Metal-Insulator transition for the Almost Mathieu Operator.

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Abstract

We prove that for Diophantine ω and almost every θ, the almost Mathieu operator, $(H_{\omega,\lambda,\theta}\Psi)(n) = \Psi(n+1) + \Psi(n-1) + \lambda \cos 2\pi(\omega n + \theta)\Psi(n)$, exhibits localization for $\lambda > 2$ and purely absolutely continuous spectrum for $\lambda < 2$. This completes the proof of (a correct version of) the Aubry-Andre conjecture.

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1 Introduction.

The almost Mathieu operator $H_{\omega,\lambda,\theta}$, acting on $\ell^2(\mathbb{Z})$ and given by

$$(H_{\omega,\lambda,\theta}\Psi)(n) = \Psi(n+1) + \Psi(n-1) + f(\omega n + \theta)\Psi(n), \tag{1.1}$$

with $f(\theta) = \lambda \cos(2\pi\theta)$,was first introduced by Peierls [39] and extensively studied in Physics and Mathematics literature since the 70th. In this paper we discuss the decomposition of spectral measures of $H_{\omega,\lambda,\theta}$. For background and some recent results on other interesting topics, not mentioned here, see [26, 37, 30, 40, 43, 32, 33, 23].

Our main result is the following

THEOREM 1 *For almost every $\omega \in \mathbb{R}, \theta \in \mathbb{R}$ the almost Mathieu operator $H_{\omega,\lambda,\theta}$ has*

1^o *for $\lambda > 2$, only pure point spectrum with exponentially decaying eigenfunctions.*

2^o *for $\lambda = 2$, purely singular-continuous spectrum.*

3^o *for $\lambda < 2$, purely absolutely continuous spectrum.*

Remark: Part 2^o is proved in [25] and stated here for completeness only.

Parts 1^o and 3^o of Theorem 1 were claimed by Aubry-Andre [1] (see also [2]) to hold for all θ and all irrational ω. It was soon realized by Avron-Simon [4], based on the Gordon Lemma [24], that 1^o does not hold for Liouville ω, for which the spectrum of $H_{\omega,\lambda,\theta}$ is purely singular continuous. It was later understood [35] that arithmetic properties of θ are also important, since for a dense G_δ of θ for any given irrational ω, operator $H_{\omega,\lambda,\theta}$ has purely singular-continuous spectrum. Thus "a.e. ω, θ" part is necessary for 1^o. We believe however that 3^o holds for all ω, θ, and 2^o for all θ and all irrational ω.

What Aubry-Andre actually proved for $\lambda > 2$, was positivity of Lyapunov exponents which, by the general Ishii-Pastur theorem (see, e.g., [11]) implies absence of absolutely continuous spectrum (for a.e. θ, and, by a recent result of Last-Simon [38], for all θ.). Their proof was made rigorous by Avron-Simon [3] and Pastur-Figotin[17], and a different proof followed from Herman [27]. Using the Dinaburg-Sinai technique [13] Bellissard-Lima-Testard [6] proved the existence of some pure point spectrum for λ very large (and of some absolutely continuous spectrum for λ very small) for a.e. ω. The first proofs of complete localization (pure point spectrum with exponentially decaying eigenfunctions) for λ large (a.e. ω, θ) are due to Sinai [42] and Fröhlich-Spencer-Wittwer [20]. Later, alternative KAM-type arguments for λ large were developed by Eliasson [16] and Goldshtein [22]. For λ very small, pure absolutely continuous spectrum was obtained by Chulaevsky-Delyon [8] using duality and the construction of Sinai [42]. Alternative proofs were given by Elliasson [15] and Goldshtein [22]. All those results use perturbative arguments and most hold for larger classes of quasiperiodic potentials (1.1)

The first nonperturbative results for $\lambda < 2$ were obtained by Last [36], who proved the existence of a large absolutely continuous component for any ω and any $\lambda < 2$ (and all θ, as shown by Gestesy-Simon [21] or as follows from [38]). In [28] a nonperturbative approach to localization was developed and pure point spectrum was proved for $\lambda \geq 15$. In [25] Gordon et al. established the dual result: pure absolutely continuous spectrum for $\lambda \leq 4/15$. In [29] the technique of [28] was combined with duality and the result of [36] to prove the existence of a large pure point component for any $\lambda > 2$, a.e. ω, θ. However, neither [36] nor [29] ruled out the existence of singular-continuous spectrum. Thus, the absence of singular continuous spectrum for $4/15 < \lambda < 15,\ \lambda \neq 2$, remained an issue, and it is solved in the present paper.

We note that 3^o immediately follows from 1^o by the strong version of duality [25], which states that for any irrational ω, if $H_{\omega,\frac{4}{\lambda},\theta}$ has only pure point spectrum for a.e. θ, then $H_{\omega,\lambda,\theta}$ has purely absolutely continuous spectrum for a.e. θ. Therefore, we will concentrate on the proof of 1^o.

We introduce the arithmetic conditions on ω, θ that will enable us to prove point spectrum. We will say that ω is Diophantine if there exist $c(\omega) > 0$ and $1 < r(\omega) < \infty$ such that

$$|\sin 2\pi j\omega| > \frac{c(\omega)}{|j|^{r(\omega)}} \tag{1.2}$$

for all $j \neq 0$. It is well known that a.e. ω is Diophantine. We define the set of resonant phases:

$$\Theta = \{\theta : \text{ the relation } |\sin 2\pi(\theta + (j/2)\omega)| < \exp(-k^{\frac{1}{2r(\omega)}}) \text{ holds for infinitely many } k\text{'s}\}. \tag{1.3}$$

It follows from the Borel-Cantelli lemma that Θ has zero Lebesgue measure. It is, however, a dense G_δ. Now we can formulate a more detailed version of part 1^o of Theorem 1.

