Asymptotic properties of the differential equation $h^3(h'' + h') = 1$

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Abstract

We derive the form of the asymptotic series, as $t \to +\infty$, for a general solution $h(t)$ of the non-linear differential equation $h(t)^3(h''(t)+h'(t)) = 1$.

1 Introduction

The purpose of this article is to describe the asymptotics of solutions of the second-order ordinary non-linear differential equation

$$h(t)^3(h''(t) + h'(t)) = 1 \tag{1}$$

with initial conditions

$$h(t_0) = h_0 > 0, \;\; h'(t_0) = h_1. \tag{2}$$

Before formulating the result let us describe our motivation and the origin of the problem which has its roots in the physical Hall effect.

In a classical mechanics description the issue is to study the dynamics of a point mass moving in a periodic planar potential and driven by an exterior electromagnetic field where the magnetic field is constant and the electric field circular and created by a linearly time dependent flux tube through the origin, see [2] for the origin of the model. The equations of motions are Hamiltonian. The time dependent Hamiltonian is

$$H(t,q,p) = \frac{1}{2}\left(p - A(q,t)\right)^2 + V(q,t) \text{ on } \mathbb{R}^2 \setminus 0 \times \mathbb{R}^2$$

with

$$A(q,t) = \left(\frac{b}{2} - \frac{et}{|q|^2}\right)(q_2, -q_1).$$

Here b and e are real parameters and V a smooth periodic function. In Newtonian form the equations of motion are

$$\ddot{q} = E(q) + b\,\mathbb{D}\dot{q} - \nabla V(q) \quad \text{in } \mathbb{R}^2 \setminus \{0\}$$

where $\mathbb{D}$ is rotation by $\pi/2$ and $E(q) = |q|^{-2}\,\mathbb{D}q = -\partial_t A(q)$.

We shall prove elsewhere that if b and e are nonzero the solutions are diffusive with or without direction depending on the direction of the fields. In this article we discuss the particular case when $e = 1, b = 0$ and $V = 0$. In polar coordinates the Hamiltonian reads

$$H(t,r,\phi,p_r,p_\phi) = \frac{1}{2}\left(p_r^2 + \frac{1}{r^2}(p_\phi + t)^2\right)$$

and the equations of motion become

$$p_\phi' = 0,\ \phi' = \frac{p_\phi + t}{r^2},\ p_r' = \frac{(p_\phi + t)^2}{r^3},\ r' = p_r\,.$$

Consequently, p_ϕ is a constant and $r'' = r^{-3}(p_\phi + t)^2$. After a shift in time we arrive at the equation

$$r'' = \frac{t^2}{r^3}\,.$$

The substitution

$$r(t) = t\,h(\ln t)$$

leads to equation (1).

In order to formulate our result we have to introduce some auxiliary notation. Let $s_{m,k} \in \mathbb{R}[a_1, a_2, \dots a_k]$ be polynomials defined as follows:

$$s_{m,k}(a_1, a_2, \dots, a_k) = \sum_{i_1+i_2+\dots+i_m=k} a_{i_1} a_{i_2} \dots a_{i_m},$$

$m = 1, 2, \dots$, $k = 0, 1, 2, \dots$. Clearly, $s_{m,k}(a_1, a_2, \dots, a_k) = 0$ if $k < m$, and we set by definition $s_{0,k} = \delta_{0,k}$. The polynomials obey the recursive rule

$$s_{m+1,k}(a_1, a_2, \dots, a_k) = \sum_{j=m}^{k-1} s_{m,j}(a_1, a_2, \dots, a_j)\, a_{k-j} \quad \text{for } m+1 \le k.$$

In the space of formal power series, $\mathbb{R}[[x]]$, it holds

$$\left(\sum_{k=1}^{\infty} a_k x^k\right)^m = \sum_{k=m}^{\infty} s_{m,k}(a_1, a_2, \dots, a_k)\, x^k, \quad m = 0, 1, 2, \dots .$$

This implies that if

$$a = \sum_{k=1}^{\infty} a_k\, x^k, f = \sum_{k=0}^{\infty} f_k\, x^k \in \mathbb{R}[[x]]$$

then

$$\sum_{m=0}^{\infty} f_m\, a^m = \sum_{k=0}^{\infty} g_k\, x^k \quad \text{where } g_k = \sum_{m=0}^{k} f_m\, s_{m,k}(a_1, a_2, \dots, a_k)\,. \tag{3}$$

Furthermore, set

$$\sigma_k^0(a_1, a_2, \dots, a_k) = \sum_{j=1}^{k} \frac{(-1)^{j+1}}{j}\, s_{j,k}(a_1, a_2, \dots, a_k) \tag{4}$$

and

$$\sigma_k^m(a_1, a_2, \dots, a_k) = \sum_{j=0}^{k} \binom{-m}{j} s_{j,k}(a_1, a_2, \dots, a_k) \quad \text{for } m \ge 1. \tag{5}$$

