Horseshoes with infinitely many branches and a characterization of Sinai-Ruelle-Bowen measures

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Abstract

Let f be a C^2 diffeomorphism of a compact riemannian manifold M^m and μ an ergodic f-invariant Borel probability with non zero Lyapunov exponents. We prove that μ is a Sinai-Ruelle-Bowen (SRB) measure if and only if we can reduce the dynamics on an invariant set of total measure to a horseshoe with infinitely many branches and variable return times. Also, and as a consequence of our approach we give a new proof of the well known Ledrappier-Young's characterization theorem [8] for hyperbolic SRB measures.

1 Introduction

A Borel probability μ preserved by a C^2 diffeomorphism of a compact riemannian manifold is a Sinai-Ruelle-Bowen (SRB) measure if it is smooth along the unstable lamination [7]. SRB measures are the best known examples of *physical measures* because they are *observable* in that their ergodic basin, the set of points $x \in M^m$ such that $1\,/\,n \sum_{k=0}^{n-1} \delta_{f^k(x)} \longrightarrow \mu$ vaguely, has positive volume [14]. A remarkable and well known theorem due to Ledrappier and Young says that μ is SRB if and only if it satisfies the Pesin entropy formula

$$h_\mu(f) = \int \sum_{\chi_i^+ > 0} \chi_i^+(x)\, dim\, E_i(x)\, d\mu(x). \tag{1}$$

See [7] and [8]. Recently, so called *horseshoes with infinitely many branches and variable return times* were introduced in [20] to provide an unified approach to several results on the existence of SRB measures and their statistical properties. This paper is aimed at proving that the presence of that generalized horseshoes in phase space is indeed a necessary and sufficient condition for the existence of SRB measures. The following is our main result.

Main Theorem *Let f be a C^2 diffeomorphism of a compact riemannian manifold M^m $(m \geq 2)$ and μ an ergodic hyperbolic Borel probability preserved by f. Then the following conditions are equivalent:*

1. *μ satisfies the Pesin entropy formula;*

2. *the conditional measures of μ respect to any measurable decomposition subordinated to the unstable Pesin's lamination are absolutely continuous respect to the riemannian volume on the leaves;*

3. *for every point $p \in supp\ \mu$ there is a regular neighborhood $\mathsf{R}(p)$ and a horseshoe with infinitely many branches and variable return times $\Omega^* \subset \mathsf{R}(p)$ induced by a family of suitable returns such that:*

   (a) $Leb_u(W^u_{loc}(x) \cap \Omega^*) > 0$ for μ-a.e. $x \in \Omega^*$ and

   (b) $\int N(x)\, dLeb_u(x) < +\infty$,

   where Leb_u is the Lebesgue measure class of $W^u_{loc}(x)$.

A somewhat detailed review of Pesin's theory will be needed to achieve these conclusions. That will lead us to a sidewalk through several classical results of the non-uniform hyperbolic systems theory in an unified setting including absolute continuity of the stable lamination, a proof of Pesin's entropy formula and Ledrappier-Young theorem on the characterization of SRB measures (cf. [7], [12]).

We note that the notion of a "horseshoe with infinitely many branches and variable return times" developed here is equivalent to that introduced in [20] although our presentation will be somewhat different at several points, as we shall see later. Also let us note that the f-invariant saturated of Ω^* has total measure since μ is ergodic, therefore we can describe the dynamics on this invariant set as a sort of "suspension" of the induced first return map to Ω^* by a return time function.

Historically, hyperbolic dynamical systems were the first examples where existence and statistical properties of SRB measures became well understood. See [9, Chapter III] and [5, Part 4]. However, to the best of my knowledge, in spite of several outstanding results appeared in the past decade proving the existence of SRB measures for some non hyperbolic systems, to give effective verifiable conditions for the existence of SRB mesures in systems which are hyperbolic in large parts of their phase space without being uniformly hyperbolic remains by large an open question.

The quadratic family and Hénon map are two of the most celebrated examples of non hyperbolic systems with SRB measures. In 1981 Jakobson proved the existence of a set of positive measure of quadratic maps exhibiting an absolutely continuous invariant measure by inducing a Markov transformation with infinitely many expanding branches and bounded distortion. For this he look at returns to suitable intervals for an special set of parameters of positive Lebesgue measure. Around 1984 Benedicks-Carlesson's work on the dynamics of the quadratic family opened a new perspective on the subject giving also a new proof of Jakobson's theorem. Later, further developments in this approach leads to the remarkable discovering that certain Hénon-like families display persistently an strange attractor for a positive set of parameters and to the proof that these invariant sets support ergodic SRB measures. Compare [3, Chapter V] and [19].

The idea to reduce Hénon attractors to a sort of topological Markov chain with countably many states has been around for some time. See for example [4]. However, as far as I know, no exposition of this fact seems to be yet available. On the other hand, as the present work suggests, new higher dimensional examples of non hyperbolic systems displaying persistently ergodic attractors with large unstable dimension may be found by inducing hyperbolicity and excluding parameters in higher dimensional settings for suitable families of endomorphisms and diffeomorphisms. For this we should investigate the concrete geometry of suitable returns to hyperbolic blocks, as Hénon and quadratic families examples shown.

In this paper we stand up in a broader, perhaps more conceptual set up, by considering the problem of inducing hyperbolicity in abstract non-uniformly hyperbolic systems. For this we took the approach of Katok-Mendoza's work [6] and [11] on the approximation of non-uniformly hyperbolic systems by sequences of large hyperbolic sets using, among other tools, pseudo-Markov property of returns to hyperbolic Pesin sets, bounded geometry of admissible manifolds, graph transforms properties, expansion along unstable admissible manifolds, volume estimates and tempered corrections due sub-exponential fluctuations along the orbits of the system.

We refer to next Section for definitions and statements.

The exposition is organized as follows. In Section 2 we make some definitions of dimension theory need for the statement of results, in Section 3 we prove Main Theorem using a technical result which shall be proved at the end, in Sections 5 and Section 6, after a review of Pesin's theory made at Section 4. In Section 5 we describe the topological picture of a horseshoe with infinitely many branches and variable return. Bounded distortion estimates follows from Pesin's theory and they are exposed at the end of Section 4. They are equivalent to Young's forward and backward distortion axioms but they are presented in a rather different way.

Acknowledgments:

Results in this paper arise from several attempts to generalize [11, Theorem 4.1 , Theorem 5.1] to higher dimensions (see below). I'm thankful to L. Mendoza for mentioned me this kind of problems.

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2 Statement of results

We shall combine inducing results with dimension theory of dynamical systems to investigate geometrical properties of hyperbolic measures. Indeed, let Ω^* be a horseshoe with infinitely many branches and variable return times. As we shall see later this set comes endowed with two invariant laminations $\mathcal{F}^s$ and $\mathcal{F}^u$, of local stable and unstable manifolds respectively endowing Ω^* with a hyperbolic product structure. Pseudo-Markov property permits to construct a nested sequence of stable cylinders $S_n(x)$ for every stable leave $\mathcal{F}^s(x)$ such that $\mathcal{F}^s(x) = \bigcap_{n>0} S_n(x)$. The reunion of all these geometrical cylinders forms a family $\wp^*$ which generates the *Borel stable sets on* R, that is, the Borel subsets which are reunion of stable leaves.

Given an admissible unstable manifold γ^u we define $\wp^*(\gamma^u) = \{P \cap \gamma^u : P \in \wp^*\}$. A countable family $\mathcal{U} = \{U_i\}_{i>0}$ $\mathcal{U} \subseteq \wp^*$ with $\operatorname{diam}(U_i) < \delta$ is called a *δ-covering by $\wp^*(\gamma^u)$-sets.* Now, given $a > 0$ we set :

$$\mathcal{D}_{a,\wp^*,\gamma^u}(X) = \lim_{\delta\to 0^+} \inf_{\mathcal{U}} \sum_{i=1}^{+\infty} \operatorname{Vol}_{\gamma^u}(U_i)^a, \tag{2}$$

where infimum is taken over δ-covering of $\wp^*(\gamma^u)$-sets. We get in this way a well defined and non-trivial measure class $\mathcal{D}_{\alpha,\gamma^u}$ and a dimensional characteristic $\dim_{\gamma^u,\wp}$ defined by

$$\dim_{\gamma^u,\wp^*}(X) = \inf\{\mathcal{D}_{\alpha,\gamma^u}(X) = 0\} = \sup\{\mathcal{D}_{\alpha,\gamma^u}(X) = +\infty\}. \tag{3}$$

We shall call these *the dynamical measure and the unstable dimension* respectively. They form a *dynamically defined Carathéodory's structure*, according to the terminology of [13, Chapter 1] and [13, Chapter 4]. The dynamical measure class generalizes to higher dimensions several well known features of the Hausdorff measure of conformal hyperbolic repellors. Even though, it is worth to mention that it does not reflect the geometry, but a dimensional property of the dynamics supported on these Cantor sets. See [13, Chapter 7].

The following result shows the connection between dimensional characteristic of the dynamics on a horseshoe with infinitely many branches, Pesin's entropy formula and smoothness of conditional measures along unstable leaves.

Theorem A *Let $0 < \rho < 1$ be given. Then, for every $p \in supp\ \mu$ there is a regular neighborhood Q and a compact set with hyperbolic product structure Ω^* contained in a slightly larger regular neighborhood $\mathsf{R} \supset \mathsf{Q}$ such that:*

1. *$\mu(\Omega^* \cap \mathsf{Q}) \geq (1-\rho)\mu(\mathsf{Q})$;*
2. *the unstable dimension of Ω^*, equals to $h_\mu(f) \,/ \int \log J^u f(x)\, d\mu(x)$;*
3. *the stable lamination $\mathcal{F}^s$ of Ω^* is absolutely continuous wrt the dynamical measure class and it has a bounded continuous Jacobian:*

$$\frac{d\,H^*\mathcal{D}_{\alpha,\overline{\gamma}^u}}{d\,\mathcal{D}_{\alpha,\gamma^u}}(x) = \left[\prod_{i=1}^{+\infty} \frac{J(T \mid \gamma^u)(T^i(x))}{J(T \mid \overline{\gamma}^u)(T^i(H(x)))}\right]^\alpha. \tag{4}$$

H is the holonomy map of the local stable lamination $\mathcal{F}^s$ defined by unstable admissible unstable sections $\gamma^u, \overline{\gamma}^u$, T is the induced first return map to Ω^* and $\alpha = \dim_u(\Omega^*)$ the unstable dimension;

4. *$\mu_{\mathcal{F}^u(z)}$, the projection of $\mu \mid \Omega^*$ along stable leaves onto an unstable manifold $\mathcal{F}^u(z)$, is equivalent to $\mathcal{D}_{\alpha,\mathcal{F}^u(z)}$;*

5. *$\mu(P) \asymp (Vol\ P)^{\dim_u(\Omega^*)}$, for every geometrical cylinder $P \in \wp^*$ bounded by a universal constant $C > 1$.*

Compare [16]. The projection $\mu_{\mathcal{F}^u(x)}$ is also called the transversal measure at x (compare [10]). This is a measure on the space of local stable leaves and it can be defined on the unstable Cantor sets defined by the intersection of a local unstable manifold with Ω^*:

$$\mu_{\mathcal{F}^u(x)}(A) = \mu\left(\bigcup_{z \in A} \mathcal{F}^s(z)\right) \qquad \text{for every Borel subset} \quad A \subset \mathcal{F}^u(x) \cap \Omega^*.$$

By the absolute continuity allows to define a transversal measure on sections $\gamma^u \cap \bigcup_{z \in \Omega^*} \mathcal{F}^s$ for every admissible unstable manifold $\gamma^u \in \Gamma^u$ (see below). In fact, $\dim_{\gamma^u,\wp^*}(\Omega^*)$ does not depend on γ^u, so unstable dimension $\dim_u(\Omega^*) = \dim_{\gamma^u,\wp^*}(\Omega^*)$ is well defined and we call it the *unstable dimension of* Ω^*. The above result also says that unstable Cantor sets are regular with respect to the dynamical measure class. This relates the smoothness properties along unstable leaves with dimension theory questions as we shall see in next Section.

The above result has an interesting geometrical interpretation for the case of surface diffeomorphisms.