THEOREM 2 *Suppose ω is Diophantine, $\theta \notin \Theta$, $\lambda > 2$. Then $H_{\omega,\lambda,\theta}$ has only pure point spectrum with exponentially decaying eigenfunctions.*

Moreover, it is possible to show that the eigenfunctions decay at exactly the Lyapunov rate. With the Lyapunov exponent $\gamma(E)$ defined by (2.2), we have the following

THEOREM 3 *With ω, λ, θ as in Theorem 2, for any eigenvalue E of operator $H_{\omega,\lambda,\theta}$ the corresponding eigenfunction Ψ_E satisfies*

$$\lim_{|n|\to\infty} \frac{\ln(\Psi_E^2(n) + \Psi_E^2(n+1))}{2|n|} = -\gamma(E) \tag{1.4}$$

Remarks.

1. As discussed above, Theorem 2 is not true for Liouville ω, nor does it hold for every θ for a given Diophantine ω. Furthermore, it is possible to obtain a complete result on localization that is true for all irrational ω (new critical constants where spectrum changes from singular-continuous to pure point will appear, depending on the exponential rate of approximation of ω by rationals [31]).

2. The set Θ, although quite small, is not optimal here. One can make it smaller by replacing $e^{-k^{\frac{1}{2r}}}$ in (1.3) by $e^{-k\epsilon}$ for small ϵ, however that requires a more delicate argument. On the other hand, replacing it with $e^{-k\epsilon}$ for sufficiently large ϵ, will lead to singular-continuous spectrum [35]. It was conjectured in [30] that there is a sharp treshhold in ϵ for the transition from pure point to singular-continuous spectrum. In any case, we did not try to optimize Θ in the present proof.

3. We prove Theorems 2 and 3 for any λ such that the Lyapunov exponent $\gamma(E) > 0$ for all E. We believe that under such condition (and ω, θ as in Theorem 2) this result should be true for operators (1.1) with rather general quasiperiodic potentials $f(\theta)$, just like positivity of the Lyapunov exponent implies zerodimensionality of spectral measures for all ω, θ [34]. While positivity of the Lyapunov exponents can easily be extracted from the present proof, we only use it, rather than prove it. Nor do we use the $\lambda > 2$ condition otherwise.

In Section 2 we prove Theorems 2 and 3 up to the main technical Lemma 4. We prove Lemma 4 in Section 3 based on two other statements, Lemma 5 and Lemma 7. The latter Lemmas are proved, correspondingly, in Sections 4 and 5. In Sections 4 and 5 we also formulate and prove some Lemmas that are more general than needed for the present proof. We believe they may be useful in certain other situations.

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2 General setup. Proof of Theorems 2 and 3

Definition. A formal solution $\Psi_E(x)$ of the equation $H_{\omega,\lambda,\theta}\Psi_E = E\Psi_E$ will be called **a generalized eigenfunction** if $|\Psi_E(x)| \leq C(1 + |x|)$ for some $C = C(\Psi_E) < \infty$.

It is well known that to prove Theorem 2 it suffices to prove that generalized eigenfunctions decay exponentially [7, 41].

We will use the notation $G_{[x_1,x_2]}(E)(x,y)$ for the Green's function $(H - E)^{-1}(x,y)$ of the operator $H_{\omega,\lambda,\theta}$ restricted to the interval $[x_1, x_2]$ with zero boundary conditions at $x_1 - 1$ and $x_2 + 1$. We now fix E, $\lambda \in \mathbb{R}$ and ω satisfying (1.2). To simplify notations, in some cases the E, λ, ω-dependence of various quantities will be omitted. We denote the one-step transfer-matrix of $H\Psi = E\Psi$ by

$$B(\theta, E) = \begin{pmatrix} E - \lambda\cos 2\pi\theta & -1 \\ 1 & 0 \end{pmatrix},$$

The $k-$step transfer-matrix is given by

$$M_k(\theta, E) = B(\theta + (k-1)\omega, E)\cdots B(\theta+\omega, E)B(\theta, E).$$

Let us denote

$$P_k(\theta, E) = \det\left[(H_{\omega,\lambda,\theta} - E)\Big|_{[0,k-1]}\right].$$

Then the $k-$step transfer-matrix can be written as

$$M_k(\theta, E) = \begin{pmatrix} P_k(\theta, E) & -P_{k-1}(\theta+\omega, E) \\ P_{k-1}(\theta, E) & -P_{k-2}(\theta+\omega, E) \end{pmatrix}. \tag{2.1}$$

The Lyapunov exponent $\gamma(E)$ is defined as

$$\gamma(E) = \lim_{n\to\infty} \frac{\int_0^1 \ln \|M_k(\theta, E)\| d\theta}{k} = \inf_k \frac{\int_0^1 \ln \|M_k(\theta, E)\| d\theta}{k}. \tag{2.2}$$

The limit exists and the last equality holds by the subadditive ergodic theorem (see, e.g., [11]).

Let $K = \{k \in \mathbb{N} : \text{ there exists } \theta \in [0,1] \text{ with } |P_k(\theta, E)| \geq \frac{1}{\sqrt{2}} e^{k\gamma(E)}\}$. By (2.1) for every $k \in \mathbb{N}$ we have that at least one of $k, k+1, k+2$ belongs to K.

Definition. Fix $E \in \mathbb{R}$, $m \in \mathbb{R}$. A point $y \in \mathbb{Z}$ will be called (m,k)-regular if there exists an interval $[x_1, x_2]$, $x_2 = x_1 + k - 1$, containing y, such that

$$|G_{[x_1,x_2]}(y, x_i)| < e^{-m|y-x_i|}, \text{ and } \operatorname{dist}(y, x_i) \geq \frac{1}{5}k;\ i = 1, 2.$$

Otherwise, y will be called (m,k)-singular.

This definition will be useful only for positive m, however, it can be used formally with negative m as well. The number $\frac{1}{5}$ here can be replaced, equivalently, by any other number smaller than $\frac{1}{4}$. The main difference between this definition and that of regularity typically used in multiscale analysis is the flexibility allowed in the choice of the interval $[x_1, x_2]$. It is going to be essential for the proof.

It is well known and can be checked easily that values of any formal solution Ψ of the equation $H\Psi = E\Psi$ at a point $x \in I \subset \mathbb{Z}$ can be reconstructed from the boundary values via

$$\Psi(x) = G_I(x, x_1)\Psi(x_1 - 1) + G_I(x, x_2)\Psi(x_2 + 1), \tag{2.3}$$

where $I = [x_1, x_2]$. This implies that if Ψ_E is a generalized eigenfunction, then every point $y \in \mathbb{Z}$ with $\Psi_E(y) \neq 0$ is (m,k)-singular for k sufficiently large: $k > k_1(E, m, \theta, y)$.

