ON INSTABILITY OF THE ABSOLUTELY CONTINUOUS SPECTRUM OF DISSIPATIVE SCHRÖDINGER OPERATORS AND JACOBI MATRICES

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Abstract. We prove the absence of the absolutely continuous spectrum for dissipative Schrodinger operators and Jacobi matrices with slowly decaying imaginary part of the potential.

1. Introduction

This paper addresses the question of stability of the absolutely continuous spectrum of one-dimensional Schrödinger operators under slowly decaying perturbations. In the self-adjoint case, the absolutely continuous spectrum is known [1, 2] to be stable under additive perturbation by a potential, that is, coincides with the positive real axis as soon as the potential is L^2. In particular, this result means that the absolutely continuous spectrum preserves when the conventional scattering theory is not applicable anymore, which, in terms of the L^p scale, is the case when the potential is not L^1. Our aim here is to show that the picture is very different for nonself-adjoint Schrödinger operators. Namely, the following two theorems hold[1],

Theorem 1. *Let $\{q_j\}$ be a bounded sequence of complex numbers, $\operatorname{Im} q_j \geq 0$, and $\operatorname{Im} q_j \to 0$, and let l be the discrete Schrödinger operator in $l^2(\mathbb{N})$ with the potential $\{q_j\}$. If the absolutely continuous subspace of l is nontrivial then $\operatorname{Im} q \in l^1$.*

Theorem 2. *Let a function $q \in L^\infty(\mathbb{R}_+)$, $\operatorname{Im} q \geq 0$, $\operatorname{Im} q(x) \to 0$ as $x \to \infty$. Define l to be the Schrödinger operator in $L^2(\mathbb{R}_+)$ with the potential q, $lu = -u'' + qu$, on the domain fixed by a self-adjoint boundary condition at zero. If the absolutely continuous subspace of l is nontrivial then $\operatorname{Im} q \in L^1$.*

One should mention that in the case when $\operatorname{Im} q \in L^1$ a local nuclear scattering theory [3, 4] applies to show that the a. c. spectrum of l coincides with that of the unperturbed operator, $\operatorname{Re} l$, similar to the situation of the self-adjoint theory. Thus, the theorems above effectively say that the existence of the a. c. spectrum in the dissipative case is equivalent to having a scattering theory.

The structure of the paper is as follows. In section 2 we define the a. c. subspace of a dissipative operator and provide necessary abstract preliminaries. Section 3 is devoted to the proof of theorem 1, and section 4 is devoted to the proof of theorem 2.

Date: January 11, 2003.

[1]precise definitions of the objects entering the formulations are given in the main text.

2. Absolutely Continuous Subspace

Let H be a Hilbert space and L be a maximal dissipative operator in H of the form $L = A + iV$, $A = A^*$, $V \geq 0$ is bounded. We shall assume that $\sigma_{ess}(L) \subset \mathbb{R}$, then the resolvent $(L-z)^{-1}$ is defined for all $z \in \mathbb{C}_+$ but countably many points as a bounded operator in H. It is also assumed throughout that the operator L is completely nonselfadjoint, that is, has no reducing subspaces on which it induces a self-adjoint operator. Define the Hardy classes of vector functions $\mathbf{H}^2_\pm$ to be the collections of functions $f : \mathbb{C}_\pm \to H$ analytic in $\mathbb{C}_\pm$, respectively, and satisfying $\sup_{\varepsilon>0} \int_{\mathbb{R}} \|f(k \pm i\varepsilon)\|^2 \, dk < \infty$.

Definition. *The absolutely continuous subspace*, N_e, of the operator L is defined as follows [3, 5]

$$N_e = \operatorname{clos}\left\{ u \in H : \sqrt{V}\,(L-z)^{-1} u \in \mathbf{H}^2_+ \right\}.$$

It is easy to verify from the definition that the subspace N_e is a regular invariant subspace of L, that is, $\overline{(L-z)^{-1} N_e} = N_e$ for all $z \in \rho(L)$, and so the restriction $L|_{N_e}$ is a closed densely defined operator in N_e.

A motivation to this definition in the framework of the theory of smooth perturbations can be found in [3, 4]. Let us just mention here that the restriction of the function $\sqrt{V}\,(L-z)^{-1} u$ to the lower half-plane belongs to $\mathbf{H}^2_-$ for any $u \in H$ [3, Theorem 1].

Another, "nonperturbative", origin of the definition is functional models[2] [5, 6, 7]. According to a fundamental theorem of Sz.-Nagy [8], the operator L admits a self-adjoint dilation, that is, there exists a self-adjoint operator $\mathcal{L}$ in a Hilbert space $\mathcal{H} \supset H$ containing H as a subspace such that for all $z \in \mathbb{C}_-$

$$(L-z)^{-1} = P_H \left(\mathcal{L} - z\right)^{-1}\big|_H ,$$

where P_H is the orthogonal projection on H in $\mathcal{H}$. The dilation is called minimal if $\bigvee_{\operatorname{Im} z \neq 0} (\mathcal{L}-z)^{-1} H = \mathcal{H}$. The subspace N_e is related to a certain invariant subspace of $\mathcal{L}$.

Theorem 2.1. [3, 7, 8] *There exists a reducing subspace, $\mathcal{H}^r$, of the minimal dilation $\mathcal{L}$ such that $N_e = \overline{P_H \mathcal{H}^r}$, and the operator $L|_{N_e}$ is quasi-similar to the a. c. self-adjoint operator $\mathcal{L}^r = \mathcal{L}|_{\mathcal{H}^r}$. Namely, the operator $W = P_H|_{\mathcal{H}^r}$ intertwines the groups generated by $L|_{N_e}$ and $\mathcal{L}^r$, $e^{itL|_{N_e}} W = W e^{it\mathcal{L}^r}$, $t \in \mathbb{R}$. The operator $L|_{N_e}$ is similar to a self-adjoint operator if and only if W is boundedly invertible.*

The spectral analysis of the restriction $L|_{N_e}$ called *the a. c. part of the operator L* has been developed in [5, 6, 8].

The triviality of the subspace N_e in the examples considered in this paper will be established on the basis of the following criterion.

Proposition 2.2. *$N_e = \{0\}$ iff for a. e. $k \in \mathbb{R}$ we have ($z = k + i\varepsilon$)*

$$D(z) \equiv \sqrt{\varepsilon}\,(L^* - z)^{-1} \sqrt{V} \xrightarrow{s} 0 \tag{2.1}$$

as $\varepsilon \downarrow 0$.

[2] The dilation theorem and theorem 2.1 are given here to clarify the definition of the a. c. subspace. They are not used in the paper.

This assertion is a partial case of results in [7]. In fact, it suffices to verify the strong convergence on a dense set, since the function $D(z)$ is bounded in $\mathbb{C}_+$, $\|D(z)\| \le 1/2$ (see [7] or the proof of lemma 2.4 below where a similar calculation is carried out). To make the exposition closed, we give here an elementary proof of the if part (which is the only actually used in applications considered in this paper).

