtrees of ϑ whose root lines are those lines. By construction $N_n(\vartheta) = N_n(\vartheta_1) + \cdots + N_n(\vartheta_m)$, so that the bounds (6.9) follow inductively by noting that for $k \geq q_{n+1}/4$ one has $8k/q_{n+1} - 1 \geq 1$.

[2] If the root line ℓ of ϑ has scale n, then we can reason as follows. Let us denote with $\ell_1, \ldots, \ell_m$ the lines on scale $\geq n$ which are the nearest to the root line of ϑ,[7] and let $\vartheta_1, \ldots, \vartheta_m$ be the subtrees with root lines $\ell_1, \ldots, \ell_m$. If $m = 0$ then (6.9) follows immediately; so let us suppose $m \geq 1$. Then the lines $\ell_1, \ldots, \ell_m$ are the entering lines of a cluster T (which can degenerate to a single point) having the root line of ϑ as the exiting line. One has $N_n(\vartheta) = 1 + N_n(\vartheta_1) + \cdots + N_n(\vartheta_m)$. In general $\tilde{m}$ subtrees among the m considered have orders $\geq q_{n+1}/4$, with $0 \leq \tilde{m} \leq m$, while the remaining $m_0 = m - \tilde{m}$ have orders $< q_{n+1}/4$. Let us numerate the subtrees so that the first $\tilde{m}$ have orders $\geq q_{n+1}/4$.

Let us distinguish the cases $k < q_{n+1}/4$ and $k \geq q_{n+1}/4$.

[6] We use that, for any tree $\vartheta \in \mathcal{T}_{k,\nu}$, each node $u \in V(\vartheta)$ carries a mode label $\nu_u = 1$, so that $\nu = k$ and $|\mathcal{T}_{k,k}| \leq 2^k 2^{2k}$. If we used a sharp multiscale decomposition, as in [33, 17, 18], we could have bound the number of elements in $\mathcal{T}_{k,k}$ with 2^{2k}.

[7] That is, such that no other line along the paths connecting the lines $\ell_1, \ldots, \ell_m$ to the root line is on scale $\geq n$.

[2.1] If $k < q_{n+1}/4$, then $\tilde{m} = 0$ and each line entering T, by Davie's lemma has a momentum which is a multiple of q_n and has a scale label n. Therefore the momentum flowing through the root line is $\nu = \nu_T + s_0 q_n$, for some $s_0 \in \mathbb{Z}_+$, with

$$\nu_T = \sum_{u \in V(T)} \nu_u. \tag{6.10}$$

Moreover also the root line of ϑ has scale n, by assumption, and momentum $\nu = s q_n$ for some $s \in \mathbb{Z}_+$, by Davie's lemma, so that $\nu_T = (s - s_0) q_n = s' q_n$, for some positive integer s'. Then $k_T \geq |\nu_T| \geq q_n$, giving:

$$N_n(\vartheta) \leq 1 + \frac{k_1 + \cdots + k_m}{q_n} \leq 1 + \frac{k - k_T}{q_n} \leq \frac{k}{q_n}, \tag{6.11}$$

and (6.9) follows.

[2.2] If $k \geq q_{n+1}/4$, assume again inductively the bounds (6.9), so that

$$N_n(\vartheta) \leq 1 + \sum_{j=1}^{\tilde{m}} \Big(\frac{k_j}{q_n} + \frac{8k_j}{q_{n+1}} - 1 \Big) + \sum_{j=\tilde{m}+1}^{m} \frac{k_j}{q_n}, \tag{6.12}$$

where k_j is the order of the subtree ϑ_j, $j = 1, \ldots, m$.

[2.2.1] If $\tilde{m} \geq 2$, then (6.9) follows immediately.

[2.2.2] If $\tilde{m} = 0$, then (6.12) gives

$$N_n(\vartheta) \leq 1 + \frac{k_1 + \cdots + k_m}{q_n} \leq 1 + \frac{k}{q_n} \leq \frac{8k}{q_{n+1}} - 1 + \frac{k}{q_n}, \tag{6.13}$$

as we are considering k such that $1 \leq 8k/q_{n+1} - 1$.