Theorem 2 will now follow from

LEMMA 4 *Suppose $\theta \notin \Theta, \omega$ satisfies (1.2). Then for every $y \in \mathbb{Z}$, $\epsilon > 0, \alpha < 2$, there exists $k_2(\theta, \omega, y, \epsilon, \alpha, E)$, such that for all $k \in K$, $k > k_2(\theta, \omega, y, \epsilon, \alpha, E)$, if x and y are both $(\gamma(E) - \epsilon, k)-$singular and $|x - y| > \frac{k+1}{2}$, then $|x - y| > k^\alpha$.*

Remarks:

1. If one somewhat relaxes the nonresonant condition on the excluded set Θ (with Θ still having measure 0, namely replacing the subexponential function by polynomial decay in (1.3)) one can obtain exponential in k repulsion of singular clusters in Lemma 4.

2. This Lemma is close in spirit to the central Lemmas of multi-scale analysis ([18]; see also [19, 20, 14]) however our proof of it does not involve multiple scales.

To complete the proof of Theorem 2, we let $E(\theta)$ be a generalized eigenvalue of $H_{\omega,\lambda,\theta}$, and $\Psi_E(x)$ the corresponding generalized eigenfunction. Assume $\Psi(0) \neq 0$. Then, by Lemma 4, if
$|x| > \max(k_1(E, \gamma(E)-\epsilon, \theta, 0), k_2(\theta, \omega, 0, \epsilon, 1.5, E))+1$, the point x is $(\gamma(E)-\epsilon, k)-$regular for some $k \in \{|x|-1, |x|, |x|+1\} \cap K \neq \emptyset$, since 0 is $(\gamma(E)-\epsilon, k)-$singular. We, therefore, obtain that there exists an interval $[x_1, x_2]$ containing x, such that

$$\frac{1}{5}(|x|-1) \leq |x_i - x| \leq \frac{4}{5}(|x|+1), \quad |G_{[x_1,x_2]}(x, x_i)| \leq e^{-(\gamma(E)-\epsilon)|x-x_i|}, \ i = 1, 2. \tag{2.4}$$

Using (2.3), we obtain the estimate: $|\Psi_E(x)| \leq 2C(\Psi_E)(2|x|+1)e^{-(\frac{\gamma(E)-\epsilon}{5})(|x|-1)}$. Since, for $\lambda > 2$, we have $\gamma(E) > 0$ for all E (note: this is the only place in the paper where we use the condition $\lambda > 2$), this implies exponential decay.
∎

Proof of Theorem 3. The fact that the lower limit of the expression in (1.4) is always not lower than $-\gamma(E)$, is a general statement, proved by Craig-Simon [10] for any quasiperiodic (with a.e. θ form true even for any ergodic) potential. Therefore, we are concerned here only with the upper bound.

It is one of the main technical points of multiscale analysis that the exponential decay of a Green's function at a scale k under certain conditions generates exponential decay with the same rate at a scale k^α. The proof is usually done using block-resolvent expansion, with the combinatorial factor being killed by the power-law growth of scales. The proof of Theorem 3 will consist, roughly, of adapting this type of argument to our situation.

We would like to note, however, that power-law repulsion of singular clusters, as in Lemma 4, although used in the following proof, is not essential here. Namely, one can prove Theorem 3 using only the fact that x is $(\gamma(E)-\epsilon,|x|)-$regular for all sufficiently large $|x|$.

Fix α,α_1 so that $1<\alpha<\alpha_1<2$. Find $\bar{k}$ such that for $k>\bar{k}$ we have $(k+3)^\alpha+4k^\alpha+5k^{\alpha-1}<k^{\alpha_1}$. Assume without loss of generality that x is positive. Let E be an eigenvalue of $H_{\omega,\lambda,\theta}$. Since $\Psi_E\in\ell^2$, (it even decays exponentially, as we already proved) we can normalize it to have an apriori bound

$$|\Psi_E(x)|\leq 1,\ x\in\mathbb{Z}. \tag{2.5}$$

Take x large enough: $|x|>k_0^\alpha$, where

$$k_0=\max(k_1(E,\gamma(E)-\epsilon,\theta,0),k_2(\theta,\omega,0,\epsilon,\alpha_1,E),\bar{k})+3.$$

Let $k\in K$ be such that

$$k^\alpha<|x|\leq(k+3)^\alpha. \tag{2.6}$$

Then $k>k_0-3$, and we have, by Lemma 4, that every $y\in[\frac{k+3}{2},k^{\alpha_1}]$ is $(\gamma(E)-\epsilon,k)-$regular. For each such y pick an interval $[x_1,x_2]$ from the definition of $(\gamma(E)-\epsilon,k)-$regularity and denote it by $I(y)$. We denote the boundary of the interval $I(y)$, the set $\{x_1,x_2\}$, by $\partial I(y)$. For $z\in\partial I(y)$ we let z' be the neighbor of z, (i.e., $|z-z'|=1$) not belonging to $I(y)$.