Proof. Given a $u \in H$, define the functions $u_\pm(z)$, $z \in \mathbb{C}_\pm$, to be the restrictions of the function $\sqrt{V}\,(L-z)^{-1}\,u$, to $\mathbb{C}_\pm$, respectively. We shall make use of the fact mentioned above that $u_- \in \mathbf{H}^2_-$ for all $u \in H$. Suppose that $u_+ \in \mathbf{H}^2_+$ for a certain $u \in H$. Define $u_\pm(k) \in L^2(\mathbb{R}, H)$ to be the boundary values of the functions $u_\pm$ on the real axis. The required assertion $u = 0$ will be established if we show that $u_+ = u_-$ in $L^2(\mathbb{R}, H)$. For all $z = k + i\varepsilon$, $\varepsilon > 0$, and $e \in H$ we have

$$|\langle u_+(z) - u_-(\overline{z}), e\rangle| = 2\varepsilon \left|\left\langle \sqrt{V}\,(L-\overline{z})^{-1}\,(L-z)^{-1}\,u, e\right\rangle\right| = \\ 2\sqrt{\varepsilon}\left|\left\langle (L-z)^{-1}\,u, D(z)e\right\rangle\right|.$$

Then, for any compact interval $\Delta \subset \mathbb{R}$

$$\int_\Delta |\langle u_+(k+i\varepsilon) - u_-(k-i\varepsilon), e\rangle|\,dk \le 2\left(\int_\Delta \|D(k+i\varepsilon)e\|^2\,dk\right)^{1/2} \\ \left(\varepsilon\int_\Delta \left\|(L-k-i\varepsilon)^{-1}\,u\right\|^2 dk\right)^{1/2}. \tag{2.2}$$

The first factor in the r. h. s. vanishes as $\varepsilon \to 0$ by the assumption and the Lebesgue theorem. Let us show that the second factor is bounded in ε. An application of the resolvent identity gives

$$\sqrt{\varepsilon}\left\|(L-z)^{-1}\,u\right\| \le \sqrt{\varepsilon}\left\|(L^*-z)^{-1}\,u\right\| + \|D(z)\|\cdot\left\|\sqrt{V}\,(L-z)^{-1}\,u\right\|.$$

To estimate the contribution of the first term in the integral, we use the inequality

$$\varepsilon\int_{\mathbb{R}} \left\|(L^*-k-i\varepsilon)^{-1}\,v\right\|^2 dk \le \pi\,\|v\|^2,\quad v \in H.$$

This inequality, first appeared in [3], follows at once from the theorem on existence of a self-adjoint dilation as applied to the maximal dissipative operator $-L^*$, if we take into account that the identity $\varepsilon\int_{\mathbb{R}} \left\|(\mathcal{A}-k-i\varepsilon)^{-1}\,v\right\|^2 dk = \pi\,\|v\|^2$ holds for any self-adjoint operator $\mathcal{A}$. Thus,

$$\varepsilon\int_{\mathbb{R}} \left\|(L-k-i\varepsilon)^{-1}\,u\right\|^2 dk \le 2\pi\,\|u\|^2 + \int_{\mathbb{R}} \left\|\sqrt{V}\,(L-k-i\varepsilon)^{-1}\,u\right\|^2 dk,$$

which is bounded by the assumption about u. We obtain that the l. h. s. of (2.2) converges to 0 which implies that $\langle u_+(k) - u_-(k), e\rangle = 0$ a. e. on Δ. From arbitrariness of Δ and e we now conclude that $u_+ = u_-$. □

In the case of trace class perturbations, $V \in \mathbf{S}^1$, a local scattering theory exists for the pair (L, A). The following assertion is implicit in [3, 4].

Theorem 2.3. *If $V \in \mathbf{S}^1$ then $\sigma\left(L|_{N_e}\right) = \sigma_{ac}(A)$ where the a. c. spectrum of the self-adjoint operator A is defined in the standard way.*

Proof (sketch). According to the results in [3, 6], the assumption about V implies that to any Borel subset $\omega \subset \mathbb{R}$ there corresponds an invariant subspace H_ω of the operator $L|_{N_e}$ such that $\sigma\left(L|_{H_\omega}\right) \subset \overline{\omega}$, and H_ω is a maximal invariant subspace with this property. Moreover, if $\omega = \cup_n \omega_n$, then $\sigma\left(L|_{H_\omega}\right) = \overline{\cup_n \sigma\left(L|_{H_{\omega_n}}\right)}$. It is shown in the course of the proof of theorem 5 in [3] that there exists a scalar contractive analytic function, m, in the upper half plane such that the bounded local wave operators $W_\pm^\omega(A, L)$ for the set $\omega = \{k \in \mathbb{R} : |m(k)| \geq c\}$ exist and are complete for any $c > 0$. Moreover, the wave operators are boundedly invertible, which means that $L|_{H_{\omega_n}}$ and the corresponding part of A are similar, and thus $\sigma\left(L|_{H_\omega}\right) \subset \sigma_{ac}(A)$. Now, taking into account that the set $\{k : m(k) = 0\}$ has Lebesgue measure zero and that $\mathcal{H}_\omega$ is trivial for a set ω of zero measure, we conclude from the additivity property above that $\sigma\left(L|_{N_e}\right) \subset \sigma_{ac}(A)$. In a similar way, one can check the inclusion in the reverse direction. $\square$

One should emphasize that this result neither follow from nor based upon the analogous assertion of the self-adjoint theory. Notice that the conclusion of the theorem holds if the condition $V \in \mathbf{S}^1$ is relaxed to $(L-z)^{-1} - (A-z)^{-1} \in \mathbf{S}^1$. It is natural to call the set $\sigma\left(L|_{N_e}\right)$ *the a. c. spectrum of the operator L.*

In general, the question of stability of the a. c. spectrum under trace class perturbations appears to be open, that is, it is not known whether $L_1 - L_2 \in \mathbf{S}^1$ implies $\sigma\left(L_1|_{N_e(L_1)}\right) = \sigma\left(L_2|_{N_e(L_2)}\right)$ for a pair of dissipative operators L_1, L_2, if neither is self-adjoint. We shall, however, need a result of this type in a special situation. The original work of this paper starts here.