Then it holds, in $\mathbb{R}[[x]]$,

$$\begin{aligned} \ln\left(1 + \sum_{k=1}^{\infty} a_k x^k\right) &= \sum_{k=1}^{\infty} \sigma_k^0(a_1, a_2, \dots, a_k)\, x^k, \\ \left(1 + \sum_{k=1}^{\infty} a_k x^k\right)^{-m} &= 1 + \sum_{k=1}^{\infty} \sigma_k^m(a_1, a_2, \dots, a_k)\, x^k \quad \text{for } m \ge 1. \end{aligned}$$

Let $\{\beta_n\}_{n=0}^{\infty}$ be a sequence of real numbers defined recursively,

$$\beta_0 = 1,$$
$$\beta_{n+1} = \left(n - \frac{3}{4}\right)\beta_n + \sum_{\substack{j,k=0\\ j+k=n+1}}^{n} \beta_j\beta_k - \sum_{\substack{j,k,\ell=0\\ j+k+\ell=n+1}}^{n} \beta_j\beta_k\beta_\ell\,. \tag{6}$$

Here are several first values:

$$\beta_1 = -\frac{3}{4},\ \beta_2 = -\frac{21}{16},\ \beta_3 = -\frac{165}{32},\ \beta_4 = -\frac{7245}{256},\ \ldots\,.$$

For a fixed constant $c \in \mathbb{R}$ we introduce a sequence of polynomials, $p_n(c;z) \in \mathbb{R}[z]$, $n \in \mathbb{Z}_+$, by the recursive rule

$$p_0(c;z) = 3z - c \tag{7}$$

and

$$\begin{aligned} p_n &= 3\,\sigma_n^0(p_0, p_1, \ldots, p_{n-1}) + \sum_{k=1}^{n-1} \frac{4^{k+1}\beta_{k+1}}{k}\,\sigma_{n-k}^k(p_0, p_1, \ldots, p_{n-k-1}) \\ &\quad + \frac{4^{n+1}\beta_{n+1}}{n}\,. \end{aligned} \tag{8}$$

This can be rewritten with the aid of the polynomials $s_{m,k}$,

$$\begin{aligned} p_n &= 3\sum_{j=1}^{n} \frac{(-1)^{j+1}}{j}\, s_{j,n}(p_0, p_1, \ldots, p_{n-1}) \\ &\quad + \sum_{k=0}^{n-1} \frac{4^{n-k+1}\beta_{n-k+1}}{n-k} \sum_{j=0}^{k} \binom{-n+k}{j} s_{j,k}(p_0, p_1, \ldots, p_{k-1}). \end{aligned}$$

For $n \geq 1$, the degree of $p_n(c;z)$ is less or equal to n (this can be easily shown by induction when using the fact that for any monomial $a_{i_1}^{s_1} a_{i_2}^{s_2} \ldots a_{i_\ell}^{s_\ell}$ occurring in $\sigma_k^m(a_1, a_2, \ldots, a_k)$ it holds $\sum i_j s_j = k$). Here are several first polynomials $p_n(z)$,

$$\begin{aligned} p_1(c;z) &= 9\,z - 21 - 3\,c, \\ p_2(c;z) &= -\frac{27}{2}\,z^2 + (90 + 9c)\,z - 228 - 30\,c - \frac{3}{2}\,c^2, \\ p_3(c;z) &= 27\,z^3 - \left(\frac{621}{2} + 27\,c\right) z^2 + (1638 + 207\,c + 9\,c^2)\,z \\ &\quad -3540 - 546\,c - \frac{69}{2}\,c^2 - c^3. \end{aligned}$$

Now we are able to formulate the result.

Theorem 1. *For any initial data* $(t_0, h_0, h_1) \in \mathbb{R} \times]0, \infty[\times \mathbb{R}$ *there exists a unique solution* $h(t)$ *to the problem (1), (2) on the real line. Moreover, there exists a constant* $c = c(t_0, h_0, h_1) \in \mathbb{R}$ *such that it holds, for all* $n \in \mathbb{Z}_+$ *and* $t \to +\infty$*:*

$$h(t) = (4t)^{1/4} \left(1 + \sum_{k=1}^{n} \frac{q_k(c; \ln(4t))}{t^k} + O\Big(\Big(\frac{\ln(t)}{t} \Big)^{n+1} \Big) \right) \tag{9}$$

where

$$q_k = \sum_{m=1}^{k} \frac{1}{4^k} \binom{\frac{1}{4}}{m} s_{m,k}(p_0, p_1, \ldots, p_{k-1}) \,.$$

The degree of $q_k(c; z)$ *is less than or equal to* k.