We now expand $\Psi(x_1-1)$ in (2.3) iterating (2.3) with $I=I(x_1-1)$, and $\Psi(x_2+1)$ using (2.3) with $I=I(x_2+1)$. Note that $I(x_i\pm1)$, $i=1,2$, are well-defined, since, by construction, $\frac{k+3}{2}<x_1-1<x_2+1<k^{\alpha_1}$, and therefore points of the form $x_i\pm1$, $i=1,2$, are $(\gamma(E)-\epsilon,k)-$regular. We continue to expand each term of the form $\Psi(z)$ in the same fashion until we arrive to z such that either $z<k$ or the number of G_I terms in the product becomes $5k^{\alpha-1}$, whichever comes first. We then obtain an expression of the form

$$\Psi(x)=\sum_{s;z_{i+1}\in\partial I(z_i')}G_{I(x)}(x,z_1)G_{I(z_1')}(z_1',z_2)\cdots G_{I(z_s')}(z_s',z_{s+1})\Psi(z_{s+1}'). \tag{2.7}$$

where in each term of the summation we have $z_i>k,\ i=1,\dots,s$, and either $0<z_{s+1}'<k,\ \ s\leq 5k^{\alpha-1}$, or $s+1=5k^{\alpha-1}$. Note, that since we bound the number of expansions by $5k^{\alpha-1}$ and since, by construction, $k>\bar{k}$, we have

$\frac{k+3}{2} < k < z_i' < x+4k^\alpha+5k^{\alpha-1} < k^{\alpha_1}$ for each $i \leq s$. Therefore each z_i', $i \leq s$, is $(\gamma(E)-\epsilon, k)-$regular, and $I(z_i')$ is well-defined. We now consider the two cases, $0 < z_{s+1}' < k$, and $s+1 = 5k^{\alpha-1}$, separately.

If $0 < z_{s+1}' < k$, we have, by the definition of regularity and (2.5),

$$|G_{I(x)}(x, z_1)G_{I(z_1')}(z_1', z_2)\cdots G_{I(z_s')}(z_s', z_{s+1})\Psi(z_{s+1}')| \leq e^{-(\gamma(E)-\epsilon)(|x-z_1|+\sum_{i=1}^{s}|z_i'-z_{i+1}|)} \leq$$

$$e^{-(\gamma(E)-\epsilon)(|x-z_{s+1}|-(s+1))} \leq e^{-(\gamma(E)-\epsilon)(x-k-5k^{\alpha-1})}.$$

If $s+1 = 5k^{\alpha-1}$, using again (2.5), the definition of regularity, and the fact that $|x - z_1| \geq \frac{k}{5}$, $|z_i - z_{i+1}| \geq \frac{k}{5}$, $i = 1, \ldots, s$, we can estimate

$$|G_{I(x)}(x, z_1)G_{I(z_1')}(z_1', z_2)\cdots G_{I(z_s')}(z_s', z_{s+1})\Psi(z_{s+1}')| \leq e^{-(\gamma(E)-\epsilon)\frac{k}{5}5k^{\alpha-1}}.$$

Using (2.6) we obtain that in either case

$$|G_{I(x)}(x, z_1)G_{I(z_1')}(z_1', z_2)\cdots G_{I(z_s')}(z_s', z_{s+1})\Psi(z_{s+1}')| \leq e^{-(\gamma(E)-\epsilon-\delta)x} \quad (2.8)$$

for any $\delta > 0$ and x sufficiently large. Finally, we observe that the total number of terms in (2.7) is bounded above by $2^{5k^{\alpha-1}}$. Combining it with (2.7), (2.8),(2.6), we obtain

$$|\Psi(x)| \leq 2^{5x^{\frac{\alpha-1}{\alpha}}} e^{-(\gamma(E)-\epsilon-\delta)x}.$$

Since δ and ϵ can be chosen arbitrarily small for sufficiently large x, this implies the needed upper bound. ∎

3 Proof of Lemma 4.

In this section we assume E fixed and drop it from most of the notations.

We start with outlining the idea of the proof. Every $(m, k)-$singular point x can be understood, very roughly, as a local center of localization, at the rate m and on the scale k. If so, for any $x_1 < x < x_1+k-1$, such that both x_1 and x_1+k-1 are sufficiently far from x, the operator $H_{\omega,\lambda,\theta}$ restricted to the interval $[x_1, x_1+k-1]$ should have an eigenvalue close to E. This could be manifested by $|P_k(\theta+x_1\omega, E)|$ being abnormally small (exponentially smaller than $e^{\gamma(E)k}$). Then, for each singular point x there exists a sufficiently long (longer than $\frac{k+1}{2}$) interval $I(x) \subset \mathbb{Z}$ of values of x_1 such that $|P_k(\theta + x_1\omega, E)|$ is small. This is made precise in Lemma 6. The main technical difficulty in the proof of this Lemma lies in establishing a uniform (in θ) upper bound on $|P_k(\theta, E)|$, that is a subject of Lemma 5, the proof of which is given in Section 4.

The rest of the proof consists mainly of establishing the following fact: if the points $\theta_1, \ldots, \theta_{k+1}$ are, in a certain sense, uniformly distributed on $[0, 1)$, and do not come abnormally close to one another, then $P_k(\theta)$ will not be small (in the sense above) at at least one of those points. Finally, for Diophantine ω, any points of the form $\theta + x_1\omega$ with x_1 running through a union of two long enough intervals, $I(x)$ and $I(y)$, are going to be uniformly distributed (Lemma 12). If x and y are not too close to each other, the intervals $I(x)$ and $I(y)$ will not intersect and their union will consist of at least $k+1$ points. Also, the Diophantine condition on ω (1.2) together with the nonresonant condition on θ (1.3) imply that if x is not very far from y, the points $\theta + x_1\omega$, with $x_1 \in I(x) \cup I(y)$, do not come too close to one another (Lemma 13). Summing it all up yields that two singular points, x and y, (if they are not very close) must be sufficiently far from each other.

We will now proceed with a formal proof.

It is a standard linear algebra (Cramer's rule) that for any $x_1,\ x_2 = x_1 + k - 1,\ x_1 \leq y \leq x_2$ we have

$$\begin{aligned}
|G_{[x_1,x_2]}(x_1, y)| &= \left| \frac{P_{x_2-y}(\theta + (y+1)\omega)}{P_k(\theta + x_1\omega)} \right|, \\
|G_{[x_1,x_2]}(y, x_2)| &= \left| \frac{P_{y-x_1}(\theta + x_1\omega)}{P_k(\theta + x_1\omega)} \right|.
\end{aligned} \tag{3.1}$$

We will need the following bound, uniform in θ, on the numerator in (3.1):

LEMMA 5 *Suppose $H_{\omega,\theta}$ is given by (1.1) with $f(\theta) = \sum_{k=0}^{p} a_k \cos^k \theta$ a trigonometric polynomial, ω any irrational. Then for every $E \in \mathbb{R}$, $\epsilon > 0$, there exists $k_f(\epsilon, E, \omega)$ such that $|P_n(\theta)| < e^{(\gamma(E)+\epsilon)n}$ for all $n > k_f(\epsilon, E, \omega)$, all θ.*

This Lemma will be proved in Section 4

Let $A_k^{z,\theta} = \{x \in \mathbb{Z} : |P_k(\theta + x\omega)| \le e^{kz}\}$. We will show that every singular point "produces" a cluster of points belonging to an appropriate A_k^z; however, for $z < \gamma(E)$, such clusters in the set A_k^z should be far apart.