Corollary B *Let f be a C^2 diffeomorphism of a compact surface preserving a hyperbolic ergodic measure μ and $\dim_{\mathcal{H}}$ the dimension of a set or a measure. Then :*

1. *the unstable dimension of Ω^* is the Hausdorff dimension of its unstable Cantor sets, i.e. $\dim_u(\Omega^*) = \dim_{\mathcal{H}}(\mu_{\mathcal{F}^u(x)})$ almost surely; also the Hausdorff dimension of Ω^* is the Hausdorff dimension of the measure μ: $\dim_{\mathcal{H}}(\Omega^*) = \dim_{\mathcal{H}}(\mu)$;*

2. *the stable lamination $\mathcal{F}^s$ of Ω^* is Lipschitz;*

3. *the transversal measure $\mu_{\mathcal{F}^u(x)}$, is equivalent to the Hausdorff measure bounded by uniform constants;*

4. *there is a uniform constant such that the following strong Frostman's distribution property holds:*

$$C^{-1} \leq \frac{\mu_{\mathcal{F}^u(x)}(B(z,r))}{r^{\dim_u(\Omega^*)}} \leq C \qquad \forall\ x \in \Omega^* \quad \text{and} \quad \forall\ z \in \Omega^* \cap \mathcal{F}^u(x).$$

In particular, the transversal measures are dimensionally exact.

We recall that $\dim_{\mathcal{H}}(\mu) = \inf\{\dim_{\mathcal{H}}(X) : \mu(X) = 1\}$ *is the Hausdorff dimension of a measure.* Young proved [18] that

$$\dim_{\mathcal{H}}(\mu) = h_\mu(f)\left[\frac{1}{\chi_\mu^+} - \frac{1}{\chi_\mu^-}\right]$$

for a hyperbolic ergodic measure preserved by a C^2 diffeomorphism of a compact surface. Similarly, Mendoza [10] proved that $\dim_{\mathcal{H}}(\mu_{\mathcal{F}^u(x)}) = h_\mu(f)\,/\,\chi_\mu^+$, where χ_μ^+ is the positive Lyapunov exponent of the measure. We may call $\dim_{\mathcal{H}}(\mu_{\mathcal{F}^u(x)})$ *the unstable dimension of* μ *and we denote it by* $\dim_u(\mu)$.

The above Corollary gives in fact an independent proof of Mendoza's formula, using the strong Frostman's distribution property stated above. First, the unstable dimension of Ω^* is equal to the unstable dimension of the measure, since

$$\dim_u(\Omega^*) = \lim_{r\to 0^+} \frac{\log\, \mu_{\mathcal{F}^u(x)}(B(z,r)\,\cap\,\mathcal{F}^u(x))}{\log\, r} = \dim_u(\mu),$$

almost surely. On the other hand, by Theorem A, $\dim_u(\Omega^*) = h_\mu(f)\,/\,\chi_\mu^+$, so proving Mendoza's formula. Similarly so $\dim_s(\Omega^*) = -h_\mu(f)\,/\,\chi_\mu^-$. Now we use the product structure in Ω^* and conclude $\dim_{\mathcal{H}}(\Omega^*) = h_\mu(f)[1\,/\,\chi_\mu^+ - 1\,/\,\chi_\mu^-]$. This relation gives a possible path to give also an independent proof of Young's formula. To prove this we have to show μ itself has some 'product structure' decomposition. Compare [12, Chapter 8, Section 26].

We conclude this Section with the following:

Corollary C *Let* μ *and* f *as above. Then, there exists a sequence of* f *horseshoes with finitely many branches* $\Omega_n^* \subset \Omega^*$ *such that:*

1. $\dim_u(\Omega_n^*) \longrightarrow \dim_u(\Omega^*)$,
2. *the Hausdorff measure of unstable Cantor sets* $\mathcal{H}_{\alpha(n)}(\Omega_n^* \cap \mathcal{F}^u(x))$ *is uniformly bounded and*
3. $C^{-1} \le \mathcal{H}_{\alpha(n)}(B(z,r)\cap\mathcal{F}^u(x))\,/\,r^{\alpha(n)} \le C$, *for* $\alpha(n) = \dim_u(\Omega_n^*)$, *for every* $0 < r < 1$ *and* $z \in \Omega^* \cap \mathcal{F}^u(x)$ *and a suitable constant* $C > 1$.

Taking the f-invariant saturate we get also sequences of hyperbolic (topologically) zero dimensional sets Ω_n such that:

- $\dim_u(\Omega_n) \longrightarrow \dim_u(\mu)$,
- $\mathcal{H}_{\alpha(n)}(\Omega_n \cap W^u(x)) \ge \gamma > 0$, for every $n > 0$ and $x \in \Omega_n$.

That gives an independent proof of [11, Theorem 4.1, Theorem 5.1], whose generalization to higher dimensions was the main motivation to the present investigation. Note the stronger control on the fractal geometrical properties of the sequence Ω_n^* contained in the section Ω^*. This will imply that their unstable Cantor sets coalesce into sets with positive Lebesgue measure if and only if

$\dim_u(\mu) = h_\mu(f) / \chi_\mu^+ = 1$. This 'condensation phenomena' of the horseshoes Ω_n^* in the unstable direction imply the observability and the absolute continuity of μ along the unstable leaves. We borrow this idea from [11, Theorem 5.1] and it will be generalized to a higher dimensional set up in next Section.

3 Proof of Main Theorem

We shall prove our main result using Theorem A. Items (1), (3), (4) and (5) in Theorem A will be proved in Section 6, after a review of Pesin's theory which shall give us the necessary bounded distortion estimates. So, let us start proving the following

Lemma 3.1.

$$\dim_u(\Omega^*) = \frac{h_\mu(f)}{\int \log\, J^u f(x)\, d\mu(x)} \tag{5}$$

Proof. Let $T : \Omega^* \circlearrowleft$ be the first return map According to Theorem A (5) for μ almost every $x \in \Omega^*$:

$$\dim_u(\Omega^*) = \lim_{n\to+\infty} \frac{\log\, \mu(S_n(x))}{\log\, \mathrm{Vol}\,(S_n(x))}.$$

Let μ^* be the restriction of μ to Ω^* normalized to a probability. μ^* is a T-invariant Borel probability. Using a local entropy approach we prove

$$\lim_{n\to+\infty} -1/n\, \log\, \mu^*(S_n(x)) = h_{\mu^*}(T), \qquad \mu - a.e.$$

On the other hand, by bounded distortion properties stated and proved in Section 3 and Section 5, we can find a constant $C > 1$ such that $C^{-1} \leq \mathrm{Vol}\,(S_n(x)) J^u T^n(z) \leq C$ for every $z \in S_n(x) \cap \mathcal{F}^u(x)$, where $J^u T$ is the unstable Jacobian of T: $J^u T = |det\, DT \mid E^u(x)|$ (see below). So, almost surely:

$$\lim_{n\to+\infty} -1/n\, \log\, \mathrm{Vol}\,(S_n(x)) = \int \log\, J^u T(z) d\mu^*(z).$$

On the other hand, let us define $R_n^* = \{x \in \Omega^* : N(x) = x\}$, where $N = N(x)$ is the first return time of x to Ω^*. Then, we have

$$\int_{\Omega^*} \log\, J^u T(z) d\mu^*(z) = \sum_{n=0}^{+\infty} \int_{R_n^*} \sum_{k=0}^{n-1} \log\, J^u f(f^k(z)) d\mu^*(z)$$

$$= \sum_{n=0}^{+\infty} \sum_{k=0}^{n-1} \int_{f^k(R_n^*)} \log\, J^u f(z) d\mu^*(z) = \int_{M^m} \log\, J^u f(z) d\mu^*(z).$$

By a theorem of Abramov [1, Theorem 1] we have $h_\mu(f) = \mu(\Omega^*) h_{\mu^*}(T)$. Therefore, the unstable dimension $\dim_u(\Omega^*)$ is equal to

$$\frac{h_{\mu^*}(T)}{\int \log\, J^u T(z) d\mu^*(z)} = \frac{\mu(\Omega^*) h_{\mu^*}(T)}{\int_{M^m} \log\, J^u f(z) d\mu(z)} = \frac{h_\mu(f)}{\int_{M^m} \log\, J^u f(z) d\mu(z)}.$$

$\square$

This kind of computations are due to P. Billingsley. See [2, pp. 136 - pp. 145]. In fact, they provide a method to calculate Hausdorff dimensions of certain dynamically defined sets and measures for surface diffeomorphisms.

Proof of Main Theorem
Let $T : \Omega^* \circlearrowleft$ be a C^2 generalized horseshoe induced in a regular neighborhood according to Theorem A. Then, the following statements are equivalent:

1. $\dim_u(\Omega^*) = 1$;
2. $\mathrm{Vol}_{\mathcal{F}^u}(\Omega \cap \mathcal{F}^u(x)) > 0$ for some local unstable manifold $\mathcal{F}^u(x)$;
3. the volume $\mathrm{Vol}_{\mathcal{F}^u}(\Omega)$ of the unstable Cantor sets $\Omega \cap \mathcal{F}^u(x)$ is uniformly bounded away from zero and $\mathcal{F}^s$ is absolutely continuous wrt Lebesgue with uniformly bounded Jacobians;
4. $\mu_{\mathcal{F}^u}$, the projection of μ onto $\mathcal{F}^u(x)$ is absolutely continuous wrt the riemannian volume along the local unstable manifolds.

In addition, if any of the above conditions hold the stable invariant lamination $\mathcal{F}^s$ is absolutely continuous wrt Lebesgue and it has a bounded Jacobian, namely

$$\frac{d\,H^*\mathcal{L}_{\overline{\gamma}^u}}{d\,\mathcal{L}_{\gamma^u}}(x) = \prod_{i=1}^{+\infty} \frac{J(T \mid \gamma^u)(T^i(x))}{J(T \mid \overline{\gamma}^u)(T^i(H(x)))} \tag{6}$$

for every pair of admissible unstable manifolds $\gamma^u, \overline{\gamma}^u \in \Gamma^u$. Here $\mathcal{L}_{\gamma^u}$ is the Lebesgue measure class of γ^u.

Compare [16, Theorem B]. Indeed, $\mathcal{D}_{1,\gamma^u}$ represents the Lebesgue measure class of γ^u. This is a straightforward consequence of the definitions and the fact that $\wp^*$ generates the stable lamination. In particular, $\mathrm{Vol}_{\gamma^u}(X) > 0$ implies $\dim_u(X) = 1$. Thus, $\dim_u(\Omega^*) = 1$ implies that μ_W, the projection of μ along $\mathcal{F}^s$ onto the local unstable manifolds, is equivalent to the restriction of the Lebesgue measure class Leb_u to $\mathcal{F}^u(x) \cap \Omega^*$, bounded by constants not depending on $x \in \Omega^*$ neither on $\mathcal{F}^u(x)$. In particular, the volume of the unstable Cantor sets $\mathcal{F}^u(x) \cap \Omega^*$ is uniformly bounded away from zero, using the absolute continuity of the lamination and that the Jacobians of the holonomy are uniformly bounded. On the other hand, if $Leb_u(\mathcal{F}^u(x) \cap \Omega^*) > 0$ for some local unstable manifold then $\dim_u(\Omega) = 1$, so the conditions are equivalent.

Now, observe that the family of local unstable manifolds $\{\mathcal{F}^u(x)\}_{x \in \Omega^*}$ defines a measurable partition and that $\mu_{\mathcal{F}^u(x)}$ are precisely the conditional measures of μ respect to that decomposition. On the other hand, according to Kac's formula $\int N(x) d\mu(x) = 1 \,/\, \mu(\Omega^*)$. Therefore, $N = N(x)$ is integrable with respect to $\mu_{\mathcal{F}^u(x)}$ almost surely. So, the return time is integrable respect to Leb_u, the measure class of $\mathcal{F}^u(x)$ for μ-a.e. $x \in \Omega^*$, since Leb_u is equivalent to $\mu_{\mathcal{F}^u(x)}$ almost surely.

Let ξ be a any measurable partition subordinated to the unstable lamination. We can suppose that ξ is increasing i.e. ξ refines $f^*\xi$ [8]. μ satisfies the Pesin entropy formula (1) if and only if $\dim_u(\Omega^*) = 1$. This is equivalent to $\mu_{\xi(z)}$ be absolutely continuous respect to the riemannian volume for almost every $z \in \Omega^*$. The set of point satisfying this condition is f-invariant since ξ is increasing, so it has total measure, by ergodicity , proving that μ is smooth along unstable leaves. Conversely, by definition μ is SRB then the conditional measures μ_ξ are absolutely continuous respect to any measurable decomposition ξ subordinated to the unstable lamination. So, let Ω^* be an induced generalized horseshoe. As Ω^* has positive measure then $\mu_{\xi(x)}(\Omega^*) > 0$ almost surely, so $Leb_u(\Omega^* \cap \mathcal{F}^(x)) > 0$ for μ-a.e. $x \in \Omega^*$, so $\dim_u(\Omega^*) = 1$ and we are done.