[2.2.3] If $\tilde{m} = 1$, then (6.12) gives

$$N_n(\vartheta) \leq 1 + \Big(\frac{k_1}{q_n} + \frac{8k_1}{q_{n+1}} - 1 \Big) + \sum_{j=2}^{m} \frac{k_j}{q_n} = \frac{k_1}{q_n} + \frac{8k_1}{q_{n+1}} + \frac{k_0}{q_n}, \tag{6.14}$$

where $k_0 = k_2 + \cdots + k_m$.

[2.2.3.1] If in such case $k_0 \geq q_{n+1}/8$, then we can bound in (6.14)

$$\frac{k_1}{q_n} + \frac{8k_1}{q_{n+1}} + \frac{k_0}{q_n} \leq \frac{k_1 + k_0}{q_n} + \frac{8(k_1 + k_0)}{q_{n+1}} - \frac{8k_0}{q_{n+1}} \leq \frac{k}{q_n} + \frac{8k}{q_{n+1}} - 1, \tag{6.15}$$

so that (6.9) follows.

[2.2.3.2] If $k_0 < q_{n+1}/8$, then, denoting with ν and ν_1 the momenta flowing through the root line ℓ of ϑ and the root line ℓ_1 of ϑ_1 respectively, one has

$$\|\omega(\nu - \nu_1)\| \leq \|\omega\nu\| + \|\omega\nu_1\| \leq \frac{1}{4q_n}, \tag{6.16}$$

as both ℓ and ℓ_1 are on scale $\geq n$ (use (4.13)). Then either $|\nu - \nu_1| \geq q_{n+1}/4$ or $\nu - \nu_1 = \tilde{s} q_n$, with $\tilde{s} \in \mathbb{Z}_+$, by Davie's lemma.

[2.2.3.2.1] If $|\nu - \nu_1| \geq q_{n+1}/4$, noting that $\nu = \nu_1 + \nu_T + \nu_0$, where $\nu_0 = s_0 q_n$ (with $s_0 \in \mathbb{Z}_+$ and $|\nu_0| \leq k_0 < q_{n+1}/8$) is the sum of the momenta flowing through the root lines of the m_0 subtrees entering T with orders $< q_{n+1}/4$ and ν_T is defined by (6.10), one has

$$k_T \geq |\nu_T| \geq |\nu - \nu_1| - |\nu_0| \geq \frac{q_{n+1}}{8}, \tag{6.17}$$

so that in (6.14) one can bound

$$\begin{aligned}\frac{k_1}{q_n} + \frac{8k_1}{q_{n+1}} + \frac{k_0}{q_n} &\le \frac{k-k_T}{q_n} + \frac{8(k-k_0-k_T)}{q_{n+1}} \\ &\le \frac{k}{q_n} + \frac{8(k-k_T)}{q_{n+1}} \le \frac{k}{q_n} + \frac{8k}{q_{n+1}} - 1,\end{aligned} \tag{6.18}$$

so that (6.9) follows again.

[2.2.3.2.2] If $\nu - \nu_1 = \tilde{s}q_n$, $\tilde{s} \in \mathbb{Z}_+$, then

$$\nu_T = \nu - \nu_1 - \nu_0 = (\tilde{s} - s_0) = sq_n, \tag{6.19}$$

where $s \in \mathbb{Z}_+$. Then $k_T \ge q_n$, so that in (6.14) one has

$$\frac{k_1}{q_n} + \frac{8k_1}{q_{n+1}} + \frac{k_0}{q_n} \le \frac{k-k_T}{q_n} - \frac{8k}{q_{n+1}} \le \frac{k}{q_n} - 1 + \frac{8k}{q_{n+1}}, \tag{6.20}$$

so implying (6.9).

This completes the proof of (6.9).

7. Analyticity domains and numerical results

A related problem, quite difficult from the mathematical point of view, is the determination of the shape of the domains of analyticity of the Lindstedt series in the variables α and ε. While studying the analyticity in α is quite interesting and some numerical results have been obtained by various authors (see for instance [42] and [48]; see also [5] for a viewpoint similar to the one taken here), we shall concentrate here on the study of the analiticity domains on the complex ε-plane.