LEMMA 6 *Suppose $y \in \mathbb{Z}$ is $(\gamma(E) - \epsilon, k)$-singular, $\epsilon < \frac{\gamma(E)}{3}$, $k > 4k_f(\frac{\epsilon}{6}, E, \omega) + 1$. Then for any x such that $y - [\frac{3}{4}]k \le x \le y - [\frac{3}{4}]k + [\frac{k+1}{2}]$, we have that x belongs to $A_k^{\gamma(E)-\frac{\epsilon}{8},\theta}$.*

Proof: Follows immediately from the definition of regularity, (3.1), and Lemma 5. ∎

Assume now that y_1 and y_2 are both $(\gamma(E) - \epsilon, k)-$singular. Assume without loss of generality that $y_2 > y_1$. We set $d = y_2 - y_1$, $x_i = y_i - [3/4k]$, $i = 1, 2$. Note that $P_k(\theta)$ is an even function of $\theta + \frac{k-1}{2}\omega$ and can be written as a polynomial of degree k in $\cos 2\pi(\theta + \frac{k-1}{2}\omega)$:
$P_k(\theta) = \sum_{j=0}^k c_j \cos^j 2\pi(\theta + \frac{k-1}{2}\omega) \stackrel{\text{def}}{=} Q_k(\cos 2\pi(\theta + \frac{k-1}{2}\omega))$. We now set

$$\theta_j = \begin{cases} \theta + (x_1 + \frac{k-1}{2} + j)\omega, \ j = 0, 1, ..., [\frac{k+1}{2}] - 1 \\ \theta + (x_2 + \frac{k-1}{2} + j - [\frac{k+1}{2}])\omega, \ j = [\frac{k+1}{2}], \ldots, k \end{cases}$$

By the assumption, $d > \frac{k+1}{2}$. This implies that all θ_j, $j = 0, \ldots, k$, are different. We have, by Lemma 6, that $|Q_k(\cos 2\pi\theta_j)| < e^{k(\gamma(E)-\epsilon/8)}$, $j = 0, \ldots, k$. We now write the polynomial $Q_k(z)$ in Lagrange interpolation form using $\cos 2\pi\theta_0, \ldots, \cos 2\pi\theta_k$:

$$|Q_k(z)| = |\sum_{j=0}^k Q_k(\cos 2\pi\theta_j) \frac{\prod_{\ell \ne j}(z - \cos 2\pi\theta_\ell)}{\prod_{k \ne j}(\cos 2\pi\theta_j - \cos 2\pi\theta_\ell)}| \tag{3.2}$$

The points θ_j, representing two long pieces of the trajectory of an ergodic rotation, are uniformly distributed on the circle, and, as we will show in Lemma 13, $\cos 2\pi\theta_j$ do not come too close to one another. We claim that this implies that the ratio of products in (3.2) does not deviate much from e^0. We make it precise in the following Lemma that will be proved in Section 5.

LEMMA 7 *Suppose $d < k^\alpha$ for some $\alpha < 2$, and θ, ω as in Theorem 2. Then for any $\epsilon > 0$ there exists $k_3(\epsilon, \omega, \alpha, \theta, y_1)$ such that for $k > k_3(\epsilon, \omega, \alpha, \theta, y_1)$ we have that for any $z \in [-1, 1]$,*

$$\frac{|\prod_{\ell \ne j}(z - \cos 2\pi\theta_\ell)|}{|\prod_{k \ne j}(\cos 2\pi\theta_j - \cos 2\pi\theta_\ell)|} \stackrel{\text{def}}{=} \frac{|I_1|}{|I_2|} \le e^{k\epsilon}$$

We can now complete the proof of Lemma 4. Set $\hat{k} = \max\{4k_{\cos}(\frac{\epsilon}{6}, E, \omega) + 1, k_3(\frac{\epsilon}{16}, \omega, \alpha, \theta, y_1)\}$. Take $\alpha < 2$, and $k \in K$, $k > \hat{k}$, and θ_0 such that $|P_k(\theta_0)| \geq \frac{1}{\sqrt{2}} \exp(k\gamma(E))$. Let $z_0 = \cos 2\pi(\theta_0 + \frac{k-1}{2}\omega)$. Suppose $d < k^\alpha$. We evaluate $Q_k(z_0)$ using (3.2) and Lemma 7 to estimate the ratio of products in (3.2). We obtain

$$\frac{1}{\sqrt{2}} e^{k\gamma(E)} \leq |Q_k(z_0)| \leq e^{k(\gamma(E) - \frac{\epsilon}{8})}(k+1)e^{\frac{k\epsilon}{16}}$$

and the contradiction proves Lemma 4 with $k_2 = \max(\hat{k}, k_4(\epsilon))$. ∎

4 Polynomials and Measure. Proof of Lemma 5.

We introduce the upper Lyapunov exponent, $\bar{\gamma}(E, \theta) = \overline{\lim}_{k\to\infty} \frac{\ln \|M_k(\theta, E)\|}{k}$. Craig-Simon [10] showed that for any ergodic potential $\bar{\gamma}(E, \theta) \leq \gamma(E)$ for a.e. θ (for all θ in the quasiperiodic case (1.1)). Thus, by (2.1), for every θ and sufficiently large n, we have $|P_n(\theta)| < e^{(\gamma(E)+\epsilon)n}$. As before, $P_k(\theta)$ is an even function of $\theta + \frac{k-1}{2}\omega$ and can be written as a polynomial of degree kp in $\cos 2\pi(\theta + \frac{k-1}{2}\omega)$. We will show that this implies that for large n, $|P_n(\theta)|$ can be bounded above uniformly in θ. The key statement is

THEOREM 8 *Let $Q(x) = \sum_{j=0}^{n} c_j x^j$ be an arbitrary n^{th} degree polynomial. Suppose $|Q(z_0)| = a^n$, some $a > 0$, $z_0 \in [-1, 1]$. Then for any $0 < b < a$ and sufficiently large n ($n > n_0(a/b)$) we have $|\{\theta \in (0, \pi) : |Q(\cos\theta)| < b^n\}| \leq c(a, b) < \pi$.*

Here and in the future we use $|\cdot|$ for Lebesgue measure. The nontrivial statement here is that the measure can be bounded by a constant not dependent on n or the polynomial Q.
Remark: The theorem is certainly true for $|\{z \in [-1, 1] : |Q(z)| < b^n\}|$ as well. We formulate it with the cosine because it is convenient for our application and also simplifies the proof.