4 A review of Pesin's theory

In this Section we review some facts of Pesin's theory which shall used later. For this we follow Katok-Mendoza's notes [6]. Some additions and changes will be noted at the proper point.

Let us begin considering a *non-atomic hyperbolic ergodic Borel probability* μ preserved by a C^2 diffeomorphism on a compact riemannian manifold M^m. By Oseledec's theorem [6, Theorem S.2.9] there is a full-measure f-invariant subset Λ in M^m and an invariant splitting $T_\Lambda M^m = E^s \oplus E^u$ into expanding and contracting subspaces. The growth rate along these subbundles are bounded from below a constant $\lambda_\mu > 1$ depending on μ. Further, the functions $p \longmapsto E^u(p)$ (resp. $E^s(p)$) are Borel measurable and the angle $\angle(E^u(p), E^s(p))$ fluctuates subexponentially along the orbits.

As related to slowly fluctuations on some quantities defined along orbits of a non-uniformly hyperbolic system, the following terminology will be frequently used: *give* $\epsilon > 0$ *we say that a Borel measurable and positive function* K_ϵ *defined over* M^m *is* ϵ*-tempered if* $(1+\epsilon)^{-1} \le K(x) \,/\, K(f(x)) \le (1+\epsilon)$ *almost surely.* The Tempering Kernel Lemma [6, Lemma S.2.12] says that given a positive measurable function K such that $\lim_{n\to\infty} 1 \,/\, n\, log K(f^n(x)) = 0$, then for every $\epsilon > 0$ there is an ϵ-tempered function K_ϵ bounding K from above.

Pesin's ϵ*-reduction Theorem* says, roughly speaking, that any measurable linear co-cycle $A : X \longrightarrow GL(n, \mathbb{R})$ over a measurable transformation $F : X \circlearrowleft$ on a Lebesgue space (X, μ) such that $\log^+ \|A^{\pm 1}(x)\|$ is integrable is equivalent to a cocycle with values in Jordan-like canonical decompositions. See [6, Theorem S.2.10].

To reduce the cocyle A to a Lyapunov-Jordan block form it is sufficient to construct a measurable *Lyapunov metric* $p \longmapsto <,>_p^*$ which makes the system "uniformly hyperbolic" on a set of total measure. For this we use Lyapunov linear coordinates $C_\epsilon(p)$ are introduced such that $< C_\epsilon(p)v, C_\epsilon(p)w >_p^* \equiv < v, w >$ for every $v\,,\, w \in \mathbb{R}^m$, where $\|C_\epsilon\|$ fluctuates subexponentially along the orbits. Stable and unstable Oseledec's invariant decomposition are orthogonal with respect to $<,>^*$ in the new linear coordinate system and Df becomes uniformly hyperbolic. Further, there is an ϵ-tempered function L_ϵ such that $1 \,/\, \sqrt{2}\|u\|_p \le \|u\|_p^* \le L_\epsilon(p)\|u\|_p$. L_ϵ measures the linear distortion of the Lyapunov metric with respect to the riemannian metric $<,>$ in M^m.

After an ϵ-reduction we can introduce *Lyapunov charts of size* $r_\epsilon(p)$ around points in Λ. Namely, we seek for local parametrizations $\Psi_p : B(0,r_\epsilon(p)) \subseteq \mathbb{R}^m \longrightarrow M^m$ defined by $\Psi_p = \exp_p \circ C_\epsilon(p)$, where C_ϵ is the Lyapunov linear coordinate change and r_ϵ a tempered function. Then $\mathrm{dist}_{C^1}(f_p, A(p)) < \epsilon$, where A_p represents the linear part at p and f_p the local representative of the diffeomorphism in Lyapunov charts and the following estimate on the metric distortion holds:

$$K^{-1} \leq \frac{\|x-y\|}{d(\psi_p(x), \psi_p(x))} \leq D_\epsilon(p) \qquad \forall\, x, y \in B(0, r_\epsilon(p)), \tag{7}$$

for a suitable constant $K > 1$ and some tempered function D_ϵ. Compare [6, Theorem S.3.1]. We use Lyapunov charts to define *regular neighborhoods of size* h as follows: $\mathsf{R}(p,h) = \Psi_p(B^s(0,hr(p)) \oplus B^u(0,hr(p)))$, where $B^u(0,r) \subseteq \mathbb{R}^u$ and $B^s(0,r) \subseteq \mathbb{R}^s$ are the open balls of center 0 and radius $r > 0$ in the standard euclidean metric. A point admitting a regular neighborhood as above is called *completely regular*. The set of completely regular points of μ has full measure. Cf. [6, pp-673].

Given some $0 < \gamma < 1$ and some ϵ-reduction we endow regular neighborhoods with families of *stable and unstable cones of width* γ [5, Definition 6.2.9]. Namely, let $\mathsf{R}(p,h)$ be a regular neighborhood of size h and define $\mathsf{K}^u_\gamma(x)$ at every point in $\Psi_p(x) \in \mathsf{R}$ setting $\mathsf{K}^u_\gamma(x) \equiv D\Psi_p(x)\{(u,v) \in E^u \times E^s : \|v\| \leq \gamma\|u\|\}$. Likewise for the stable ones. With these cone families we define *admissible stable and unstable submanifolds of dispersion γ [6, Definition S.3.4], Γ^u and* Γ^s, respectively. $\gamma^u \in \Gamma^u$ is the image under the Lyapunov chart Ψ_p of a graph $\overline{\gamma}^u = Graph(\phi)$ of a C^1 map $\phi : B^u(0,r(p)) \longrightarrow B^s(0,r(p))$ whose tangent spaces lies inside the unstable cones in $\mathbb{R}^m$. In the terminology of [9, Lemma 13.4] γ^u is the image under Ψ_p of a *graph with dispersion γ in $\mathbb{R}^m$*. Likewise stable admissible submanifolds. We denote $\Gamma^u(p,h)$ the set of (u,γ)-admissible submanifolds in $\mathsf{R}(p,h)$, $\Gamma^u(p)$ the admissible unstable manifolds in $\mathsf{R}(p)$ and $\overline{\Gamma}^u(0,p)$ the space of graphs in $B^u(0,r(p)) \times B^s(0,r(p))$ with bounded dispersion. Similarly so their stable counterparts (cf. [6, pp-675]). Every $\gamma^u \in \Gamma^u$ and $\gamma^s \in \Gamma^s$ intersect transversally at a single point x with an angle $\angle(T_x\gamma^u, T_x\gamma^s)$ which is uniformly bounded from below. Notice that *the geometry of graphs with dispersion γ is bounded.* In fact, let $\overline{g}_{ij}$ be the riemannian metric of $\overline{\gamma}^u$, a graph with dispersion γ, then $\overline{g}_{ij}(x) = \delta_{ij} + o(\gamma^2)$. The property of bounded geometry of graphs with small dispersion is transferred to admissible manifolds $\gamma^u = \Psi_p(\overline{\gamma}^u)$ with slowly varying corrections along the orbit, due to properties of Lyapunov charts.

Lemma 4.1. *Let $\epsilon > 0$ be a positive number, p a completely regular point and $\mathsf{R}(p)$ a regular neighborhood, $\gamma^u \in \Gamma^u(p)$ and $\overline{\gamma}^u \in \overline{\Gamma}^u(0,p)$ such that $\gamma^u = \Psi_p(\overline{\gamma}^u)$. Then there is an ϵ-tempered positive function $K_0(p) = K_0(p,\gamma,\epsilon) > 1$ such that the ratios:*

$$\frac{\overline{g}_{ij}(x)}{g_{ij}(\Psi_p(x))}, \qquad \frac{\overline{V}_{\gamma^u}(x)}{V_{\gamma^u}(\Psi_p(x))}, \qquad \textit{and} \qquad \frac{dist_{\gamma^u}(\Psi_p(x), \Psi_p(y))}{\|x-y\|} \tag{8}$$

are bounded in the interval $[K_0^{-1}(p), K_0(p)]$ *for every* $x, y \in \overline{\gamma}^u$, *where* g_{ij} *is the intrinsic riemannian metric and* V_{γ^u} *the intrinsic volume given by the inmersion in* (M^m, g).

The above result is an example of the following useful **principle of tempered bounded geometry of admissible manifolds**: *properties proved for graphs with small dispersion* $\overline{\gamma}^u \in \overline{\Gamma}^u(0,p)$ *translate into properties of admissible manifolds in* M^m, $\gamma^u \in \Gamma^u(p,h)$, *with corrections bounded by tempered functions.*

Admissible submanifolds associated to a suitable ϵ-reduction are preserved by iterations under f if their dispersion is small enough and if they are near enough to a hyperbolic orbit.

Lemma 4.2. *There is a universal constant* $0 < \gamma_0 < 1$ *such that for every* $\gamma \in (0, \gamma_0]$ *there exists* $\epsilon = \epsilon(\gamma) \in (0,1)$ *depending on* γ *and an* ϵ*-reduction such that for every* $0 < h \leq 1$ *and for every* $\gamma^u \in \Gamma^u(p, h/2)$ *the image* $f(\gamma^u) \cap \mathrm{R}(f(p), h/2)$ *is an admissible unstable manifold with dispersion* $< \gamma$.

This gives a well defined *graph transform* $f_* : \Gamma^u(p, h/2) \longrightarrow \Gamma(f(p), h/2)$.

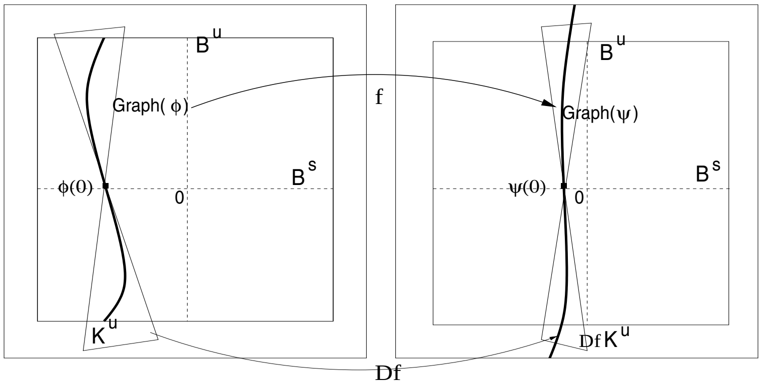

Figure 1: Geometry of Graph Transform

Lemma 4.3. *There exist* $\lambda > 1$ *depending on* λ_μ, γ *and* ϵ *such that for every* $\gamma^u \in \Gamma^u(p, h/2)$ *there is a disc* $\mathbb{D}^u \subset \gamma^u$ *such that the map* $f : \mathbb{D}^u \longrightarrow f_*\gamma^u$ is an expanding diffeomorphism *with a correction due the tempered bounded geometry of admissible manifolds, that is:*

$$\frac{d_u(f(x), f(y))}{K_1(p) d_u(x,y)} \geq \lambda > 1 \qquad \text{for every} \quad x \neq y \in \mathbb{D}^u. \tag{9}$$

Here $K_1 = K_1(p, \gamma, \epsilon) > 1$ *is a suitable tempered correction and* d_u *is the distance along admissible unstable manifolds given by the intrinsic riemannian metric. Further, the dispersion of* $f_*\gamma^u$ *is* $< \lambda^{-1}\gamma$.

Lemma 4.2 and Lemma 4.3 are straightforward consequences of the following result and from tempered bounded geometry principle.

Lemma 4.4. *Let $f : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ be a C^1 map satisfying the following conditions*

a) $f(0) = 0$*;*

b) there exists $\lambda > 1$ such that, if $f(u, v) = (Au + h_1(u, v), Bu + h_2(u, v))$ then $\|A\|, \|B^{-1}\| < \lambda^{-1}$.

Then there are real numbers $0 < \gamma_0$, $\epsilon_0 < 1$ such that for every $\gamma \in (0, \gamma_0]$ there exists $0 < \epsilon(\gamma) \leq \epsilon_0$ such that, whenever

1. $0 < \epsilon < \epsilon(\gamma)$, $\|D_z h\| < \epsilon$ *for every* $z \in \mathbb{R}^m$ *and*
2. $\lambda^{-1} < 1 - 4\epsilon < 1 < 1 + 4\epsilon < \lambda$,

the following holds: given $\phi \in C^1(B^u(\delta), B^s(\delta))$ with $\|D\phi\| \leq \gamma$ then

$$f(Graph(\phi)) = Graph(\psi)$$

and ψ is determined by the following set of conditions:

1. $f(\phi(w), w) = (\psi(w'), w')$ *where* $w' = \xi(w)$*, being* $\xi := B + h_2$*;*
2. ξ *is expanding with* $\|\xi(w) - \xi(w')\| \geq (1 + 2\epsilon)\|w - w'\|$*;*
3. $\|\psi(0)\| \leq (1 - 2\epsilon)\|\phi(0)\|$*;*
4. $\|D\psi\| \leq \lambda^{-1} \cdot \gamma$.