The existence or not of an invariant curve $\mathcal{C}_\varepsilon(\omega)$ – and of quasi-periodic solutions running on it – for the standard map is related to the existence or not of solutions of the form given by (2.2), (2.3), with irrational ω. Of course the Lindstedt series (2.6) can *diverge*, even though the conjugating function u is still analytic in ε, just as the function $f(z) = 1/(1+z^2)$ is real analytic for all real values of z, even beyond the radius of convergence of its Taylor series, which is 1. So another quantity must be introduced to characterize the real value of ε at which a given invariant curve, of given rotation number ω, breaks: this is called the *critical function*.

For the standard map, simple symmetry considerations show that positive and negative values of ε give the same critical function (in absolute value). So we define

$$\varepsilon_{\rm c}(\omega) = \sup\{\varepsilon' \ge 0 : \forall \varepsilon'' < \varepsilon' \quad \mathcal{C}_{\varepsilon''}(\omega) \text{ exists and is analytic }\}. \tag{7.1}$$

If we were to consider the GSM, then one would have to distinguish the two cases $\varepsilon > 0$ and $\varepsilon < 0$.

Clearly $\rho(\omega) \le \varepsilon_{\rm c}(\omega)$, and from the physical point of view it is more interesting $\varepsilon_{\rm c}(\omega)$, which gives the actual threshold at which the quasiperiodic solution of given rotation number ceases to exist: $\rho(\omega)$, in a sense, is an artifact of the Taylor expansion, which is forced to converge in a disk and therefore its convergence is obstructed by the singularity nearest to the origin, even though it may be in the complex ε-plane. The discrepancy between the radius of convergence and the critical function, in other terms, is determined by the shape of the domains of analyticity in the complex ε-plane.

A detailed study of the analytic properties of the Lindstedt series is clearly beyond the current state of the art in the field. We can though perform a numerical study of the Lindstedt series, and try to derive some information on the nature and shape of its domains of analyticity.

Such information can be obtained by computing numerically the coefficients of the Lindstedt series for selected, non-trivial values of α, by determining its Padé approximants and by finding their poles: this is the standard method used to numerically gain some insight on the nature of the singularities of analytic functions.

Quite briefly, a Padé approximant of order $[N/M]$ to a function $f(z)$, analytic in a neighborhood of the origin, is a rational function $P(z)/Q(z)$, where $P(z)$ and $Q(z)$ are polynomials of degree, respectively, N and M, such that its first $N+M+1$ Taylor coefficients coincide with those of $f(z)$. We do not include here a detailed explanation of the numerical methodology we used, directing the interested reader to the original papers (in particular [6, 5, 12, 13, 14, 4]) and to the vast amount of literature available on the theory of Padé approximants (see in particular [2]). We just limit ourselves to say that to obtain a reasonable reliability of the results, very high precision is needed in the computation of the coefficients of the Lindstedt series, of the Padé approximants and in the determination of their poles, especially if the order of the Padé approximants is high. In practice, we used between 32 and 480 digits for Padé approximants of orders ranging from a few tens up to 240.

Since [6] and [5], it is believed that the Lindstedt series for the quasi-periodic solutions has a natural boundary of analyticity on the complex ε-plane, or, in other terms, that its domain of analyticity is bounded by a continuous curve with dense singularities, obstructing analytic continuation. This natural boundary appears, numerically, to be independent on α (except of course in the trivial cases $\alpha = 0$ and $\alpha = \pi$, when the Lindstedt series is identically zero). For the above mentioned symmetry properties, such a curve must be symmetric with respect both to the real and imaginary ε-axis. The radius of convergence is then given by the distance of the natural boundary from the origin, while the critical function is given by its intersection with the positive real ε-axis.

For the standard map, when ω is the golden mean $\gamma = (\sqrt{5}-1)/2$ it was found in [6] that there is indeed a natural boundary, and its shape seems to be circular. Qualitatively, this seems to be the case for the standard map each time that the continued fraction expansion of the rotation number has small partial quotients (the best case being then just the golden mean). This implies that at least approximately for such rotation numbers ω the quantities $\varepsilon_{\rm c}(\omega)$ and $\rho(\omega)$ are equal.