To complete the proof of Lemma 5 we set $A_n = \{\theta \in (0, \pi) : \text{for all } k > n, |P_k(\cos\theta)| \leq e^{(\gamma(E)+\epsilon/2)k}\}$. Then $|A_n^c|$ goes to 0 as $n \to \infty$, however if there exists θ such that $|P_k(\cos\theta)| > e^{(\gamma(E)+\epsilon)k}$, some $k > n$, then $|A_n^c| \geq |\{\theta \in (0, \pi) : |P_k(\cos\theta)| > e^{(\gamma(E)+\epsilon/2)k}\}| \geq \pi - c(e^{\frac{\gamma(E)+\epsilon}{p}}, e^{\frac{\gamma(E)+\epsilon/2}{p}}) > 0$. This contradiction proves Lemma 5. ∎

Proof of Theorem 8. We start with the following elementary Lemma, proven below:

LEMMA 9 *For any measurable set $B \in \mathbb{R}$, $|B| = m$, any $n \in \mathbb{N}$, and any $\delta < m/n$ one can find $x_1, \ldots, x_{n+1} \in B$ such that $|x_i - x_j|/\delta \in \mathbb{N}$ for all $i \neq j$.*

In other words, one can find $n+1$ elements of an arithmetic progression with step δ belonging to B. We will now prove Theorem 8. Let $B = \{\theta \in (0,\pi) : |Q(\cos\theta)| < b^n\}$. Let $m = |B|$. We fix $\delta < \frac{m}{n}$, and pick, according to Lemma 9, $x_1, \ldots, x_{n+1} \in B$, elements of an arithmetic progression with step δ. We write $Q(x)$ in a Lagrange interpolation form using $\cos x_1, \ldots, \cos x_{n+1}$. Then

$$a^n = |Q(z_0)| = \left|\sum_{j=1}^{n+1} Q(\cos x_j)\frac{\prod_{k\neq j}(z_0 - \cos x_k)}{\prod_{k\neq j}(\cos x_j - \cos x_k)}\right| < (n+1)b^n \max_j \left|\frac{\prod_{k\neq j}(z_0 - \cos x_k)}{\prod_{k\neq j}(\cos x_j - \cos x_k)}\right| \tag{4.1}$$

We denote the last ratio by I and will estimate separately the numerator and denominator in I. First,

$$\exp(\sum_{j\neq k} \ln|z_0 - \cos x_j|) \leq \exp(\sum_{j=0}^{n} \ln|1 - \cos(\pi - j\delta)|) \leq \exp(1/\delta \int_0^{n\delta} \ln(1+\cos x)dx + \ln 2). \tag{4.2}$$

Similarly,

$$\exp(\sum_{j\neq k} \ln|\cos x_k - \cos x_j|) \geq \exp(\sum_{\substack{j = -[\frac{n+1}{2}] \\ j \neq 0}}^{[\frac{n+1}{2}]} \ln|\cos(\pi/2 + j\delta)|) \geq \exp(2/\delta \int_0^{\frac{n+1}{2}\delta} \ln\sin x dx) \tag{4.3}$$

We set $m_1 = n\delta$, and, combining (4.2), and (4.3) we obtain

$$I \leq 2\exp\left(\frac{n}{m_1}\left(-\int_{m_1}^{\pi} \ln(1+\cos x)dx + 2\int_{m_1/2}^{\pi/2} \ln\sin x dx\right) - \ln\sin\frac{m_1}{2}\right) \tag{4.4}$$

where we used the elementary identity (5.8). We set

$$g(m) = \exp(1/m(-\int_m^{\pi} \ln(1+\cos x)dx + 2\int_{m/2}^{\pi/2} \ln\sin x dx).$$

It is easily verified that g is a monotone decreasing function on $(0, \pi]$, and $g(\pi) = 1$. For $m_1 \geq \frac{\pi}{3}$ we have $\sin \frac{m_1}{2} \geq 1/2$ and we can rewrite (4.1) as

$$\frac{a}{b}\left(\frac{1}{4(n+1)}\right)^{\frac{1}{n}} < g(m_1)$$

which implies $m_1 \leq g^{-1}\left(\frac{a}{b}(\frac{1}{4(n+1)})^{\frac{1}{n}}\right)$ and, finally, for any ϵ and sufficiently large n, we obtain $m \leq (1+\epsilon)g^{-1}(\frac{a}{b})$. ∎

Proof of Lemma 9. We will need an elementary

LEMMA 10 *Let $A = \cup_{i=1}^{\infty} A_i$, A_i are measurable sets, $i \in \mathbb{N}$, and suppose every $x \in A$ belongs to no more than k different A_j's. Then $|A| \geq \frac{1}{k}\sum_{i=1}^{\infty}|A_i|$.*

Proof. We use induction in k. Set $B_j = A_j \cap (\cup_{i=1}^{j-1} A_i)$, $j = 2, \ldots,$ and write

$$\sum_{i=1}^{\infty}|A_i| = |A_1| + |A_2 \backslash A_1| + \cdots + |A_n \backslash \cup_{i=1}^{n-1} A_i| + \cdots + \sum_{i=2}^{\infty}|B_i| =$$

$$= |A| + \sum_{i=2}^{\infty}|B_i| \leq |A| + (k-1)|A| = k|A|$$

since if $x \in \cap_{j=1}^{k} B_{i_j}$, $2 \leq i_1 < i_2 < \cdots < i_k$, then x necessarily belongs to $\cap_{j=0}^{k} A_{i_j}$ for some $i_0 < i_1$, which is a contradiction. ∎