Moreover, there exists $\lambda = \lambda(\lambda(\mu), \gamma, \epsilon) > 1$ such that

$$\|f(w) - f(w')\| \geq \lambda\|w - w'\|$$

for every $w, w' \in Graph(\phi)$, i.e.

$$f \mid Graph(\phi) : \mathbb{D} \subseteq Graph(\phi) \longrightarrow Graph(\psi)$$

is λ-expanding and onto for some u-dimensional disc $\mathbb{D}$ contained in $Graph(\phi)$.

Remark 4.1. It is worth to note that the above estimations are somewhat different to those in [6, Proposition S.3.5]. We prove the above lemma by solving a system of elementary inequalities.

We endow $\Gamma^u(p, h)$ (resp. $\Gamma^s(p, h)$) with an structure of Banach space by identification with $C^1(B^u(0, 1), B^s(0, 1))$ with the C^1 norm. More precisely, we associate to every $\phi \in C^1(B^u(0, hr(p)), B^s(0, hr(p)))$ a function $\overline{\phi} \in C^1(B^u(0, 1), B^s(0, 1))$ with $\|D\overline{\phi}\| \leq \gamma$ by linearly rescaling its domain and rank: $\overline{\phi}(x) = (hr(p))^{-1}\phi(hr(p)x)$ (we can suppose that $0 < r(p) < 1$ almost surely). The following result, due to Aleeksev and Moser, says that f_*, the graph transform introduced above, is a contraction.

Lemma 4.5. (Aleeksev-Moser) *There exists* $0 < \theta < 1$ *only depending on the* ϵ*-reduction and* γ *such that*

$$dist_{C^1}(f_*(\gamma_1^u), f_*(\gamma_2^u)) \leq \theta \; dist_{C^1}(\gamma_1^u, \gamma_2^u). \tag{10}$$

Cf. [5, Lemma 6.2.17]. We underline that the contraction constant θ does not depend of the point where the graph transform is applied. By Luzin-Egoroff's theorem there are large hyperbolic non-invariant sets which shall be called *hyperbolic Pesin sets* (cf. [6, Theorem S.4.3]).

Lemma 4.6. *Given* $0 < \delta < 1$ *there exists a compact subset* Λ_δ *with* $\mu(\Lambda_\delta) \geq 1 - \delta$ *such that: (a) the functions* $C_\epsilon : M \longrightarrow GL(m, \mathbb{R})$, $r_\epsilon : M \longrightarrow (0, +\infty)$ *and* $L_\epsilon, D_\epsilon, K_\epsilon : M \longrightarrow (1, +\infty)$ *are continuous when restricted to* Λ_δ *and (b)* $T_pM = E^s(p) \oplus E^u(p)$ *is a continuous splitting over* Λ_δ*, where* $K = \max\{K_1, K_2\}$ *introduced in (8) and (9).*

We denote L_δ, D_δ the maximum of L_ϵ and D_ϵ respectively over Λ_δ and r_δ, K_δ the minimum of the respective functions.

Definition 4.1. We denote $\tilde{\mathsf{R}}(p, h) := \Psi_p(B^s(0, hr_\delta) \oplus B^u(0, hr_\delta))$ and call it the *h-reduced regular neighborhood* centered at p.

Lyapunov charts depend continuously on points in a given hyperbolic Pesin set (cf. [6, Proposition S.4.5]). Thus, although changes in the center of a Lyapunov coordinate system may increase the dispersion, this deformation can be controlled when p runs in a hyperbolic Pesin set. Namely,

Lemma 4.7. ([6, Corollary S.4.6]) *Let* Λ_δ *be given and* $0 < h < 1$*. Then, given* $1 < \beta \leq 2$ *there exists* $\kappa = \kappa(\delta, \beta, h) > 0$ *such that if* $p, , q \in \Lambda_\delta$, $d(p, q) < \kappa$ *and* $\gamma^u \in \Gamma^u(p, h)$ *then* $\Psi_p(\Psi_q^{-1}(\gamma^u)) \cap \tilde{\mathsf{R}}(q, h/2))$ *is part of an admissible* $(u, \beta\gamma)$*-submanifold near* p.

Definition 4.2. A compact connected subset subset $S \subseteq \mathsf{R}(p)$ is an *admissible stable cylinder* if it admits a foliation by admissible stable submanifolds and if its unstable sections $S \cap \gamma^u$ are convex sets for every $\gamma^u \in \Gamma^u(\mathsf{R})$. Unstable cylinders are defined similarly.

A map C^1 $T : S \longrightarrow U$ *between stable and unstable cylinders is hyperbolic* if it preserves strictly the cone families: $\mathsf{K}^s(T(x)) \subseteq \operatorname{int} DT(x)\mathsf{K}^s(x)$ and $DT(x)\mathsf{K}^u(x) \subseteq \operatorname{int} \mathsf{K}^u(T(x))$.

Lemma 4.8. (Hyperbolic Return Lemma) *Let* Λ_δ *be a hyperbolic Pesin set. Then, for every* $0 < h \leq 1$ *there exists* $\kappa = \kappa(\delta, h) > 0$ *with the following property.*

Given $p, q \in \Lambda_\delta$ *and* $x \in \Lambda_\delta$ *returning to* Λ_δ *between* m *iterates and such that* $d(x, p) < \kappa$ *and* $d(f^m(x), q) < \kappa$*, then there are stable and unstable cylinders* $S \subseteq \tilde{\mathsf{R}}(p, h)$ *and* $U \subseteq \tilde{\mathsf{R}}(q, h)$ *containing* x *and* $f^m(x)$ *respectively such that:*

1. $f^m : S \longrightarrow U$ *is hyperbolic;*

2. *diam* $(f^i(S)) \leq D_\delta \max\{diam(\tilde{\mathsf{R}}(p,h))\theta^i, diam(\tilde{\mathsf{R}}(f^m(p),h))\theta^{m-i}\}$ *for - every* $i = 0, \cdots, m$*;*

3. *for every* $\gamma^u \in \Gamma^u(p))$ *it holds that* $f^m : \gamma^u \cap S(x) \longrightarrow f_*^n \gamma^u$ *is an expanding diffeomorphism, that is:*

$$d_u(f^m(z), f^m(z')) \geq K_\delta \lambda^m \, d_u(z, z')$$

for every $z, z' \in \gamma^u \cap S(x)$*, being* d_u *the intrinsic riemannian metric along admissible unstable manifolds.*

$S(x)$ *is the connected component of* $\mathsf{R}(p,h) \cap f^{-m}(\mathsf{R}(q,h)$ *containing* x *and* $U(x)$ *is the the connected component of* $f^m(\mathsf{R}(p,h)) \cap \mathsf{R}(q,h)$ *containing* $f^m(S(x))$*.*

Previous Lemma explains what is the *pseudo-Markov property of hyperbolic Pesin sets covered by regular neighborhoods.* See [Definition S.4.15] and [Theorem S.4.16] in [6]. Hyperbolic Return Lemma is a consequence that expansion along admissible unstable overflows the slowly subexponential corrections of Lyapunov charts, when one return to a hyperbolic non-invariant Pesin set, so that small pieces of admissible unstable manifolds can be uniformly expanded onto admissible manifolds of uniformly bounded size in finitely many iterates.

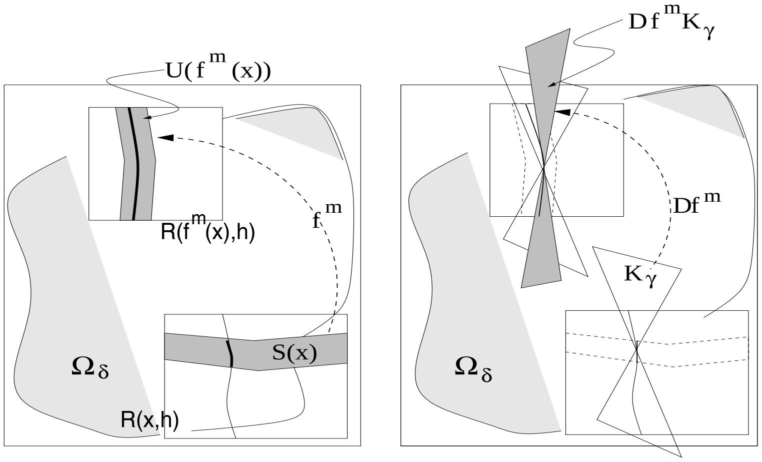

Figure 2: A hyperbolic return to a Pesin set

Now we recall for later use several well known results of Pesin's theory.

Katok's Closing Lemma *Let* Λ_δ *be a hyperbolic Pesin set and let* $\epsilon > 0$ *be given. Then there exists* $\beta = \beta(\epsilon, \delta) < \kappa$ *with the followinbg property: suppose that* x *and* $f^m(x)$ *belongs to* Λ_δ *and that* $dist(x, f^m(x)) < \beta$*, then there is a hyperbolic periodic point* $z \in M^m$ *such that* $dist(f^j(x), f^j(z)) < \epsilon$ *for* $j = 0, \cdots, m$*.*

Proof. (Outline) First we choose $0 < h < 1$ such that

$$\max\{\text{diam}\,(\tilde{\mathsf{R}}(p,h)), \text{diam}\,(\tilde{\mathsf{R}}(f^m(p),h))\} \leq \epsilon\,/\,2D_\delta$$

and $\beta < \kappa$ such that $B(p,\beta) \subset \tilde{\mathsf{R}}(p,h/2)$. Then, by Hyperbolic Return Lemma there is a nested sequence of unstable (resp. stable) cylinders defined inductively as follows: $U_{n+1} = f^m(S_0 \cap U_n)$ (resp. $S_{n+1} = f^{-m}(U_0 \cap S_n)$) for $n \geq 0$. Now, $W^u(z,\mathsf{R}) = \bigcap_{n\geq 1} U_n$ (resp. $W^u(z,\mathsf{R}) = \bigcap_{n\geq 1} U_n$) are invariant unstable (resp. stable) manifold of of period m and "size" R. That is, $W^u \subset f^m(W^u)$ and $f^m(W^s) \subset W^s$. Actually, f^m expands (resp. contracts) uniformly W^u (resp. W^s). This follows from the properties of the graph transform. Moreover, W^u and W^s intersect transversally at a unique point z, since they are admissible manifolds. Clearly z is a hyperbolic periodic point of period m, since W^s and W^u are invariant. This shows, in particular that $W^u = W^u(z,\mathsf{R})$ (resp. $(W^u = W^s(z,\mathsf{R}))$. Now, also using Hyperbolic Return Lemma we can prove that the orbit of z ϵ shadows the periodic pseudo-orbit $p, \cdots f^m(p)$. $\square$

Compare [6, Theorem S.4.13]. We use similar arguments to prove

Pesin's Stable Manifold Theorem *Given $\epsilon > 0$ there exist measurable positive tempered functions ρ and $\beta > 1$ such that for almost every point $p \in \Lambda_\delta$ there is a unique local invariant C^2 manifold*

$$W^s_{loc}(p) = \{q \in M^m :\ d(f^n(p), f^n(q)) \leq \rho(f^n(p))\ \ \forall\ n \geq 0\ \},$$

and such that $d(f^n(p), f^n(q)) \leq \beta(f^n(p))\lambda^{-n} d(p,q)$ for every $q \in W^s_{loc}(p)$ and $f(W^s_{loc}(p)) \subset W^s_{loc}(f(p))$.
A similar result holds true when we consider f^{-1} yielding local unstable manifolds.

