Orbital Angular Momentum in Nelson's Stochastic Mechanics

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Abstract

We provide numerical examples of integer-valued functionals of Nelson's stochastic processes. More precisely, we consider stochastic motion, according to Nelson's form of Newton's second law of dynamics, in a magnetic field having an axis z of cylindrical symmetry and a gradient in the direction of this axis. We show that there are two sets of functionals of the stochastic process having the same law as the component of quantum mechanical angular momentum along z. The functionals of the first set involve the z coordinate of the process and correctly model the behaviour of the "needle of the measuring apparatus". The functionals of the second set involve the coordinates in a plane orthogonal to z and strongly suggest the possibility of a stochastic model of the collapse of the "system" toward the state indicated by the "needle".

1. Introduction

Of the many possible pedagogical introductions to stochastic mechanics [1], the one that best fits with our expository strategy was given by Nelson himself in [2]. We reproduce it here with some notational change and some minor addition.

Consider the stochastic process $\{q_0(t), t \geq 0\}$ defined by $q_0(t) = X_0 + \sigma \cdot W(t)$, where $\sigma > 0$ is a constant, $X_0 \sim N(0,\alpha)$ (namely X_0 is a normal random variable of mean 0 and variance α) and $\{W(t)\ , t \geq 0\}$ is a standard Brownian motion (namely, a normal process with $W(0) = 0$, and covariance function given by $\operatorname{cov}(W(s)\ ,\ W(t)) = \min(s,t)$).

Define, for diffusion processes $q(t)$ with constant and assigned diffusion coefficient $\sigma^2 \equiv \lim_{h \to 0^+} E_t\left((q(t+h) - q(t))^2 / h\right)$, the mean forward velocity and the mean backward velocity, respectively by:

$$b(t, q(t)) \equiv \lim_{h \to 0^+} E_t\left(\frac{q(t+h) - q(t)}{h}\right)$$

$$b^*(t, q(t)) \equiv \lim_{h \to 0^+} E_t\left(\frac{q(t) - q(t-h)}{h}\right)$$

(Here E_t indicates the conditional expectation given the value of the process at time t).

The above limits are easily shown to exist for the process $q_0(t)$: they are given, respectively, by:

$$b_0(t, q_0(t)) = 0$$

and

$$b_0^*(t, q_0(t)) = \frac{\sigma^2 \cdot q_0(t)}{\alpha + \sigma^2 \cdot t} \quad .$$

Having set, more generally, for a function $F(t, q(t))$,

$$DF(t, q(t)) = \lim_{h \to 0^+} E_t\left(\frac{F(t+h, q(t+h)) - F(t, q(t))}{h}\right)$$

$$D^*F(t, q(t)) \equiv \lim_{h \to 0^+} E_t\left(\frac{F(t, q(t)) - F(t-h, q(t-h))}{h}\right)$$

one can inquire about the mean acceleration of the process, defined by

$$a(t, q(t)) = \frac{D^* b(t, q(t)) + D b^*(t, q(t))}{2}.$$

The limit defining the above quantity is easily shown to exist for the process $q_0(t)$ and to be given explicitly by:

$$a_0\left(t, q_0(t)\right) = -\frac{\sigma^4 \cdot q_0(t)}{2 \cdot \left(\alpha + \sigma^2 \cdot t\right)^2}$$

The process $q_0(t)$ (namely Brownian motion with initial condition $N(0,\alpha)$) appears therefore as a particular solution of the problem of finding the diffusion processes $q(t)$ satisfying the conditions:

$$\begin{cases} \lim\limits_{h \to 0^+} E_t\left(\dfrac{\left(q(t+h) - q(t)\right)^2}{h} \right) = \sigma^2 \\ \dfrac{D^* D q(t) + D D^* q(t)}{2} = -\dfrac{\sigma^4 \cdot q(t)}{2 \cdot \left(\alpha + \sigma^2 \cdot t\right)^2} \end{cases} \tag{1}$$

As an exercise in the theory of diffusion processes one can pose the problem of finding <u>all</u> the solutions of the problem posed by system (1). Such a problem is easily solved according to the following three steps:

Step 1: Introduce the current velocity field $v(t,x) = \left(b(t,x) + b^*(t,x)\right)/2$ and ask for those solutions of (1) which have the same velocity field

$$v_0(t,x) = \frac{\sigma^2 \cdot x}{2 \cdot \left(\alpha + \sigma^2 \cdot t\right)}$$

as the process $q_0(t)$.

It is easy to show that this intermediate problem has a countable infinity $q_n(t)$, $n = 0,1,\ldots$ of solutions, the probability density of $q_n(t)$ being given by

$$\rho_n(t,x) = \frac{1}{\sqrt{2 \cdot \pi \cdot \left(\alpha + \sigma^2 \cdot t\right)}} \cdot \exp\left(-\frac{x^2}{2 \cdot \left(\alpha + \sigma^2 \cdot t\right)} \right) \cdot \frac{1}{2^n \cdot n!} \cdot H_n\left(\frac{x}{\sqrt{2 \cdot \left(\alpha + \sigma^2 \cdot t\right)}} \right)^2$$

where H_n is the $n-th$ Hermite polynomial.

Step 2: Introduce the osmotic "velocity" field $u(t,x) = \left(b(t,x) - b^*(t,x)\right)/2$ and write system (1) as a system of differential equations in u and v:

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} + \dfrac{\partial v(t,x)}{\partial x} u(t,x) + \dfrac{\partial u(t,x)}{\partial x} v(t,x) + \dfrac{\sigma^2}{2} \dfrac{\partial^2 v(t,x)}{\partial x^2} = 0 \\[2ex] \dfrac{\partial v(t,x)}{\partial t} + \dfrac{\partial v(t,x)}{\partial x} v(t,x) - \dfrac{\partial u(t,x)}{\partial x} u(t,x) - \dfrac{\sigma^2}{2} \dfrac{\partial^2 u(t,x)}{\partial x^2} = a(t,x) \end{cases} \tag{2}$$

where, in the case at hand:

$$a(t,x) = a_0(t,x) = -\frac{\sigma^4 \cdot x}{2 \cdot \left(\alpha + \sigma^2 \cdot t\right)^2} \tag{3}$$

As it is well known [1], setting

$$a(t,x) = -\frac{\partial V(t,x)}{\partial x},$$

$$u(t,x) = \frac{\partial R(t,x)}{\partial x},$$

$$v(t,x) = \frac{\partial S(t,x)}{\partial x},$$

system (2) transforms into the <u>linear</u> equation

$$i \cdot \sigma^2 \cdot \frac{\partial \varphi}{\partial t} = -\frac{\sigma^4}{2} \cdot \frac{\partial^2 \varphi}{\partial x^2} + V(t,x) \cdot \varphi \tag{4}$$

in the unknown function $\varphi(t,x) = e^{R(t,x) + i \cdot S(t,x)}$. In particular, problem (1) reduces to the linear equation

$$i \cdot \sigma^2 \cdot \frac{\partial \varphi}{\partial t} = -\frac{\sigma^4}{2} \cdot \frac{\partial^2 \varphi}{\partial x^2} + \frac{\sigma^4 \cdot x^2}{4 \cdot \left(\alpha + \sigma^2 \cdot t\right)^2} \cdot \varphi \tag{5}$$