Lemma 2.4. *Let L and $\widetilde{L}$ be two dissipative operators such that $\widetilde{L} - L = i\Gamma$ with a $\Gamma \geq 0$, $\Gamma \in \mathbf{S}^1$. Then $N_e(\widetilde{L}) = \{0\}$ implies $N_e(L) = \{0\}$.*

Proof. Let $V = \operatorname{Im} L$, $\widetilde{V} = \operatorname{Im} \widetilde{L}$. Applying the criterion of proposition 2.2 to $\widetilde{L}$, one concludes that $\sqrt{\varepsilon}\left(\widetilde{L}^* - z\right)^{-1}\sqrt{\Gamma} \xrightarrow{s} 0$ for a. e. $k \in \mathbb{R}$, since $\sqrt{\Gamma} = \sqrt{\widetilde{V}}T$ with a bounded operator T. Let us first show that $\sqrt{\varepsilon}\left(L^* - z\right)^{-1}\sqrt{\Gamma} \xrightarrow{s} 0$ for a. e. $k \in \mathbb{R}$. We have,

$$\sqrt{\varepsilon}\left(\widetilde{L}^* - z\right)^{-1}\sqrt{\Gamma} = \sqrt{\varepsilon}\left(L^* - z\right)^{-1}\sqrt{\Gamma} \cdot G(z),$$

where

$$G(z) = I + i\sqrt{\Gamma}\left(\widetilde{L}^* - z\right)^{-1}\sqrt{\Gamma}.$$

Then the following calculation shows that $G(z)$ is a contractive function in $\mathbb{C}_+$,

$$\begin{aligned} I - G^*(z)G(z) = I - &\left(I - i\sqrt{\Gamma}\left(\widetilde{L} - \overline{z}\right)^{-1}\sqrt{\Gamma}\right)\left(I + i\sqrt{\Gamma}\left(\widetilde{L}^* - z\right)^{-1}\sqrt{\Gamma}\right) = \\ &i\sqrt{\Gamma}\left(\widetilde{L} - \overline{z}\right)^{-1}\left(\left(\widetilde{L}^* - z\right) - \left(\widetilde{L} - \overline{z}\right) + i\Gamma\right)\left(\widetilde{L}^* - z\right)^{-1}\sqrt{\Gamma} = \\ &\sqrt{\Gamma}\left(\widetilde{L} - \overline{z}\right)^{-1}\left(2\varepsilon + 2V + \Gamma\right)\left(\widetilde{L}^* - z\right)^{-1}\sqrt{\Gamma} \geq 0. \end{aligned} \tag{2.3}$$

Notice that the condition on Γ implies that $\sqrt{\Gamma}\left(\widetilde{L}^{*}-z\right)^{-1}\sqrt{\Gamma}\in\mathbf{S}^{1}$. Thus, $G(z)$ is a contractive analytic operator-function in $\mathbb{C}_{+}$ such that $I-G(z)\in\mathbf{S}^{1}$. Hence, by the Sz.-Nagy theorem [8], it admits scalar multiple, that is, there exists a scalar contractive analytic function, $g\not\equiv 0$, such that $G(z)\Omega(z)=g(z)\cdot I$ for a certain bounded analytic function Ω. By the operator Fatou theorem [8] it follows that the strong boundary values, $G^{-1}(k)=\operatorname{s-lim}_{\varepsilon\downarrow 0}G^{-1}(k+i\varepsilon)$ of the function G^{-1} exist for a. e. $k\in\mathbb{R}$. From this we infer that for a. e. $k\in\mathbb{R}$

$$\sqrt{\varepsilon}\,(L^{*}-z)^{-1}\sqrt{\Gamma}=\sqrt{\varepsilon}\left(\widetilde{L}^{*}-z\right)^{-1}\sqrt{\Gamma}\cdot G^{-1}(z)\xrightarrow[\varepsilon\downarrow 0]{s}0.$$

One can now verify the condition (2.1) for the operator L. We have,

$$\sqrt{\varepsilon}\,(L^{*}-z)^{-1}\sqrt{V}=\sqrt{\varepsilon}\left(\widetilde{L}^{*}-z\right)^{-1}\sqrt{V}-i\sqrt{\varepsilon}\,(L^{*}-z)^{-1}\sqrt{\Gamma}\cdot Q(z),\tag{2.4}$$

where $Q(z)=\sqrt{\Gamma}\left(\widetilde{L}^{*}-z\right)^{-1}\sqrt{V}$ is a bounded analytic function in $\mathbb{C}_{+}$ since $Q(z)=T_{1}\cdot\sqrt{\widetilde{V}}\left(\widetilde{L}^{*}-z\right)^{-1}\sqrt{\widetilde{V}}\cdot T_{2}$ with bounded T_{1}, T_{2}, and a calculation similar to (2.3) shows that the middle factor is a contraction. The first term in the r. h. s. of (2.4) converges strongly to zero by the assumption, and an application of the Fatou theorem to the function $Q(z)$ shows that the second term converges strongly to zero as well. □

3. Discrete Schrödinger Operator

Let $\{q_{n}\}$ be an arbitrary bounded sequence of complex numbers with $\operatorname{Im}q_{n}\geq 0$. Define $\Theta(z)$ and $\Phi(z)$, $z\in\mathbb{C}$, to be the solutions of the difference equation

$$u_{n+1}+u_{n-1}+q_{n}u_{n}=zu_{n}\tag{3.1}$$

for $n>1$ with the initial conditions $\Theta_{1}=0$, $\Theta_{2}=-1$, $\Phi_{1}=1$, $\Phi_{2}=z-q_{1}$.

The discrete Schrödinger operator, l, acts in the Hilbert space $H=l^{2}(\mathbb{N})$ and has the form

$$l=S+S^{*}+Q,$$

where $(Su)_{n}=u_{n+1}$, $(Qu)_{n}=q_{n}u_{n}$. Then l is a bounded dissipative operator which is completely nonself-adjoint if $\operatorname{Im}q\not\equiv 0$. For $z\in\mathbb{C}_{-}$ let $u(z)=(l-z)^{-1}\delta_{1}$ where $H\ni\delta_{1}=(1,0,\dots)$, so $u(z)$ is an l^{2} - solution to (3.1) for $n>1$. The function u satisfies

$$u(z)=-\Theta(z)+m(z)\Phi(z)$$

where $m(z)\overset{def}{=}\left\langle (l-z)^{-1}\delta_{1},\delta_{1}\right\rangle$ is a minus Nevanlinna function in $\mathbb{C}_{-}$ ($\operatorname{Im}m(z)\leq 0$). By general properties of Nevanlinna functions, the finite boundary values, $m(k)=\lim\limits_{\varepsilon\downarrow 0}m(k-i\varepsilon)$, exist for almost all $k\in\mathbb{R}$. At such k's, the solution $u(k)$, $u_{j}(k)=\lim\limits_{\varepsilon\downarrow 0}u_{j}(k-i\varepsilon)$, to (3.1) with $z=k$ is defined.