Proof. (Outline: the C^1 part) Fix a hyperbolic Pesin set Λ_δ and $p \in \Lambda_\delta$ and consider consecutive first return times n_i, i.e. $f^{n_i}(p) \in \Lambda_\delta$ and $f^k(p) \notin \Lambda_\delta$ for $n_i < k < n_{i+1}$ with $n_0 = 0$. By Hyperbolic Return Lemma for every $i > 0$ there are cylinders $S_i \subset \tilde{\mathsf{R}}(p)$ and $U_i \subset \tilde{\mathsf{R}}(f^{n_i}(p))$ containing p and $f^{n_i}(p)$ respectively such that $f^{n_i} : S_i \longrightarrow U_i$ is hyperbolic. Further, $S_{i+1} \subseteq S_i$ so they form a nested sequence which converges to $W^s(p,\mathsf{R}) := \bigcap_{i=0}^{+\infty} S_i$ in the C^1 topology. Therefore, $W^s(p,\mathsf{R})$ is an admissible stable manifold in $\mathsf{R}(p)$. Moreover, $d(f^{n_i}(p), f^{n_i}(q)) \leq D_\delta \lambda^{-N_i} d(p,q)$ for every $q \in W^s(p,\mathsf{R})$, where $N_i = \sum_{j\leq i} n_j$. By graph transform properties $W^s(p,\mathsf{R}(p))$ is locally invariant. Moreover, there is a tempered function $\beta : \Lambda_\delta \longrightarrow (1,+\infty)$ such that $d(f^n(p), f^n(q)) \leq \beta(f^n(p))\lambda^{-n} d(p,q)$. Also notice that $p \in \Lambda_\delta \longmapsto W^s(p,\mathsf{R}(p))$ is continuous in the C^1 topology of admissible manifolds. Now we pass to the limit considering a disjoint sequence of hyperbolic Pesin sets Λ_i such that $\mu(\bigcup_{i=1}^n \Lambda_i) \longrightarrow 0^+$. We obtain in this way a measurable family of local stable manifolds passing by almost every point in M^m and subordinated to the regular neighborhoods provided by Lyapunov charts and Pesin's ϵ-reduction. $\square$

Compare [13]. The following is a sort of *non uniform volume distortion lemma.*

Lemma 4.9. *Let $0 < h < 1$ be given. Then, for every $\epsilon > 0$ there is an ϵ-tempered correction $K_2 > 1$ with the following properties:*

1. *let $\gamma^u \in \Gamma^u(p, h/2)$ be an admissible unstable manifold, $m > 0$ and $\mathbb{D}^u_m \subset \gamma^u$ such that $f^m : \mathbb{D}^u_m \longrightarrow f^m_* \gamma^u$ is an expanding map. Then,*

$$\left| \frac{Jac\,(f^n \mid \gamma^u)(z)}{Jac\,(f^n \mid \gamma^u)(z')} \right| \le \exp[K_2(f^n(p))\, d_u(f^n(z), f^n(z'))] \qquad \forall\ z, z' \in \mathbb{D}^u_m\,; \tag{11}$$

2. *let $\gamma^u(x), \gamma^u(y) \in \Gamma^u(p, h/2)$ be two admissible unstable manifolds passing by x and y and suppose that they are in the same local stable manifold. We define*

$$h(x, y) = \prod_{i=1}^{+\infty} \frac{Jac(f \mid \gamma^u(x))(f^i(x))}{Jac(f \mid \gamma^u(y))(f^i(y))}.$$

Then, $\log h(x, y) \le K_2(p)\, d_s(x, y)$ and for every $x' \in \gamma^u(x)$ and every $y' \in \gamma^u(y)$ whcih are in the same local stable manifold it holds

$$\log \frac{h(x, y)}{h(x', y')} \le K_2(p) \max\{d_u(x, x'), d_u(y, y')\}; \tag{12}$$

3. *for every $z, z' \in M^m$ resting in the same local unstable manifold we have*

$$\Delta(z, z') = \prod_{i=1}^{+\infty} \frac{J^u f(f^{-i}(z))}{J^u f(f^{-i}(z'))} \le \exp(K_2(p) d_u(z, z')). \tag{13}$$

Jacobians and distances d_u along unstable admissible manifolds are computed with the intrinsic riemannian metric.

Proof of (1): we shall prove part (1) considering graphs with small dispersion. Next we shall use tempered bounded geometry principle to draw our conclusions. For this we let $\gamma^u = Graph(\phi)$ a graph in $B^u(0, r(p)) \times B^s(0, r(p))$ whose dispersion is bounded by $0 < \gamma < 1$. We let also $f_n : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ be the representative of f in Lyapunov charts at $f^n(p)$, that is:

$$f_n(x, y) = \Psi^{-1}_{f^{n+1}(p)} \circ f \circ \Psi_{f^n(p)}(x, y)$$

whenever $(x, y) \in B^u(0, r(f^n(p))) \times B^s(0, r(f^n(p)))$. We can write $f_n(x, y) = (A^s_n \cdot x + \phi^s_n(x, y), A^u_n \cdot y + \phi^u_n(x, y))$, where A_n is the representative of the linear part is the Lyapunov coordinates. Now let $F_n = f_n \circ \cdots \circ f_0$. We have a well defined graph transform $(F_n)_* \gamma^u$ and a disc $\mathbb{D}^u_n \subset \gamma^u$ such that $F_n : \mathbb{D}^u_n \longrightarrow (F_n)_* \gamma^u$ is an expanding diffeomorphism. We can write the restriction of F_n to γ^u as $F_n(x) = F_n(x, \phi(x))$, for some $\mathbb{D}_n(0) \subset B^u(0, r(p))$. Therefore, $F_n(x) = L_n \cdot x + \psi_n(x)$, for a matrix L_n with $\|L_n\| \ge \lambda^n$ and ψ_n and a non-linear part such that $\|D\psi_n(x)\|$ is bounded from above by a constant only depending on γ, the expansion λ along admissible manifolds and the ϵ-reduction. Now, let $J_n(x) = det(L_n + D\psi_n(x))$ be the expansion coefficient of the volume under iterations by F_n, then:

$$\frac{J_n(x)}{J_n(y)} = \frac{det(Id + L_n^{-1}D\psi_n(x))}{det(Id + L_n^{-1}D\psi_n(x))} \leq 1 + C_1\|L_n^{-1}D\psi_n(x) - L_n^{-1}D\psi_n(y)\|,$$

for a suitable $C_1 = C_1(\gamma, \lambda)$. We can write $[f_n \mid \mathbb{D}_n^u](x) = M_n \cdot x + \xi_n(x)$ for a matrix M_n and a non linear perturbation ξ_n, for $x \in \mathbb{D}_n(0)$, so

$$F_n(x) = \prod_{k=0}^{n-1} M_{n-k} \cdot x + \sum_{i=1}^{n-1} \prod_{k=0}^{n-i-1} M_{n-k} \cdot \xi_{i+1} \circ F_i(x).$$

Thus, $\|L_n^{-1}(D\psi_n(x) - D\psi_n(y)\| \leq \sum_{i=1}^{n-1} \lambda^{-n}\lambda^{n-i-2}\gamma\|F_i(x) - F_i(y)\|$ and then

$$\begin{aligned}\|L_n^{-1}(D\psi_n(x) - D\psi_n(y)\| &\leq \sum_{i=1}^{n-1} \lambda^{-(i+2)}\gamma\lambda^{-(n-i)}\|F_n(x) - F_n(y)\| \\ &\leq \gamma n \lambda^{-(n+2)}\|F_n(x) - F_n(y)\| \leq C_2(\gamma, \lambda)\|F_n(x) - F_n(y)\|,\end{aligned}$$

for a suitable constant C_2. From this we conclude

$$\frac{J_n(x)}{J_n(y)} \leq \exp(K'(f^n(p))d_u(F^n(x), F^n(y)), \tag{14}$$

where $K' = K'(p)$ is a correction due to the tempered bounded geometry of small dispersion graphs.

Proof of (2): take a point p in a hyperbolic Pesin set Λ_δ and consider the continuous lamination of stable local manifolds passing by $x \in \mathsf{R}(p) \cap \Lambda_\delta$. We define, for every $x \in \mathsf{R}(p) \cap \Lambda_\delta$, a set $\mathcal{W}(x)$ formed by the u-planes $E \subset \mathsf{K}^u(y)$ contained in the unstable cones at points $y \in W^s(x, \mathsf{R}(p))$. We introduce $\mathbf{W}(p, \delta)$ the reunion of all $\mathcal{W}(x)$ passing by points $x \in \mathsf{R}(p) \cap \Lambda_\delta$. This is a compact subset of $\mathbf{G}^u(p)$ the Grassmannian of u planes passing by points in $\mathsf{R}(p)$. Now let us consider $J(E) = det|Df \mid E|$. In fact, $J : \mathbf{G}^u(p) \longrightarrow \mathbb{R}$ is a C^1 smooth function so its derivative DJ is bounded and we can define $K''(p) = \sup\{\|D\log J(E)\| : E \in \mathbf{G}^u(p)\}$. Further, K'' is a tempered function, i.e. $\lim_{n\to+\infty} 1/n \log K''(f^n(p)) = 0$. This can be seen as follows: let γ^u an admissible unstable manifold tangent to $E \in \mathbf{G}^u(p)$. Using Lyapunov local coordinates we can check that $J(E)$ comparable to $\sqrt{det < \partial_{x_i}\alpha, \partial_{x_j}\alpha >}$, bounded by a tempered correction, where $\alpha(x) = (x, \phi(x))$ is the natural parametrization of $\overline{\gamma}^u$ the small dispersion graph which represents γ^u in these coordinates. Observe also that $< \partial_{x_i}\alpha, \partial_{x_j}\alpha >= \delta_{ij}+ < \partial_{x_i}\phi, \partial_{x_j}\phi >$. A further elementary computation shows that $\|D\log J(E)\|$ is comparable with $r(p)$, the radius of the Lyapunov chart, bounded by a tempered correction.

Now, by graph transform contraction property, there is $0 < \theta < 1$ such that $d(f_*E, f_*E') \leq \theta\, d(E, E')$ for every $E, E' \in \mathcal{W}(x)$. So, $|J(f_*E) - J(f_*E')| \leq$

$K''(f(p))d(f_*E, f_*E') \leq K''(f(p))\theta\, d(E,E')$.Hence,

$$\log \prod_{i=1}^{+\infty} \frac{Jac(f \mid \gamma^u(x))(f^i(x))}{Jac(f \mid \gamma^u(y))(f^i(y))} \leq \sum_{i=1}^{+\infty} K''(f^i(p))\theta^n\, d(T_x\gamma^u(x), T_y\gamma^u(y)).$$

Take $\epsilon > 0$ such that $(1+\epsilon)\theta \leq \theta_0 < 1$ and K''_ϵ an ϵ-tempered upper bound function for K''. We conclude that $\log h(x,y) \leq K'''(p)\theta_0(1-\theta_0)^{-1}d_s(x,y)$, for some tempered correction K'''. Here d_s is, as usual, the distance along the local stable manifold W^s_{loc} passing by x and y with the intrinsic riemannian metric. Then, $\log h(x,y) \leq K'''(p)\theta_0(1-\theta_0)^{-1}\, d_s(x,y)$ and

$$\begin{aligned} \log \frac{h(x,y)}{h(x',y')} &\leq K'''(p)\theta_0(1-\theta_0)^{-1}[d_s(x,y) - d_s(x',y')] \\ &\leq K'''(p)\theta_0(1-\theta_0)^{-1}\max\{d_u(x,x'), d_u(y,y')\}, \end{aligned} \tag{15}$$

using triangular inequality.

To prove (3) we argue as we did above using the fact that W^u_{loc} are backward invariant C^1 submanifolds. Compare [7, Proposition 3.1]. To complete the proof of the above lemma we take the maximum of tempered corrections at (14) and (15). QED

Lemma 4.10. *There is a tempered function $K_3 = K_3(p,\epsilon,\gamma) > 1$ such that for every stable cylinder $S \subset \mathsf{R}(p) \cap f^{-n}\mathsf{R}(f^n(p))$ it holds*

1. *$K_3^{-1} \leq Vol_{\gamma^u}(S)Jac(f^n \mid \gamma^u)(x) \leq K_3$, for every $\gamma^u \in \Gamma^u(p,h/2)$, $n > 0$ and $x \in \gamma^u \cap S$;*
2. *$K_3^{-1} \leq Vol_{\gamma^u(x)}(S) \,/\, Vol_{\gamma^u(y)}(S) \leq K_3$ for any pair of unstable admissible manifolds $\gamma^u(x)$ and $\gamma^u(y)$ passing by points $x, y \in S$ in the same stable local manifold, i.e. $W^s(x,\mathsf{R}) = W^s(x,\mathsf{R})$;*
3. *$K_3^{-1} \leq Vol(S)Jac(f^n \mid \gamma^u)(x) \leq K_3$ for every $\gamma^u \in \Gamma^u(p,h/2)$, $x \in S \cap \gamma^u$.*

Proof. As $f^n : S \cap \gamma^u \longrightarrow f^n_*\gamma^u$ is onto, for any $\gamma^u \in \Gamma^u(p)$ so we get

$$\inf_{\gamma^u \in \Gamma^u(f^n(p))} \mathrm{Vol}_u(\gamma^u) \leq \mathrm{Vol}_{\gamma^u}(f^n(S \cap \gamma^u)) \leq \sup_{\gamma^u \in \Gamma^u(f^n(p))} \mathrm{Vol}_u(\gamma^u).$$