The Brownian motion $q_0(t)$, as already pointed out in [2], appears, therefore, as a particular solution of (5). Other solutions of (5) have, in fact, been found in Step 1: they are given explicitly by:

$$\varphi_n(t,x) = \frac{1}{\left(2 \cdot \pi \cdot \left(\alpha + \sigma^2 \cdot t\right)\right)^{1/4}} \cdot \frac{1}{\left(2^n \cdot n!\right)^{1/2}} \cdot H_n\left(\frac{x}{\sqrt{2 \cdot \left(\alpha + \sigma^2 \cdot t\right)}}\right) \cdot$$
$$\cdot \exp\left(-\frac{x^2}{4 \cdot \left(\alpha + \sigma^2 \cdot t\right)} + i \cdot \left(\frac{x^2}{4 \cdot \left(\alpha + \sigma^2 \cdot t\right)} - \frac{1 + 2 \cdot n}{4} \cdot \ln\left(\frac{\alpha + \sigma^2 \cdot t}{\alpha}\right)\right)\right)$$

Step 3: The correspondence set in the previous step between solutions $q(t)$ of (1) and solutions $\varphi(t,x)$ of (5) requires in particular that $q(t)$ have probability density $\rho(t,x)=|\varphi(t,x)|^2$. Because of the linearity of equation (5) and of orthonormality and completeness of Hermite functions, every solution of problem (1) corresponds to a normalised linear combination of the above solutions $\varphi_n(t,x)$ (with coefficients determined, for instance, by the initial conditions $v(0,x)$ and $\rho(0,x)$ imposed on the current velocity field and on the probability density field).

The paradigmatic example proposed in [2], the presentation of which has been given -and somewhat enriched- above, helps set the stage of the problem we wish to address here.

As we have explicitly observed in our particular example, the solution of the problem of finding the diffusion processes having an assigned diffusion coefficient and an assigned mean acceleration, written here as $a(t,x)=-\frac{1}{m}\frac{\partial V(t,q(t))}{\partial x}$, is equivalent to the solution of the problem

$$i\cdot(m\cdot\sigma^2)\cdot\frac{\partial\varphi}{\partial t}=-\frac{(m\cdot\sigma^4)}{2}\cdot\frac{\partial^2\varphi}{\partial x^2}+V(t,x)\cdot\varphi;$$

it is, therefore, a fact that the mathematical problem of solving in $L^2(\mathbb{R})$ the Schrödinger equation

$$i\cdot\hbar\cdot\frac{\partial\varphi}{\partial t}=-\frac{\hbar^2}{2\cdot m}\cdot\frac{\partial^2\varphi}{\partial x^2}+V(t,x)\cdot\varphi$$

is equivalent to the mathematical problem of solving, in the class of diffusion processes, the system

$$\begin{cases}\lim\limits_{h\to0^+}E_t\left(\dfrac{(q(t+h)-q(t))^2}{h}\right)=\dfrac{\hbar}{m}\\[2ex] m\cdot\dfrac{D^*Dq(t)+DD^*q(t)}{2}=-\dfrac{\partial V(t,q(t))}{\partial x}\end{cases}$$

Much harder is the physical question whether quantum mechanics can be replaced by a stochastic description in terms of stochastic processes. The paradigmatic example discussed above helps focusing some of the issues raised by this question; for instance, one would not say that the Brownian motion $\{q_0(t),t\geq0\}$ is a quantum mechanical phenomenon only because it appears as a particular solution of the linear problem (5). The main point is (even not to consider the issue of the numerical value of the diffusion coefficient) that there seems to be, in the phenomenological context described by the Brownian motion $q_0(t)$, no useful role for other solutions $q_n(t)$ of the same problem (5); moreover, the fact that problem (5) is linear is of no practical relevance because there seems to be no useful application, for instance, of the processes corresponding to linear combinations of the functions $\varphi_n(t,x)$.

For the same reason, the observation made in Step 1 that problem (5), under the additional constraint $v(t,x) = v_0(t,x)$, admits a countable infinity of solutions does <u>not</u> mean that there is a "quantized" observable - number of nodes in the probability density - of relevance in the phenomenological context of Brownian motion.

The point is here that quantum mechanics is not only the (unrestrictedly linear) Schrödinger theory of the evolution of the initial condition $\varphi(t_0,x)$ at a given initial instant t_0 into $\varphi(t,x)$ at any instant t, but also an interpretative scheme of the meaning of $\varphi(t_0,x)$ at each fixed t_0.

This interpretative scheme is by no means exhausted by the statement that $|\varphi(t_0,x)|^2$ is the probability density of the position observable (on this statement quantum mechanics and stochastic mechanics agree, essentially by construction of the latter).

Part of the interpretative scheme of $\varphi(t_0,\cdot)$ is, for instance, also the statement, that we make here in the simplest possible context of a particle moving on the real line $\mathbb{R}$, that the random variable, call it $p(\varphi(t_0,\cdot))$, defined by having characteristic function

$$\chi_{p(\varphi(t_0,\cdot))}(\tau) = \int_{-\infty}^{+\infty} \overline{\varphi(t_0,x)} \cdot \varphi(t_0,x+\tau)dx$$

has some relevance in the description of the behaviour of the process having current velocity field and probability density field at time t_0 determined by $\varphi(t_0,\cdot)$ and that, for "some" reason, it deserves the name of "linear momentum".

In much the same way, part of the interpretative scheme of $\varphi(t_0,\cdot)$ is also the statement, that we make here in the simplest possible context of a particle moving on the plane $\mathbb{R}^2$, that the random variable, call it $L_3(\varphi(t_0,\cdot))$, defined by having characteristic function

$$\chi_{L_3(\varphi(t_0,\cdot))}(\tau) = \int_0^{+\infty} r \cdot dr \int_0^{2\cdot\pi} d\vartheta \;\; \overline{\varphi(t_0,x_1(r,\vartheta),x_2(r,\vartheta))} \cdot \varphi(t_0,x_1(r,\vartheta+\tau),x_2(r,\vartheta+\tau))$$

has some relevance in the description of the behaviour of the system described at time t_0 by $\varphi(t_0,\cdot)$ and that, for "some" reason, it deserves the name of "orbital angular momentum with respect to the origin".

The question we raise in this paper is the following: having fixed t_0, having fixed $\varphi(t_0,\cdot)$, having determined, according to the usual procedure, the current velocity field $v(t_0,x)$ and the probability density field $\rho(t_0,x)$, is there a diffusion process $\{q(t), t \geq t_0\}$ evolving from the initial condition determined by $v(t_0,x)$ and $\rho(t_0,x)$, a suitable functional of which has the same law as $L_3(\varphi(t_0,\cdot))$? Having found such a process $\{q(t),t \geq t_0\}$ and such a functional $L(q(t)\ ,t \geq t_0)$ does the "force" $m \cdot a(t,q(t))$ under which this process evolves justify the attribution to $L(q(t)\ ,t \geq t_0)$ and, therefore to $L_3(\varphi(t_0,\cdot))$, of the name "orbital angular momentum with respect to the origin" ? By this we mean: under <u>classical</u> evolution $\{x(t)\ ,\ t \geq t_0\}$ under the force $m \cdot a(t,x(t))$, does $L(x(t)\ ,t \geq t_0)$ assume the value of orbital angular momentum competing to the initial condition $x(t_0)\ ,\dot{x}(t_0)$?

Figure 1 shows an example of the fact that the above questions admit a positive answer; in particular it shows that there do exist integer-valued functionals of Nelson's processes. It refers to a numerical simulation of the stochastic process

$$q(t) = (q_1(t), q_2(t), q_3(t))$$

corresponding, in the sense of Nelson [1, 2, 3], to a Schrödinger wave function evolving in a magnetic field having a gradient in the x_3 direction.

