Qualitative properties of stationary measures for three-dimensional Navier–Stokes equations

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Abstract

The paper is devoted to studying the distribution of stationary solutions for 3D Navier–Stokes equations perturbed by a random force. Under a non-degeneracy assumption, we show that the support of such a distribution coincides with the entire phase space, and its finite-dimensional projections are minorised by a measure possessing an almost surely positive smooth density with respect to the Lebesgue measure. Similar assertions are true for weak solutions of the Cauchy problem with a regular initial function. The results of this paper were announced in the short note [Shi06b].

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Contents

0 Introduction

Let us consider the 3D Navier–Stokes (NS) system

$$\partial_t u + \langle u, \nabla \rangle u - \nu \Delta u + \nabla p = f(t, x), \quad \operatorname{div} u = 0, \quad x \in \mathbb{T}^3, \tag{0.1}$$

where $\mathbb{T}^3$ denotes the 3D torus, $u = (u_1, u_2, u_3)$ and p are unknown velocity field and pressure of the fluid, $\nu > 0$ is the viscosity, and f is an external force. In what follows, we assume that f is the time derivative of a random process with independent increments and sufficiently non-degenerate distribution in the space variables. Our aim is to study qualitative properties of the law of stationary weak solutions for (0.1). This question has significant importance in applications for at least two reasons. First, it is widely believed that stationary solutions corresponding to small values of viscosity can be used to describe turbulent behaviour of solutions. And, second, under some additional assumptions, a large class of weak solutions for (0.1) converge to a stationary solution as time goes to infinity. Before turning to a description of the contents of this paper, let us recall some earlier results on 3D stochastic NS equations.

Existence of weak solutions for the Cauchy problem and of stationary solutions, as well as some a priori estimates for them, was established by Bensoussan, Temam [BT73], Vishik, Komech, Fursikov [VKF79], Capiński, Gątarek [CG94], Flandoli, Gątarek [FG95] and others. A first result showing the mixing character of 3D NS dynamics under non-degenerate random forcing was obtained by Flandoli [Fla97]. He proved that if the noise is effective in all Fourier modes, then the support of any "admissible" weak solution coincides with the entire phase space. In the case of a rough white noise, Da Prato and Debussche [DD03] constructed a Markov semigroup concentrated on weak solutions of the 3D NS equations and established a mixing property for it. Under similar conditions, Odasso [Oda06] proved that any solution obtained as a limit of Galerkin approximations converges exponentially to a stationary solution. Flandoli and Romito [FR06] have constructed a Markov selection of weak solutions and proved the irreducibility and strong Feller property for it, provided that the random perturbation is sufficiently rough. The results of this paper show that, in the case of periodic boundary conditions, non-degeneracy of the noise with respect to the first few Fourier modes ensures mixing character of the dynamics.

We now describe in more details the main result of this paper. Let us assume that the right-hand side of (0.1) has the form

$$f(t,x) = h(x) + \sum_{j=1}^{\infty} b_j \dot\beta_j(t) e_j(x), \tag{0.2}$$

where h is a deterministic function belonging to the space $L^2 = L^2(\mathbb{T}^3, \mathbb{R}^3)$ of square-integrable vector fields on $\mathbb{T}^3$, $\{e_j\}$ is a trigonometric basis in L^2, $b_j \geq 0$ are some constants going to zero sufficiently fast, and $\{\beta_j\}$ is a sequence of independent standard Brownian motions. As is shown in [VKF79, FG95], problem (0.1), (0.2) has a stationary weak solution $u(t)$ defined for $t \geq 0$. Let μ be its distribution. Thus, μ is a probability measure on the Hilbert space $H \subset L^2$ of divergence-free square-integrable vector fields. The following theorem is a simplified version of the main result of this paper.

Main Theorem. *There is an integer $N \geq 1$ not depending on h and ν such that if*

$$b_j \neq 0 \quad \textit{for } j = 1, \dots, N,$$

then the following assertions hold.

(i) *Any ball in the space $H^1 \cap H$ has a positive μ-measure. In particular, the support of μ coincides with H.*

(ii) *Let F be a finite-dimensional subspace. Then the projection of μ to F can be minorised by a measure of the form $\rho_F(y)\ell_F(dy)$, where $\ell_F(dy)$ denotes the Lebesgue measure on F and ρ_F stands for a smooth function that is positive almost everywhere.*

It should be mentioned that if we restrict ourselves to the family $\mathcal{M}$ of stationary measures obtained as limits of Galerkin approximations, then the measure of balls in $H^1 \cap H$ can be minorised uniformly with respect to $\mu \in \mathcal{M}$, and the function ρ_F can be chosen independently of $\mu \in \mathcal{M}$. Furthermore, the proof given in Section 3 does not use the Gaussian structure of the noise η, and therefore a similar result is true for other types of random perturbations, such as random kick forces,

$$\eta(t,x) = \sum_{k=1}^{\infty} \eta_k(x)\delta(t-k), \tag{0.3}$$

or piecewise-constant stochastic processes,

$$\eta(t,x) = \sum_{k=1}^{\infty} \eta_k(x) I_{[k-1,k)}(t). \tag{0.4}$$

Here $\delta(t)$ denotes the Dirac measure concentrated at zero, $I_{[k-1,k)}(t)$ stands for the indicator function of the interval $[k-1,k)$, and $\{\eta_k\}$ is a sequence of i.i.d. random variables in appropriate functional space. These questions and similar problems for 3D NS equations in other domains will be addressed in a subsequent publication.

The Main Theorem formulated above is related to some earlier results established in the 2D case. Namely, using the Malliavin calculus, it was shown by Mattingly and Pardoux [MP06] that, in the case of NS equations perturbed by a degenerate white noise force, the law of finite-dimensional projections of solutions possesses a positive smooth density with respect to the Lebesgue measure. In [AKSS06], the authors have proved a weaker version of that result for various types of (non-Gaussian) perturbations, including random forces of the form (0.3) and (0.4). The proofs in the present paper are based on a combination of the methods developed in [AKSS06] and some properties of 3D NS equations. The most important of them are the controllability, regularity of the resolving operator on strong solutions, and weak-strong uniqueness. Roughly speaking, we show that a large part of weak (stationary) solutions consists of strong solutions, and therefore the law of weak solutions is minorised by that of strong solutions. Combining this with the property of approximate controllability, we obtain assertion (i) (cf. [Fla97]). Furthermore, applying a general result on the image of probability measures under a smooth mapping to strong solutions (this can be done due to solid controllability of NS equations, see Subsection 1.4) and using a simple localisation argument, we establish the second part of the theorem. We refer the reader to Section 3.1 for a more detailed description of the scheme of the proof.

The paper is organised as follows. In Section 1, we have compiled some preliminaries. Exact formulation of the main theorem is given in Section 2. The third section is devoted to the proofs. In the appendix, we establish some auxiliary results.

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Notation

Let $J \subset \mathbb{R}$ be a closed interval, let $D \subset \mathbb{R}^3$ be a bounded domain with smooth boundary ∂D, and let X be a Banach space. We shall use the following functional spaces.

$H^s(D)$ is the Sobolev space of order s on D.

$H^s(D, \mathbb{R}^3)$ is the space of vector functions (u_1, u_2, u_3) whose components belong to $H^s(D)$. In the case $s = 0$, we obtain the usual Lebesgue space $L^2(D, \mathbb{R}^3)$.

$H_0^1(D, \mathbb{R}^3)$ is the space of functions in $H^1(D, \mathbb{R}^3)$ that vanish on ∂D.

$C(X)$ is the space of real-valued continuous functions on X.

$C^k(J,X)$, $0 \le k \le \infty$, is the space of k times continuously differentiable functions $f : J \to X$. In the case $k = 0$, we shall write $C(J,X)$.

$L^p(J,X)$ is the space of Bochner-measurable functions $f : J \to X$ such that

$$\|f\|_{L^p(J,X)} := \left(\int_J \|f(t)\|_X^p dt\right)^{1/p} < \infty.$$

$L^p_{\mathrm{loc}}(J,X)$ is the space of functions $f : J \to X$ whose restriction to any compact interval $I \subset J$ belongs to $L^p(I,X)$.