Lemma 3.1. *For a. e. $k\in\mathbb{R}$ the sum $\sum_{j}\operatorname{Im}q_{j}\left|u_{j}(k)\right|^{2}$ is finite, and*

$$\sup_{0<\varepsilon<1}\varepsilon\left\|u(k-i\varepsilon)\right\|_{H}^{2}<\infty.\tag{3.2}$$

Proof. Both assertions result from consideration of the limit $\varepsilon \downarrow 0$ in the identity ($z = k - i\varepsilon$)

$$\sum_{j=1}^{\infty} (\operatorname{Im} q_j + \varepsilon) |u_j(z)|^2 = -\operatorname{Im} m(z).$$

To verify this identity we notice that the l. h. s. equals to

$$\operatorname{Im} \langle (l - z)u(z), u(z) \rangle = \operatorname{Im} \langle \delta_1, u(z) \rangle = \operatorname{Im} \overline{m(z)} = -\operatorname{Im} m(z).$$

□

Our aim now is to apply the criterion from proposition 2.2. For a function Ψ analytic in $\mathbb{C}_-$ introduce the notation $\widetilde{\psi}(z) = \overline{\psi(\overline{z})}$. On calculating the Wronskian, $W[\Phi(z), u(z)] \equiv \Phi_{j-1}(z)u_j(z) - \Phi_j(z)u_{j-1}(z) = 1$, we obtain

$$\left[(l^* - z)^{-1} f\right]_j = \widetilde{\Phi}_j(z) \sum_{n=j}^{\infty} \widetilde{u}_n(z) f_n + \widetilde{u}_j(z) \sum_{n=1}^{j-1} \widetilde{\Phi}_n(z) f_n, \ z \in \mathbb{C}_+.$$

Let $D(z)$, $z = k + i\varepsilon$, be the operator defined in (2.1) with $L = l$.

Lemma 3.2. *If $\sqrt{\operatorname{Im} q}\, \Phi(k) \notin l^2$ for a. e. $k \in M$ for a set M of positive Lebesgue measure, then $D(k + i\varepsilon) \xrightarrow[\varepsilon\downarrow 0]{s} 0$ for a. e. $k \in M$.*

Proof. We shall actually show that $D(k + i\varepsilon) \xrightarrow{s} 0$ for all $k \in M$ such that the assumption holds and $m(k)$ exists and is finite, and so (3.2) is satisfied. Notice that the set $\mathcal{D} = \left\{ v \in l^2(\mathbb{N}) : v \text{ is compactly supported, and} \sum_j v_j \sqrt{\operatorname{Im} q_j} \widetilde{\Phi}_j(k) = 0 \right\}$ is dense in l^2 when $\sqrt{\operatorname{Im} q}\, \Phi(k) \notin l^2$. Hence, it is enough to check that $D(k + i\varepsilon) f \to 0$ for $f \in \mathcal{D}$.

One has for $z = k + i\varepsilon$,

$$\left[(l^* - z)^{-1} \sqrt{\operatorname{Im} l} f\right]_j = \widetilde{\Phi}_j(z) \sum_{n=j}^{\infty} \sqrt{\operatorname{Im} q_n} \widetilde{u}_n(z) f_n + \widetilde{u}_j(z) \sum_{n=1}^{j-1} \sqrt{\operatorname{Im} q_n}\, \widetilde{\Phi}_n(z) f_n.$$

The first term in the r. h. s. vanishes for j's to the right of the support of f. Considering that $\widetilde{u}_n(z)$ is bounded in ε for each n, one concludes that this term is uniformly bounded in ε and j when j ranges over compacts. From this we infer that

$$D(z) f = g_\varepsilon \sqrt{\varepsilon} \widetilde{u}(z) + o(1), \ \ \varepsilon \to 0,$$

where

$$g_\varepsilon = \sum_{n=1}^{\infty} \sqrt{\operatorname{Im} q_n}\, \widetilde{\Phi}_n(k + i\varepsilon) f_n = O(\varepsilon)$$

since $g_0 = 0$ by the choice of the set $\mathcal{D}$, and all $\widetilde{\Phi}_n(z)$ are analytic in z. Taking into account that $\varepsilon \left\| \widetilde{u}(k + i\varepsilon) \right\|^2$ is bounded as $\varepsilon \to 0$ by (3.2), we obtain that $D(z) f \to 0$. □

Theorem 3.3. *Let $\{q_j\}$ be a bounded sequence of complex numbers, $\operatorname{Im} q_j \geq 0$, such that $\sigma_{ess}(l) \subset \mathbb{R}$. Then $N_e \neq \{0\} \Longrightarrow \operatorname{Im} q \in l^1$.*

Proof. For a. e. $k \in \mathbb{R}$ we have

$$1 = W[\widetilde{\Phi}(k), \widetilde{u}(k)] \equiv \widetilde{\Phi}_{j-1}(k)\widetilde{u}_j(k) - \widetilde{\Phi}_j(k)\widetilde{u}_{j-1}(k), \ j > 1. \tag{3.3}$$

Multiplying this by $\alpha_j \alpha_{j-1}$, $\alpha_j \stackrel{def}{=} \sqrt{\operatorname{Im} q_j}$, and summing over j, we obtain

$$\sum^n \alpha_j \alpha_{j-1} \leq \sum^n \alpha_{j-1} |\Phi_{j-1}(k)| \cdot \alpha_j |u_j(k)| + \sum^n \alpha_j |\Phi_j(k)| \cdot \alpha_{j-1} |u_{j-1}(k)| \leq$$
$$2 \left(\sum^n \operatorname{Im} q_j |\Phi_j(k)|^2 \right)^{1/2} \left(\sum^n \operatorname{Im} q_j |u_j(k)|^2 \right)^{1/2} \leq C \left(\sum^n \operatorname{Im} q_j |\Phi_j(k)|^2 \right)^{1/2},$$

where the first assertion of lemma 3.1 has been taken into account. We obtain that if $\sum_j \alpha_j \alpha_{j-1} = \infty$ then $\sum_j \operatorname{Im} q_j |\Phi_j(k)|^2 = \infty$ for a. e. $k \in \mathbb{R}$, and the lemma just proved implies that $N_e = \{0\}$.

The general case is reduced to the one considered above, since if $\operatorname{Im} q \notin l^1$, then $\sum_j \widetilde{\alpha}_j \widetilde{\alpha}_{j-1} = \infty$, $\widetilde{\alpha}_j \equiv \sqrt{\operatorname{Im} \widetilde{q}_j}$, for $\widetilde{q} = q + id$ with a nonnegative $d \in l^1$. Indeed, by the uniform boundedness principle, $\sqrt{\operatorname{Im} q} \notin l^2$ implies that there exists a $\rho \in l^2$ such that $\sum_j \rho_j \alpha_j = \infty$, and, obviously, the ρ can be chosen to be nonnegative. Now one can set $d_j = \rho_{j-1}^2$, for $\widetilde{\alpha}_j \widetilde{\alpha}_{j-1} \geq \rho_{j-1} \alpha_{j-1}$. An application of lemma 2.4 completes the proof. □

It should be mentioned that in the case $\operatorname{Im} q \in l^1$ the imaginary part of l is of the trace class, and therefore $\sigma\left(l|_{N_e}\right) = \sigma_{ac}(\operatorname{Re} l)$. Thus the problem of description of the a. c. spectrum of dissipative discrete Schrödinger operator is solved completely.