Remarks. (i) Several first polynomials $q_k(c; z)$ are

$$\begin{aligned} q_1(c; z) &= \frac{3}{16} z - \frac{1}{16} c, \\ q_2(c; z) &= -\frac{27}{512} z^2 + \left(\frac{9}{64} + \frac{9}{256} c \right) z - \frac{21}{64} - \frac{3}{64} c - \frac{3}{512} c^2, \\ q_3(c; z) &= \frac{189}{8192} z^3 - \left(\frac{135}{1024} + \frac{189}{8192} c \right) z^2 + \left(\frac{549}{1024} + \frac{45}{512} c + \frac{63}{8192} c^2 \right) z \\ &\quad - \frac{57}{64} - \frac{183}{1024} c - \frac{15}{1024} c^2 - \frac{7}{8192} c^3. \end{aligned}$$

(ii) In the final step of the proof, in Subsection 4.5, we shall show the following invariance property of the asymptotic expansion. Set

$$A_n(c; t) = (4t)^{1/4} \left(1 + \sum_{k=1}^{n} \frac{q_k(c; \ln(4t))}{t^k} \right),$$

with $n \in \mathbb{Z}_+$ and $c, t \in \mathbb{R}$. Then for all $s \in \mathbb{R}$ it holds true that

$$A_n(c; t + s) = A_n(c - 4s; t) + t^{1/4} O\Big(\Big(\frac{\ln(t)}{t} \Big)^{n+1} \Big) \quad \text{as } t \to +\infty.$$

The remainder of the paper contains all necessary steps to prove Theorem 1. We shall proceed as follows. In Section 2 we show the completeness and derive the first term of the asymptotic series. In Section 3 we make use of the fact that the second order differential equation can be reduced to a first order differential equation and we investigate the asymptotic properties of the latter equation. These results are used in Section 4 to derive the asymptotic properties of the original second order differential equation and this way we complete the proof of Theorem 1. Section 5 contains an additional remark on the asymptotics of the Lambert function.

2 Basic properties of the differential equation

The differential equation (1) is equivalent to the dynamical system

$$(x', y') = \left(y, \frac{1}{x^3} - y\right) \quad \text{on } M = \,]0, \infty[\, \times \mathbb{R}. \tag{10}$$

Proposition 2. *The flow of (10) is complete and so for all initial data* $(t_0, h_0, h_1) \in \mathbb{R} \times \,]0, \infty[\, \times \mathbb{R}$ *there exists a unique globally defined positive solution* h *of (1) with initial conditions* $h(t_0) = h_0$*,* $h'(t_0) = h_1$*.*

Proof. We use the following criterion (c.f. [1, chap. 2.1.20]): *The flow of a* C^1 *vector field* ξ *on a manifold* M *is complete if there is a proper map* $f \in C^1(M, \mathbb{R})$ *which meets the estimate*

$$\exists A > 0, B > 0, \forall p \in M, \quad |\xi \cdot f(p)| \leq A\,|f(p)| + B.$$

In our case $M = \,]0, \infty[\, \times \mathbb{R}$, $\xi = y\partial_x + (x^{-3} - y)\,\partial_y$ and we choose $f(x, y) = x^2 + y^2 + x^{-2}$. With this choice we have

$$|\xi \cdot f(x, y)| = |2xy - 2y^2| \leq x^2 + 3y^2 \leq 3\,f(x, y).$$

Moreover, for any bounded set $S \subset \mathbb{R}$ the inverse image $f^{-1}(S)$ is bounded and separated from the border of the half-plane M: there exists $\varepsilon > 0$ such that $f^{-1}(S) \subset [\,\varepsilon, \infty\,] \times \mathbb{R}$. This implies that f is in fact a proper map and the proposition is proven. □

Proposition 3. *Let* $(x(t), y(t))$*, with* $t \in [\,t_0, \infty[\,$*, be a solution of the dynamical system (10). Then there exists* $T \in [\,t_0, \infty[\,$ *such that*

$$\forall t, s, t \geq s \geq T, \quad \sqrt{2}\,(t + c(s))^{1/4} \leq x(t) \leq y(s) + \sqrt{2}\,(t + c(s))^{1/4}$$

where $c(s) = \frac{1}{4}\,x(s)^4 - s$*.*

Proof. Set (in this proof) $g(x, y) = x^{-3} - y$ and

$$g_1(x, y) = -3\,\frac{y}{x^4} - \frac{1}{x^3} + y = -\left(1 + 3\,\frac{y}{x}\right) g(x, y) - 3\,\frac{y^2}{x}$$

on M. Thus $x' = y$, $y' = g(x, y)$ and $\frac{d}{dt} g(x(t), y(t)) = g_1(x(t), y(t))$.

We shall show first that $y'(t)$ is negative for all sufficiently large t. Note that

$$\forall (x, y) \in M, \quad g(x, y) \leq 0 \Longrightarrow g_1(x, y) < 0.$$

Thus if $y'(s) \leq 0$ then $y'(t) < 0$ for all $t > s$. Hence it is sufficient to show that there is at least one t such that $y'(t) \leq 0$. Suppose the contrary. Since it holds

$$y(t) = y(t_0) + \frac{t - t_0}{x(t_0)^3} - \int_{t_0}^{t} \left(3\,\frac{t-s}{x(s)^4} + 1 \right) y(s)\,ds$$

there exists $s \geq t_0$ such that $y(s) > 0$. Then both $x(t)$ and $y(t)$ are increasing positive functions on the interval $[\,s, \infty[$ and, in addition, $x(t) < y(t)^{-1/3}$. So the function $g(t) = g(x(t), y(t))$ obeys

$$\forall t \geq s, \quad g'(t) = -\left(1 + 3\,\frac{y(t)}{x(t)}\right) g(t) - 3\,\frac{y(t)^2}{x(t)} < -3\,y(s)^{7/3} < 0$$

which clearly contradicts the assumption $g(t) = y'(t) > 0$ for all t.