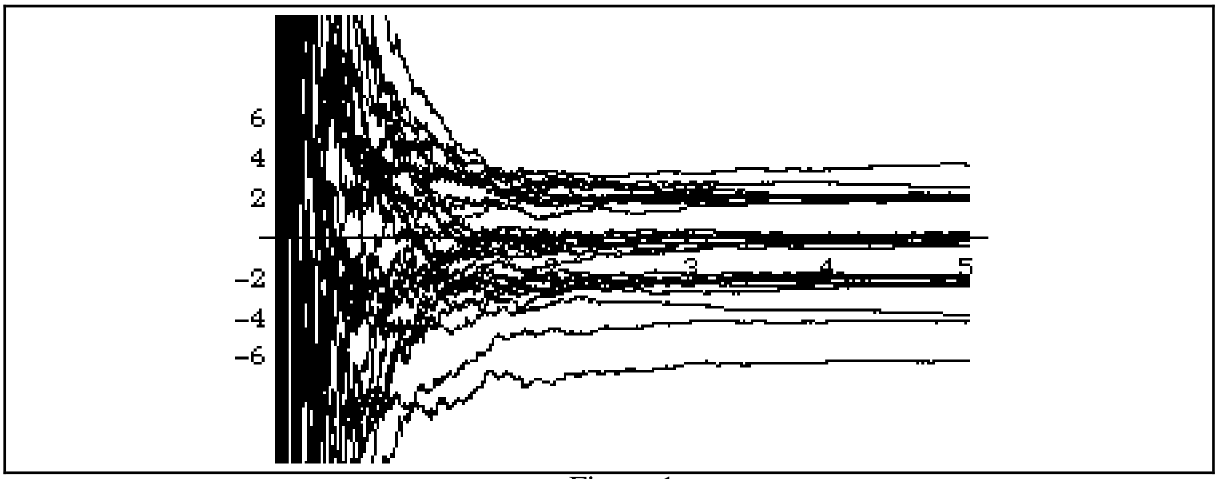

Figure 1

$\dfrac{q_3(t)}{\varepsilon \cdot t^2/2}$ *as a function of* t

We shall give a detailed description of the model - an idealised version of a Stern-Gerlach apparatus [4] - in Section 2. Here we wish to point out that Figure 1 shows that the average acceleration ~ $q_3(t)/t^2$ in the direction of the field gradient tends to a discrete random variable as t increases; this fact closely parallels, in the case of orbital angular momentum considered here, the results obtained by Faris [5] in the case of spin angular momentum. As only a configurational observable $q_3(t)$ is involved in this phenomenon, and as the predictions of quantum mechanics and stochastic mechanics coincide on such an observable, it will be, in Section 2, a simple exercise in quantum mechanics to prove, in our context, the generality of the limiting behaviour shown in Figure 1.

The configuration space for the process we consider is $\mathbb{R}^3$, as opposed to $\mathbb{R}^3 \times SU(2)$ in the case of Faris [5]. The simplicity we gain in this more elementary context allows us to explore the question whether there are, beyond $q_3(t)/t^2$, other functionals of the processes exhibiting the same asymptotic behaviour. Figure 2 shows that this is the case: it follows the motion in the x_1, x_2 plane in terms of the time dependence of a stochastic analogue

$$\lambda_3(t) = q_1(t) \cdot p_2(t, q(t)) - q_2(t) \cdot p_1(t, q(t))$$

of the x_3 component of classical canonical angular momentum and shows that also $\lambda_3(t)$ tends to a discrete random variable. Section 3 will be devoted to a construction, in a simple context, of the process $q(t)$ and to the definition of the random variables $p_j(t, q(t))$ in terms of the current velocity field associated with $q(t)$ and of the vector potential $A(x)$.

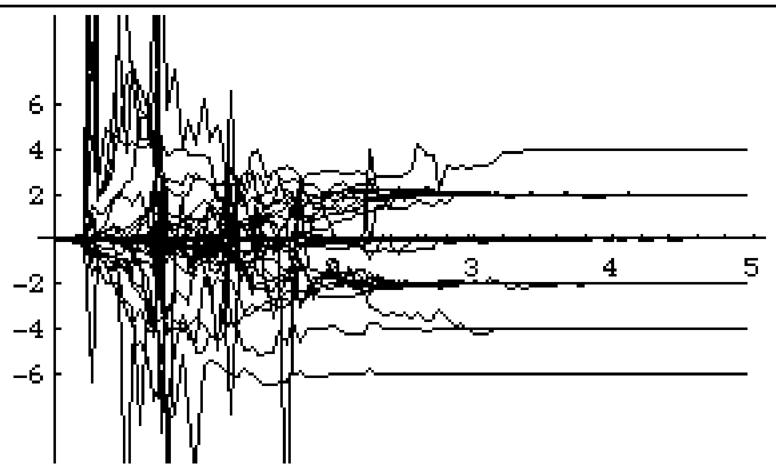

Figure 2

$q_1(t) \cdot p_2(t, q(t)) - q_2(t) \cdot p_1(t, q(t))$ *as a function of* t

In Section 4 we shall comment on the line of reasoning leading to the previous observations, in the framework of the following statement by R. Feynman and A. Hibbs: "Indeed all measurements of quantum mechanical systems could be made to reduce eventually to position and time measurements (e.g. the position of a needle on a meter or the time of flight of a particle). Because of this possibility a theory formulated in terms of position measurements is complete enough in principle to describe all phenomena" [6, p. 96]. We shall argue that, in modelling the measurement of a component of orbital angular momentum, the obvious pointer variable $q_3(t)/t^2$ is not the only configurational observable that conforms to the above point of view; we shall give, indeed, evidence of the fact that, though $\lambda_3(t)$ makes reference to the current velocity field of the process, its limit as $t \to +\infty$ coincides with the limit of the purely configurational random variable

$$\Lambda_3(t) = \frac{1}{t}\int_0^t q_1(s) \cdot dq_2(s) - q_2(s) \cdot dq_1(s) + \frac{1}{t}\int_0^t \left(q_1(s) \cdot A_2(q(s)) - q_2(s) \cdot A_1(q(s))\right) \cdot ds$$

This last observation supports the conjecture, advanced in [7] in an attempt of understanding Bohr quantization in the stochastic context, that in Nelson's stochastic mechanics one can give a Keplerian definition of angular momentum in terms of "stochastic area per unit time", with, of course, the purely classical correction given by the field dependent addendum.