If ξ is a random variable, then $\mathcal{D}(\xi)$ denotes its distribution. If X is a Polish space, $x_0 \in X$, and $r > 0$, then we denote by $B_X(x_0,r)$ (respectively, $\dot{B}_X(x_0,r)$) the closed (open) ball in X of radius r centred at x_0.

1 Preliminaries

1.1 Weak and strong solutions for Navier–Stokes equations

Let $D \subset \mathbb{R}^3$ be a bounded domain with C^2-smooth boundary ∂D. Consider the 3D Navier–Stokes (NS) equations

$$\dot{u} + \langle u, \nabla \rangle u - \nu \Delta u + \nabla p = h(x), \quad \operatorname{div} u = 0, \quad x \in D, \tag{1.1}$$

where $u = (u_1, u_2, u_3)$ and p are unknown velocity field and pressure, $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product in $\mathbb{R}^3$, and $h \in L^2(D, \mathbb{R}^3)$ is a given function. We introduce the spaces

$$H = \{u \in L^2(D,\mathbb{R}^3) : \operatorname{div} u = 0 \text{ in } D, \langle u, \boldsymbol{n} \rangle|_{\partial D} = 0\},$$
$$V = H_0^1(D,\mathbb{R}^3) \cap H, \quad U = H^2(D,\mathbb{R}^3) \cap V,$$

where $\boldsymbol{n}$ stands for the outward unit normal to ∂D. It is well known (e.g., see [Tem79]) that H is a closed vector space in $L^2(D,\mathbb{R}^3)$, and we denote by Π the orthogonal projection in $L^2(D,\mathbb{R}^3)$ onto H. The Navier–Stokes system (1.1) is equivalent to the following evolution equation in H obtained formally by applying Π to the first relation in (1.1):

$$\dot{u} + \nu L u + B(u) = h. \tag{1.2}$$

Here $L = -\Pi\Delta$, $B(u) = B(u,u)$, $B(u,v) = \Pi\{(u,\nabla)v\}$, and we use the same notation for the right-hand side h and its projection to H.

In what follows, we shall need also the following NS type equation:

$$\dot{v} + \nu L v + B(v+z) = h. \tag{1.3}$$

Here z is a given function belonging to the space $\mathcal{Y} := C(\mathbb{R}_+, V) \cap L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$. Let $J = [0,T]$, let $(\cdot,\cdot)$ be the scalar product in $L^2(D,\mathbb{R}^3)$, and let $\|\cdot\|$ be the corresponding norm. We denote by $\|u\|_V = \|\nabla u\|$ the norm in the space V.

Definition 1.1. A function $v \in \mathcal{X}_J := L^\infty_{\text{loc}}(J, H) \cap L^2_{\text{loc}}(J, V)$ is called a *weak solution* for (1.3) if it possesses the following properties.

(i) Equation (1.3) holds in the sense of distributions, that is, for any divergence-free vector field $\varphi \in C_0^\infty(J \times D, \mathbb{R}^3)$, we have

$$\int_J \bigl(-(v, \dot\varphi) + \nu(v, L\varphi) + (B(v+z), \varphi) - (h, \varphi)\bigr)\, ds = 0. \tag{1.4}$$

(ii) The function v satisfies the energy inequality[1]

$$\frac{1}{2}\|v(t)\|^2 + \nu \int_0^t \|v(s)\|_V^2 ds + \int_0^t (B(v+z, z), v)\, ds \\ \leq \frac{1}{2}\|v(0)\|^2 + \int_0^t (h, v)\, ds, \quad t \in J. \tag{1.5}$$

Note that if $v \in \mathcal{X}_J$ satisfies (1.3), then $\dot v \in L^1_{\text{loc}}(J, H^{-1})$, where H^{-1} is the dual space for $H_0^1(D, \mathbb{R}^3)$. It follows that v is a weakly continuous function of time with range in H, and therefore all terms in (1.5) are well defined.

Definition 1.2. A function $v \in \mathcal{Y}_J := C(J, V) \cap L^2_{\text{loc}}(J, U)$ is called a *strong solution* for (1.3) if it satisfies (1.3) in the sense of distributions (see property (i) in Definition 1.1).

Note that if $v \in \mathcal{Y}_J$ is a strong solution for (1.3), then the following energy equality holds for it:

$$\frac{1}{2}\|v(t)\|^2 + \nu \int_0^t \|v(s)\|_V^2 ds + \int_0^t (B(v+z, z), v)\, ds = \frac{1}{2}\|v(0)\|^2 + \int_0^t (h, v)\, ds. \tag{1.6}$$

Indeed, a standard limiting argument shows that relation (1.4) remains valid for any function $\varphi \in \mathcal{Y}_J$ such that $\dot\varphi \in L^2(J, H)$ and $\varphi(0) = \varphi(T) = 0$. Let us fix any $t > 0$ and consider the sequence $\varphi_k = \chi_k v$, where $\chi_k \in C^\infty(\mathbb{R})$ are arbitrary functions such that $0 \leq \chi_k \leq 1$,

$$\chi_k(s) = 1 \quad \text{for } \tfrac{1}{k} \leq s \leq t - \tfrac{1}{k}, \qquad \chi_k(s) = 0 \quad \text{for } s \leq 0 \text{ or } s \geq t.$$

Writing identity (1.4) with $\varphi = \varphi_k$ and using the relation $(B(v+z, v), v) = 0$, we obtain

$$-\int_0^t (v, \partial_s(\chi_k v))\, ds + \int_0^t \chi_k(s)\bigl(\nu\|v\|_V^2 + (B(v+z, z), v) - (h, v)\bigr)\, ds = 0. \tag{1.7}$$

It is easily seen that

$$-\int_0^t (v, \partial_s(\chi_k v))\, ds = -\frac{1}{2}\int_0^t \bigl(\partial_s(\chi_k\|v\|^2) + \dot\chi_k\|v\|^2\bigr)\, ds \to \frac{1}{2}\bigl(\|v(t)\|^2 - \|v(0)\|^2\bigr)$$

[1]This inequality is obtained formally by taking the scalar product of (1.3) with v, integrating in time, and replacing $=$ by $\leq$.

as $k \to \infty$. Passing to the limit in (1.7) as $k \to \infty$, we arrive at (1.6).

The following proposition establishes a weak-strong uniqueness for solutions of (1.3).

Proposition 1.3. *Let $v \in \mathcal{X}_J$ and $\tilde v \in \mathcal{Y}_J$ be, respectively, weak and strong solutions for* (1.3) *such that $v(0) = \tilde v(0)$. Then $v = \tilde v$.*

In the case $z = 0$, a proof of Proposition 1.3 can be found in [Soh01]. For the reader's convenience, we outlined the proof for the general case in the Appendix.

1.2 Admissible weak solutions for stochastic Navier–Stokes equations

Let us consider 3D Navier–Stokes equations perturbed by a random force:

$$\dot u + \nu L u + B(u) = h + \eta(t). \tag{1.8}$$

Here $h \in H$ is a deterministic function and η is a random process white in time and regular in the space variables. In what follows, we always assume that η satisfies the following assumption.

Condition 1.4. There exists a Hilbert–Schmidt operator $Q : H \to V$ and an H-valued cylindrical Wiener process ζ defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a right-continuous filtration $\{\mathcal{F}_t\}_{t\geq 0}$ such that

$$\eta(t) = \frac{\partial}{\partial t} Q\zeta(t). \tag{1.9}$$

The following lemma gives an alternative description of random processes satisfying Condition 1.4; its proof is given in the Appendix (see Section 4.2).