The above proof extends straightforwardly to the case of general Jacobi matrix with asymptotically constant *real* weight. Namely, given a bounded sequence, $\{\rho_n\}$, of real nonzero numbers, define the Jacobi matrix

$$J = \Lambda S + S^* \Lambda + Q,$$

where $(\Lambda u)_n = \rho_n u_n$, and Q is as before. Then J is a bounded dissipative operator, and the corresponding version of theorem 3.3 reads as,

Theorem 3.4. *Let $\rho_n \to c \neq 0$ and $\{q_j\}$ be a bounded sequence of complex numbers, $\operatorname{Im} q_j \geq 0$, such that $\sigma_{ess}(J) \subset \mathbb{R}$. Then $N_e(J) \neq \{0\} \Longrightarrow \operatorname{Im} q \in l^1$.*

Indeed, assuming without loss of generality that $\rho_1 = 1$, one can define the functions Θ, Φ as solutions of the corresponding equations with the same boundary conditions as above and let $u(z) = (J - z)^{-1} \delta_1$. The proof then proceeds verbatim until (3.3), where we will have ρ_{j-1} in the l. h. s. at the place of 1. Still, it shows that the r. h. s. is bounded away from zero, since so is ρ_j, and the proof proceeds as before.

It is not clear whether the methods above can be extended to cover the case of complex weight ρ. This leaves open the question whether there exist dissipative Jacobi matrices with nontrivial a. c. subspace in the case when $|\operatorname{Im} \rho| + \operatorname{Im} q \notin l^1$.

4. Continuous case

Let $q \in L^\infty(\mathbb{R}_+)$ be a function satisfying $\operatorname{Im} q(x) \geq 0$ a. e. and h be an arbitrary real number. Define $\Theta(x,z)$ and $\Phi(x,z)$ to be the solutions to

$$-u'' + qu = zu, \ z \in \mathbb{C}, \tag{4.1}$$

with the Cauchy data $\Phi(0,z) = \Theta'(0,z) = 0$, $\Phi'(0,z) = -1$, $\Theta'(0,z) = 1$. Let $u_h = \Theta - h\Phi$.

The Schrödinger operator l, $ly = -y'' + qy$, is defined in the Hilbert space $H = L^2(\mathbb{R}_+)$ on the domain fixed by the self-adjoint boundary condition $y'(0) = hy(0)$. Then l is a maximal dissipative operator of the class considered in section 2, provided that $\operatorname{Im} q(x)$ decays at infinity in an appropriate sense, for instance, if $\operatorname{ess\,sup}_{x>A} \operatorname{Im} q(x) \to 0$ as $A \to \infty$, which guarantees that $\sigma_{ess}(L) \subset \mathbb{R}$. We shall assume this condition satisfied throughout.

The following lemma is a continuous analog of lemma 3.1.

Lemma 4.1. *For all $z \in \mathbb{C}_-$ there exists exactly one linear independent L^2-solution, $u(x,z)$, to (4.1), which can be chosen in the form*

$$u(x,z) = \Theta(x,z) + m(z)\Phi(x,z)$$

where m is an analytic function in $\mathbb{C}_-$. For almost all $k \in \mathbb{R}$ the function m has finite boundary values, $m(k) = \lim_{\varepsilon \downarrow 0} m(k - i\varepsilon)$, the solution $u(x,k) = \lim_{\varepsilon\downarrow 0} u(x, k - i\varepsilon)$ satisfies

$$\int \operatorname{Im} q(x) \, |u(x,k)|^2 \, dx < \infty,$$

and

$$\sup_{0<\varepsilon<1} \varepsilon \, \|u(k - i\varepsilon)\|_H^2 < \infty. \tag{4.2}$$

Proof. First, $\Phi(x,z)$ cannot be L^2, otherwise it belongs to the domain of the dissipative Schrödinger operator $\widetilde{l}$ with the same potential and the Dirichlet boundary condition at zero, and therefore $(\widetilde{l} - z)\Phi = 0$ from whence we infer that $\Phi = 0$, for $z \in \mathbb{C}_-$. This implies the first assertion of the lemma.

Consider the identity,

$$0 = \int_0^x (-u'' + (q - z)u) \, \overline{u} \, dt = -u'(x)\overline{u(x)} - m(z) + \int_0^x (q - z) \, |u|^2 \, dt + \int_0^x |u'|^2 \, dt.$$

Notice that u and u' vanish as $x \to \infty$ since $u \in \mathcal{D}(l_0^*)$ where l_0 is the restriction of l to functions satisfying the boundary condition $y(0) = y'(0) = 0$, and therefore $u', u'' \in L^2$. Thus, the first term in the r. h. s. goes to 0 as $x \to \infty$, and on taking the imaginary part, we obtain

$$\operatorname{Im} m(z) = \int_0^\infty (\operatorname{Im} q(t) + \varepsilon) \, |u(t,z)|^2 \, dt, \ z = k - i\varepsilon. \tag{4.3}$$

Alternatively, this formula can be established on the basis of nesting circles analysis [9]. It shows that $m(z)$ is a Nevanlinna function in $\mathbb{C}_-$ ($\operatorname{Im} m(z) \geq 0$), and therefore it has finite boundary values a. e. on the real axis. The rest of assertions now follow immediately from consideration of the limit $\varepsilon \to 0$ in the formula above. $\square$

The resolvent of the operator l^* for $z \in \mathbb{C}_+$ has the form

$$\left((l^* - z)^{-1} f\right)(x) = \tag{4.4}$$
$$\frac{-1}{h+m(z)} \left[\widetilde{u}(x,z) \int_0^x \widetilde{u_h}(s,z) f(s) ds + \widetilde{u_h}(x,z) \int_x^\infty \widetilde{u}(s,z) f(s) ds\right],$$

where the notation $\widetilde{\psi}(z) = \overline{\psi(\overline{z})}$ is used. In the following lemma $D(z)$ is defined by (2.1) with $L = l$.

Lemma 4.2. *If $\sqrt{\operatorname{Im} q}\, u_h(\cdot,k) \notin L^2$ for a. e. $k \in M$ for a set M of positive Lebesgue measure, then $D(k+i\varepsilon) \xrightarrow[\varepsilon\downarrow 0]{s} 0$ for a. e. $k \in M$.*

Proof of this lemma is completely analogous to that of lemma 3.2, so we just sketch it. The only difference is that we shall establish the limit $D(k+i\varepsilon) \xrightarrow{s} 0$ for those $k \in M$ at which the finite limit $\lim_{\varepsilon\downarrow 0} m(k - i\varepsilon)$ exists *and is not equal to* $-h$, to guarantee the boundedness of the scalar factor in (4.4). Since the set of k such that $m(k) = -h$ has measure zero by the uniqueness theorem for Nevanlinna functions, this does not affect the conclusion.