2. The model

The quantum mechanical predictions on the result of a measurement of the observable

$$L_3 = \frac{1}{i}\left(x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}\right)$$

on a spinless particle in the state described by the wave function $\psi_0(x_1, x_2, x_3)$ are summarised by the random variable, that we shall call $L_3(\psi_0)$, having characteristic function

$$\chi_{L_3(\psi_0)}(\tau) \equiv E\left(e^{i\cdot\tau\cdot L_3(\psi_0)}\right) = \left\langle \psi_0, e^{i\cdot\tau\cdot L_3} \psi_0 \right\rangle.$$

If one introduces the Hamiltonian

$$H_{\omega,\varepsilon} = h_{1,2} + h_3 \tag{6}$$

where

$$h_{1,2} = \frac{1}{2} p_1^{\ 2} + \frac{1}{2} p_2^{\ 2} + \frac{1}{2}\omega^2 \cdot \left(q_1^{\ 2} + q_2^{\ 2}\right)$$

$$h_3 = \frac{1}{2} p_3^{\ 2} - \varepsilon \cdot q_3 \cdot \left(q_1 \cdot p_2 - q_2 \cdot p_1\right)$$

and

$$p_j = \frac{1}{i}\frac{\partial}{\partial x_j}, \quad j = 1,\ 2,\ 3 \quad ,$$

and sets

$$\psi_t = e^{-i\cdot t\cdot H_{\omega,\varepsilon}} \psi_0 ,$$

it is elementary to check that

$$\left\langle \psi_0, e^{i\cdot\tau\cdot L_3} \psi_0 \right\rangle = \lim_{t\to\infty} \left\langle \psi_t, \exp\left(i \cdot \tau \cdot \frac{q_3}{\varepsilon \cdot t^2/2} \right) \psi_t \right\rangle \tag{7}$$

The proof of this statement requires only the expansion

$$\psi_0(x_1,x_2,x_3)=\sum_{n_1,n_2}\frac{\varphi_{n_1,n_2}(x_1,x_2)}{\sqrt{2\cdot\pi}}\int_{-\infty}^{+\infty}\gamma_{n_1,n_2}(p)\cdot e^{i\cdot p\cdot x_3}dp$$

of the initial condition ψ_0 in terms of a complete orthonormal set $\left\{\varphi_{n_1,n_2}\right\}_{\substack{n_1=0,1,\ldots\\ n_2=0,1,\ldots}}$ of simultaneous eigenfunctions of $h_{1,2}$ and L_3 satisfying

$$\begin{aligned} h_{1,2}\varphi_{n_1,n_2} &= (n_1+n_2+1)\cdot\omega\cdot\varphi_{n_1,n_2} \\ L_3\varphi_{n_1,n_2} &= (n_1-n_2)\cdot\varphi_{n_1,n_2}. \end{aligned}$$

The observation that, on each component of angular momentum n_1-n_2, the evolution according to $H_{\omega,\varepsilon}$ reduces to the easily solvable problem of one-dimensional motion in a constant force field $f_{n_1,n_2}=\varepsilon\cdot(n_1-n_2)$ gives, then, the explicit expression

$$\begin{aligned} &\left\langle\psi_t,\exp\left(i\cdot\tau\cdot\frac{q_3}{\varepsilon\cdot t^2/2}\right)\psi_t\right\rangle= \\ &=\sum_{n_1,n_2}e^{i\cdot\tau\cdot(n_1-n_2)}\cdot\int_{-\infty}^{+\infty}\overline{\gamma_{n_1,n_2}(p)}\cdot\gamma_{n_1,n_2}\left(p-\frac{\tau}{\varepsilon\cdot t^2/2}\right)\cdot e^{2\cdot i\cdot\left(\frac{p\cdot\tau}{\varepsilon\cdot t}-\frac{\tau^2}{\varepsilon^2\cdot t^3}\right)}dp \end{aligned} \tag{8}$$

from which the limit relation (7) immediately follows.

It is instructive to consider the particular case of a Gaussian initial distribution

$$\psi_0(x_1,x_2,x_3)=\prod_{j=1}^{3}\frac{1}{(2\cdot\pi\cdot\alpha_j)^{1/4}}\exp\left(-\frac{x_j^{\,2}}{4\cdot\alpha_j}\right) \tag{9}$$

(all numerical examples in this paper will refer to this particular case; moreover, we shall set, in what follows, $m=1,\ \hbar=1$). The right hand side of (8) can be explicitly computed in this case:

$$\left\langle\psi_t,\exp\left(i\cdot\tau\cdot\frac{q_3}{\varepsilon\cdot t^2/2}\right)\psi_t\right\rangle=\chi_{L_3(\psi_0)}(\tau)\cdot\exp\left(-\frac{\tau^2}{2}\cdot\frac{4\cdot\alpha_3^{\,2}+t^2}{\varepsilon^2\cdot\alpha_3\cdot t^4}\right) \tag{10}$$

where

$$\chi_{L_3(\psi_0)}(\tau) = \frac{1}{\sqrt{1 + \frac{(\alpha_1 - \alpha_2)^2}{4 \cdot \alpha_1 \cdot \alpha_2 \cdot sin^2 \tau}}}$$

Namely, for each finite t, the distribution of the pointer variable $\frac{q_3}{\varepsilon \cdot t^2/2}$ in the state ψ_t coincides with the distribution of a random variable $\xi_3(t)$ that can be written as $L_3(\psi_0)$ plus an independent normal random variable of mean 0 and variance:

$$\frac{4 \cdot \alpha_3^{\ 2} + t^2}{\varepsilon^2 \cdot \alpha_3 \cdot t^4} = \frac{1}{\left(\varepsilon \cdot t^2/2\right)^2}\left((\Delta q_3)^2 + t^2 \cdot (\Delta p_3)^2\right).$$

Here Δq_3 and Δp_3 are, respectively, the standard deviations of q_3 and p_3 in the state ψ_0. This situation is depicted in Figure 3.

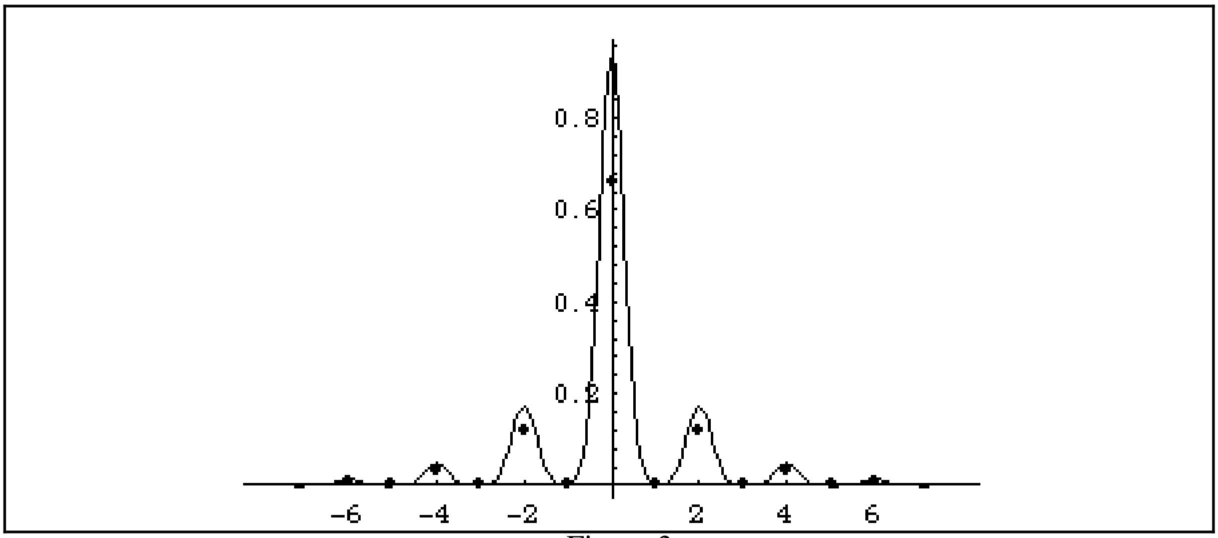

Figure 3

The dots represent the probability mass function of $L_3(\psi_0)$*;*

the continuous graph is the probability density function of ξ_3 *at time* $t = 4$.

(the values of the parameters are the same as in Figures 1, 2 and are summarised in Table 1).