Lemma 1.5. *A random process η satisfies Condition 1.4 if and only if it is representable in the form*

$$\eta(t) = \sum_{j,k=1}^{\infty} b_{jk} \dot\beta_j(t) f_k, \tag{1.10}$$

where $\{f_k\}$ is an orthonormal basis in V, $\{\beta_j\}$ is a sequence of independent standard Brownian motions on $\mathbb{R}_+$, and $\{b_{jk}\}$ is a family of real numbers satisfying the condition

$$\sum_{j,k=1}^{\infty} b_{jk}^2 < \infty.$$

We now recall the concept of an admissible weak solution for (1.8). To this end, we first define the Ornstein–Uhlenbeck process

$$z(t) = \int_0^t e^{-\nu(t-s)L} Q d\zeta(t). \tag{1.11}$$

Relation (1.11) implies that z is a V-valued Gaussian process whose almost every trajectory belongs to the space $\mathcal{Y} = C(\mathbb{R}_+, V) \cap L^2_{\text{loc}}(\mathbb{R}_+, U)$ and satisfies the Stokes equation

$$\dot{u} + \nu L u = \eta(t). \tag{1.12}$$

Definition 1.6. An H-valued random process $u(t)$ is called an *admissible solution* for (1.8) if it is representable in the form

$$u(t) = v(t) + z(t), \tag{1.13}$$

where $v(t)$ is an H-valued $\mathcal{F}_t$-progressively measurable random process whose almost every trajectory is a weak solution for (1.3) on the half-line $\mathbb{R}_+$.

Definition 1.7. An H-valued random process $u(t)$ is called an *admissible weak solution* for (1.8) if there is a process $\{\tilde{\eta}(t), t \geq 0\}$ satisfying Condition 1.4 such that $u(t)$ is an admissible solution for (1.8) with η replaced by $\tilde{\eta}$.

Definition 1.8. An admissible weak solution $u(t)$ for (1.8) is said to be *stationary* if its distribution does not depend on t:

$$\mathcal{D}(u(t)) = \mu \quad \text{for all } t \geq 0.$$

In this case, μ is called a *stationary measure* for (1.8).

Note that, in Definition 1.8, we do not require u to be a stationary process. The following proposition is essentially established in [VF88, CG94, FG95] (see also [Rom01] for the existence of a suitable weak solution).

Proposition 1.9. *Suppose that $h \in H$ and Condition 1.4 is fulfilled. Then Eq.* (1.8) *has at least one stationary measure $\mu \in \mathcal{P}(H)$ such that*

$$\mathfrak{m}(\mu) := \int_H \|v\|_V^2 \mu(dv) < \infty. \tag{1.14}$$

Note that, in [VF88, CG94, FG95], the authors do not state explicitly the fact that the energy inequality (1.5) holds. However, they construct a solution as a pointwise limit of Galerkin approximations, which satisfy an energy inequality of the form (1.5). It is not difficult to see that one can pass to the limit in those inequalities. Also note that if μ is a stationary measure obtained as a limit of Galerkin approximations, then $\mathfrak{m}(\mu)$ is bounded by a constant depending only on ν, D, h, and η.

1.3 Controllability properties of Navier–Stokes equations

In this subsection, we have compiled some recent results on controllability for the NS system (1.8) supplemented with the initial condition

$$u(0) = u_0, \tag{1.15}$$

where $u_0 \in V$. We first introduce some notations.

Let $h \in H$ be a function, let $T > 0$ be a constant, and let $J = [0, T]$. For any $u_0 \in V$, we denote by $\Theta_T(h, u_0)$ the set of functions $\eta \in L^2(J_T, H)$ for which problem (1.8), (1.15) has a unique solution $u \in \mathcal{Y}_J$. Using the implicit function theorem, it can be shown that (see Theorem 1.8 in [Shi06a])

$$\mathcal{D}_T(h) := \{(\eta, u_0) \in L^2(J_T, H) \times V : \eta \in \Theta_T(h, u_0)\} \tag{1.16}$$

is an open subset of $L^2(J_T, H) \times V$, and the operator $\mathcal{R}$ taking $(\eta, u_0) \in \mathcal{D}_T$ to the solution $u \in \mathcal{Y}_J$ of (1.8), (1.15) is locally Lipschitz continuous. We denote by $\mathcal{R}_t$ the restriction of $\mathcal{R}$ to the time $t \in J$.

Let $E \subset U$ and $F \subset H$ be finite-dimensional subspaces and let $\mathsf{P}_F : H \to H$ be the orthogonal projection onto F. In the next definition, we assume that η is an E-valued control function.

Definition 1.10. Equation (1.8) is said to be *approximately controllable in time T* if for any $u_0, \hat{u} \in V$ and any $\varepsilon > 0$ there is $\eta \in \Theta_T(h, u_0) \cap C^\infty(J, E)$ such that

$$\|\mathcal{R}_T(\eta, u_0) - \hat{u}\|_V < \varepsilon. \tag{1.17}$$

Equation (1.8) is said to be *solidly F-controllable in time T* if for any $u_0 \in V$ and any $R > 0$ there is a constant $\delta > 0$ and a compact set $\mathcal{K}$ in a finite-dimensional subspace $X \subset C^\infty(J, E)$ such that $\mathcal{K} \subset \Theta_T(h, u_0)$, and for any continuous mapping $\Phi : \mathcal{K} \to F$ satisfying the inequality

$$\sup_{\eta \in \mathcal{K}} \|\Phi(\eta) - \mathsf{P}_F \mathcal{R}_T(\eta, u_0)\|_F \le \delta, \tag{1.18}$$

we have $\Phi(\mathcal{K}) \supset B_F(R)$.

For any finite-dimensional subspace $G \subset U$, we denote by $\mathcal{F}(G)$ the largest vector space $G_1 \subset U$ such that any element $\eta_1 \in G_1$ is representable in the form

$$\eta_1 = \eta - \sum_{j=1}^{k} \lambda_j B(\zeta^j),$$

where $\eta, \zeta^1, \dots, \zeta^k \in G$ are some vectors and $\lambda_1, \dots, \lambda_k$ are non-negative constants. Since B is a quadratic operator continuous from U to $H^1(D, \mathbb{R}^3)$, we see that $\mathcal{F}(G) \subset U$ is a well-defined vector space of finite dimension. Also note that $\mathcal{F}(G) \supset G$.

We now define a sequence of subspaces $E_k \subset U$ by the rule

$$E_0 = E, \quad E_k = \mathcal{F}(E_{k-1}) \quad \text{for } k \ge 1, \quad E_\infty = \bigcup_{k=1}^{\infty} E_k. \tag{1.19}$$

The following result is established in [Shi06a, Shi06c].

Proposition 1.11. *Let $E \subset U$ be a finite-dimensional subspace such that E_∞ is dense in H. Then the following assertions take place for any $h \in H$, $T > 0$, and $\nu > 0$.*

(i) *Equation* (1.8) *is approximately controllable in time* T *by an* E*-valued control.*

(ii) *For any finite-dimensional subspace* $F \subset H$, *Eq.* (1.8) *is solidly* F*-controllable in time* T *by an* E*-valued control.*

1.4 Analyticity of the resolving operator

Let X and Y be Banach spaces and let $\mathcal{D} \subset X$ be an open set. Recall that a continuous function $f : \mathcal{D} \to Y$ is said to be *analytic* if for any $x_0 \in \mathcal{D}$ there is a constant $r > 0$ such that

$$f(x) = f(x_0) + \sum_{m=1}^{\infty} L_m(x - x_0) \quad \text{for } x \in B_X(x_0, r), \tag{1.20}$$

where $L_m : X \to Y$ is an m-linear operator depending on x_0, and series (1.20) converges regularly. The latter means that

$$\sum_{m=1}^{\infty} \| L_m \| r^m < \infty,$$

where $\| \cdot \|$ stands for the norm of an m-linear operator (see [VF88] for more details).

Let us fix an interval $J = [0, T]$ and consider Eq. (1.3), in which $h \in H$ and $z \in \mathcal{Y}_J$ are given functions. We denote by $\mathcal{C}_T(h)$ the set of functions $(z, v_0) \in \mathcal{Y}_J \times V$ for which Eq. (1.3) has a unique solution $v \in \mathcal{Y}_J$ satisfying the initial condition

$$v(0) = v_0. \tag{1.21}$$

Theorem 1.8 in [Shi06a] implies that $\mathcal{C}_T(h)$ is an open subset of $\mathcal{Y}_J \times V$ and the operator $\mathcal{S}$ taking $(z, v_0) \in \mathcal{C}_T(h)$ to the solution $v \in \mathcal{Y}_J$ is locally Lipschitz continuous. The following proposition can be proved by the methods used in [Kuk82] (see also [Brz91]).