It is instead quite interesting to prove for $\varepsilon_{\rm c}(\omega)$ a result analogous to (3.1) for the radius of convergence, possibily with a different value of β. In particular, a value of $\beta = 1$ if often quoted as a conjecture, like in [49] and [20]. The original motivation for (3.1) was the analogy with Siegel's problem [55] quoted in Section 3, in which case Yoccoz [60] proved an analogous result for Siegel's radius, with $\beta = 1$: the value of β was considered to be the order of the recurrence defining the dynamics. But soon the numerical works of [49] and [13] showed that actually one had to distinguish between the two thresholds, as they appear to behave differently as ω tends to a rational value. This justifies the interest in considering the whole of the domain of analyticity of u in ε *when the rotation number is close to rationals*. Notice that until recently a lot of confusion has arisen by confounding the two different thresholds $\rho(\omega)$ and $\varepsilon_{\rm c}(\omega)$, especially since the semistandard map and Siegel's problem have just one critical threshold. So if the analogy is the only motivation for a form of Bryuno's interpolation for the critical function of the standard map, it is by no means obvious that the analogy must be with both and not just one threshold.

In [4] the poles of Padé approximants of high order (up to [240/240]) were computed using high precision arithmetics (up to 480 digits), in the complex ε-plane for the conjugating function u, when the rotation number is close to selected rational numbers: the distribution of poles in the complex plane should model the shape of the natural boundary of u in ε.

Before explaining the numerical results, we must give at least an intuitive definition of what means that an irrational number ω is "close" to a rational p/q: in fact, in a simple metric sense, all irrationals are arbitrarily close to infinitely many rationals. Given an irrational number $\omega \in [0,1]$, let $[0, a_1, a_2, \ldots]$ be its continued fraction expansion; we shall say that ω is "close" to the rational number $[0, a_1, \ldots, a_N]$ if a_{N+1} is "large"; with this notion of being close, the only way for an

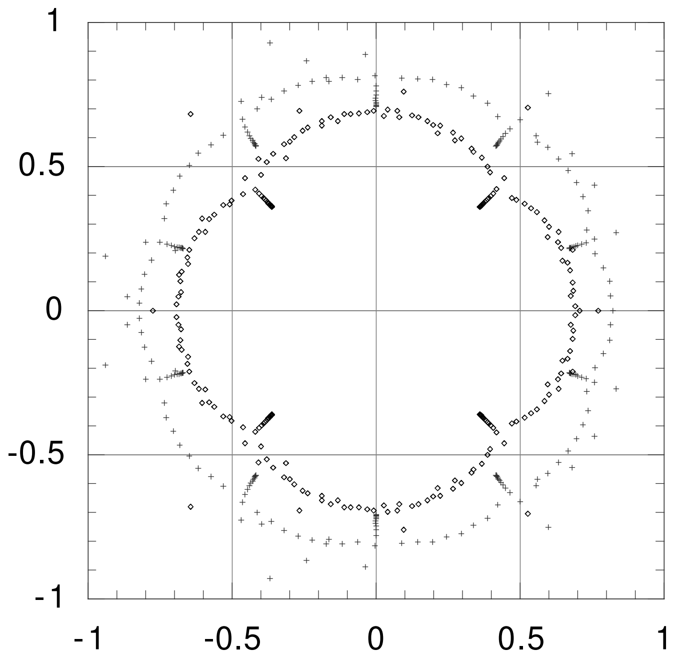

Figure 7: Poles of the Padé approximant [240/240], $\omega = [0, 2, 2, 12, 1^\infty]$, represented by squares, and of the Padé approximant [240/240], $\omega = [0, 2, 10, 1^\infty]$, represented by crosses; both at $\alpha = 1.0$. Note the 10-lobed and 4-lobed shape of the natural boundaries, as the rotation numbers are close respectively to 2/5 and 1/2.

irrational number to be really close to *all* its approximants whould be to have a very rapidly increasing sequence of partial quotients a_k, that is basically not to be a Bryuno number.

In particular, we took sequences of rotation numbers of the form

$$\omega_j = [0, a_1, \ldots, a_N, j, 1^\infty],$$

which for j large are close, in the sense described above, to the rational $p/q = [0, a_1, \ldots, a_N]$ and no others. Note that the "tail" $[1^\infty]$ assures that the sequence is close to p/q in the sense used in Section 3, *i.e.* that we are considering the limit superior of rotation numbers tending to the rational value p/q. In this case, we see that the domain of analyticity is made of $2q$ lobes, separated by 2q lines of singularities, arranged at first approximation as the $2q$-th roots of -1: these are what we call *dominant* singularities. For example, in Figure 7 we see the poles of the Padé approximants [240/240], at $\alpha = 1$, for $\omega = [0, 2, 10, 1^\infty]$ and for $\omega = [0, 2, 2, 12, 1^\infty]$, *i.e.* close respectively to 1/2 and 2/5. We clearly see respectively 4 and 10 lobes separated by the same number of lines of singularities.