$\omega = 2 \cdot \pi$	$\varepsilon = 1$	$\alpha_1 = 4$	$\alpha_2 = 64$	$\alpha_3 = 1$

Table 1

3. The stochastic process

The considerations of the previous Section strictly parallel Feynman's time-of-flight analysis of linear momentum [6], that in the present notational set-up can be summarised by the following relation between characteristic functions:

$$\left\langle \psi_0, e^{i\cdot\tau\cdot p_j}\psi_0 \right\rangle = \lim_{t\to\infty}\left\langle e^{-i\cdot\tau\cdot H_{0,0}}\psi_0, \exp\left(i\cdot\tau\cdot\frac{q_j}{t}\right)e^{-i\cdot\tau\cdot H_{0,0}}\psi_0 \right\rangle, \tag{11}$$

$H_{0,0}$ being the free Hamiltonian.

As it is well known, the simple statement (11) has evolved, after the germinal work of Shucker [8], who translated it in terms of the pathwise asymptotics of Nelson's processes associated with the free Schrödinger evolution, into a powerful stochastic model of quantum scattering by a scalar potential [9,10]. In the same spirit, by studying, in this Section, Nelson's process associated with $\psi_t = e^{-i\cdot t\cdot H_{\omega,\varepsilon}}\psi_0$ (for the simple choice (9) of ψ_0 dictated by criteria of feasibility of numerical simulations) we intend to contribute to a preliminary, heuristic understanding of some aspects of stochastic motion in a vector potential.

For an analysis of unsurpassed clarity of the Hamiltonian approach to stochastic mechanics needed in the case, considered here, of velocity dependent forces, we refer the reader to [11]. Here we just recall the basic facts we need in order to draw Figure 1, namely in order to define and simulate the stochastic process

$$q(t) = \left(q_1(t), q_2(t), q_3(t)\right)$$

corresponding to $\psi_t = e^{-i\cdot t\cdot H_{\omega,\varepsilon}}\psi_0$, with $H_{\omega,\varepsilon}$ given by (1) and ψ_0 given by (4).

The probability density of $q(0)$ is :

$$\rho(0,x) = \left|\psi_0(x)\right|^2$$

where, of course, we use the notation $x = \left(x_1, x_2, x_3\right)$.

The components of the mean forward velocity field are given, for $j = 1,2,3$, by:

$$b_j(t,x) = \operatorname{Re}\left(\frac{1}{\psi_t(x)}\cdot\frac{\partial\psi_t(x)}{\partial x_j}\right) + \operatorname{Im}\left(\frac{1}{\psi_t(x)}\cdot\frac{\partial\psi_t(x)}{\partial x_j}\right) - A_j(x)$$

The vector potential $A(x)$ is given by:

$$\begin{cases} A_1(x) = -\varepsilon\cdot x_2\cdot x_3 \\ A_2(x) = \varepsilon\cdot x_1\cdot x_3 \\ A_3(x) = 0 \end{cases}$$

The system of stochastic differential equations is, for $j = 1,2,3$:

$$dq_j(t) = b_j(t, q(t)) \cdot dt + dw_j(t) \tag{12}$$

Here $dt > 0$, and the independent Brownian motions $w_j(t)$ are such that

$$E(dw_j(t)) = 0 \text{ and } E(dw_j(t) \cdot dw_k(t)) = \delta_{\mathrm{j,k}} \cdot dt$$

The construction leading to the numerical simulation of (12) and, therefore, to the sample paths of Figure 1 is completed by the following explicit expression for the solution, under the initial condition (9), of the Schrödinger equation $i \cdot d\psi_t/dt = H_{\omega,\varepsilon}\psi_t$:

$$\psi_t(x) = \sum_{n=-\infty}^{+\infty} c\left(t, x_3; 2 \cdot \varepsilon \cdot n, \sqrt{\alpha_3}\right) \cdot K_n(t, x_1, x_2).$$

Here

$$c(t,z;f,\sigma) = \frac{1}{(2 \cdot \pi)^{1/4} \sqrt{\sigma + i \cdot t/(2 \cdot \sigma)}} \cdot \exp\left(-\frac{\left(z - f \cdot t^2/2\right)^2}{4 \cdot \sigma^2 + 2 \cdot i \cdot t} + i \cdot \left(z \cdot f \cdot t - f^2 \cdot t^3/6\right) \right)$$

and

$$K_n(t, x_1, x_2) = i^n \cdot a_1(t) \cdot a_2(t) \cdot \exp\left(-\left(d_1(t) + d_2(t)\right) \cdot \left(x_1{}^2 + x_2{}^2\right)/4\right) \cdot$$
$$\cdot J_n\left(i \cdot \left(d_1(t) - d_2(t)\right) \cdot \left(x_1{}^2 + x_2{}^2\right)/4\right) \cdot \left(\frac{x_1 + i \cdot x_2}{\sqrt{x_1{}^2 + x_2{}^2}} \right)^{2 \cdot n}$$

Here J_n is a Bessel function of the first kind and, for $j = 1,2$, $\beta_j = 1/(2 \cdot \alpha_j)$,

$$a_j(t) = \left(\beta_j/\pi\right)^{1/4} \cdot \left(\frac{\omega}{\omega \cdot \cos(\omega \cdot t) + i \cdot \beta_j \cdot sin(\omega \cdot t)} \right)^{1/2}$$

$$d_j(t) = \frac{\beta_j \cdot \omega^2}{\left(\omega^2 \cdot \cos^2(\omega \cdot t) + \beta_j^2 \cdot \sin^2(\omega \cdot t)\right)} +$$
$$- \frac{i \cdot \omega \cdot \left(\beta_j^2 - \omega^2\right) \cdot \sin(\omega \cdot t) \cdot \cos(\omega \cdot t)}{\left(\omega^2 \cdot \cos^2(\omega \cdot t) + \beta_j^2 \cdot \sin^2(\omega \cdot t)\right)}$$

As $h_{1,2}$ and h_3 commute, the above solution can be easily found by observing that a Gaussian initial condition in the x_1,x_2 plane evolves, under a harmonic potential, into a Gaussian wave function; this wave function is, in turn, expanded as

$$\exp\left(-\left(d_1\cdot x_1^{\ 2}+d_2\cdot x_2^{\ 2}\right)/2\right)=\exp\left(-\frac{d_1+d_2}{4}\cdot r^2\right)\cdot\sum_{n=-\infty}^{+\infty}e^{i\cdot 2\cdot n\cdot\vartheta}\cdot i^n\cdot J_n\left(i\frac{d_1-d_2}{4}\cdot r^2\right),$$

where, r and ϑ are the polar coordinates in the plane x_1,x_2; the above expansion reduces then the problem of motion in the x_3 direction to a sequence of elementary problems in a constant force, with a Gaussian initial condition. As Figure 3 shows, for our choice of parameters only terms with n ranging from -3 to $+3$ contribute significantly to the above expansion, and are, in fact, taken into account in our numerical simulations.