Proposition 1.12. *For any* $h \in H$, $T > 0$, *and* $\nu > 0$, *the resolving operator* $\mathcal{S} : \mathcal{C}_T(h) \to \mathcal{Y}_T$ *is analytic.*

1.5 Decomposable measures

Let Z be a separable Banach space and let $\lambda \in \mathcal{P}(Z)$.

Definition 1.13. The measure λ is said to be *decomposable* if there are two sequences of closed subspaces $\{F_n\}$ and $\{G_n\}$ such that the following properties hold.

(i) For any $n \geq 1$, we have $F_n \subset F_{n+1}$, and the union $\cup_n F_n$ is dense in Z.

(ii) For any $n \geq 1$, the space Z can be decomposed into the direct sum of F_n and G_n,

$$Z = F_n \dotplus G_n, \tag{1.22}$$

and the measure λ is representable as

$$\lambda = \mathsf{P}_n\lambda \otimes \mathsf{Q}_n\lambda, \tag{1.23}$$

where P_n and Q_n are the projections associated with decomposition (1.22).

Example 1.14. Let $\lambda \in \mathcal{P}(Z)$ be a non-degenerate centred Gaussian measure, that is, a probability Borel measure on Z such that for any continuous functional $\ell \in Z^*$ the image of μ under ℓ is a centred Gaussian measure on the real line, and the support of λ coincides with Z. We claim that λ is decomposable. Indeed, let $H(\lambda)$ be the Cameron–Martin space for λ (see Section 2.1 in [Bog98]), let $\{e_j\}$ be an orthonormal basis in $H(\lambda)$, and let $\{\xi_j\}$ be a sequence of scalar i.i.d. random variables on the same probability space such that the distribution of ξ_j is a standard Gaussian measure on $\mathbb{R}$. Then, by Theorem 3.4.4 in [Bog98], the series

$$\sum_{j=1}^{\infty} \xi_j(\omega)e_j$$

converges almost surely in Z, and the distribution of its sum $\xi(\omega)$ coincides with λ. Let us set

$$F_n = \operatorname{span}\{e_j, 1 \leq j \leq n\}, \quad G_n = \overline{\operatorname{span}\{e_j, j \geq n+1\}},$$

where $\overline{B}$ denotes the closure of B in the space Z. Let us show that properties (i) and (ii) of Definition 1.13 hold.

By Theorem 3.5.1 in [Bog98], the support of λ coincides with the closure of $H(\lambda)$ in Z. By assumption, we have $\operatorname{supp}\lambda = Z$, and therefore the vector space $\cup_n F_n = \operatorname{span}\{e_j, j \geq 1\}$ is dense in Z. To prove (ii), we fix any integer $n \geq 1$ and note that $F_n \cap G_n = \{0\}$. Therefore decomposition (1.22) will be established if we show that any vector $z \in Z$ is representable in the form $z = y_n + z_n$, where $u_n \in F_n$ and $v_n \in G_n$. This fact is obvious for elements of $H(\lambda)$ and can be proved by a simple approximation argument for any $z \in Z$. Furthermore, to prove (1.23), we write

$$\xi(\omega) = \sum_{j=1}^{n} \xi_j(\omega)e_j + \sum_{j=n+1}^{\infty} \xi_j(\omega)e_j =: \eta_n(\omega) + \zeta_n(\omega).$$

The construction implies that $\mathcal{D}(\eta_n) = \mathsf{P}_n\lambda$ and $\mathcal{D}(\zeta_n) = \mathsf{Q}_n\lambda$. Since η_n and ζ_n are independent, we obtain (1.23)

In what follows, we shall deal with measures $\lambda \in \mathcal{P}(Z)$ satisfying the following condition:

Condition 1.15. The measure λ is decomposable in the sense of Definition 1.13, its support coincides with Z, and for any $n \geq 1$ the projection $\mathsf{P}_n\lambda$ possesses a positive continuous density with respect to the Lebesgue measure on F_n.

Example 1.16. Let λ be a Gaussian measure on a separable Banach space X. Denote by Z the support of λ. Then Z is also a separable Banach space, and the restriction of λ to Z is a non-degenerate Gaussian measure. Furthermore, any finite-dimensional projection of λ possesses a positive smooth density with respect to the Lebesgue measure. Thus, any Gaussian measure satisfies Condition 1.15.

2 Main results

2.1 Formulations

Let us consider the NS system (1.8), where $h \in H$ is a deterministic function and η is a random process satisfying Condition 1.4. In what follows, we assume that h and η are fixed and do not trace the dependence of various parameters on them. Recall that the concept of stationary measure for (1.8) is introduced in Definition 1.8. For any finite-dimensional space F, we denote by ℓ_F the Lebesgue measure on it. If μ_1 and μ_2 are two measures such that $\mu_1(\Gamma) \geq \mu_2(\Gamma)$ for any measurable set Γ, then we write $\mu_1 \geq \mu_2$. The following theorem is the main result of this paper.

Theorem 2.1. *Suppose that the image of the operator Q in Condition 1.4 contains a finite-dimensional subspace $E \subset U$ for which the vector space E_∞ defined in* (1.19) *is dense in H. Let $\mu \in \mathcal{P}(H)$ be a stationary measure for* (1.8) *such that*

$$\mathfrak{m}(\mu) = \int_H \|v\|_V^2 \mu(dv) \leq \mathfrak{m}_0, \tag{2.1}$$

where $\mathfrak{m}_0 > 0$ is a constant. Then the following assertions take place.

- **(i)** *For any ball $B \subset V$ there is a constant $p(B, \mathfrak{m}_0) > 0$ such that*

$$\mu(B) \geq p(B, \mathfrak{m}_0). \tag{2.2}$$

  In particular, the support of μ coincides with H.

- **(ii)** *Let $F \subset H$ be a finite-dimensional subspace and let μ_F be the projection of μ to F. Then there is a function $\rho_F \in C^\infty(F)$ depending only on $\mathfrak{m}_0$ such that $\mu_F \geq \rho_F \ell_F$ and $\rho_F(y) > 0$ for ℓ_F-almost every $y \in F$.*

A proof of Theorem 2.1 is given in the next section. It is based on an auxiliary result, which is of independent interest. Before formulating it, we make two remarks.

Remark 2.2. Analysing the proof given below, it is not difficult to see that Theorem 2.1 remains valid for any admissible weak solution u of Eq. (1.8) such that $\mathbb{P}\{u(0) \in V\} > 0$. In particular, we can take a solution of the Cauchy problem (1.8), (1.15) with any deterministic initial function $u_0 \in V$.

Remark 2.3. For a general bounded domain $D \subset \mathbb{R}^3$, the condition of density of E_∞ in the space H is difficult to check. However, Theorem 2.1 is valid for any three-dimensional torus, and it is shown in [Shi06a] that if E contains the first N eigenvalues of the Stokes operator, then E_∞ is dense in H for sufficiently large N. We thus obtain a "uniform" version of the Main Theorem stated in the Introduction.

We now turn to the auxiliary result needed in the proof of Theorem 2.1. Let Z be a separable Banach space, let V be a Polish space, and let F be a finite-dimensional vector space. For any points $z_0 \in Z$ and $u_0 \in V$ and positive constants r_1 and r_2, we set

$$D(z_0, u_0) = \dot{B}_Z(z_0, r_1) \times \dot{B}_V(u_0, r_2).$$

Let $f : D(z_0, u_0) \to F$ be a continuous mapping. For any $\lambda \in \mathcal{P}(Z)$ and $\mu \in \mathcal{P}(V)$, denote by $f_*(\lambda, \mu)$ the image under f of the restriction of the product measure $\lambda \otimes \mu$ to $D(z_0, u_0)$. The following theorem is a modified version of a result established in [AKSS06]; its proof is given in the next subsection.