Let now $T \geq t_0$ be such that $y'(t) = x(t)^{-3} - y(t) < 0$ for all $t > T$ and fix $s \geq T$. For any $t > T$ we have $(\frac{1}{4}x(t)^4)' = x(t)^3 y(t) > 1$. Consequently, if $t \geq s$ then

$$x(t) \geq \sqrt{2}\left(t + \frac{1}{4}\,x(s)^4 - s\right)^{1/4}.$$

To show the other inequality set, for $t \geq s$, $z(t) = x(t) - \sqrt{2}\,(t+c)^{1/4}$ where $c = c(s)$. We find that

$$\begin{aligned}
(e^t z')' &= e^t(z' + z'') = e^t\left(\frac{1}{x^3} - (\sqrt{2}\,(t+c)^{1/4})^{-3} + \frac{3\sqrt{2}}{16}\,(t+c)^{-7/4}\right) \\
&\leq \frac{3\sqrt{2}}{16}\,e^t\,(t+c)^{-7/4}.
\end{aligned}$$

It follows that

$$z'(t) \leq e^{s-t} z'(s) + \frac{3\sqrt{2}}{16}\,e^{-t} \int_s^t e^u\,(u+c)^{-7/4}\,du$$

and

$$\begin{aligned}
z(t) &\leq z(s) + \left(1 - e^{s-t}\right) z'(s) + \frac{3\sqrt{2}}{16} \int_s^t e^{-\tau} \int_s^\tau e^u\,(u+c)^{-7/4}\,du\,d\tau \\
&\leq z(s) + z'(s) + \frac{3\sqrt{2}}{16} \int_s^t \left(1 - e^{u-t}\right)\,(u+c)^{-7/4}\,du \\
&\leq z(s) + z'(s) + \frac{3\sqrt{2}}{16}\,\frac{4}{3}\,(s+c)^{-3/4} \\
&= z(s) + x'(s).
\end{aligned}$$

But $z(s) = 0$ and so $x(s) \leq y(s) + \sqrt{2}\,(t+c)^{1/4}$. $\square$

Corollary 4. *If $h(t)$ is a solution of (1) on $[\,t_0, \infty[$ with the initial conditions $h(t_0) = h_0 > 0$, $h'(t_0) = h_1$, then there exists $T \geq t_0$ such that $h'(t) > 0$ for all $t > T$.*

Corollary 5. *If $h(t)$ is a solution of (1) on $[\,t_0, \infty[$ with the initial conditions $h(t_0) = h_0 > 0$, $h'(t_0) = h_1$, then*

$$h(t) = \sqrt{2}\, t^{1/4} + O(1) \quad \text{as } t \to +\infty.$$

Remark. This means that if we restrict ourselves in what follows to the initial condition (2) with $h_1 > 0$ we don't loose the generality as far as the asymptotics is concerned. Furthermore, owing to the invariance of the differential equation in time we can set $t_0 = 0$. This fact will be used in the course of the proof. First we verify Theorem 1 for the particular case when $t_0 = 0$ and $h_1 > 0$ and then, in Subsection 4.5, we shall extend the result to the general initial condition.

3 A reduced differential equation of first order

In accordance with the remark at the end of Section 2 we assume that $t_0 = 0$ and $h_1 > 0$. The second-order differential equation equation (1) is invariant in t and this is why it can be reduced to a first-order differential equation. Actually, using the substitution $h(t) = \left(G^{-1}(4t)\right)^{1/4}$, $z_0 = 4/h_0^4$ and $g_0 = h_0^3 h_1$ where

$$G(x) = \int_{h_0^4}^{x} \frac{ds}{g\left(\frac{4}{s}\right)}$$

we arrive at a first-order nonlinear differential equation, namely

$$\left(1 - \frac{3}{4} z\, g(z) - z^2 g'(z)\right) g(z) = 1, \;\; g(z_0) = g_0. \tag{11}$$

We shall carry out the computations relating (1), (2) and (11) later, in Subsection 4.1. Here we concentrate on the study of (11) on the interval $[\,0, z_0\,]$, assuming that $z_0 > 0$ and $g_0 > 0$.

3.1 Domain of the left maximal solution $g(z)$

We shall need two equivalent forms of the differential equation, namely

$$g'(z) = \frac{1}{z^2}\left(1 - \frac{1}{g(z)}\right) - \frac{3}{4}\frac{g(z)}{z} \tag{12}$$

and

$$\left(z^{3/4}g(z)\right)' = z^{-5/4}\left(1 - \frac{1}{g(z)}\right) . \tag{13}$$

Remark. In what follows we use repeatedly the following elementary argument: *if ψ and φ are two differentiable functions on $]a, b[$ and the equality $\psi(z) = \varphi(z)$ implies $\psi'(z) < \varphi'(z)$, for all $z \in]a, b[$, then the two functions coincide in at most one point $z \in]a, b[$.*

Proposition 6. *Let $g(z)$ be the left maximal solution of (11). Then $g(z)$ is defined and positive on $]0, z_0]$, and*

$$\lim_{z \downarrow 0} g(z) = 1 . \tag{14}$$

Proof. Let γ be the minimal non-negative number such that $g(z)$ is defined on $]\gamma, z_0]$. Equation (11) clearly excludes the possibility that $g(z) = 0$ for some $z \in]\gamma, z_0]$. So $g(z)$ is positive on $]\gamma, z_0]$. Our goal is to show that $\gamma = 0$ and (14) holds true. We split the proof into six claims.