Figure 4 gives a fairly intuitive stochastic mechanical description of "space quantization" in the x_3 direction in terms of sample paths of the stochastic process $b_3(t,q(t))$ (mean forward velocity in the x_3 direction):

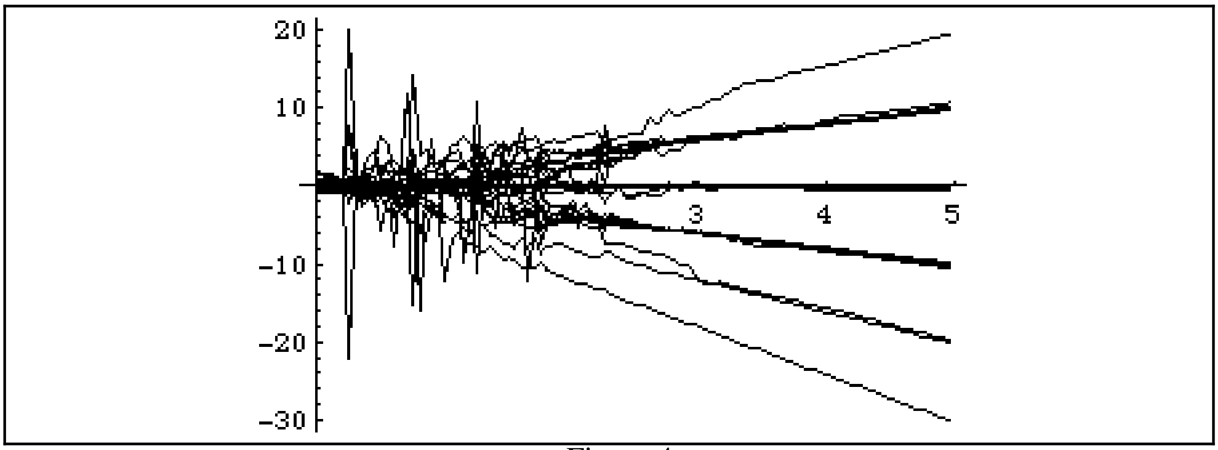

Figure 4
$b_3(t,q(t))$ as a function of t.

Figure 2 follows, in an analogous way, the motion in the x_1,x_2 plane. In order to motivate the introduction of the quantity plotted there, we recall, first of all, an elementary fact about the operator L_3: if $\psi_t(x)=\exp(R(t,x)+i\cdot S(t,x))$, with $R(t,x)$ and $S(t,x)$ real functions, then

$$\frac{1}{\psi_t(x)}\cdot L_3\psi_t(x)=x_1\frac{\partial S(t,x)}{\partial x_2}-x_2\frac{\partial S(t,x)}{\partial x_1}-i\cdot\left(x_1\frac{\partial R(t,x)}{\partial x_1}-x_2\frac{\partial R(t,x)}{\partial x_2}\right)$$

If, therefore, at some point x, it is, for some real m,

$$\frac{1}{\psi_t(x)} \cdot L_3 \psi_t(x) = m, \tag{13}$$

it follows that

$$x_1 \frac{\partial S(t,x)}{\partial x_2} - x_2 \frac{\partial S(t,x)}{\partial x_1} = m$$

and

$$\left(x_1 \frac{\partial R(t,x)}{\partial x_1} - x_2 \frac{\partial R(t,x)}{\partial x_2} \right) = 0$$

We recall that $u(t,q(t)) = grad\, R(t,q(t))$ has, in stochastic mechanics, the meaning of osmotic "velocity", while $v(t,q(t)) = grad\, S(t,q(t)) - A(t,q(t))$ has the meaning of current velocity. It is therefore natural, in perfect analogy with the classical canonical formalism, to introduce the canonical momentum random variables $p(t,q(t)) = v(t,q(t)) + A(t,q(t)) = grad\, S(t,q(t))$ and to define $\lambda_3(t,q(t)) \equiv q_1(t) \cdot p_2(t,q(t)) - q_2(t) \cdot p_1(t,q(t))$ as the stochastic analogue of canonical angular momentum (Busch constant of motion for a classical system moving in a magnetic field with rotational symmetry around the x_3 axis).
As t increases, the components of ψ_t belonging to different eigenvalues m of L_3 become sharply separated along the x_3 direction; this fact has the consequence (because, for large t, (13) is, to a high degree of approximation, locally satisfied) that the contribution of osmotic velocity to angular momentum becomes negligible, while the contribution of current velocity tends to a discrete random variable: this is precisely what is shown by the sample paths of $\lambda_3(t,q(t))$ plotted in Figure 2. Figure 2' follows, for completeness, the moment, with respect to the x_3 axis, of osmotic "velocity".

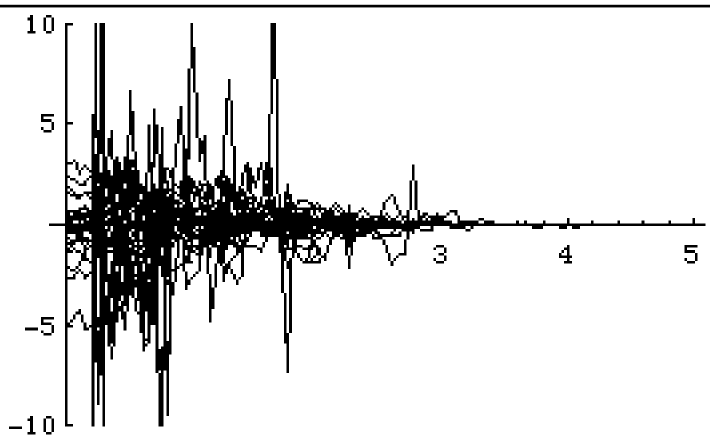

Figure 2'
$q_1(t) \cdot u_2(t,q(t)) - q_2(t) \cdot u_1(t,q(t))$ *as a function of* t.

As the contribution of osmotic velocity is vanishingly small for large t, it is no surprise that, as Figure 2'' shows, the only contribution to stochastic canonical angular momentum comes, asymptotically, from the mean forward velocity.

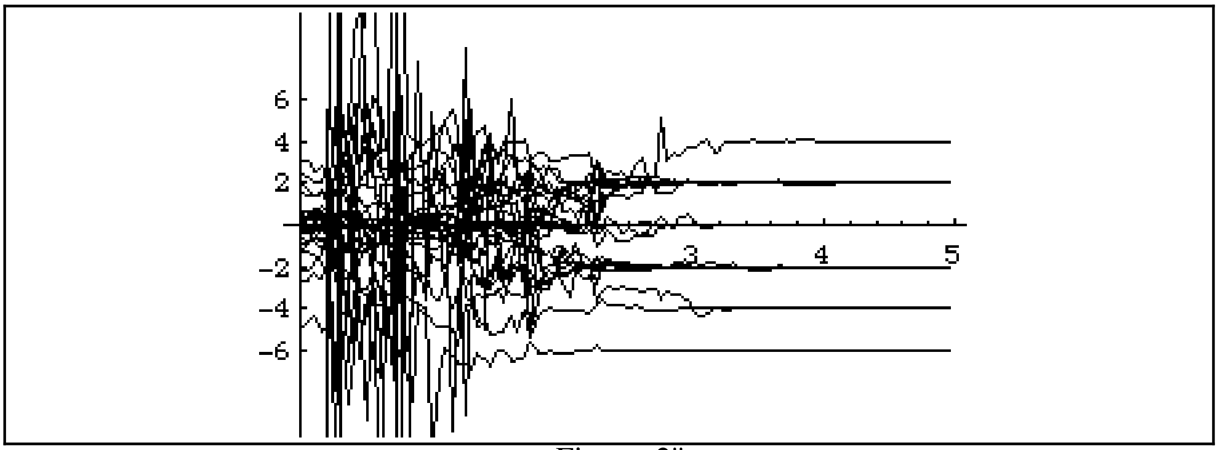

Figure 2''
$q_1(t)\cdot\left(b_2\left(t,q(t)\right)+A_2\left(q(t)\right)\right)-q_2(t)\cdot\left(b_1\left(t,q(t)\right)+A_1\left(q(t)\right)\right)$ *as a function of* t.