Theorem 2.4. *Let $f : D(z_0, u_0) \to F$ be a continuous mapping such that, for any $u \in \dot{B}_V(u_0, r_2)$, the function $f(\cdot, u) : \dot{B}_Z(z_0, r_1) \to F$ is continuously differentiable, the derivative $(D_z f)(z, u)$ is continuous on $D(z_0, u_0)$, and the image of the linear operator $(D_z f)(z_0, u_0)$ coincides with the entire space F. Let $\lambda \in \mathcal{P}(Z)$ and $\mu \in \mathcal{P}(V)$ be two measures such that Condition 1.15 holds for λ, and $\operatorname{supp} \mu = V$. Then there is a function $\rho \in C(F)$ such that $\rho > 0$ in a neighbourhood of $y_0 = f(z_0, u_0)$ and*

$$f_*(\lambda, \mu) \geq \rho \ell_F. \tag{2.3}$$

Furthermore, there is an open ball $B \subset V$ centred at u_0 and a compactly supported function $\psi \in C(F \times B)$, both of them not depending on μ, such that

$$\rho(y) = \int_B \psi(y, u)\, \mu(du) \quad \textit{for } y \in F, \tag{2.4}$$

$$\psi(y_0, u_0) > 0. \tag{2.5}$$

2.2 Proof of Theorem 2.4

We repeat the scheme used in [AKSS06] for the case of analytic functions and measures on Hilbert spaces. By assumption, the image of $A := (D_z f)(z_0, u_0)$ coincides with F and $\cup_n F_n$ is dense in Z. Therefore we can find an integer $m \geq 1$ and a subspace $F_m^1 \subset F_m$ of dimension $\dim F$ such that $A(F_m^1) = F$. Let us denote by $F_m^2 \subset F_m$ any subspace such that $F_m = F_m^1 \dotplus F_m^2$. Combining this with (1.22), we obtain the direct decomposition

$$Z = F_m^1 \dotplus F_m^2 \dotplus G_m.$$

For $z \in Z$, we shall write $z = (z_m, z_m') = (z_m^1, z_m^2, z_m')$, where $z_m^1 \in F_m^1$, $z_m^2 \in F_m^2$, $z_m \in F_m$, and $z_m' \in G_m$. Applying the implicit function theorem to the function $f(z,u)$ in the neighbourhood of (z_0,u_0), we can find open balls

$$V_1 \subset F_m^1, \quad V_2 \subset F_m^2, \quad V_3 \subset G_m, \quad B \subset V$$

such that, for any $z_m^2 \in V_2$, $z_m' \in V_3$, and $u \in B$, the mapping $f(\cdot, z_m^2, z_m', u)$ is a diffeomorphism of V_1 onto its image $W(z_m^2, z_m', u)$. Let $g(\cdot, z_m^2, z_m', u)$ be the inverse mapping, so that for $z_m^1 \in V_1$ and $y \in W(z_m^2, z_m', u)$, we have

$$y = f(z_m^1, z_m^2, z_m', u) \quad \text{if and only if} \quad z_m^1 = g(y, z_m^2, z_m', u). \tag{2.6}$$

Let us fix some bases in F_m^1 and F and denote by $d(z_m^1, z_m^2, z_m', u)$ the determinant of the derivative $(D_{z_m^1} f)(z_m^1, z_m^2, z_m', u)$. Now note that, in view of (1.23), the product measure $\lambda \otimes \mu$ on the space $Z \times V$ can be written as

$$(\lambda \otimes \mu)(dz, du) = \rho_m(z_m)\, dz_m\, \lambda_m'(dz_m')\, \mu(du)$$

where dz_m denotes the Lebesgue measure on F_m, ρ_m is the density of $\mathsf{P}_m\lambda$ with respect to dz_m, and $\lambda_m' = \mathsf{Q}_m\lambda$. It follows that if $\chi(z,u)$ is a continuous function on $Z \times V$ with support in the set $S := V_1 \times V_2 \times V_3 \times B$, then the image of the truncated measure $\chi(\lambda \otimes \mu)$ under f admits the representation (cf. Sections 2.2 and 2.3 in [AKSS06])

$$f_*\big(\chi(\lambda \otimes \mu)\big)(dy) = \left\{ \int \frac{(\widetilde{\chi}\tilde{\rho}_m)(y, z_m^2, z_m', u)}{|\tilde{d}(y, z_m^2, z_m', u)|} dz_m^2 \lambda_m'(dz_m')\mu(du) \right\} \ell_F(dy), \tag{2.7}$$

where the integral is taken over $V_2 \times V_3 \times B$,

$$\tilde{\rho}_m(y, z_m^2, z_m', u) = \rho_m(g(y, z_m^2, z_m', u), z_m^2),$$

and the functions $\widetilde{\chi}, \tilde{d}$ are defined in a similar way. Let us choose a continuous function χ supported by S and equal to 1 in the neighbourhood of (z_0, u_0) and denote

$$\psi(y,u) = \int_{V_2 \times V_3} \frac{(\widetilde{\varphi}\tilde{\rho}_m)(y, z_m^2, z_m', u)}{|\tilde{d}(y, z_m^2, z_m', u)|} dz_m^2 \lambda_m'(dz_m'), \quad (y,u) \in F \times B. \tag{2.8}$$

Then it follows from (2.7) that inequality (2.3) holds with the function ρ defined by (2.4). The continuity of the functions ψ and ρ is obvious from the explicit formulas for them. To prove (2.5), it suffices to note that the support of the measure $dz_m^2 \otimes \lambda_m'$ is the entire space $F_m^2 \times G_m$, and the integrand in (2.8) with $y = y_0$ and $u = u_0$ is positive on an open set. Finally, the positivity of ρ in the neighbourhood of y_0 follows from (2.5) and the fact that the support of μ coincides with V. The proof is complete.

3 Proof of the main theorem

In this section, we present a proof of Theorem 2.1. To make the main ideas more transparent, we first prove a weaker version of our result. Namely, we establish assertions (i) and (ii) of Theorem 2.1 for a given stationary solution, without caring about the uniformity of estimates. In the second subsection, we show how to modify the proof to obtain the result in full generality.

3.1 Simplified version: non-uniform estimates

We first explain the main idea. By definition, if $\mu \in \mathcal{P}(H)$ is a stationary measure for (1.8), then there is an admissible weak solution $u(t)$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with right-continuous filtration $\mathcal{F}_t$ such that $\mathcal{D}(u(t)) = \mu$ for any $t \geq 0$. Let us represent u in the form (1.13), where $z(t)$ is the Ornstein–Uhlenbeck process defined by (1.11) and $v(t)$ is an $\mathcal{F}_t$-progressively measurable random process whose almost every trajectory is a weak solution of (1.3). Let $J = [0, 1]$ and let

$$\Omega_0 = \big\{\omega \in \Omega : u(0) \in V, z \in C(\mathbb{R}_+, V) \cap L^2_{\mathrm{loc}}(\mathbb{R}_+, U)\big\}.$$

Then $\Omega_0 \in \mathcal{F}$ and $\mathbb{P}(\Omega_0) = 1$. If $\omega \in \Omega_0$, and Eq. (1.3) has a strong solution $\hat{v} \in \mathcal{Y}_J$ satisfying the initial condition $\hat{v}(0) = u_0$, where $u_0 = u(0)$, then Proposition 1.3 implies that

$$u(t) = z(t) + \mathcal{S}_t(z, u_0) =: \mathcal{T}_t(z, u_0) \quad \text{for } t \in J. \tag{3.1}$$

Here $\mathcal{S}$ denotes an operator taking $(z, v_0) \in \mathcal{Y}_J \times V$ to the solution $v \in \mathcal{Y}_J$ of problem (1.3), (1.21) and $\mathcal{S}_t$ stands for the restriction of $\mathcal{S}$ to the time t. What has been said implies that[2]

$$\mu(\Gamma) = \mathbb{P}\{u(1) \in \Gamma\} \geq \mathbb{P}\{\mathcal{T}_1(z, u_0) \in \Gamma\} \quad \text{for any } \Gamma \in \mathcal{B}(H). \tag{3.2}$$

Thus, Theorem 2.1 will be established if we show that assertions (i) and (ii) are valid for the distribution of the random variable $\mathcal{T}_1(u_0, z)$. The first of them is a simple consequence of the approximate controllability of the NS system, while the other will follow from the solid controllability in finite-dimensional projections and Theorem 2.4.

We now turn to the accurate proof. It is divided into several steps.