Notice that for any compactly supported f the $D(z)f$ is a continuous function of x, which is $O\left(\sqrt{\varepsilon}\right)$ uniformly on compacts. Then,

$$D(z)f = g_\varepsilon \sqrt{\varepsilon}\,\widetilde{u}(\cdot,z) + o(1), \quad \varepsilon \to 0,$$

where

$$g_\varepsilon = -\frac{1}{h + m(k+i\varepsilon)} \int_0^\infty \sqrt{\operatorname{Im} q(s)}\, \widetilde{u_h}(s, k+i\varepsilon) f(s) ds.$$

Now choose f to satisfy $g_0 = 0$, then $g_\varepsilon = o(1)$, and thus $D(z)f \to 0$. Since the set of such f's is dense in L^2 when $\sqrt{\operatorname{Im} q} u_h(\cdot,k) \notin L^2$, the proof is completed. $\square$

Before proving the main result (theorem 2 in introduction) we would like to give a motivation of it on the level of the WKB asymptotics for the solutions. The formal WKB asymptotic for solutions, $u_\pm$, of (4.1) with $z = k > 0$ has the form

$$u_\pm \sim \exp\left[\pm i\left(\sqrt{k}x - \frac{1}{\sqrt{k}} \int^x q(\xi) d\xi\right)\right], \quad x \to \infty. \tag{4.5}$$

For $\operatorname{Im} q \notin L^1$ the solution u_+ is growing, $|u_+| \sim \exp\left(\frac{1}{\sqrt{k}} \int q(\xi) d\xi\right)$, while u_- vanishes. Notice that if the WKB asymptotics holds for given $k > 0$, and v, a solution to (4.1) with $z = k > 0$, satisfies $\sqrt{\operatorname{Im} q} v \in L^2$, then it must be a multiple of u_-, for

$$\int^\infty \operatorname{Im} q\, |u_+|^2 dx \sim \int^\infty \operatorname{Im} q(x) \exp\left(C \int^x \operatorname{Im} q(\xi) d\xi\right) dx =$$
$$\exp\left(C \int^\infty \operatorname{Im} q(x) dx\right) = \infty.$$

Let the WKB asymptotic hold for a. e. $k > 0$. It follows that the solutions $u_h(\cdot,k)$ and $u(\cdot,k)$ must coincide for a. e. $k \in \mathbb{R}_+$ whenever u exists and $\sqrt{\operatorname{Im} q}\, u_h(\cdot,k) \in L^2$.

This is, however, only possible on the set of k such that $m(k) = -h$, which has zero measure, hence $\sqrt{\operatorname{Im} q}\, u_h(\cdot, k) \notin L^2$ for a. e. $k > 0$. We have thus established the following proposition. Let us call the function q *a WKB-class potential* if the asymptotic (4.5) holds for a. e. $k > 0$, and $\sigma_{ess}(l) \subset \mathbb{R}_+$.

Proposition 4.3. *Let q be a WKB-class potential. Then $N_e \neq \{0\} \Longrightarrow \operatorname{Im} q \in L^1$.*

A simple sufficient condition for a function q to be a WKB-class potential is $q(x) \to 0$, and $q' \in L^1$ [10], which in fact secures the asymptotic (4.5) for all $k > 0$. In particular, this shows immediately that the operator with the potential $q(x) = ix^{-\beta}$, $0 < \beta \leq 1$, has trivial a. c. subspace. Notice that the eigenvalues of this operator only accumulate at 0 by a complex scaling argument (a remark due to S. Naboko). One should mention here that the instability phenomenon described in this paper is likely to be a specific feature of dissipative Schrödinger operators. In the general nonself-adjoint case, a potential of the WKB - class will have all solutions of (4.1) bounded if its imaginary part is conditionally integrable. This suggests that in this case the a. c. subspace can be nontrivial even if $\operatorname{Im} q \notin L^1$.

We shall now develop an analog of the subordinacy theory [11] for the dissipative Schrödinger operator. It will be used (in one direction) in our proof of the main result.

Recall [11] that a solution u_1 to (4.1) is called *subordinate* if

$$\frac{\|u_1\|_N}{\|u_2\|_N} \to 0$$

as $N \to \infty$ for any solution u_2 linear independent of u_1. Here $\|\cdot\|_N \equiv \|\cdot\|_{L^2(0,N)}$.

Theorem 4.4. *Given a set M of positive Lebesgue measure, the following assertions are equivalent*

(*i*) *$D(k + i\varepsilon) \xrightarrow[\varepsilon\downarrow 0]{s} 0$ for a. e. $k \in M$.*

(*ii*) *For a. e. $k \in M$ there exists a subordinate solution.*

The proof of this theorem is subdivided into several assertions.

Lemma 4.5. (*i*) *is equivalent to saying that*

$$\varepsilon \|u(k - i\varepsilon)\|^2 \to 0 \tag{4.6}$$

for a. e. $k \in M$.

Proof. Let k be such that $m(k)$ exists and $m(k) \neq -h$. As we have seen in the proof of lemma 4.2, for any compactly supported $f \in H$ $(z = k + i\varepsilon)$

$$D(z)f = \left(-\frac{1}{h + m(z)} \int_0^\infty \widetilde{u_h}(s, z)\sqrt{\operatorname{Im} q(s)} f(s) ds\right) \sqrt{\varepsilon}\, \widetilde{u}(\cdot, z) + o(1).$$

This formula immediately shows that if (4.6) is satisfied, then $D(z)f \to 0$.

To establish the reverse implication, we choose f to be $u_h(x, k)\chi(x)$ where χ is the indicator of an interval Δ large enough. Then,

$$D(z)f = \left(-\frac{1}{h + m(k)} \int_\Delta |\widetilde{u_h}(s, k)|^2 \sqrt{\operatorname{Im} q(s)} ds\right) \sqrt{\varepsilon}\, \widetilde{u}(\cdot, z) + o(1).$$

Considering that the constant factor in parentheses is nonzero, we obtain that $D(z)f \to 0$ implies (4.6). □

Proposition 4.6. *If $\varepsilon \|u(k - i\varepsilon)\|^2 \to 0$ for a certain $k \in \mathbb{R}$ such that $m(k)$ exists and $m(k) \neq h$, then $u(x, k)$ is a subordinate solution.*

The proof of the corresponding assertion of the self-adjoint theory [11] carries over the situation under consideration without any modification. In fact, it can be slightly simplified on the basis of the following

Lemma 4.7. *Let y_0 be any solution to (4.1) with $z = k \in \mathbb{R}$ such that $r = y_0'(0)/y_0(0) \in \mathbb{R}$ (or $y_0(0) = 0$) and y be arbitrary solution to (4.1). Then $|y(x)| \le |Cy_0(x)|$ for x large enough, provided that $\operatorname{Im} q \not\equiv 0$.*