(i) $\exists \xi \in]\gamma, z_0]$ s.t. $g(\xi) \geq 1$.

Suppose that $g(z) < 1$, $\forall z \in]\gamma, z_0]$. Then, by (12), $g'(z) < 0$ on $]\gamma, z_0]$ and so there exists $\lim_{z \downarrow \gamma} g(z) = g_1$ with $g_0 < g_1 \leq 1$. Hence, by minimality of γ, it should hold $\gamma = 0$. According to (13),

$$\left(z^{3/4}g(z)\right)' < 0 \Longrightarrow g(z) > g_0\left(\frac{z_0}{z}\right)^{3/4} , \quad \forall z \in]0, z_0],$$

a contradiction. In the following choose $\xi \in]\gamma, z_0]$ to be the largest number such $g(\xi) \geq 1$.

(ii) $g(z) > 1$, $\forall z \in]\gamma, \xi[$.

Actually, $g(z) = 1$ implies $g'(z) < 0$.

(iii) $\gamma = 0$.

If $\gamma > 0$ then, by (13), $|(z^{3/4}g(z))'| \leq \gamma^{-5/4} < \infty$ for all $z \in]\gamma, \xi[$. This means that $z^{3/4}g(z)$ is absolutely continuous on $]\gamma, \xi[$ and so $\lim_{z \downarrow \gamma} g(z)$ exists and is finite, a contradiction with the minimality of γ.

(iv) $\exists \eta \in]0, \xi[$ s.t. $g'(z) \neq 0$, $\forall z \in]0, \eta[$.

Equalling the RHS of (12) to zero we get a quadratic equation with respect to $g(z)$. Its solution is a couple of functions,

$$\varphi_1(z) = \frac{2}{1+\sqrt{1-3z}}, \quad \varphi_2(z) = \frac{2}{1-\sqrt{1-3z}},$$

defined on the interval $]0,\xi[\cap]0,\frac{1}{3}[$. Clearly, $\varphi_1'(z) > 0$ and $\varphi_2'(z) < 0$ everywhere on that interval. This implies that the RHS of (12) vanishes in a point z from that interval if and only if $g(z) = \varphi_1(z)$ or $g(z) = \varphi_2(z)$ and in such a case either $0 = g'(z) < \varphi_1'(z)$ or $0 = g'(z) > \varphi_2'(z)$. Thus $\varphi_1(z)$ coincides with $g(z)$ in at most one point z, and the same is true for $\varphi_2(z)$. Consequently, there exists $\eta \in]0,\xi[\cap]0,\frac{1}{3}[$ such that $g'(z)$ doesn't vanish on $]0,\eta[$. Choose η having this property.

(v) $g'(z) > 0$, $\forall z \in]0,\eta[$.

If $g'(z) < 0$, $\forall z \in]0,\eta[$, then $g_1 = \lim_{z\downarrow 0} g(z)$ exists (finite or infinite) and $g_1 > g(\eta) > 1$. On the other hand, in virtue of (12),

$$z^2 g'(z) = 1 - \frac{1}{g(z)} - \frac{3}{4}\, z\, g(z) < 0\,. \tag{15}$$

Equality (13) implies that $(z^{3/4} g(z))' > 0$ on $]0,\eta[$ and so

$$g(z) < g(\eta)\left(\frac{\eta}{z}\right)^{3/4}, \quad \forall z \in]0,\eta[\,.$$

Consequently, $\lim_{z\downarrow 0} z\, g(z) = 0$. Sending z to 0 in (15) gives

$$1 - \frac{1}{g_1} \leq 0\,,$$

a contradiction.

(vi) $\lim_{z\downarrow 0} g(z) = 1$.