If we consider together the plots obtained from a given Padé approximant with the same rotation number ω but varying the value of the angle α, we observe that the shape of the domain of analyticity is basically the same near the lines of the dominant singularities, while some minor changes appear elsewhere. As the domain should take into account all values of α we can superimpose all these plots: for instance for $\omega = [0, 3, 12, 1^\infty]$ we obtain Figure 8. We should also recall that the dependence on α is expected to be irrelevant for the domain of analyticity of the *full* conjugating function $u(\alpha, \varepsilon, \omega)$, so that, by representing the latter as a rational function (as it is implicit by using Padé approximants), it is reasonable that a slight dependence becomes observable.

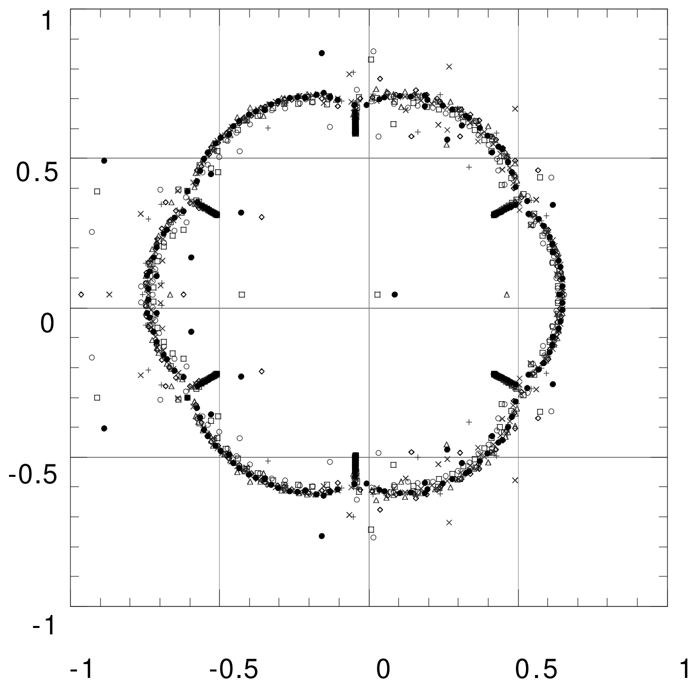

Figure 8: Poles of the Padé approximant [240/240], $\omega = [0, 3, 12, 1^\infty]$, for several values of α. The poles corresponding to $\alpha = 1.0$ are represented by black circles, while the other six white indicators (square, triangle, cross, plus, circle and rhombus) represent the poles corresponding to $\alpha = 2$, 3, 5, 6, 80, 100. Spurious pole/zero pairs have *not* been deleted.

By considering different continued fraction expansions it is possible to see the effect of being close to two different rational numbers. For example, in Figure 9 we see the poles of the Padé approximants of order [240/240], at $\alpha = 1.0$, for $\omega = [0, 12, 1^\infty]$ and for $\omega = [0, 12^\infty]$: while the first rotation number can be considered close to 0/1, the second is also close to 1/12. We see that while for the first only the effect of the dominant singularities is apparent (see the two-lobed shape of the domain), in the case of the second the two lobes themselves are cut by smaller lines of singularities, corresponding to 1/12: we call these *subdominant* singularities. Note that only the subdominant singularities near the real axis are really detectable in this case, since close to the imaginary axis they are masked by the deep cuts caused by the dominant singularities. To see further order of singularities would require much higher order in the Padé approximants, far beyond our availability of computing power.