Proof. Notice that $y_0(x)$ does not vanish for x large enough, since otherwise y_0 would be an eigenfunction of a completely nonself-adjoint dissipative Schrödinger operator on an interval corresponding to a real eigenvalue, and therefore zero identically. It is then enough to establish the estimate for the solution

$$y_1(x) = \left(\int_a^x \frac{d\xi}{y_0^2(\xi)} \right) y_0(x)$$

linear independent with y_0. On multiplying the equation (4.1) by $\overline{y_0}$ and integrating from 0 to x, we obtain,

$$- y_0'(x)\overline{y_0(x)}\Big|_0^x + \int_0^x |y_0'|^2 \, dt + \int_0^x (q - k) |y_0|^2 \, dt = 0.$$

Taking the imaginary part and noticing that $y_0'(0)\overline{y_0(0)} = r |y_0(0)|^2$ is real, we derive that

$$\operatorname{Im} \left[y_0'(x)\overline{y_0(x)} \right] = \int_0^x \operatorname{Im} q \, |y_0|^2 \, dt. \tag{4.7}$$

Introduce the new variables R, Ψ, $R > 0$, by $y_0 = Re^{i\Psi}$. On calculating, the l. h. s. in (4.7) is $R^2(x)\Psi'(x)$, and therefore $\Psi' > 0$ for x large enough and the phase Ψ can be chosen to be a continuous monotonic increasing function of x. Denote the r. h. s. of (4.7) by $\rho(x)$. We have,

$$\int^x \frac{d\xi}{y_0^2(\xi)} = \int^x e^{-2i\Psi(\xi)} \frac{\Psi'(\xi) d\xi}{R^2(\xi)\Psi'(\xi)} = \int^{\Psi(x)} \frac{e^{-2i\Psi} d\Psi}{\rho(\xi(\Psi))}.$$

The nonnegative function $\zeta(s) = 1/\rho(\xi(s))$ is monotonic decreasing, which implies that the integral in the r. h. s. is uniformly bounded, and the result follows. □

This lemma shows, in particular, that the special solutions Θ and Φ estimate each other at large x, $\Theta(x) \asymp \Phi(x)$.

Proof of proposition 4.6. The solution $u(x, z)$ satisfies the following integral equation,

$$v(x) = \Theta(x, k) + m(z)\Phi(x, k) + i\varepsilon \left[\Theta(x, k) \int_0^x \Phi(s, k)v(s)ds - \Phi(x, k) \int_0^x \Theta(s, k)v(s)ds \right].$$

This equation can be iterated in $L^2(0, A(\varepsilon))$ to give

$$\|u(\cdot,z)\|_2 \geq \|u(\cdot,z)\|_{A(\varepsilon)} \asymp \|\Theta(\cdot,k)+m(z)\Phi(\cdot,k)\|_{A(\varepsilon)} = \\ \|u(\cdot,k)+(m(z)-m(k))\Phi(\cdot,k)\|_{A(\varepsilon)} = \\ \|u(\cdot,k)\|_{A(\varepsilon)} + o\left(\|\Phi(\cdot,k)\|_{A(\varepsilon)}\right),$$

provided that

$$\varepsilon \|\Phi(\cdot,k)\|_{A(\varepsilon)} \|\Theta(\cdot,k)\|_{A(\varepsilon)} < \frac{1}{2}. \tag{4.8}$$

Let C be any constant such that $|\Theta(x)| \leq C|\Phi(x)|$ for x large enough. Define $A(\varepsilon)$ by $\|\Phi(\cdot,k)\|_{A(\varepsilon)} = 1/\sqrt{2C\varepsilon}$. Then (4.8) is satisfied, and $A(\varepsilon)$ is monotone increasing to infinity since otherwise Φ would be an eigenfunction of a completely nonself-adjoint dissipative Schrödinger operator corresponding to a real eigenvalue, and therefore zero identically. We finally obtain that

$$\frac{\|u(\cdot,k)\|_{A(\varepsilon)}}{\|\Phi(\cdot,k)\|_{A(\varepsilon)}} \leq \frac{\|u(\cdot,z)\|}{\|\Phi(\cdot,k)\|_{A(\varepsilon)}} + o(1) = \sqrt{2C\varepsilon}\,\|u(\cdot,z)\| + o(1) \to 0$$

under the assumption imposed. □

Proposition 4.8. *If (4.1) has a subordinate solution for a certain $z = k \in \mathbb{R}$ such that $m(k)$ exists and $m(k) \neq -h$, then this solution is a multiple of $u(x,k)$.*

The proof of this assertion can also be taken verbatim from the corresponding assertion in the self-adjoint theory [11], so we omit it.

Proof of theorem 4.4. It remains to show that if $u(\cdot,k)$ is a subordinate solution, then (4.6) is satisfied. We shall in fact show that, moreover,

$$\int_0^\infty \operatorname{Im} q(x)\,|u(x,k)|^2\,dx = \operatorname{Im} m(k)$$

(see (4.3)). The derivation is based on an argument similar to that used in the proof of lemma 4.7. Again, multiplying (4.1) by $\overline{u}$, integrating and taking the imaginary part, we obtain the following identity,

$$-\operatorname{Im}\left[u'(x,k)\overline{u(x,k)}\right] = \operatorname{Im} m(k) - \int_0^x \operatorname{Im} q(t)\;|u(t,k)|^2\,dt.$$

The constant term in the r. h. s. comes from the substitution in the lower limit of the off-integral term. Introduce the new variables R, Ψ, $R > 0$, by $u = Re^{-i\Psi}$. Then the l. h. s. becomes $R^2(x)\Psi'(x)$, hence $R^2(x)\Psi'(x) = \rho(x)$ where $\rho(x)$ is positive, provided that $\operatorname{Im} q$ is not compactly supported. Here we took (4.3) into account. It follows that the phase Ψ can be chosen to be a continuous monotonic increasing function of x.