From Claim (v) follows that $g_1 = \lim_{z\downarrow 0} g(z)$ exists and $1 \leq g_1 < g(\eta)$. Suppose that $g_1 > 1$. Then one concludes from (12) that there exists $\delta \in]0,\eta[$ s.t.

$$g'(z) > \frac{d}{z^2},\ \forall z \in]0,\delta[\,, \quad \text{where } d = \frac{1}{2}\left(1 - \frac{1}{g_1}\right) > 0.$$

This implies

$$g(z) < g(\delta) + \frac{d}{\delta} - \frac{d}{z},\ \forall z \in]0,\delta[\,,$$

a contradiction. $\square$

Corollary 7. *The maximal solution $g(z)$ satisfies the integral identity*

$$g(z)^2 = 2z^{-3/2} \int_0^z s^{-1/2}(g(s) - 1)\, ds, \quad \forall z \in]0, z_0\,]. \tag{16}$$

Proof. Rewrite (12) as

$$(z^{3/2}g^2)' = 2z^{-1/2}(g-1),$$

and integrate from 0 to z when taking into account (14). $\square$

3.2 Asymptotics of the left maximal solution $g(z)$ at $z = 0$

Lemma 8. *If $\beta \geq 0$ then*

$$e^{-\frac{1}{z}} \int_z^1 s^{\beta-2} e^{\frac{1}{s}}\, ds = O(z^\beta) \quad \textit{as } z \downarrow 0\,.$$

Proof.

$$\begin{aligned} e^{-\frac{1}{z}} \int_z^1 s^{\beta-2} e^{\frac{1}{s}}\, ds &= e^{-\frac{1}{z}} \left(\int_1^{\frac{1}{2z}} + \int_{\frac{1}{2z}}^{\frac{1}{z}} \right) u^{-\beta} e^u \, du \\ &\leq e^{-\frac{1}{z}} \left(e^{\frac{1}{2z}} - e \right) + (2z)^\beta e^{-\frac{1}{z}} \left(e^{\frac{1}{z}} - e^{\frac{1}{2z}} \right). \end{aligned}$$

$\square$

Notation. $(\alpha)_0 = 1$, $(\alpha)_j = \alpha(\alpha+1)\ldots(\alpha+j-1)$.

Lemma 9. *Let $\delta > 0$, $\nu \in \mathbb{R}$. Then the expression*

$$z^{-\nu} e^{-1/z} \int_z^\delta s^{\nu-2} e^{1/s}\, ds\,,$$

regarded as a function in the variable z, has the asymptotic series, as $z \downarrow 0$,

$$\sum_{j=0}^{\infty} (\nu)_j z^j\,.$$

Proof. It suffices to show that it holds, for all $n \in \mathbb{Z}_+$,

$$\begin{aligned} e^{-\frac{1}{z}} \int_z^\delta s^{\nu-2} e^{\frac{1}{s}}\, ds &= -e^{\frac{1}{\delta}-\frac{1}{z}} \sum_{j=0}^{n-1} (\nu)_j \delta^{\nu+j} + \sum_{j=0}^{n-1} (\nu)_j z^{\nu+j} \\ &\quad + (\nu)_n e^{-\frac{1}{z}} \int_z^\delta s^{\nu+n-2} e^{\frac{1}{s}}\, ds\,. \end{aligned}$$

Actually, according to Lemma 8 this means that the relation

$$z^{-\nu}e^{-\frac{1}{z}}\int_z^\delta s^{\nu-2}e^{\frac{1}{s}}\,ds=\sum_{j=0}^{n-1}(\nu)_j z^j+O(z^n)$$

holds true for all $n\geq-\nu$, and so for all $n\in\mathbb{N}$.

We shall proceed by induction in n. For $n=0$ this is a trivial equality. The induction step $n\to n+1$:

$$\begin{aligned} e^{-\frac{1}{z}}\int_z^\delta s^{\nu+n-2}e^{\frac{1}{s}}\,ds &= -e^{-\frac{1}{z}}\int_z^\delta s^{\nu+n}\left(e^{\frac{1}{s}}\right)'\,ds \\ &= -e^{-\frac{1}{z}}[s^{\nu+n}e^{\frac{1}{s}}]_z^\delta+(\nu+n)e^{-\frac{1}{z}}\int_z^\delta s^{\nu+n-1}e^{\frac{1}{s}}\,ds\,. \end{aligned}$$

□

Let us consider equation (11) (without the initial condition) in the space of formal power series $\mathbb{C}[[z]]$. Its solution

$$\tilde{g}(z)=\sum_{k=0}^\infty \alpha_k z^k\in\mathbb{C}[[z]] \tag{17}$$

is unique, with the coefficients being determined by the recursive relation

$$\alpha_0=1,\quad \alpha_{k+1}=\left(\frac{1}{2}k+\frac{3}{4}\right)\sum_{j=0}^k\alpha_j\alpha_{k-j}\quad\text{for } k\geq 0\,. \tag{18}$$

Several first coefficients are

$$\alpha_0=1,\ \alpha_1=\frac{3}{4},\ \alpha_2=\frac{15}{8},\ \alpha_3=\frac{483}{64},\ldots\,.$$

Proposition 10. *The left maximal solution $g(z)$ has an asymptotic series, as $z\downarrow 0$, that is equal to*

$$\sum_{k=0}^\infty \alpha_k z^k\,.$$

Proof. We have to show that, for all $n\in\mathbb{Z}_+$,

$$g(z)=\sum_{k=0}^n\alpha_k z^k+o(z^n)\,. \tag{19}$$

We shall proceed by induction in n. The case $n=0$ means that $\lim_{z\downarrow 0} g(z) = 1$ and is covered by Proposition 6. Let us suppose that (19) is valid for some $n \in \mathbb{Z}_+$. Denote (in this proof)

$$a_- = \liminf_{z\downarrow 0} \frac{g(z) - \sum_{k=0}^{n} \alpha_k z^k}{z^{n+1}} .$$

Similarly, a_+ designates the limes superior, as $z \downarrow 0$, of the same function. Thus the induction step $n \to n+1$ means to verify that $a_- = a_+ = \alpha_{n+1}$. We shall do it in three steps.