Step 1. Let us denote by $\lambda \in \mathcal{P}(\mathcal{Y}_J)$ the law of the restriction of the Ornstein–Uhlenbeck process (1.11) to the interval J and let $Z = \operatorname{supp} \lambda$. It is well known that λ is a Gaussian measure on $\mathcal{Y}_J$. As was explained in Example 1.16, the measure $\lambda \in \mathcal{P}(Z)$ satisfies Condition 1.15. We claim that, for any Borel subsets $B_1 \subset Z$ and $B_2 \subset V$ satisfying the inclusion

$$B_1 \times B_2 \subset \mathcal{C}_1(h), \tag{3.3}$$

we have

$$\mu \geq \mathcal{T}_{1*}(\lambda_{B_1}, \mu_{B_2}), \tag{3.4}$$

where the right-hand side of (3.4) stands for the image under $\mathcal{T}_1$ of the restriction of the product measure $\lambda \otimes \mu$ to the set $B_1 \times B_2$.

To prove (3.4), let us take any Borel set $\Gamma \subset H$ and write

$$\mu(\Gamma) = \mathbb{P}\{u(1) \in \Gamma\} \geq \mathbb{P}\{z \in B_1, u_0 \in B_2, u(1) \in \Gamma\}, \tag{3.5}$$

[2]Inequality (3.2) is not entirely accurate, because the operator $\mathcal{T}_1$ is defined only on an open subset of $\mathcal{Y}_J \times V$.

where $u_0 = u(0)$. In view of the weak-strong uniqueness (see Proposition 1.3) and inclusion (3.3), we have

$$u(1) = \mathcal{T}_1(z, u_0) \quad \text{for } z \in B_1,\ u_0 \in B_2.$$

Substituting this relation into (3.5) and using the independence of z and u_0, we obtain

$$\begin{aligned}\mu(\Gamma) &\geq \mathbb{P}\{z \in B_1, u_0 \in B_2, \mathcal{T}_1(z, u_0) \in \Gamma\}\\ &= \mathbb{E}\,\mathbb{P}\{z \in B_1, u_0 \in B_2, \mathcal{T}_1(z, u_0) \in \Gamma \,|\, \mathcal{F}_0\}\\ &= \mathbb{E}\Big(I_{B_2}(u_0)\,\mathbb{E}\left\{I_{B_1}(z) I_\Gamma(\mathcal{T}_1(z, v))\right\}\big|_{v=u_0}\Big),\end{aligned} \tag{3.6}$$

where I_Γ denotes the indicator function of Γ. Now note that

$$\mathbb{E}\left\{I_{B_1}(z) I_\Gamma(\mathcal{T}_1(z, v))\right\} = \mathcal{T}_{1*}(\lambda_{B_1}, v)(\Gamma) \quad \text{for any } v \in B_2, \tag{3.7}$$

where $\mathcal{T}_{1*}(\lambda_{B_1}, v)$ denotes the image of the restriction of λ to B_1 under the mapping $\mathcal{T}_1(\cdot, v)$. Substituting (3.7) into (3.6), we derive

$$\mu(\Gamma) \geq \mathbb{E}\big\{I_{B_2}(u_0)\mathcal{T}_{1*}(\lambda_{B_1}, u_0)(\Gamma)\big\} = \mathcal{T}_{1*}(\lambda_{B_1}, \mu_{B_2})(\Gamma).$$

Since Γ was arbitrary, we arrive at the required inequality (3.4).

Step 2. We now show that for any $\hat{u}_0 \in V$ and any ball $B \subset V$ there is $\hat{z} \in \operatorname{supp} \lambda$ such that

$$\mathcal{T}_1(\hat{z}, \hat{u}_0) \in B. \tag{3.8}$$

Indeed, in view of Proposition 1.11, for any $\hat{u}_0 \in V$ there exists $\hat{\eta} \in C^\infty(J, E)$ such that

$$\mathcal{R}_1(\hat{\eta}, \hat{u}_0) \in B. \tag{3.9}$$

Let us set

$$\hat{z}(t) = \int_0^t e^{-\nu(t-s)L}\hat{\eta}(s)\,ds.$$

It is a matter of direct verification to show that

$$\mathcal{R}_1(\hat{\eta}, \hat{u}_0) = \hat{z}(1) + \mathcal{S}_1(\hat{z}, \hat{u}_0) = \mathcal{T}_1(\hat{z}, \hat{u}_0), \tag{3.10}$$

and therefore (3.8) follows immediately from (3.9). To prove that $\hat{z} \in \operatorname{supp} \lambda$, first note that

$$\hat{z} = (M \circ I)(\hat{\eta}), \tag{3.11}$$

where we set

$$(I\xi)(t) = \int_0^t \xi(s)\,ds, \quad (M\xi)(t) = \int_0^t e^{-\nu(t-s)L}\partial_s\xi(s)\,ds. \tag{3.12}$$

Let us denote by $\mathcal{W}$ the law of the restriction of the process $Q\zeta$ to the interval J. Thus, $\mathcal{W}$ is a Gaussian measure on the space $L^2(J, V)$. In view of (1.11) and the second relation in (3.12), we have $\lambda = M_*(\mathcal{W})$, where $M_*(\mathcal{W})$ stands for the image of $\mathcal{W}$ under the linear operator M. By assumption, the image of Q contains E, and therefore $I\hat{\eta} \in \operatorname{supp} \mathcal{W}$. Recalling (3.11), we see that $\hat{z}$ is contained in the support of $M_*(\mathcal{W})$.

Step 3. Let us prove that

$$\mu(B) > 0 \quad \text{for any ball } B \subset V. \tag{3.13}$$

Fix a ball B and a point $\hat{u}_0 \in \operatorname{supp} \mu \cap V$. We choose $\hat{z} \in \operatorname{supp} \lambda$ such that (3.9) holds. Since the function $\mathcal{T}_1 : \mathcal{C}_1(h) \to V$ is continuous (see Proposition 1.12), we can find balls $B_1 \subset \mathcal{Y}_J$ and $B_2 \subset V$ satisfying (3.3) such that

$$\mathcal{T}_1(z, u_0) \in B \quad \text{for } z \in B_1,\ u_0 \in B_2, \tag{3.14}$$

$$\lambda(B_1) > 0, \quad \mu(B_2) > 0. \tag{3.15}$$

Combining (3.4) and (3.14), we obtain

$$\mu(B) \geq \mathcal{T}_{1*}(\lambda_{B_1}, \mu_{B_2})(B) = \lambda(B_1)\mu(B_2).$$

In view of (3.15), this implies the required inequality (3.13).

Step 4. We now turn to the proof of assertion (ii). Let us fix an arbitrary finite-dimensional subspace $F \subset H$. We claim that there is a sequence of balls $B_j = B_F(y_j, r_j)$ and functions $\varphi_j \in C^\infty(F)$ such that

$$\ell_F(F \setminus F_0) = 0, \tag{3.16}$$

$$\mu \geq \varphi_j \ell_F \quad \text{for all } j \geq 1, \tag{3.17}$$

$$\varphi_j(y) > 0 \quad \text{for } y \in B_j, \tag{3.18}$$

where $F_0 = \cup_j B_j$. If this claim is established, then the required result can be proved by the following simple argument.

Without loss of generality, we assume that $0 \leq \varphi_j \leq 1$. Let $\chi_j \in C^\infty(F)$ be such that $0 \leq \chi_j \leq 1$,

$$\chi_j(y) = 0 \text{ for } y \notin B_F(y_j, 2r_j), \qquad \chi_j(y) = 1 \text{ for } y \in B_F(y_j, r_j).$$

Let $\{\varepsilon_j\}$ be a sequence of positive constants such that

$$\sum_{j=1}^{\infty} \varepsilon_j = 1, \quad \sum_{j=1}^{\infty} \varepsilon_j \|\chi_j \varphi_j\|_{C^j} < \infty,$$

where $\|\cdot\|_{C^j}$ stands for the usual norm in the space of bounded C^j-smooth functions on F with bounded derivatives up to the order j. Then the function

$$\rho(y) := \sum_{j=1}^{\infty} \varepsilon_j \chi_j(y) \varphi_j(y)$$

is infinitely smooth and $\rho(y) > 0$ for $y \in F_0$. It follows from (3.16) that $\rho > 0$ almost everywhere on F. Furthermore, inequality (3.17) implies that

$$\mu(\Gamma) \geq \int_\Gamma \varphi_j(y)\ell_F(dy) \tag{3.19}$$

for any $j \geq 1$ and $\Gamma \in \mathcal{B}(F)$. Multiplying (3.19) by ε_j and summing up the resulting inequalities, we obtain

$$\mu(\Gamma) \geq \int_\Gamma \rho(y)\,\ell_F(dy) \quad \text{for any } \Gamma \in \mathcal{B}(F).$$

whence it follows that $\mu \geq \rho\ell_F$.