We then have for a solution v linear independent of u,

$$v(x) = \left(\int_a^x \frac{d\xi}{u^2(\xi)}\right) u(x) = \left(\int^x e^{2i\Psi(\xi)} \frac{\Psi'(\xi) d\xi}{R^2(\xi)\Psi'(\xi)}\right) u(x) = \left(\int^{\Psi(x)} e^{2i\Psi} \zeta(\Psi) d\Psi\right) u(x),$$

where $\zeta(s) = 1/\rho(\xi(s))$ is a positive monotone increasing function. It follows that $\zeta(s) \to \infty$, otherwise the integral would be bounded and v would be bounded above by u, which contradicts the assumption that u is subordinate. Thus, $\int_0^x \operatorname{Im} q\, |u|^2\, dt \to \operatorname{Im} m(k)$ as $x \to \infty$, as required. Since the assertion is trivial in the case of compactly supported $\operatorname{Im} q$, we are done. $\square$

We are now able to prove theorem 2 from Introduction. The proof involves a Wronskian based argument similar to that used in the discrete case, and a spectral averaging argument for the self-adjoint problem corresponding to $\operatorname{Re} q$ from paper [12]. It is hardly likely that an arguing for an individual k can give the result in the continuous case in full generality. We shall need the following

Lemma 4.9. *If $\sqrt{\operatorname{Im} q}\, u_h(\cdot, k) \in L^2$ for all $k \in M$, $M \subset \mathbb{R}$, then for a. e. $k \in M$ the equation*

$$-y'' + \operatorname{Re} q\, y = ky \tag{4.9}$$

has solutions $y_{1,2}$ such that $u_h = y_1(1 + o(1))$, $u = y_2(1 + o(1))$ as $x \to \infty$.

Proof. Let ψ_1, ψ_2 be arbitrary solutions to (4.1) with unit Wronskian. Rewriting the equation (4.9) in the matrix form,

$$Y' = \begin{pmatrix} 0 & 1 \\ \operatorname{Re} q - k & 0 \end{pmatrix} Y, \ \ Y = \begin{pmatrix} y \\ y' \end{pmatrix},$$

and applying the variation of the parameter formula, we obtain the equation

$$Z' = i \begin{pmatrix} \operatorname{Im} q\, \psi_1\psi_2 & \operatorname{Im} q\, \psi_2^2 \\ -\operatorname{Im} q\, \psi_1^2 & -\operatorname{Im} q\, \psi_1\psi_2 \end{pmatrix} Z, \tag{4.10}$$

where

$$Y = \begin{pmatrix} \psi_1 & \psi_2 \\ \psi_1' & \psi_2' \end{pmatrix} Z.$$

Recall that by lemma[3] 4.1 $\sqrt{\operatorname{Im} q}\, u(\cdot, k) \in L^2$ for a. e. $k \in \mathbb{R}$. This and the assumption mean that any solution to (4.1) is square-summable with the weight $\operatorname{Im} q$ for a. e. $k \in M$. It follows that the matrix in (4.10) is L^1, and the Levinson theorem applies to give the result. $\square$

Proof of theorem 2. Let $\operatorname{Im} q \notin L^1$. Suppose by contradiction that $D(k) \neq 0$, and so $\sqrt{\operatorname{Im} q}\, u_h(\cdot, k) \in L^2$, for a. e. $k \in M$ for a set $M \subset \sigma_{ess}(l)$ of positive Lebesgue measure. Let $y_{1,2}(\cdot, k)$ be the solutions to (4.9) defined in lemma 4.9 for a. e. $k \in M$.

[3]Alternatively, for $k \in M$ one can refer to lemma 4.7.

Define μ to be the a. c. component of the spectral measure for $\mathrm{Re}\, l$. Fix arbitrary c_0, N and A such that the set

$$I = \left\{ k \in M\colon \ |m(k)| \le c_0, \ \left\| \sqrt{\mathrm{Im}\, q} y_1(\cdot, k) \right\|^2 + \left\| \sqrt{\mathrm{Im}\, q} y_2(\cdot, k) \right\|^2 \le N \right\} \cap [-A, A]$$

has positive μ - measure. This is possible since, as we shall show later, $\mu(M) > 0$. Let $\rho(k) = 1 + \max\{|y_{1,2}(0,k)|, |y'_{1,2}(0,k)|\}$, and $T(x,k)$ be the transfer matrix for the equation (4.9). Then, $|y'_{1,2}(x,k)| \le \rho(k) \|T(x,k)\|$. Fix an arbitrary nonnegative $f \in L^2$ such that $\sqrt{\mathrm{Im}\, q} f \notin L^1$. On calculating the Wronskian of y_1, y_2 at infinity, we have

$$|W[y_1, y_2]| = |W[u_h, u]| = |h + m(k)|.$$

Multiplying this by $(\rho(k))^{-1} \sqrt{\mathrm{Im}\, q} f$ and integrating, we obtain

$$\int_I \frac{|h + m(k)|}{\rho(k)} d\mu(k) \int^x f \sqrt{\mathrm{Im}\, q}\, dt \le C \left(\int_I d\mu(k) \int^x \mathrm{Im}\, q \, \left(|y_1|^2 + |y_2|^2 \right) dt \right)^{1/2} \left(\int^x dt\, |f(t)|^2 \int_I \|T(t,k)\|^2 d\mu(k) \right)^{1/2}.$$

We shall now use the fact that $\int_I \|T(t,k)\|^2 d\mu(k)$ is bounded uniformly in t. This follows from theorem 2.1C in [12] and a standard estimate for solutions of a second-order differential equation, see, for instance the proof of Lemma 3.1 in [13], which says that

$$|g'(x)|^2 \le C(1 + k^2) \int_{x-1}^{x+1} |g(s)|^2 ds \tag{4.11}$$

for any solution g to (4.9) with a constant C depending on the potential only. This gives

$$C \int^x f \sqrt{\mathrm{Im}\, q}\, dt \le C \left(\int_I d\mu(k) \int^x \mathrm{Im}\, q \, \left(|y_1|^2 + |y_2|^2 \right) dt \right)^{1/2} \left(\int^x |f|^2 dt \right)^{1/2} \le C\sqrt{N} \|f\|,$$

and so the integral in the l. h. s. is bounded, a contradiction.

It remains to verify that $\mu(M) > 0$. Suppose, by contradiction, that $\mu(M) = 0$. Then (4.9) has a subordinate solution for a. e. $k \in M$ with respect to the Lebesgue measure due to the Gilbert - Pearson theory [11]. In turn, lemma 4.9 shows that the equation (4.1) with $z = k$ has a subordinate solution for a. e. $k \in M$ as well. By theorem 4.4, this implies that $D(k) = 0$ a. e. on M, a contradiction. $\square$

The condition on the real part of the potential in this theorem can apparently be substantially relaxed up to the requirement of uniform local L^2 - boundedness which secures [12, 13] the bound (4.11) for the derivatives and the possibility to apply the spectral averaging argument. It is, however, not clear whether the result holds if $\mathrm{Re}\, q$ is merely assumed to be limit-point.

5. Conclusions

As is well-known [8], triviality of the space N_e implies that the characteristic function of the operator L is inner. So far, the spectral analysis of operators with inner characteristic functions has only been developed effectively for the discrete part of the spectrum in the case of trace class perturbations [14]. The examples considered in this paper suggest, however, that the appearance of a nontrivial "singular" inner function is typical in applications when the perturbation decays slowly, and the trace class theory is not applicable.

Acknowledgements. The author is indebted to S. Naboko for important remarks, encouragement and critical reading of the manuscript.

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