Also, by the above lemma, the ratios

$$\frac{\int_{S\cap\gamma^u} Jac(f^n \mid \gamma^u)(z) d\mathrm{Vol}_{\gamma^u}(z)}{\mathrm{Vol}_{\gamma^u}(S)Jac(f^n \mid \gamma^u)(x)}$$

are bounded between

$$K_2(f^n(p))^{-1}\mathrm{diam}_{\gamma^u}(\mathsf{R}(f^n(p)) \quad \text{and} \quad K_2(f^n(p))\mathrm{diam}_{\gamma^u}(\mathsf{R}(f^n(p)).$$

Hence, the following tempered function is an upper bound for the products $\mathrm{Vol}_{\gamma^u}(S) Jac(f^n \mid \gamma^u)(x)$:

$$K_3' = K_2(f^n(p)) \mathrm{diam}_{\gamma^u}(\mathsf{R}(f^n(p)) \frac{\sup_{\gamma^u \in \Gamma^u(f^n(p))} \mathrm{Vol}_u(\gamma^u)}{\inf_{\gamma^u \in \Gamma^u(f^n(p))} \mathrm{Vol}_u(\gamma^u)}.$$

A similar computation gives a tempered lower bound. To prove (2) we first observe that for every $x, y \in S$ in the same stable local manifold, the ratios $\mathrm{Vol}_{\gamma^u(x)}(S) \,/\, \mathrm{Vol}_{\gamma^u(y)}(S)$ are bounded in the interval

$$\frac{Jac(f^n \mid \gamma^u)(y)}{Jac(f^n \mid \gamma^u)(x)} \, [\, (K_3')^{-1}(p) \, , \, K_3'(p) \,],$$

and that, by Lemma 4.9(2) we see that

$$\frac{Jac(f^n \mid \gamma^u)(y)}{Jac(f^n \mid \gamma^u)(x)} \leq h(x,y) \leq K_2(p) \, \mathrm{diam}_s(\mathsf{R}(p)),$$

where diam_s is the diameter along the stable manifold $W^s(x, \mathsf{R})$ computed with the intrinsic riemannian metric. (3) is proved in a similar way observing that $\mathrm{Vol}\,(S)$ is comparable with the volume of unstable sections $S \cap \gamma^u$ up to tempered corrections. K_3 is the maximum of the above corrections. $\square$

We collect in a single tempered function $K = K(p, \epsilon, \gamma)$ the several corrections introduced along this Section. That gives us a control on the geometry and forward and backward volume distortion on iterates of admissible and locally invariant unstable manifolds through a single tempered correction parameter.

Lemma 4.11. (Bounded Distortion Lemma) *For any given δ there is a hyperbolic Pesin set Λ_δ and $K_\delta > 1$ such that or every $p \in \Lambda_\delta$:*

1. *for every $\gamma^u \in \Gamma^u(p, h/2))$ and $m > 0$ such that $f^m(p) \in \Lambda_\delta$*

   $$\log \left| \frac{Jac\,(f^n \mid \gamma^u)(z)}{Jac\,(f^n \mid \gamma^u)(z')} \right| \leq K_\delta \, d_u(f^n(z), f^n(z')),$$

   for every $z, z' \in \mathbb{D}^u_m$, where $\mathbb{D}^u_m \subset \gamma^u$ is such that $f^m : \mathbb{D}^u_m \longrightarrow f^m_* \gamma^u$ is an expanding map;

2. *$\log h(x,y) \leq K_\delta \, d_s(x,y)$, for every $\gamma^u(x), \gamma^u(y) \in \Gamma^u(p,h)$ such that $W^s(x, \mathsf{R}(p)) = W^s(y, \mathsf{R}(p))$. Also, for every $x' \in \gamma^u(x)$ and $y' \in \gamma^u(y)$ such that $W^s(x', \mathsf{R}(p,h)) \cap \gamma^u(y)$ it holds that*

   $$\log \frac{h(x,y)}{h(x',y')} \leq K_\delta \max\{d_u(x,x'), d_u(y,y')\};$$

3. *$\log \Delta(z,z') \leq K_\delta d_u(z,z')$;*

4. *let $S \subset \mathsf{R}(p) \cap f^{-n} \mathsf{R}(f^n(p))$ be an stable cylinder produced some hyperbolic return, then the have the following bounds:*

(a) $K_\delta^{-1} \leq Vol_{\gamma^u}(S) Jac(f^n \mid \gamma^u)(x) \leq K_\delta$*;*

(b) $K_\delta^{-1} \leq Vol \overline{\gamma}^u(S) \,/\, Vol_{\gamma^u}(S) \leq K_\delta$*;*

(c) $K_\delta^{-1} \leq Vol(S) \,/\, Vol_{\gamma^u}(S) \leq K_\delta$ *and*

(d) $K_\delta^{-1} \leq Vol(S) Jac(f^n \mid \gamma^u)(x) \leq K_\delta$

for every pair of admissible unstable sections $\gamma^u, \overline{\gamma}^u \in \Gamma^u(p)$*,* $n > 0$ *and* $x \in \mathbb{D}_n^u = \gamma^u \cap S$ *a disc such that* $f^n : \mathbb{D}_n^u \longrightarrow f_*^n \gamma^u$ *is an expanding diffeomorphism.*

5 Horseshoes: a topological picture

We fix throughout this Section a hyperbolic Pesin set Λ_δ for an ergodic hyperbolic Borel probability μ, $0 < h < 1$, $p \in \Lambda_\delta$ a point in the support of the measure μ and $\tilde{\mathsf{R}}(p,h)$ an h-reduced regular neighborhood. Let also $h' = h'(\kappa)$ be such that $\tilde{\mathsf{R}}(p,h') \subseteq \tilde{\mathsf{R}}(p,h)$ and diam $\tilde{\mathsf{R}}(p,h') < \kappa$. These regular neighborhoods will be held fixed and shall denoted by $\mathsf{R} = \tilde{\mathsf{R}}(p,h)$ and $\mathsf{Q} = \tilde{\mathsf{R}}(p,h')$, respectively. Now, let $0 < \rho < 1$ and choose $\delta > 0$ and $0 < h < 1$ such that $\mu(\Lambda_\delta \cap \mathsf{Q}) \geq (1-\rho)\mu(\mathsf{Q})$.

Proposition 5.1. *There is a maximal compact set* $\Omega^* \subseteq \mathsf{R}$ *such that* $\mu(\Omega^*) \geq (1-\rho)\mu(\mathsf{Q})$ *with a* hyperbolic product structure*, that is:*

1. *there are two continuous laminations* $\mathcal{F}^s$ *and* $\mathcal{F}^u$ *such that* dim $\mathcal{F}^s(x)$ *and* dim $\mathcal{F}^u(x)$ *are admissible manifolds for every* $x \in \Omega^*$

2. $\mathcal{F}^u(f^n(x)) \subset f^n(\mathcal{F}^u(x))$ *and* $f^n(\mathcal{F}^s(x)) \subset \mathcal{F}^s(f^n(x))$ *whenever* $x \in \Omega^*$ *and* $f^n(x) \in \Omega^*$*; further,* $d(f^n(p), f^n(q)) \longrightarrow 0^+$ *exponentially fast for every pair of points* $p, q \in \mathcal{F}^s(x)$*,* $x \in \Omega^*$*; likewise for backward orbits of points in* $\mathcal{F}^u$*-leaves;*

3. $\mathcal{F}^s(x)$ *intersects* $\mathcal{F}^u(y)$ *transversally at a unique point with angle uniformly bounded away from zero, for every* x *and* y *in* Ω^**;*

4. $\Omega^* = \bigcup \mathcal{F}^s \cap \bigcup \mathcal{F}^u$*.*

Lemma 5.1. *There is a countable dense subset of hyperbolic periodic points* $\{z_i\}$ *such that* $\mathsf{Q} \cap \Lambda_\delta \subseteq \overline{\{z_1, z_2, \cdots\}}$

Proof. Let $\wp_n$ be a sequence of decompositions $\wp_n$ of Q into compact sets with non overlapping interiors such that diam $(\wp_n) \longrightarrow 0^+$ as n goes to infinite (take for example images by Lyapunov charts of decompositions into dyadical cubes in $\mathbb{R}^m$). Using Katok's closing lemma argument we find a hyperbolic periodic point $z \in \wp_n(x)$, for every $x \in \mathsf{Q} \cap \Lambda_\delta$ and $n > 0$ such that $\mu(\wp_n(x)) > 0$. Let $\{z_1, z_2, \cdots\}$ be the set of hyperbolic periodic points obtained in this way, indexed in increasing order according to their prime period. Clearly $\{z_1, z_2, \cdots\}$ is dense in $\mathsf{Q} \cap \Lambda_\delta$. $\square$

Lemma 5.2. *Let $E \subseteq \mathsf{Q}$ be a finite subset of hyperbolic periodic points. Then, there exists a finite disjoint collection of cylinders $\{S(z)\}_{z\in E}$ and $\{U(z)\}_{z\in E}$ and hyperbolic branches $T_z : S(z) \longrightarrow U(z)$ generating a hyperbolic horseshoe $\Lambda^*(E)$ which contains E.*

Proof. By Hyperbolic Return Lemma there exists for each $z \in E$ a nested sequence of stable cylinders converging to $W^s(z, \mathsf{R})$, the local stable manifold of z. Indeed, $W^s(z,\mathsf{R}) = \bigcap_{k>0} f^{-km(z)} S_0(z)$, where $f^{m(z)} : S_0(z) \longrightarrow U_0(z)$ is provided by the pseudo-Markov property in the Hyperbolic Return Lemma. Then we can choose a finite disjoint collection of stable cylinders $\{S(z)\}_{z\in E}$ mapped hyperbolically to a finite disjoint collection of unstable cylinders, namely: for each $z \in E$ there exists $k(z) > 0$ such that $\{S(z)\}_{z\in E}$ is a disjoint family where $S(z) = f^{-kn(z)} S_0(z)$ and $n(z)$ is the prime period of z. In addition, $f^{2kn(z)} : S(z) \longrightarrow U(z)$ is hyperbolic. Then, $\Lambda^*(E) = \bigcap_{n\in\mathbb{Z}} T^{-n}(\bigcup_{z\in E} S(z))$ is a hyperbolic horseshoe with finitely many branches as claimed. □

Proof of Proposition 5.1
Given $n > 0$ one construct a horseshoe with finitely many branches Ω^*_n such that $\{z_1, \cdots, z_n\} \subseteq \Omega^*_n$. $\overline{\bigcup_n \Omega^*_n}$ satisfies conditions (1), (2), (3) and (4) in the proposition. In fact, every $\mathcal{F}^s$-leave (resp.$\mathcal{F}^u$-leave) is the limit, in the C^1-topology, of a suitable sequence of local stable manifolds of points in $\bigcup_n \Omega^*_n$. Hyperbolic return lemma and properties of the graph transform implies that these local stable manifolds have the required properties since they are admissible. Similarly so unstable leaves.

A simple verification shows that the reunion of an increasing sequence of hyperbolic product structures defined by admissible manifolds in R is again a hyperbolic product structure. Therefore, there is a maximal member in the collection of all hyperbolic product structures. QED.

Ω^* is the homoclinic closure of all transversal homoclinic points related to points x at the hyperbolic Pesin set Λ_δ whose local invariant manifolds passing by R are admissible.

The following Lemma describes Ω^* as a horseshoe with infinitely many hyperbolic branches.