4. Discussion

We summarise the considerations made up to this point into the statements:

$$L_3\left(\psi_0\right)=\lim_{t->+\infty}\frac{q_3(t)}{\varepsilon\cdot t^2/2} \tag{14}$$

$$L_3\left(\psi_0\right)=\lim_{t->+\infty}q_1(t)\cdot b_2\left(t,q(t)\right)-q_2(t)\cdot b_1\left(t,q(t)\right)+\varepsilon\cdot q_3(t)\cdot\left(q_1(t)^2+q_2(t)^2\right) \tag{15}$$

where both limits are to be taken in law.

Statement (14) is just a rephrasing of Feynman's assertion reported in the Introduction: here the coordinate q_3 at a fixed, large, value of t plays the role of "needle" giving information about the rotational state of motion in the plane x_1,x_2 of the "system" that one would, classically, describe by the canonical coordinates q_1,p_1,q_2,p_2.

Statement (15) refers to the effects of the measuring procedure on the "system" itself. We have been careful, at the end of the previous Section, to stress the fact that (15), though expressed here in the notational scheme of stochastic mechanics, is just a consequence of the fact that the components of ψ_t belonging to different eigenvalues of L_3 become sharply separated along the x_3 direction. It is, nevertheless, interesting to observe that (15) - as opposed to (14) - makes a subtle reference to values of the process $q(t)$ at two different instants, due to the fact that:

$$b_j(t,q(t)) \equiv \lim_{h\to 0^+} E_t\left(\frac{q_j(t+h)-q_j(t)}{h}\right).$$

The same is true, incidentally, for the statement

$$L_3(\psi_0) = \lim_{t->+\infty} \frac{b_3(t,q(t))}{\varepsilon \cdot t} \tag{16}$$

suggested by Figure 4.

It is a long-standing problem in stochastic mechanics to understand the role of the assignment it attempts of a joint probability law to $q(t)$ and $q(t')$ for $t \neq t'$ in terms of transition probabilities.

Are such transition probabilities just "a kind of generalised gauge variables necessary to express the dynamical content of the theory in the simple and unifying form of stochastic variational principles" [12]; or do they, as relation (15) seems to suggest, play a role in modelling the measurement process of non configurational observables ?

Here we wish to pursue this point of view a little further, adding the conjecture, that figure 2" seems to support, that the convergence in (15) is almost sure. If this is case, we can write:

$$\begin{aligned}
L_3(\psi_0) &= \lim_{t->+\infty} q_1(t)\cdot b_2(t,q(t)) - q_2(t)\cdot b_1(t,q(t)) + \varepsilon\cdot q_3(t)\cdot\left(q_1(t)^2+q_2(t)^2\right) = \\
&= \lim_{t->+\infty} \frac{1}{t}\int_0^t \left[q_1(s)\cdot b_2(s,q(s)) - q_2(s)\cdot b_1(s,q(s)) + \varepsilon\cdot q_3(s)\cdot\left(q_1(s)^2+q_2(s)^2\right)\right]ds = \\
&= \lim_{t->+\infty}\left\{\frac{1}{t}\int_0^t q_1(s)\cdot dq_2(s) - q_2(s)\cdot dq_1(s) + \frac{1}{t}\int_0^t\left[\varepsilon\cdot q_3(s)\cdot\left(q_1(s)^2+q_2(s)^2\right)\right]ds\right\}
\end{aligned} \tag{17}$$

In writing the above relation we have taken into account the fact that

$$\begin{aligned}
\Delta(t) &\equiv \frac{1}{t}\int_0^t q_1(s)\cdot dq_2(s) - q_2(s)\cdot dq_1(s) - \frac{1}{t}\int_0^t \left[q_1(s)\cdot b_2(s,q(s)) - q_2(s)\cdot b_1(s,q(s))\right] ds = \\
&= \frac{1}{t}\int_0^t q_1(s)\cdot dw_2(s) - q_2(s)\cdot dw_1(s)
\end{aligned}$$

is a random variable of mean 0 and variance

$$\mathrm{var}(\Delta(t)) = \frac{1}{t^2}\int_0^t \left[\left(q_1(s)^2+q_2(s)^2\right)\right]ds;$$

we also point out that the harmonic term in

$$h_{1,2} = \frac{1}{2}p_1^{\ 2} + \frac{1}{2}p_2^{\ 2} + \frac{1}{2}\omega^2 \cdot \left(q_1^{\ 2} + q_2^{\ 2}\right)$$

has been introduced into the model just in order to prevent spreading of the wave function in the x_1, x_2 plane.

The intuition that (17) provides on the time progress of the process of measurement of $L_3(\psi_0)$ is shown by Figure 5 and 6. We recall, for comparison, that for the analogous classical system, canonical angular momentum is a linear combination of area per unit time spanned by the vector $(q_1(t), q_2(t))$ and magnetic flux linked to the trajectory.

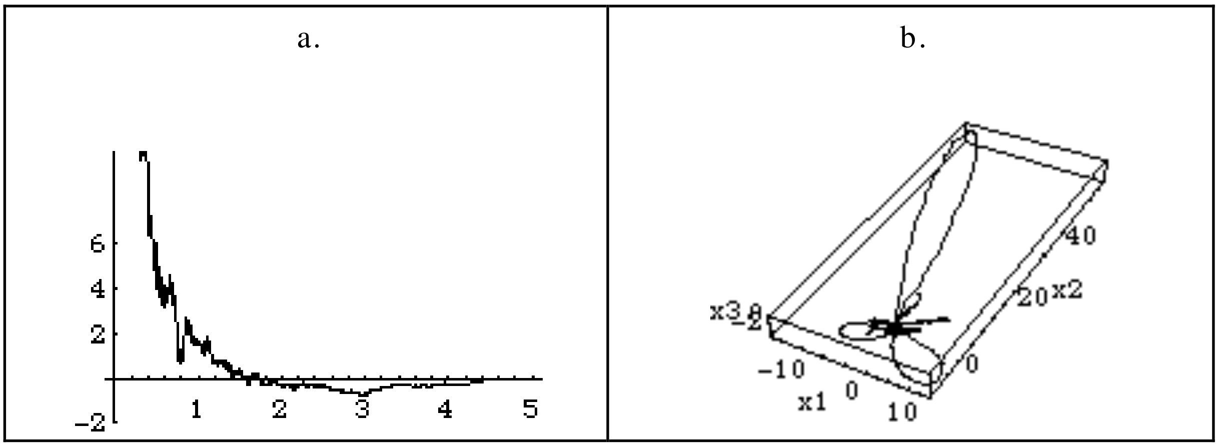

Figure 5

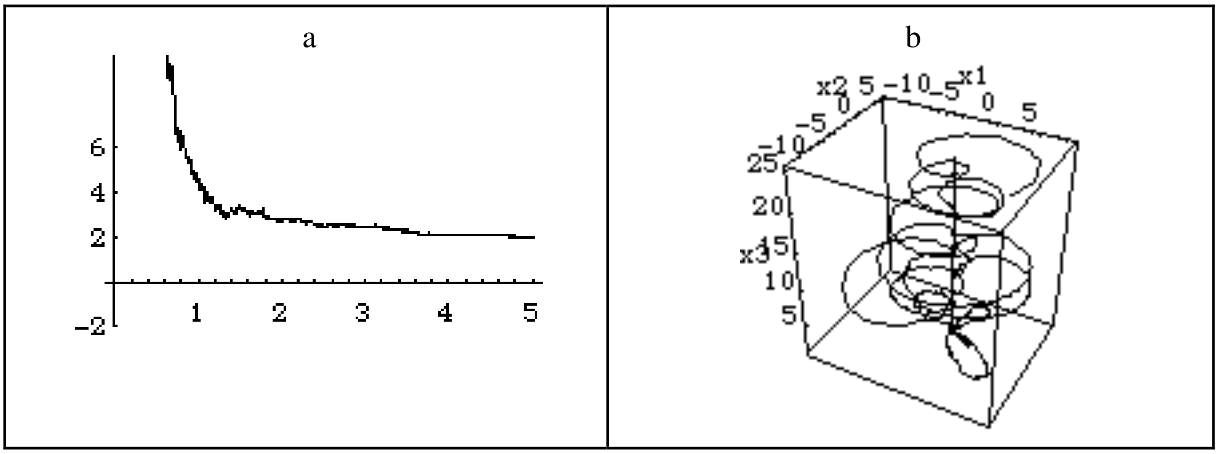

Figure 6

The behaviour of the pointer variable $\dfrac{q_3(t)}{\varepsilon \cdot t^2/2}$ *as a function of* t *shown in* a. *corresponds to the sample path* $(q_1(t), q_2(t), q_3(t))$ *shown in* b..