Step 5. To construct sequences $\{B_j\}$ and $\{\varphi_j\}$ satisfying (3.16) – (3.18), it suffices to prove that for any integer $r > 0$ there are countably many balls $B_k^r \subset F$ and functions $\varphi_k^r \in C^\infty(F)$ such that (3.17) and (3.18) hold with B_j and φ_j replaced by B_k^r and φ_k^r, respectively, and

$$\ell_F\big(B_F(r) \setminus \cup_k B_k^r\big) = 0. \tag{3.20}$$

If such sequences are constructed, then we can take the union with respect to all positive integers r and k to obtain the required sequences $\{B_j\}$ and $\{\varphi_j\}$.

Step 6. Let us set

$$f(z, u_0) = \mathsf{P}_F \mathcal{T}_1(z, u_0) \quad \text{for } (z, u_0) \in \mathcal{C}_1(h), \tag{3.21}$$

where z and u_0 are regarded as deterministic functions. By Proposition 1.12, the function f is analytic on its domain of definition $\mathcal{C}_1(h)$. We wish to apply Theorem 2.4 to f.

Let us fix a constant $r > 0$ and a point $\hat{u}_0 \in V \cap \operatorname{supp}\mu$. We claim that there is a finite-dimensional subspace $Z_0 \subset Z$ and an open subset $\mathcal{O} \subset Z_0$ such that $\mathcal{O} \subset \mathcal{C}_1(h)$ and

$$f(\mathcal{O}, \hat{u}_0) \supset B_F(r). \tag{3.22}$$

Indeed, by Proposition 1.11 (ii), there is a compact subset $\mathcal{K}$ in a finite-dimensional space $X_0 \subset C^\infty(J, E)$ such that $\mathcal{K} \subset \Theta_1(h, \hat{u}_0)$ and

$$\mathsf{P}_F \mathcal{R}_1(\mathcal{K}, \hat{u}_0) \supset B_F(r). \tag{3.23}$$

Let us denote by Z_0 the image of X_0 under the linear operator M defined in (3.12). Then Z_0 is a finite-dimensional subspace of Z and $M(\mathcal{K})$ is a compact subset of Z_0 contained in $\mathcal{C}_1(h)$. Furthermore, it follows from (3.10) and (3.23) that

$$f(M(\mathcal{K}), \hat{u}_0) \supset B_F(r).$$

Since $\mathcal{C}_1(h)$ is open, we conclude that (3.22) holds for a small neighbourhood $\mathcal{O}$ of the compact set $M(\mathcal{K})$.

Thus, the image of the smooth mapping

$$f(\cdot, \hat{u}_0) : \mathcal{O} \to F \tag{3.24}$$

contains the ball $B_F(r)$. By the Sard theorem, almost every point of $B_F(r)$ is regular[3] (see [Ste83]). Applying Theorem 2.4, for almost every $y_0 \in B_F(r)$ we can construct a function $\rho_{y_0} \in C(F)$ and a closed ball $B_{y_0} \subset F$ centred at y_0 such that

$$f_*(\lambda_{B_1}, \mu_{B_2}) \geq \rho_{y_0} \ell_F, \qquad \rho_{y_0}(y) > 0 \quad \text{for } y \in B_{y_0},$$

where $B_1 \subset \mathcal{O}$ and $B_2 \subset H$ are some balls such that $B_1 \times B_2 \subset \mathcal{C}_1(h)$. It is clear that χ_{y_0} can be minorised by a function $\varphi_{y_0} \in C^\infty(F)$ that is positive on B_{y_0}. Recalling inequality (3.4), we see that

$$\mu_F \geq \rho_{y_0} \ell_F \geq \varphi_{y_0} \ell_F. \tag{3.25}$$

We can now complete the construction of B_k^r and φ_k^r by a standard argument. Namely, let $\{K_n\}_{n \geq 1}$ be a sequence of compact subsets of $B_F(r)$ such that any point $y_0 \in K_n$ is regular for (3.24) and $\ell_F\big(B_F(r) \setminus \cup_n K_n\big) = 0$. Each set K_n can be covered by finitely many closed balls B_{in}, $i = 1, \dots, I_n$, such that

$$\mu_F \geq \varphi_{in} \ell_F \quad \text{for } i = 1, \dots, I_n,$$

where $\varphi_{in} \geq 0$ is a continuous function on F that is positive on B_{in}. The required families B_k^r and φ_k^r can be obtained by taking the union of B_{in} and φ_{in} over all $i = 1, \dots, I_n$ and $n \geq 1$. This completes the proof of assertions (i) and (ii) for a given stationary measure.

3.2 General case: uniform estimates

The derivation of uniform estimates is based on the following simple result. It shows that a compact subset of V carries some uniformly positive parts of all stationary measures μ satisfying (2.1).

Proposition 3.1. *For any $\mathfrak{m}_0 > 0$ there is a compact set $\mathcal{A} \subset V$ and a constant $\delta > 0$ such that*

$$\mu(\mathcal{A}) \geq \delta \quad \textit{for any stationary measure } \mu \textit{ satisfying } (2.1). \tag{3.26}$$

Proof. We first note that inequality (3.4) proved for $t = 1$ is true for any time $t = s > 0$. Namely, let us set $J_s = [0, s]$ and denote by λ^s the law for the restriction of the Ornstein–Uhlenbeck process (1.11) to the interval J_s and by $Z_s \subset C(J_s, V) \cap L^2(J_s, U)$ the support of λ_s. Then for any Borel subsets $B_1 \subset Z_s$ and $B_2 \subset V$ satisfying the inclusion $B_1 \times B_2 \subset \mathcal{C}_s(h)$ we have

$$\mu \geq \mathcal{T}_{s*}(\lambda^s_{B_1}, \mu_{B_2}). \tag{3.27}$$

[3]Recall that a point $y_0 \in F$ is said to be regular for (3.24) if the rank of the derivative $(D_z f)(z_0, \hat{u}_0)$ is maximal for any point $z_0 \in \mathcal{O}$ such that $f(z_0, \hat{u}_0) = y_0$.

Let $B_2 \subset V$ be a ball such that $\mu(B_2) \geq 1/2$ for any μ satisfying (2.1). Standard local existence results for the NS-type system (1.3) imply that we can choose $s > 0$ and a compact set $B_1 \subset Z_s$ of positive λ^s-measure such that $B_1 \times B_2 \subset \mathcal{C}_s(h)$ (for instance, see [FK64] or [Tay97]). Furthermore, it follows from the regularising property of the resolving operator for (1.3) that $\mathcal{T}_s(B_1 \times B_2)$ is contained in a compact subset $\mathcal{A}$ of V. Inequality (3.27) now implies that

$$\mu(\mathcal{A}) \geq \lambda^s(B_1)\mu(B_2) \geq \frac{1}{2}\lambda^s(B_1).$$

It remains to note that the right-hand side of this inequality is positive and does not depend on μ. $\square$

We now turn to the proof of (2.2). Repeating the argument used in Step 3 of Subsection 3.1, for any $\hat{u}_0 \in \mathcal{A}$ we can find open balls $B_1(\hat{u}_0) \subset \mathcal{Y}_J$ and $B_2(\hat{u}_0) \subset V$ such that $B_1(\hat{u}_0) \times B_2(\hat{u}_0) \subset \mathcal{C}_1(h)$ and

$$\mathcal{T}_1(z, u_0) \in B \quad \text{for } z \in B_1(\hat{u}_0),\ u_0 \in B_2(\hat{u}_0), \tag{3.28}$$

$$\lambda(B_1(\hat{u}_0)) > 0. \tag{3.29}$$

The family $\{B_2(\hat{u}_0), \hat{u}_0 \in \mathcal{A}\}$ forms an open covering for the compact set $\mathcal{A}$, and we can choose a finite subcovering $\{B_2^i\}_{i=1}^N$. Denote by $\{B_1^i\}_{i=1}^N$ the corresponding set of balls in the space $\mathcal{Y}_J$.