Lemma 5.3. *There is a countable collection of disjoint sets $\Omega^*_i \subseteq \Omega^*$ such that:*

1. *$\mu(\Omega^* - \bigcup_i \Omega^*_i) = 0$;*
2. *for each Ω^*_i there is a return time n_i such that $f^{n_i}(\Omega^*_i) \subseteq \Omega^*$; further, for every $n > 0$, $\#\{i : n_i = n\}$ is finite;*
3. *for every $x \in \Omega^*_i$ we have $W^s(x,\mathsf{R}) \cap \Omega^* \subseteq \Omega^*_i$ and $W^u(f^{n_i}(x),\mathsf{R}) \cap \Omega^* \subseteq f^{n_i}(\Omega^*_i)$, that is, Ω^*_i is saturated in the stable direction and $f^{n_i}(\Omega^*_i)$ is saturated in the unstable direction.*

Proof. As $\mu(\Omega^*) > 0$ we can decompose it into disjoint subsets according to their first past return time to Ω^*, namely: R_n is the set of points $x \in \Omega^*$ such that

$f^{-n}(x) \in \Omega^*$ and $f^{-k}(x) \notin \Omega^*$ for $0 < k < n$. Each R_n decomposes into a finite family of disjoint subsets saturated in the stable direction. Indeed, let $\Omega^s = \bigcup_{x\in\Omega^*} \mathcal{F}^s(x)$ be the reunion of all the stable leaves passing by points in Ω^* and let $n > 0$ a return time. Then $f^{-n}\Omega^s$ is a compact set and is a disjoint reunion of stable leaves, since $f^{-n}\mathcal{F}^s(x)$ overflows $\mathcal{F}^s(f^{-n}x)$. In fact, as $f^{-n}\mathsf{R} \cap \mathsf{R}$ decomposes into finitely many connected components, say S_i^n, $i = 1, \cdots, N_n$, which are stable cylinders and such that f^n maps S_i^n hyperbolically onto an unstable cylinder we can define $\Omega_i^* = S_i^n \cap \Omega^*$ concluding that $f^{-n}\Omega^s \cap \Omega^* = \bigcup_{i=1}^{N_n} \Omega_i$. $\square$

The above subsets Ω_i^* and $f^{n_i}(\Omega_i^*)$ are called s-sets and u-sets respectively, according to Young's terminology. They provide a decomposition of the hyperbolic product structure in Ω^* satisfying axioms **(P1)** and **(P2)** in [20]. We also notice for later use that $\Omega_n^* = \bigcup_{n_i \le n} \Omega_i$ is an ordinary horseshoe with finitely many hyperbolic branches. So we have the following

Corollary 5.1. *There is a sequence of horseshoes with finitely many branches $\Omega_n^* \subseteq \Omega^*$ such that $\Omega^* = \overline{\bigcup_n \Omega_n^*}$.*

6 Proof of Theorem A

As Ω^* has positive measure we can induce a first return map $T : \Omega \circlearrowleft$, defined by $T(x) = f^{N(x)}(x)$ where $f^k(x) \notin \Omega^*$ for $0 < k < N(x)$. T extends to the invariant laminations. Further, for every stable leave $\mathcal{F}^s(x)$, $x \in \Omega^*$ there is a nested sequence of stable cylinders $S_n(x) \subset \mathsf{R}$ such that $\mathcal{F}^s(x) = \bigcap_{n>0} S_n(x)$. The reunion of these stable cylinders forms a family $\wp^*$ which generates the Borel stable sets on R. So, given an admissible unstable manifold γ^u we define a C-structure as we mentioned at the Introduction.

Proposition 6.1. *Let $\Omega^* \subset \mathsf{R}$ be a maximal subset with hyperbolic product structure constructed as before. Then, the following holds:*

1. *the stable lamination $\mathcal{F}^s$ is absolutely continuous wrt the measure classes $\{\mathcal{D}_{\alpha,\gamma^u}\}_{\gamma^u\in\Gamma^u}$ and it has a continuous and bounded Jacobian; further,*

$$\frac{d\, H^*\mathcal{D}_{\alpha,\overline{\gamma}^u}}{d\,\mathcal{D}_{\alpha,\gamma^u}}(x) = \left[\prod_{i=1}^{+\infty} \frac{J(T \mid \gamma^u)(T^i(x))}{J(T \mid \overline{\gamma}^u)(T^i(H(x)))}\right]^\alpha, \tag{16}$$

   where $H_{\gamma^u,\overline{\gamma}^u}$ is the holonomy map of the local stable lamination $\mathcal{F}^s$ defined by unstable admissible unstable sections $\gamma^u, \overline{\gamma}^u \in \Gamma^u(\mathsf{R})$;

2. *there is a unique ergodic T invariant measure μ^* such that $\mu^*_{\mathcal{F}^u(z)}$, the projection along stable leaves onto any unstable manifold $\mathcal{F}^u(z)$, is equivalent to $\mathcal{D}_{\alpha,\mathcal{F}^u(z)}$;*

3. *there is a constant $C > 1$ only depending in the distortion, the bounded geometry of the admissible manifolds and the hyperbolic Pesin set Λ_δ such that*

$$C^{-1} \leq \frac{\mu_{\mathcal{F}^u(z)}(P)}{(Vol\, P)^{\dim_u(\Omega^*)}} \leq C \qquad \forall\ P \in \wp^*. \tag{17}$$

To prove this we follow the arguments in [16]. First, we project the dynamics on Ω^* along stable leaves to get an expanding transformation. We obtain in this way an asymptotic measure, by ([17, Theorem 8]). Next we pull back this asymptotic measure to get a T-invariant μ^* with the stated properties. Bounded distortion lemmas proved before give the necessary estimates to follow this program.

So, given $\mathcal{F}^u(x)$ and $\mathcal{F}^u(T(x))$ we define $\tau(x) = H \circ T(x)$, where H is the holonomy transformation defined by these unstable sections. Let $\Omega^u(z) = \Omega \cap \mathcal{F}^u(z)$ be the unstable Cantor set intersected by $\mathcal{F}^u(x)$. We define $d^*(x,y) = \mathrm{dist}_{\mathcal{H}}(\mathcal{F}^s(x), \mathcal{F}^s(y))$, where $\mathrm{dist}_{\mathcal{H}}$ is the Hausdorff distance of sets. (Ω^u, d^*) is a compact metric space. It is inmediate to verify that τ satisfies conditions (I) and (III) in [17], that is: it is a topologically exact covering expanding map with countable discrete fibers.

A key step is to prove that the dynamical measure classes $\mathcal{D}_{\alpha,\mathcal{F}^u(x)}$ for $x \in \Omega^*$ are non trivial. For this we show first that this is true for horseshoes with finitely many branches (well known) and then we show that Ω^* is the limit of an increasing sequence of horseshoes with finitely many branches, using the same principle of Corollary 5.1. The Gibbs measures μ_n^* for the potentials $\phi_n = -\dim_u(\Omega_n^*)$ supported on Ω_n^*, maximizes dimension in the unstable direction. Actually, they all share regularity properties stated in Proposition 6.1 and $\mu_n^*(P) \asymp \mathrm{Vol}\,(P)^{\dim_u(\Omega_n^*)}$, bounded by uniform constant not depending on n. We pass to the limit getting $\dim_u(\Omega_n^*) \longrightarrow \alpha$ and find two (possibly different) Borel probability measures ν^* and ν_* such that $\nu_*(int\, P) \leq C\,\mathrm{Vol}\,(P)^\alpha$ and $\mathrm{Vol}\,(P)^\alpha(P) \leq \nu^*(P)$ for every $P \in \wp^*$ where . Using an argument similar to Frostman's lemma we conclude that the measure classes in dimension α are bounded away from zero and infinite. In particular, $\alpha = \dim_u(\Omega^*)$ and $\dim_u(\Omega_n^*) \longrightarrow \dim_u(\Omega^*)$.

Volume estimates in Bounded Distortion Lemma will imply that the holonomy canonical transformations of the stable lamination are absolutely continuous respect to the dynamical measure class. Next we use that the measure class is conformal compute its Jacobian. Using this we prove that τ has a Jacobian wrt $\mathcal{D}_{\alpha,\mathcal{F}^u(x)}$. Indeed,

$$J_\alpha \tau(x) = \frac{d\, H^* \mathcal{D}_{\alpha,\mathcal{F}^u}(x)}{d\, \mathcal{D}_{\alpha,\mathcal{F}^u}(T(x))}(T(x))[J^u T(x)]^\alpha. \tag{18}$$

Let $\mathcal{L}_\psi h(x) = \sum_{\{y:\tau(y)=x\}} e^{\psi(y)} h(y)$ be the Perron-Ruelle-Frobenius transfer operator of the potential $\psi = -\log J_\alpha \tau$. It holds that $\mathcal{L}_\psi^* \mathcal{D}_{\alpha,\mathcal{F}^u(x)} = \mathcal{D}_{\alpha,\mathcal{F}^u(x)}$, where $\mathcal{L}_\psi^*$ is the adjoint of the transfer operator acting over the Borel measures (see [17, Lemma 14]).

An standard argument shows that there is a constant $C > 1$ such that

$$\sum_{y \in \tau^{-n} x} e^{S_n \psi(y)} \leq C \quad \forall\, x \in \Omega^u. \tag{19}$$

Also, for every x and y in Ω^u we look for a constant

$$C_\psi(x,y) = \sup_{n>0} \sup_{y \in \tau^{-n} x} \sum_{i=0}^{n-1} \left[\psi(\tau^i(x)) - \phi(\tau^i(y))\right] \tag{20}$$

such that $C_\psi(x,y) \longrightarrow 0$ as $d(x,x') \longrightarrow 0$. Compare [16] and [17].

Lemma 6.1. *There exists a constant $C > 1$ such that*

$$J_\alpha \tau^n(x) \,/\, J_\alpha \tau^n(y) \leq \exp(C\, d^*(\tau^n(x), \tau^n(y)))$$

whenever $x, y \in \mathbb{D}_n^u$, where $T^n : \mathbb{D}_n^u \subset \mathcal{F}^u(x) \longrightarrow \mathcal{F}^u(T(x))$ is expanding.

Proof. Bounded Distortion Lemma will give us the desired estimate. In fact let us denote

$$J^u(x) = Jac\,(T \mid \mathcal{F}^u(x))(x) \quad \text{and} \quad h(z) = \prod_{i=1}^{+\infty} \frac{J^u(T^i z)}{J^u(T^i H(z))}.$$

With this notation we can write

$$\frac{J_\alpha \tau^n(x)}{J_\alpha \tau^n(y)} = \left[\prod_{k=0}^{n-1} \frac{h(T(\tau^k(x)))}{h(T(\tau^k(y)))} \cdot \frac{J^u(\tau^k(x))}{J^u(\tau^k(y))}\right]^\alpha. \tag{21}$$

Using the Bounded Distortion Lemma we see that

$$\begin{aligned} \sum_{k=0}^{n-1} |\log\, J^u(\tau^k(x)) - \log\, J^u(\tau^k(y))| &\leq K_\delta \sum_{k=0}^{n-1} \lambda^{-n+k} d^*(\tau^n(x), \tau^n(y)) \\ &\leq K_\delta (1 - \lambda^{-1}) d^*(\tau^n(x), \tau^n(y)). \end{aligned} \tag{22}$$

Also, by Bounded Distortion Lemma $\log\, h(T(\tau^k(x))) \,/\, h(T(\tau^k(y)))$ is bounded from above $K_\delta \max\{d_u(T(\tau^k(x)), T(\tau^k(y)))\,,\, d_u(\tau^{k+1}(x), \tau^{k+1}(y))\}$, and this, in turn, is limited by $K_\delta^2 d^*(\tau^{k+1}(x), \tau^{k+1}(y))$, since d_u is comparable with d^* with a correction bounded by K_δ. Thus, for every $n > 0$ we have

$$\begin{aligned} \log \left[\prod_{k=0}^{n-1} \frac{h(T(\tau^k(x)))}{h(T(\tau^k(y)))}\right] &\leq K_\delta^2 \sum_{k=0}^{n-1} \lambda^{-n-k-1} d^*(\tau^n(x), \tau^n(y)) \\ &\leq K_\delta^2 (1 - \lambda^{-1}) d^*(\tau^n(x), \tau^n(y)). \end{aligned} \tag{23}$$

Therefore, $J_\alpha \tau^n(x) \,/\, J_\alpha \tau^n(y) \leq C d^*(\tau^n(x), \tau^n(y))$ absorbing different constants in $C > 1$. ☐

Now, given $x\,,\,y\,\in\,\Omega^u$ we can order $\tau^{-n}x\,=\,\{x_i\}_{i>0}$ and $\tau^{-n}y\,=\,\{y_i\}_{i>0}$ in such way that $\mathbb{D}^u_n(x_i)=\mathbb{D}^u_n(y_i)$ for every $i>0$. So, by Lemma 6.1

$$\log\frac{J_\alpha\tau^n(x_i)}{J_\alpha\tau^n(y_i)}\leq C\|x-y\|\qquad\forall\ i>0.$$

Hence, $C_\psi(x,y)\leq C\,\|x-y\|$. This proves (20).

We can use [17, Theorem 8] and the above conditions to find a (unique) τ invariant Borel probability $\mu_{\mathcal{F}^u}$ equivalent to the dynamical measure class and such that

$$\mu_{\mathcal{F}^u}(B)=\lim_{n\to+\infty}\frac{\mathcal{D}_{\alpha,\mathcal{F}^u}(\tau^{-n}B)}{\mathcal{D}_{\alpha,\mathcal{F}^u}(\Omega^u)}$$

fore every Borel subset B. $\mu_{\mathcal{F}^u}$ has a density bounded away from zero and infinite, so it is equivalent to the dynamical measure class. See also [17, Corollary 12]. Next, we pull back $\mu_{\mathcal{F}^u}$ to a T-invariant ergodic Borel probability μ^* supported on Ω^* following [9, Theorem 2.1]. This proves the above proposition. The following lemma and Proposition 6.1 will complete the proof of Theorem A.