(i) $a_- \leq \alpha_{n+1} \leq a_+$.

If $b < a_-$ then there exists $\delta > 0$ s.t. $g(z) > \sum_{k=0}^{n} \alpha_k z^k + b\, z^{n+1}$, $\forall z \in]0, \delta[$. Further, from the recurrent relation (18) one finds that

$$\tilde{g}(z)^2 = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \alpha_j \alpha_{k-j} \right) z^k = \sum_{k=0}^{\infty} \frac{2}{k+\frac{3}{2}}\, \alpha_{k+1} z^k \,, \tag{20}$$

and the assumption (19) implies that the asymptotics of $g(z)^2$ is given by a truncation of the formal power series $\tilde{g}(z)^2$, namely

$$g(z)^2 = \sum_{k=0}^{n} \frac{2}{k+\frac{3}{2}} \alpha_{k+1} z^k + o(z^n) \,.$$

Combining this with (16) leads to the conclusion that, $\forall z \in]0, \delta[$,

$$\begin{aligned} \sum_{k=0}^{n} \frac{2}{k+\frac{3}{2}}\, \alpha_{k+1} z^k + o(z^n) &> 2\, z^{-3/2} \int_0^z s^{-1/2} \left(\sum_{k=1}^{n} \alpha_k s^k + b\, s^{n+1} \right) ds \\ &= 2 \sum_{k=1}^{n} \frac{\alpha_k}{k+\frac{1}{2}}\, z^{k-1} + 2\, \frac{b}{n+\frac{3}{2}}\, z^n \,. \end{aligned}$$

Hence $b \leq \alpha_{n+1}$ for all $b < a_-$ and consequently $a_- \leq \alpha_{n+1}$. The inequality $a_+ \geq \alpha_{n+1}$ can be proven symmetrically.

(ii) Let $\lambda > 0$ be a fixed parameter. In the second step we introduce an auxiliary function $\varphi(z)$ whose choice depends on whether $n=0$ or $n>0$. In the former case we set

$$\begin{aligned} \varphi(z) &= \frac{1}{\lambda} z^{-\frac{3}{4}} e^{-\frac{1}{\lambda z}} \int_z^1 s^{-\frac{5}{4}} e^{\frac{1}{\lambda s}}\, ds \\ &= (\lambda z)^{-\frac{3}{4}} e^{-\frac{1}{\lambda z}} \int_{\lambda z}^{\lambda} s^{-\frac{5}{4}} e^{\frac{1}{s}}\, ds \end{aligned} \tag{21}$$

and in the latter one

$$\varphi(z) = z^{-\frac{3}{4}} e^{-\frac{1}{z}} \int_z^1 \sum_{k=0}^{n+1} \varphi_k \, s^{\frac{3}{4}+k-2} \, e^{\frac{1}{s}} \, ds \tag{22}$$

where

$$\begin{aligned} \varphi_0 &= \alpha_0, \\ \varphi_k &= \alpha_k - \left(k - \frac{1}{4}\right) \alpha_{k-1} \quad \text{for } 1 \le k \le n, \\ \varphi_{n+1} &= \lambda \, \alpha_{n+1} - \left(n + \frac{3}{4}\right) \alpha_n. \end{aligned} \tag{23}$$

Observe that $\varphi_1 = \alpha_1 - \frac{3}{4}\alpha_0 = 0$. In the case $n = 0$, $\varphi(z)$ solves the differential equation

$$\varphi'(z) = \frac{1}{\lambda \, z^2}(\varphi(z) - 1) - \frac{3}{4}\frac{\varphi(z)}{z}, \tag{24}$$

while in the case $n > 0$, $\varphi(z)$ solves

$$\varphi'(z) = \frac{1}{z^2}(\varphi(z) - 1) - \frac{3}{4}\frac{\varphi(z)}{z} - \sum_{k=2}^{n+1} \varphi_k z^{k-2}. \tag{25}$$

We claim that, in the both cases, the asymptotic behaviour of $\varphi(z)$, as $z \downarrow 0$, is given by

$$\varphi(z) = \sum_{k=0}^{n} \alpha_k z^k + \lambda \, \alpha_{n+1} z^{n+1} + o(z^{n+1}). \tag{26}$$

Actually, equality (26) follows directly from Lemma 9. In more detail, Lemma 9 gives, for the case $n > 0$,

$$\begin{aligned} \varphi(z) &= \sum_{k=0}^{n+1} \varphi_k z^k \sum_{j=0}^{n+1-k} \left(k + \frac{3}{4}\right)_j z^j + o(z^{n+1}) \\ &= \sum_{m=0}^{n+1} \sum_{k=0}^{m} \varphi_k \left(k + \frac{3}{4}\right)_{m-k} z^m + o(z^{n+1}). \end{aligned} \tag{27}$$

The coefficients φ_k, as given in (23), have been chosen so that the asymptotics (26) is satisfied. This is to say that equalling (27) to (26) leads to a system of linear equations on the coefficients φ_k whose unique solution is exactly (23) as follows from the identity

$$\sum_{k=0}^{m}\alpha_k\left(k+\frac{3}{4}\right)_{m-k}-\sum_{k=1}^{m}\alpha_{k-1}\left(k-\frac{1}{4}\right)\left(k+\frac{3}{4}\right)_{m-k}=\alpha_m.$$

The case $n=0$ is even more straightforward.