To further characterize the nature of the boundary is very difficult at this stage. In fact, these findings could be explained in several different ways. For example, it could be that *all* rational approximants contribute singular lines to the shape of the boundary, which therefore has a fractal structure that the relatively low order of the Padé approximants is not able to detect. Or just the first few approximants actually contribute, giving a smooth boundary except for a finite number of cusps. Or the boundary could actually be completely smooth, and the observed singular lines are just artifact of the Padé approximants method due to the presence of branch points *inside* the domain of analyticity. While we tend to unbalance ourselves toward the second hypothesis, none of the three can be ruled out. To set the question from the numerical point of view would require moreover far too many computing resources to be feasible.

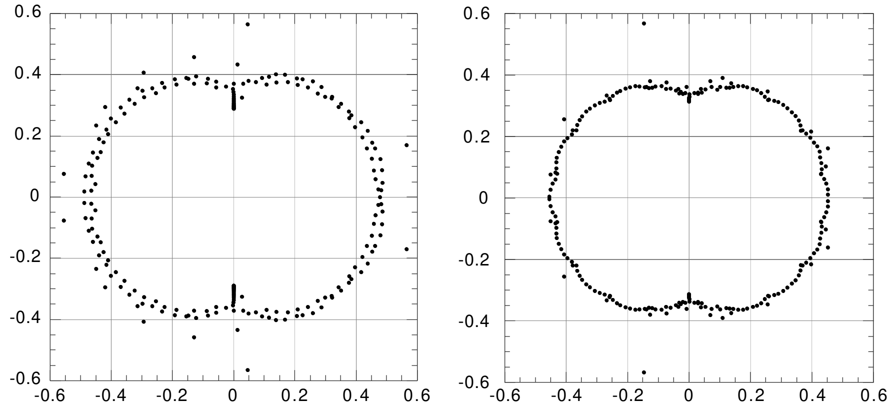

Figure 9: On the left, poles of the Padé approximant [240/240], $\omega = [0, 10, 1^\infty]$, and on the right, Poles of the Padé approximant [240/240], $\omega = [0, 10^\infty]$; both at $\alpha = 1$. Note the presence of the subdominant singularities in the second graph.

A much harder problem, even from the numerical point of view, is to derive as precise results for the critical function $\varepsilon_c(\omega)$ as for the radius of convergence $\rho(\omega)$. Currently in fact the only methods we can use to compute numerically the critical function are the above mentioned Padé approximant method, the frequency map method, and Greene's method. All three bear some difficulty, and moreover they share a common problem, that is the need to use high precision arithmetic to make the computations when ω is close enough to a rational value.

In particular, one would like to check the conjecture that

$$|\log \varepsilon_c(\omega) + \beta_c B_1(\omega)| \leq C, \tag{7.2}$$

for some constant C and some value of β_c: the common hypothesis is $\beta_c = 1$.

The frequency map method, due to Laskar [45], has been used to compute the critical function for the standard map by Carletti and Laskar [20]. We refer the reader to their article for further details on the method and the results. We note only a few points. First, the frequency map method does not give estimates of $\varepsilon_c(\omega)$ for *given* values of ω, but rather gives the "inverse function": that is, it estimates those values of ω for which an invariant curve with the given value of ε exists. In this way, we lose control on the arithmetic properties of the rotation numbers which we are considering, since they are known only within a numerical error. Secondly, also this method faces a precision problem: as explained in [20], the closer one gets to a rational number, the larger becomes the number of digits necessary for the intermediate calculations. In fact, they have reached as close as 1/4000 from the rational values (according to their tables). Note that the results published by [20] appear to confirm (7.2) with $\beta_c = 1$ with a relative error of the order of up to 10%, so it is necessary to consider more "singular" rotation numbers to get better results.

The above described Padé method permits the computation of the critical function by looking at the intersection of the natural boundary with the real ε-axis. But a problem arises here too: in fact, the shape of the natural boundary is very well modeled around the dominant singularities (that is, the poles of the Padé approximants tend to accumulate near such singularities) and the closer one gets to the rational value for the rotation number, the stronger such a phenomenon is; so if one gets really close to p/q, one shall see basically only the contribution of the dominant singularities, with the rest of the boundary approximated by just a few scattered points, hiding all structure. To overcome this phenomenon, one should use very high order Padé approximants, and therefore very high precision in the arithmetic: this implies very long computing times and a very large memory requirement for the computer. Moreover, to compute the roots of polynomials of very high degree, with very large coefficients as it turns out to be the case, is quite a difficult task from the point of view of the numerical analysis. Practically, all these difficulties make it unfeasible with current technology to go much beyond Padé approximants of order in the range of a couple of hundreds, which gives us quite unreliable estimates of the critical function already for rotation numbers like $\omega = [0, 50, 1^\infty]$.