Lemma 6.2. *Let μ^* be the asymptotic measure provided by Proposition 6.1. Then,*

$$\mu^*=\frac{\mu\mid\Omega^*}{\mu(\Omega^*)}.$$

Proof. We shall prove that there is a sequence of horseshoes $\Omega^*_n\subset\Omega^*$ and asymptotic measures μ^*_n supported on Ω^*_n such that $\mu^*_n\longrightarrow\mathsf{M}^*$, the restriction of μ to Ω^* normalized to a probability. For this we let $\{\phi_i\}_{i>0}$ a enumerable dense subset of continuous functions on Ω^*. Then we define $G_{\rho,n,N}$ a set of quasi-generic points for M^* as the set of points $x\in\mathsf{R}$ such that

$$\left|\frac{1}{m}\sum_{i=0}^{m-1}\phi_i(T^i(x))-\int\phi_i(z)\,d\mathsf{M}^*(z)\right|<\rho\quad i=1,\cdots,n\ \ \forall\ m\geq N.\tag{24}$$

Let also $\epsilon>0$ such that $|\phi_i(x)-\phi_i(y)|<\rho$ for $i=1,\cdots,n$ whenever $d(x,y)<\epsilon$. We claim that there exists $N=N(\rho,n,\epsilon)>0$ such that $G_{\rho,n,m}$ is saturated in the stable direction for $m\geq N$. In fact, for every $\epsilon>0$ there exists $N_0=N_0(\epsilon)>0$ such that $d(T^i(x),T^i(y))<\epsilon$ for every $i\geq N_0$ whenever $\mathcal{F}^s(x)=\mathcal{F}^s(y)$. Here we use the rate of contraction along stable leaves provided by the hyperbolic product structure in Ω^*. Thus, suppose that $y\in\mathcal{F}^s(x)$, then

$$\left|\frac{1}{m}\sum_{j=0}^{m-1}\phi_i(T^j(x))-\frac{1}{m}\sum_{j=0}^{m-1}\phi_i(T^j(y))\right|\leq\frac{2N_0\,\max_{1\leq i\leq n}\|\phi_i\|_\infty}{m}+\frac{m-N_0}{m}\rho.$$

So, for $N=N(\rho,n,\epsilon)>0$ we have

$$\left|\frac{1}{m}\sum_{j=0}^{m-1}\phi_i(T^j(x))-\frac{1}{m}\sum_{j=0}^{m-1}\phi_i(T^j(y))\right|\leq 2\rho\qquad\forall\ m\geq N.\tag{25}$$

Thus, if $x \in G_{\rho,n,N}$ implies that $\mathcal{F}^s(x) \subset G_{3\rho,n,N}$.

Now, for given ρ and n it holds $\mathsf{M}^*(G_{\rho,n,m}) \longrightarrow 1$ as $m \longrightarrow +\infty$. This is true since μ restricted to Ω^* is T invariant and ergodic. So, let $0 < \delta < 1$ and choose $N = N(\delta)$ the least integer such that $G_{\rho,n,m}$ is saturated in the stable direction and $\mathsf{M}^*(G_{\rho,n,m}) \geq 1 - \delta$ for every $m \geq N$ and $\Omega^*_{N,\rho} = \bigcup_{i=1}^N \Omega_i$ has $\mathsf{M}^*(G_{\rho,n,N} \Delta \Omega^*_{N,\rho}) = 0$, where Δ is the set-theoretic symmetric difference. This is possible since $\mu(\Omega^* - \bigcup_i \Omega_i) = 0$. $\Omega^*_{N,\rho}$ is a horseshoe with finitely many hyperbolic branches. Further, $\mu^*_{N,\rho}$, the Gibbs measure for the potential $\phi^* = -\dim_u(\Omega^*_{N,\rho})$ is the (unique) asymptotic measure the class of the dynamical measure. Now, take a generic point $z \in \Omega^*_{N,\rho}$ for $\mu^*_{N,\rho}$ and choose $m \geq N$ such that

$$\left| \frac{1}{m} \sum_{j=0}^{m-1} \phi_i(T^j(z)) - \int \phi_i(x)\, d\mu^*_{N,\rho}(x) \right| < \rho \quad \text{for } i = 1, \cdots, n,$$

and $z' \in G_{\rho,n,N}$ such that $\mathcal{F}^s(z) = \mathcal{F}^s(z')$. Therefore, using (24) and (25) we get for N large enough and for every $i = 1, \cdots, n$

$$\left| \int \phi_i(z)\, d\mathsf{M}^*(z) - \int \phi_i(z)\, d\mu^*_{N,\rho}(z) \right| < 4\rho. \tag{26}$$

From the above arguments we get a sequence of horseshoes Ω^*_n with finitely many branches and asymptotic measures μ^*_n such that $\mu^*_n \longrightarrow \mathsf{M}^*$ vaguely. We conclude that M^* is absolutely continuous with respect to the dynamical measure class, using as we did before, a Frostman's Lemma type argument. Also we get $\alpha = \dim_u(\Omega^*)$. Therefore, M^* is absolutely continuous respect to μ^*, the asymptotic measure constructed in Proposition 6.1, so they are equal, by ergodicity. □

Proof of Corollary B

Using Bounded Distortion Lemma and volume estimates we show that for every $n \geq 0$ and $P \in \wp^*$ such that $T^n(P)$ is an unstable cylinder (i.e. such that $T^n(P)$ has full width) diam $(P \cap W^u(x, \mathsf{R})) \asymp \|J^u T^n(x)\|^{-1}$ bounded by constant not depending on x, P neither n. Now, given $r > 0$ and $x \in \Omega^*$ we define $n = n(x, r) > 0$ as the minimum positive integer satisfying $(J^u T^n(x))^{-1} \leq r$ and $(J^u T^{n-1} f(x))^{-1} > r$. Therefore,

$$B(x, C^{-1}\, r) \cap \gamma^u \subseteq S_n(x) \cap \gamma^u \cap \gamma^u \subseteq B(x, C\, r) \cap \gamma^u \tag{27}$$

for every $0 < r < 1$ and $x \in \Omega^*$ and some universal constant $C > 1$ only depending on the distortion and the bounded geometry of admissible manifolds, where $W^s(x, \mathsf{R}) = \bigcap_{n \geq 0} S_n(x)$ is a nested sequence of stable cylinders converging to the local stable manifold, $n = n(x, r)$ and $T^n(S_n(x) \cap \gamma^u)$ is an admissible unstable manifold.

We use this to define Moran's covers $\wp_r$ for every $0 < r < 1$ associated to $\wp$ according to Cf. [12, Chapter 7]. Moran's covers satisfies the following finite multiplicity property: there exists a universal constant $M > 1$ such that

$B(x,r) \cap \gamma^u$ intersects at most M atoms in the family $\wp_r(\gamma^u)$, for every $x \in \Lambda$, $0 < r < 1$ and admissible unstable manifold γ^u. M should depend in principle on γ^u, however, by geometry of admissible manifolds shows that this dependence can be dropped out.

By Theorem A transversal measures satisfy

$$\mu_{\mathcal{F}^u(x)}(P) \asymp \operatorname{diam}\left(P \cap W^u(x, \mathsf{R})\right)^{\dim_{\mathcal{D}}(\Omega^*)},$$

where $\dim_{\mathcal{D}}(\Omega^*)$ is the dynamical dimension of Ω^*, so for every $P \in \wp^*$, hence

$$\mu_{\mathcal{F}^u(x)}(B(x,r)) \asymp \operatorname{diam}\left(B(x,r) \,\cap\, \gamma^u)\right)^{\dim_{\mathcal{D}}(\Omega^*)} \asymp r^{\dim_{\mathcal{D}}(\Omega^*)},$$

for every $x \in \Omega^*$, $0 < r < 1$, using the bounded geometry of admissible manifolds. So, $\dim_{\mathcal{H}}(\Omega^* \cap W^u(x, \mathsf{R}) == \dim_{\mathcal{D}}(\Omega^*) = \dim_u(\Omega^*)$, and the dynamical measure, the Hausdorff measure and the transversal measure are in the same measure class and the transversal measure $\mu_{\mathcal{F}^u(x)}$ has the strong uniform distribution property $\mu_{\mathcal{F}^u(x)}(B(x,r)) \asymp r^{\dim_u(\Omega^*)}$, for every $x \in \Omega^*$ and $0 < r < 1$, bounded by a universal constant.

In particular, the holonomies have a bounded Jacobian respect to the Hausdorff measure. Finally, by Frostman's uniform distribution property and the existence of a bounded Jacobian with respect to the Hausdorff measure for the holonomies of the stable lamination $\mathcal{F}^s$ we see that

$$\|x-y\|^\alpha \asymp \mathcal{H}_{\alpha,\gamma^u}([x,y]) \asymp \mathcal{H}_{\alpha,\overline{\gamma}^u}([H(x),H(y)]) \asymp \|H(x)-H(y)\|^\alpha,$$

where $\alpha = \dim_{\mathcal{H}}(\Omega \cap W)$, $\mathcal{H}_{\alpha,\gamma^u}$ is the restriction of the Hausdorff measure to the admissible unstable manifold γ^u and $H = H_{\gamma^u,\overline{\gamma}^u}$ the holonomy map defined by γ^u and $\overline{\gamma}^u$. Constants are uniform, by previous remarks and the bounded geometry of admissible manifolds. Hence,

$$\frac{\|H(x)-H(y)\|}{\|x-y\|} \asymp 1 \qquad \forall\, x,y \in \gamma^u \,\cap\, \bigcup_{z\in\Omega} W^s(z,\mathsf{R}),$$

meaning that $C^{-1}\|x-y\| \le \|H(x)-H(y)\| \le C\|x-y\|$, $\mathcal{F}^s$ is Lipschitz. QED.

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References

[1] Abramov, L.M. : On the entropy of an induced map. Translations of the AMS 49, 167 -170 (1966).

[2] Billingsley, P.: Ergodic Theory and Information. Wiley & Sons. 1965.

[3] De Melo, W. - Van Strien : One Dimensional Dynamics. Springer-Verlag. 1993.

[4] Newhouse, S. - Jakobson, M. : On the strucuture of non hyperbolic attractors. Preprint.

[5] Katok, A. and Hasselblat, B. : Introduction to the Modern Theory of Dynamical Systems, Encyclopedia of Math. and its Applications. Vol. 54 . Cambridge University Press, 1995.

[6] Katok, A. - Mendoza, L. : Dynamical systems with non-uniformly hyperbolic behavior. Supplement to Introduction to the Modern Theory of Dynamical Systems, by Katok, A. and Hasselblat, B..

[7] Ledrappier, F. : Propietés ergódiques des mesures de Sinai. Publ. Mathematiques IHES, 59 (1984).

[8] Ledrappier, F. - L.-S. Young : Entropy of diffeomorphisms I, II. Ann. of Math., 1985.

[9] Mañé, R : Ergodic Theory and Differentiable Dynamics. Springer-Verlag. 1987.

[10] Mendoza, L. : The entropy of C^2 surface diffeomorphisms in terms of Hausdorff dimension and a Lyapunov exponent. Erg. Th. and Dyn. Sys. (1985), 5, 273 - 283.

[11] Mendoza, L. : Ergodic attractors for diffeomorphisms of surfaces. Journal of the London Math. Soc. , 1988.

[12] Pesin, Ya. : Characteristic Lyapunov exponents and smooth ergodic theory, Russian Math. Surveys 32 (1977) 55-114.

[13] Pesin, Ya. : Dimension Theory and Dynamical Systems. Chicago Lectures in Mathematics.Chicago, 1997.

[14] Pugh, C. - Shub, M. : Ergodic Attractors. Transactions of the AMS. Vol. 312, No. 1, 1989.

[15] Sanchez-Salas, Fernando J. : Higher dimensional versions of the Folklore Theorem, I. preprint.

[16] Sanchez-Salas, Fernando J. : Higher dimensional versions of the Folklore Theorem, II. preprint.

[17] Walters, P. : Invariant Measures and Equilibrium States for some mappings which expands distances. Trans. American Math. Soc., Volume 236, February 1978.

[18] Young, L. -S.: Dimension, entropy and Lyapunov exponents, Erg. Th. and Dyn. Sys. (1982), 2, 109 - 124.

[19] Young, L. -S.: Ergodic Theory of Attractors. Proc. ICM-94, v.2, 1221-1229. Birkhäuser 1995.

[20] Young, L. -S.: Statistical Properties of Dynamical Systems with Some Hyperbolicity. Annals of Math. 147 (1998), 585 - 650.