To complete the proof of Lemma 9 we let $A = [0, \delta]$ and set $A_i = (B - i\delta) \cap A$, $i \in \mathbb{Z}$. Assume the statement of the Lemma is false, which means every $x \in A$ belongs to no more than n different A_j's. Then, by Lemma 10,

$$\frac{m}{n} > \delta = |A| \geq |\cup_{i=-\infty}^{\infty} A_i| \geq \frac{1}{n}\sum_{i=-\infty}^{\infty}|A_i| = \frac{1}{n}\sum_{i=-\infty}^{\infty}|B \cap (A + i\delta)| = \frac{m}{n}$$

This contradiction proves the Lemma. ∎

5 Uniform distribution and small denominators. Proof of Lemma 7.

Let $\phi(n)$ be a monotone increasing function, $\phi(n) = o(n)$. We will say that $x_1, \ldots, x_n$ are $\phi-$uniformly distributed on $[a, b]$ if for any $f \in C([a, b])$ we have $|\sum_{i=1}^n f(x_i) - \frac{n}{b-a}\int_a^b f(x)dx| \le \phi(n)\mathrm{Var}f$.

Clearly, if $x_1, \ldots, x_n$ are $\phi-$uniformly distributed and $y_1, \ldots, y_n$ are $\psi-$uniformly distributed, then $x_1, \ldots, x_n, y_1, \ldots, y_n$ are $\phi(n)+\psi(n)-$uniformly distributed. We will say that $f \in C([a, b])$ is normal if it takes each value no more than p times (some $p = p(f)$), and the family $\{\ln(z - f(x))\}_{z\in[\min f, \max f]}$ is absolutely equicontinuous, by which we mean that for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|\int_{|z-f(x)|<\delta} \ln|z - f(x)|dx| < \epsilon/2 \text{ for all } z \in [\min f, \max f]. \tag{5.1}$$

It can be easily verified that every analytic function (in particular, $\cos 2\pi x$) is normal.

LEMMA 11 *Suppose $f(x)$ is normal, $\psi(n) = o(n), \phi(n) = o(n)$. Then for every $\epsilon > 0$ and n sufficiently large ($n > n_f(\epsilon, \phi, \psi)$)for any $x_1, \ldots, x_n$ that are $\phi-$uniformly distributed on $[a, b]$ and for any $j_1, \ldots, j_{\psi(n)} \in [1, \ldots, n]$ we have:*

$$\begin{aligned}
\sum_{\substack{i=1\\ i\neq j_1,\ldots,j_{\psi(n)}}}^{n} \ln|z - f(x_i)| &\le \frac{n}{b-a}\left(\int_a^b \ln|z - f(x)|dx + \epsilon\right)\\
\sum_{\substack{i=1\\ i\neq j_1,\ldots,j_{\psi(n)}}}^{n} \ln|z - f(x_i)| &\ge \frac{n}{b-a}\left(\int_a^b \ln|z - f(x)|dx - \epsilon\right) + \qquad (5.2)\\
&+ \phi(n)p(f)\ln \min_{\substack{i=1,\ldots,n\\ i\neq j_1,\ldots,j_{\psi(n)}}} |z - f(x_i)|
\end{aligned}$$

Proof. For $0 < \delta < 1$ put $g_\delta(z) = \begin{cases} \ln|z|, & |z| > \delta \\ \ln\delta, & |z| \le \delta. \end{cases}$ Given $\epsilon > 0$, fix $\delta(\epsilon)$ such that (5.1) is satisfied. Then, with $\delta = \delta(\epsilon)$, we obtain:

$$\sum_{\substack{i=1\\ i\neq j_1,\ldots,j_{\psi(n)}}}^{n} \ln|z - f(x_i)| \le \sum_{\substack{i=1\\ i\neq j_1,\ldots,j_{\psi(n)}}}^{n} g_\delta(|z - f(x_i)|) \le$$

$$\leq \frac{n}{b-a}\int_a^b g_\delta(|z-f(x)|)dx + \phi(n)\mathrm{Var} g_\delta(|z-f(x)|) - \psi(n)\ln\delta \leq$$

$$\leq \frac{n}{b-a}(\int_a^b \ln(|z-f(x)|)dx+\epsilon/2)+\phi(n)p(f)(-\ln\delta+\ln(\max_x|z-f(x)|))-\psi(n)\ln\delta \leq$$

$$\leq \frac{n}{b-a}(\int_a^b \ln(|z-f(x)|)dx+\epsilon), \text{ for } n > n_1(\epsilon,\phi,\psi,f).$$

Let $\min_{\substack{i=1,\ldots,n\\ i\neq j_1,\ldots,j_{\psi(n)}}} |z-f(x_i)| = c < 1$. For an estimate from below we write:

$$\sum_{\substack{i=1\\ i\neq j_1,\ldots,j_{\psi(n)}}}^{n} \ln|z-f(x_i)| = \sum_{\substack{i=1\\ i\neq j_1,\ldots,j_{\psi(n)}}}^{n} g_c(|z-f(x_i)|) \geq$$

$$\geq \frac{n}{b-a}\int_a^b g_c(|z-f(x)|)dx - \psi(n)\max_x \ln|z-f(x)| - \phi(n)\mathrm{Var}_x g_c(|z-f(x)|) \geq$$

$$\geq \frac{n}{b-a}(\int_a^b \ln|z-f(x)|dx-\epsilon)+\phi(n)p(f)\ln c, \text{ for } n > n_2(\epsilon,\phi,\psi) \ \blacksquare$$

Let $\{x\}$ denote the fractional part of x. We will use the following

LEMMA 12 *For any ω satisfying (1.2) and any $x \in \mathbb{R}$, points $\{x\}, \{x+\omega\}, \ldots, \{x+(n-1)\omega\}$ are $\phi-$uniformly distributed on $[0,1]$, with $\phi(n) = c_1(\omega)n^{1-r(\omega)^{-1}}\ln n$. Here $r(\omega)$ is given by (1.2).*