As to the "physical reality" of trajectories such as those shown in Figures 5 and 6 , we refer the reader to the cautionary remarks in the beautifully lucid introduction of Reference [13]. We just wish to point out that, as it is clear from the analysis of the measurement process given by Pauli [14], the case of scattering by a scalar potential V does not exhaust the instances in which a particle description of quantum behaviour is asymptotically appropriate; the case of motion in a vector potential A deserves a similar attention. Also in this case, therefore, a comparison with the paths-of-physical-particles picture of stochastic mechanics is in order.

Reference [13] makes it clear (through the sharp statement that, for those scattering diffusions $q(t)$ that admit a final linear momentum $p_+ = \lim_{t\to\infty} q(t)/t$, any random variable measurable with respect to the tail σ–algebra is a function of p_+) that no contradiction emerges in the case of a scalar potential.

If equalities (15), (16), (17) can, as equality (14), be proven to be true beyond the heuristic level at which we have introduced them, the question arises of characterising the random variables which are measurable with respect to the tail σ–algebra generated by the process $q(t)$ studied in Section 3.

Stated in physical terms this question can be reformulated as: What else, beyond a component of angular momentum, can be "measured" through the long-time behaviour of a Nelson process in an inhomogeous magnetic field ? Making the obvious answer (energy of the motion in the x_1, x_2 plane, as suggested by the quantum mechanical statement of simultaneous measurability of $h_{1,2}$ and L_3) precise requires further research. We are presently studying a stochastic analogue of the classical adiabatic invariant $\oint p_1 \cdot dq_1 + p_2 \cdot dq_2$, carrying energy information in much the same way as $\Lambda_3(t)$ carries angular momentum information. In carrying out this program it is not any more sufficient to consider the Hamiltonian

$$H_{\omega,\varepsilon} = \frac{1}{2}p_1^{\ 2} + \frac{1}{2}p_2^{\ 2} + \frac{1}{2}\omega^2 \cdot \left(q_1^{\ 2} + q_2^{\ 2}\right) + \frac{1}{2}p_3^{\ 2} - \varepsilon \cdot q_3 \cdot \left(q_1 \cdot p_2 - q_2 \cdot p_1\right) \qquad (6')$$

adopted up to this point. In describing the coupling of the harmonic "system" q_1 , q_2 with the "apparatus" q_3 through the vector potential

$$\begin{cases} A_1(x) = -\varepsilon \cdot x_2 \cdot x_3 \\ A_2(x) = \varepsilon \cdot x_1 \cdot x_3 \\ A_3(x) = 0 \end{cases}$$

the Hamiltonian (6') neglects, in fact, the quadratic term

$$k_{\varepsilon^2} = \frac{1}{2} \cdot \varepsilon^2 \cdot q_3^{\ 2} \cdot \left(q_1^{\ 2} + q_2^{\ 2}\right)$$

responsible of transfer of energy between "system" and "apparatus".

Figure 7 gives (in a simple adiabatic approximation) a hint of the new phenomena (relevant to the problem of understanding the meaning itself of "quantization" and "superposition" in a stochastic context) that appear in stochastic evolution under $H_{\omega,\varepsilon} + k_{\varepsilon^2}$: the dominant new feature is now that the motion in the x_1, x_2 plane visits in turn, at random times, the possible "quantized" values of $q_1(t) \cdot p_2(t, q(t)) - q_2(t) \cdot p_1(t, q(t))$ present in the initial "superposition" of states, alternating these visits with transient episodes similar to the ones depicted in the initial part of Figure 2.

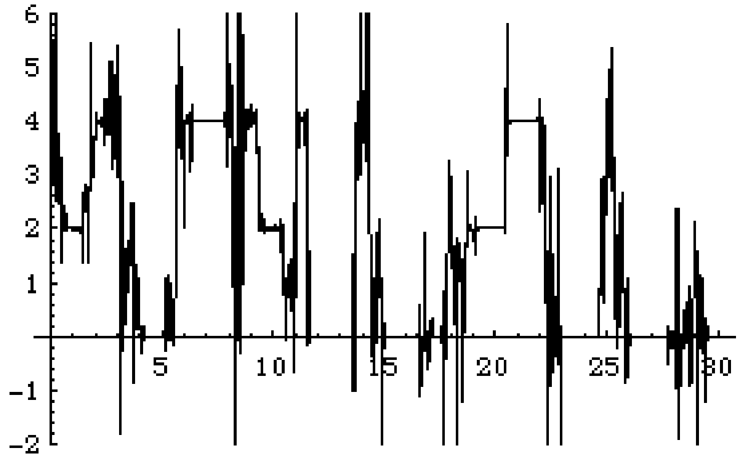

Figure 7

$q_1(t) \cdot p_2(t, q(t)) - q_2(t) \cdot p_1(t, q(t))$ as a function of t for evolution under $H_{\omega,\varepsilon} + k_{\varepsilon^2}$ from the initial condition given by a superposition, with equal weights, of $\varphi_{0,0}$, $\varphi_{2,0}$, $\varphi_{4,0}$

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References

[1] E. Nelson: "Dynamical Theories of Brownian Motion" Princeton University Press, 1967.

[2] E. Nelson: Bulletin of the American Mathematical Society, **84**, 121-124, 1970.

[3] E. Nelson: "Quantum Fluctuations" Princeton University Press, 1985.

[4] W. Gerlach, O. Stern: Annalen der Physik. **LXXIV**, 673-699, 1924.

[5] W. Faris: Foundations of Physics, **12**, 1-26, 1982.

[6] R. Feynman, A. Hibbs: "Quantum Mechanics and Path Integrals", McGraw-Hill, 1965.

[7] D. de Falco: Letters in Mathematical Physics, **14**, 41-45, 1987.

[8] D. Shucker: Journal of Functional Analysis, **38**, 146-155, 1980.

[9] E. Carlen: Annales de l'Institut Henri Poincaré, **42**, 407-428, 1985.

[10] A. Posilicano, S. Ugolini: "The Flux-across-surfaces Theorem: a Probabilistic Proof", Preprint, Università dell'Insubria, 1999.

[11] F. Guerra: Journal de Physique, **34**, C1-95 - C1-98, 1973.

[12] F. Guerra: "The Problem of the Physical Interpretation of Nelson Stochastic Mechanics as a Model for Quantum Mechanics", Preprint, Università di Roma "La Sapienza", 1997.

[13] E. Carlen: Lecture Notes in Physics, **262**, 139-147, 1986.

[14] W. Pauli: "General Principles of Quantum Mechanics" (English translation), Springer-Verlag, 1980.