Now let $\mu \in \mathcal{P}(H)$ be a stationary measure for (1.8) that satisfies (2.1). Then it follows from (3.26) that

$$\mu(B_2^j) \geq N^{-1}\delta \quad \text{for some } j.$$

Combining this inequality with (3.28) and (3.29), we see that

$$\mu(B) \geq \mathcal{T}_{1*}(\lambda_{B_1^j}, \mu_{B_2^j}) \geq \lambda(B_1^j)\,\mu(B_2^j) \geq N^{-1}\delta \min_{1\leq i\leq N} \lambda(B_1^i).$$

It remains to note that the right-hand side of this inequality is positive and does not depend on μ.

Let us prove assertion (ii) with a function ρ_F not depending on μ. As is shown in Steps 4–6 of Subsection 3.1, it suffices to prove that for any $r > 0$ and almost every point $y_0 \in B_F(r)$ there is a function $\varphi_{y_0} \in C^\infty(F)$, depending only on $\mathfrak{m}_0$ and positive at y_0, such that

$$\mu_F \geq \varphi_{y_0}\ell_F. \tag{3.30}$$

Repeating the argument used in Step 6 of Subsection 3.1 and applying Theorem 2.4, for almost every $y_0 \in B_F(r)$ we can construct open balls $B_1(y_0) \subset Z$ and $B_{y_0} \subset B_2(y_0) \subset V$, a constant $\delta > 0$, and a compactly supported function $\psi_{y_0} \in C(F \times B_{y_0})$ such that

$$\psi_{y_0}(y_0, u_0) > 0 \tag{3.31}$$

$$f_*(\lambda_{B_1(y_0)}, \mu_{B_2(y_0)}) \geq \ell_F \int_{B_{y_0}} \psi_{y_0}(y, u)\mu(du), \tag{3.32}$$

where u_0 denotes the centre of B_{y_0}. Let us fix a closed ball $Q_{y_0} \subset B_{y_0}$ centred at u_0 and set

$$\delta(y) := \inf_{u \in Q_{y_0}} \psi_{y_0}(y, u).$$

It is clear that δ is a Borel function. In view of (3.31), we can choose Q_{y_0} so small that

$$\delta(y) \geq \delta_0 \quad \text{for } y \in \mathcal{O}_{y_0}, \tag{3.33}$$

where $\delta_0 > 0$ is a constant and $\mathcal{O}_{y_0}$ is a ball centred at y_0. Combining (3.4), (2.2), and (3.32), we obtain

$$\mu_F(dy) \geq p(Q_{y_0}, \mathfrak{m}_0)\, \delta(y)\, \ell_F(dy). \tag{3.34}$$

In view of (3.33), we can minorise $p(Q_{y_0}, \mathfrak{m}_0)\delta$ by a function $\varphi_{y_0} \in C^\infty(F)$ that is positive at y_0. Inequality (3.30) is now implied by (3.34). This completes the proof of Theorem 2.1 in the general case.

4 Appendix

4.1 Proof of Proposition 1.3

A standard limiting argument shows that if $v \in \mathcal{X}_J$ is a weak solution for (1.3), then relation (1.4) is true for any function $\varphi \in \mathcal{Y}_J$ such that $\dot\varphi \in L^2_{\mathrm{loc}}(J, H)$ and $\varphi(0) = \varphi(T) = 0$. In particular, we can take $\varphi = \chi_k \tilde v$, where $\{\chi_k\}$ is the sequence of functions defined in Section 1.1. We thus obtain

$$-\int_0^t (v, \partial_s(\chi_k \tilde v))\, ds + \int_0^t \chi_k(s)\big(\nu(v, \tilde v)_V + (B(v+z), \tilde v) - (h, \tilde v)\big)\, ds = 0. \tag{4.1}$$

Furthermore, a similar argument shows that, in identity (1.4) for the strong solution $\tilde v$, we can take $\varphi = \chi_k v$. This results in

$$-\int_0^t (\tilde v, \partial_s(\chi_k v))\, ds + \int_0^t \chi_k(s)\big(\nu(\tilde v, v)_V + (B(\tilde v+z), v) - (h, v)\big)\, ds = 0. \tag{4.2}$$

Taking the sum of (4.1) and (4.2) and passing to the limit as $k \to \infty$, we obtain

$$\begin{aligned}(v(t), \tilde v(t)) + \int_0^t \big(2\nu(\tilde v, v)_V + (B(v+z), \tilde v) + (B(\tilde v+z), v)\big)\, ds \\ = (v(0), \tilde v(0)) + \int_0^t (h, \tilde v + v)\, ds.\end{aligned} \tag{4.3}$$

Adding together relations (1.5) and (1.6) (with v replaced by $\tilde v$), subtracting (4.3), and carrying out some simple transformations, we derive

$$\frac{1}{2}\|w\|^2 + \int_0^t \big(\nu\|w\|_V^2 + (B(w, z+\tilde v), w)\big)\, ds \leq 0, \tag{4.4}$$

where $w = v - \tilde{v}$. In view of standard estimates for the nonlinear term B and Sobolev embedding theorems, we have

$$\begin{aligned}|(B(w, z + \tilde{v}), w)| &\leq C_1 \|w\|_{L^3}^2 \big(\|\nabla z\|_{L^3} + \|\nabla \tilde{v}\|_{L^3}\big)\\ &\leq C_2 \|w\|_V \|w\| \big(\|z\|_U + \|\tilde{v}\|_U\big)\\ &\leq \nu \|w\|_V^2 + C_3 \|w\|^2 \big(\|z\|_U^2 + \|\tilde{v}\|_U^2\big).\end{aligned}$$

Substituting this inequality into (4.4), we see that

$$\|w(t)\|^2 \leq C_4 \int_0^t \|w(s)\|^2 \big(\|z(s)\|_U^2 + \|\tilde{v}(s)\|_U^2\big)\, ds.$$

Application of the Gronwall inequality shows that $w = v - \tilde{v} \equiv 0$.

4.2 Proof of Lemma 1.5

Let η be a random process satisfying Condition 1.4 and let ζ be the corresponding H-valued cylindrical Wiener process. Then there is an orthonormal basis $\{e_j\}$ in H and a sequence of independent standard Brownian motions $\{\beta_j\}$ such that

$$\zeta(t) = \sum_{j=1}^{\infty} \beta_j(t) Q e_j,$$

where the series converges in $L^2(\Omega \times [0, T], H)$ for any $T > 0$. Using the polar decomposition for Q (see [RS80]), we can rewrite ζ as

$$\zeta(t) = \sum_{j=1}^{\infty} \beta_j(t) A U e_j = \sum_{j,k=1}^{\infty} b_k \beta_j(t) (U e_j, f_k)_V f_k, \tag{4.5}$$

where $U : H \to V$ is a partial isometry, A is a Hilbert–Schmidt selfadjoint operator in V with eigenbasis $\{f_k\}$ and eigenvalues $\{b_k\}$, and $(\cdot, \cdot)_V$ denotes the scalar product in V. Setting $b_{jk} = b_k (U e_j, f_k)_V$ and differentiating (4.5) with respect to t, we obtain (1.10).

Conversely, suppose that η is representable in the form (1.10). Choose an arbitrary orthonormal basis $\{e_j\}$ in H and define a Hilbert–Schmidt operator $Q : H \to V$ by the relations

$$Q e_j = \sum_{k=1}^{\infty} b_{jk} f_k, \quad j \geq 1.$$

We can rewrite (1.10) as

$$\eta(t) = \sum_{j=1}^{\infty} \dot{\beta}_j(t) Q e_j.$$

This is equivalent to representation (1.9).

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