This is a slightly different form of a known statement (see, e.g., [33]). For the reader's convenience we provide a proof here:
Proof. Let p_n/q_n be the sequence of continued fraction approximants of ω. Let $n(k)$ be such that $q_{n(k)} \leq k < q_{n(k)+1}$. We will use r for $r(\omega)$. Writing $k = b_n q_n + b_{n-1}q_{n-1} + \cdots + b_1 q_1 + b_0$ and using the Denjoy-Koksma inequality (see, e.g., Lemma 4.1, Ch. 3 [9]), we get

$$\left|\sum_{j=0}^{k-1} f(\{\theta+j\omega\}) - k\int_0^1 f(\theta)d\theta\right| \leq (b_0+\cdots+b_n)\mathrm{Var}(f) \leq \left(\sum_{i=0}^{n}\left[\frac{q_{i+1}}{q_i}\right]\right)\mathrm{Var}(f). \tag{5.3}$$

Since (1.2) implies $q_{i+1} < \frac{q_i^r}{c}$, where $c = \frac{c(\omega)}{2\pi}$, we have $\frac{q_{i+1}}{q_i} < \frac{q_{i+1}}{(cq_{i+1})^{1/r}} = \frac{q_{i+1}^{1-1/r}}{c^{1/r}}$. The right-hand side of (5.3) can now be estimated as

$$\leq \left(c^{-1/r}\sum_{i=1}^{n(k)} q_i^{1-1/r} + \frac{k}{q_{n(k)}}\right)\mathrm{Var}(f) < \left(c^{-1/r}n(k)q_{n(k)}^{1-1/r} + \frac{k}{q_{n(k)}}\right)\mathrm{Var}(f).$$

Since $k < q_{n(k)+1} \leq \frac{q^r_{n(k)}}{c}$, we have $q_{n(k)} \geq (ck)^{1/r}$, and $\frac{k}{q_{n(k)}} \leq c^{-1/r}k^{1-1/r}$. Also, for any ω, $n(k) \leq \frac{2\ln q_n(k)}{\ln 2}$ (see, e.g. Ch. 4 [9]). Thus we can continue our estimate as

$$\leq (c_1 k^{1-1/r} \ln k)\mathrm{Var}(f). \ \blacksquare$$

Besides Lemma 12, the Diophantine properties of ω and the arithmetic properties of θ will only play a role in the following simple Lemma that allows us to estimate the small denominators in (3.2),

LEMMA 13 *Suppose $\theta \notin \Theta$, ω satisfies (1.2), then for k sufficiently large ($k > k_5(y_1, \theta, \omega)$),*

$$|\cos 2\pi\theta_j - \cos 2\pi\theta_\ell| \geq \exp\left(-(d + k/4 + 1)^{\frac{1}{2r(\omega)}}\right) \frac{c(\omega)}{(d + k + 1)^{r(\omega)}}. \tag{5.4}$$

Proof. Define $i_j \in \mathbb{Z}$ by $i_j = \frac{\theta_j - \theta}{\omega} - (x_1 + \frac{k-1}{2})$, $j = 0, \ldots, k$. Then

$$i_j - x_1 \in [[-3k/4], \ldots, [-k/4] + 1] \cap [d - [3k/4], \ldots, d - [k/4] + 1]. \tag{5.5}$$

We write

$$|\cos 2\pi\theta_j - \cos 2\pi\theta_\ell| = 2|\sin 2\pi(\theta + x_1\omega + \frac{k - 1 + i_j + i_\ell}{2}\omega) \sin 2\pi(i_j - i_\ell)\omega|. \tag{5.6}$$

Since $\theta \notin \Theta$ we have that for sufficiently large $j : j > k_6(\theta, \omega)$

$$|\sin 2\pi(\theta + j/2)\omega| > \exp(-j^{\frac{1}{2r(\omega)}}). \tag{5.7}$$

Therefore we can combine (1.2),(5.7),(5.5), and (5.6) to obtain the statement of the Lemma with $k_5(y_1, \theta, \omega) = k_6(\theta + y_1\omega, \omega)$. $\blacksquare$

Proof of Lemma 7. Set $\phi(n) = 2c_1(\omega)(\frac{n}{2})^{1-r(\omega)^{-1}} \ln \frac{n}{2}$. According to Lemma 12 and a remark in the beginning of this section points $\theta_0, \ldots, \theta_n$ are $\phi-$uniformly distributed on $[0, 1]$. Thus we can apply Lemma 11 with $f(x) = \cos x$, $\psi(n) = 1$, $\phi(n)$. The special property of the cosine that leads to spectral uniformity in energy (absence of mobility edge, in particular) is that

$$\int_0^1 \ln|z - \cos 2\pi x|dx = -\ln 2 \text{ for any } z \in [-1, 1] \tag{5.8}$$

The important part is that the integral in (5.8) is constant over z, not the actual value. In particular, it implies that $|I_1|$ and $|I_2|$ are of the same order. Precisely, we obtain, by Lemma 11, (5.8) and (5.4), that for k large enough, $k > \max\left(n_{\cos}(\frac{\epsilon}{3}, \phi(n), 1), k_5(y_1, \theta, \omega)\right)$, we have

$$|I_1| \leq \exp\left[(k+1)\left(-\ln 2 + \frac{\epsilon}{3}\right)\right] \tag{5.9}$$

and

$$|I_2| \geq \exp S \tag{5.10}$$

where

$$S = (k+1)\left(-\ln 2 - \frac{\epsilon}{3}\right) - 2\phi(k+1)\left((d + k/4 + 1)^{(2r(\omega))^{-1}} - \ln \frac{c(\omega)}{(d+k+1)^{r(\omega)}}\right).$$

Suppose $d \leq k^\alpha$, some $\alpha < 2$. Then we can write $|I| = \frac{|I_1|}{|I_2|} \leq \exp((k+1)\frac{2\epsilon}{3} + o(k))$ where $o(k)$ depends only on ω, α. This proves Lemma 7. ∎

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