Apparently the best method so far to compute the critical function for *given* rotation numbers is Greene's method [41], which consists in considering the stable periodic orbits with the rational approximants p_k/q_k of ω as rotation numbers, and looking at the values ε_k at which they lose stability: according to a conjecture by Greene [41], such values should tend to the critical function $\varepsilon_c(\omega)$ as k tends to infinity. In other words, the breakdown of an invariant curve with given irrational rotation number comes with the loss of stability of the nearby periodic orbits, whose period is a (high order) approximant of the rotation number of the curve. We do not enter here in the details of the practical implementation of the method; we note however that also Greene's method needs high precision arithmetic to be of practical value when ω is very close to a rational number. In fact, the stability or not of a given periodic orbit is decided by looking upon the trace of the tangent dynamics along the orbit, that is, if (x_i, y_i), $i = 1, \ldots, q$, is the periodic orbit,

$$\mathcal{T} = \operatorname{tr} \prod_{i=1}^{q} \begin{bmatrix} 1 + \varepsilon \cos x_i & 1 \\ \varepsilon \cos x_i & 1 \end{bmatrix}. \tag{7.3}$$

Now, it turns out that, when the periodic orbit has a rotation number p/q which is a high order approximant of a rotation number close to a rational p'/q' (that is, practically, when $p/q \approx p'/q'$, $q' \ll q$), then the diagonal elements of the product matrix are two very large numbers, almost equal in absolute value and with opposite sign; this numerical instability can be defeated trivially, but effectively, using enough precision in the calculation of the coefficients of the product matrix. In this way, using up to a few hundred digits of precision and a few days of computer time on a DEC Alpha processor, we could compute the critical threshold for rotation numbers up to $[0, 50000, 1^\infty]$. The amount of data we have collected so far does not give us yet a clear result on the validity of (7.2) and in particular of the value of β_c, so we refer the reader to a forthcoming paper.

8. Final remarks

As we have now seen, after more than 20 years the standard map (and its generalizations) still present us with deep and challenging problems. While some of the facts previously only numerically observed or conjectured have been now proved – for example the scaling of the radius of convergence or Bryuno's interpolation for the radius of convergence of the Lindstedt series *for the standard map* – most of the other problems must still be considered as open.

Of course, the first problem that comes to mind is the proof of the existence of a natural boundary of analyticity for the Lindstedt series of the standard map and the generalized standard map. The fact that generically the shape is quite far from circular just shows that the problem is quite hard, and no established techniques exist to address it.

The only general rigorous result on the analyticity domain of the Lindstedt series for the SM concerns the radius of convergence (see theorem 1 in Section 3). In [20], by using also results by Treshchëv and Zubelevich [57], some rigorous information is provided for the critical function for rotation numbers close to the fundamental resonance $\omega = 0$. But we lack of a complete description of the analyticty domain, and for general rotation numbers no result exists about the possibility of interpolating (or not) the value of the critical function through the Bryuno function (see also the conjecture discussed in Section 7). Notice that for what matters both the shape of the analyticity domains and the interpolation of the critical function in terms of the Bryuno's function there is a lack of understanding even at the level of the numerical analysis.

Even for the radius of convergence, rigorous results stop at the SM. We do not know, for example, if the radius of convergence has an interpolation formula in the case of the GSM. Actually, the results stated in theorem 5 suggest that no interpolation formula like (3.1) holds for general perturbations in (2.16); however one could ask if some other interpolation formula (*i.e.* in terms of some other function than the Bryuno function, or anything else) could be possible. Less ambitiously one could ask simply if it is possible to obtain an optimal bound representing both an upper and a lower bound. It could be interesting to investigate such a possibility in concrete examples (as a trigonometric polynomial with two harmonics).

Finally, we remark that, as noted in Sections 3 and 7 if we take the limit $\omega \to p/q$ for the rescaled function $\bar{u}$ on the reals along some particular sequence of Bryuno numbers, we find numerically the same results as considering the limit from the complex ω-plane. It should be interesting to have a mathematical explaination for such a fact.

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