(iii) $a_+\leq\alpha_{n+1}$ and $a_-\geq\alpha_{n+1}$.

Let us show the first inequality, the other one can be proven analogously. We shall need the asymptotics of the function

$$\begin{aligned}\frac{(g(z)-1)^2}{g(z)\,z^2} &= \left(\sum_{k=0}^{n}\alpha_k z^k+o(z^n)\right)^{-1}\left(\sum_{k=1}^{n}\alpha_k z^{k-1}+o(z^{n-1})\right)^2\\ &= \sum_{k=0}^{n-1}\gamma_k z^k+o(z^{n-1})\,.\end{aligned}$$

Again, assumption (19) implies that $\sum_{k=0}^{n-1}\gamma_k z^k$ is a truncation of the power series

$$\begin{aligned}\frac{(\tilde{g}(z)-1)^2}{\tilde{g}(z)\,z^2} &= (\tilde{g}-1)\tilde{g}'+\frac{3}{4}\frac{\tilde{g}(\tilde{g}-1)}{z}\\ &= \frac{1}{2}(\tilde{g}^2)'-\tilde{g}'+\frac{3}{4}\frac{\tilde{g}^2-\tilde{g}}{z}\,.\end{aligned}$$

Here we have used that $\tilde{g}$ solves (12). Combining (17) and (20) one arrives at the formula

$$\gamma_k=\alpha_{k+2}-\left(k+\frac{7}{2}\right)\alpha_{k+1}. \tag{28}$$

Now we can compare the functions $g(z)$ and $\varphi(z)$. Let us choose $\lambda>1$ in (21) resp. (22). Suppose that $\varphi(z)=g(z)$ at some point z. Then, owing to (12) and (24), it holds

$$\varphi'(z)-g'(z)=\frac{1}{\lambda z^2 g(z)}(g(z)-1)(g(z)-\lambda),$$

when $n=0$, and using (25), (23) and (28),

$$\begin{aligned}\varphi'(z)-g'(z) &= \frac{1}{z^2 g(z)}(g(z)-1)^2-\sum_{k=2}^{n+1}\varphi_k z^{k-2}\\ &= (1-\lambda)\alpha_{n+1}z^{n-1}+o(z^{n-1}),\end{aligned}$$

when $n > 0$. In any case, there exists $\delta > 0$ s.t. $\varphi(z) = g(z)$ implies $\varphi'(z) < g'(z)$, $\forall z \in]0, \delta[$ (in the case $n = 0$, we need also that $g(z) > 1$ if z is sufficiently close to 0, see Proposition 6). This means that the functions $g(z)$ and $\varphi(z)$ coincide in at most one point $z \in]0, \delta[$. Furthermore, we have already shown that $a_- \leq \alpha_{n+1}$, and so, using also (26), we conclude that

$$\liminf_{z \downarrow 0} \frac{g(z) - \varphi(z)}{z^{n+1}} = a_- - \lambda\, a_{n+1} \leq (1-\lambda)\alpha_{n+1} < 0.$$

Thus there exists a sequence $\{z_n\}$ s.t. $z_n \downarrow 0$ and $g(z_n) < \varphi(z_n)$, $\forall n$. Consequently, $g(z) < \varphi(z)$ on a right neighbourhood of 0. Hence, in virtue of (26),

$$a_+ \leq \limsup_{z \downarrow 0} \frac{\varphi(z) - \sum_{k=0}^{n} \alpha_k z^k}{z^{k+1}} = \lambda\, \alpha_{n+1}.$$

The claim is a consequence of the limit $\lambda \downarrow 1$. $\square$

Corollary 11. *The left maximal solution $g(z)$, after having been defined at $z = 0$ by $g(0) = 1$, belongs to $C^\infty([0, z_0])$.*

Proof. Observe that consecutive differentiation of equation (12) jointly with Proposition 10 imply that, for any $m \in \mathbb{Z}_+$, $z^{m+1} g^{(m)}(z)$ has an asymptotic series at $z = 0$ which we shall call $\sum_{k=0}^{\infty} \alpha_k^m z^k$ .We have to show that $g \in C^m$, $\forall m$, and this in turn amounts to showing that $\alpha_k^m = 0$ for $k < m+1$. Let us proceed by induction in m. The case $m = 0$ was the content of Proposition 6. Assume now that $g \in C^m$. Then $\alpha_k^m = 0$ for $k < m+1$, and the mean value theorem implies that

$$\liminf_{z \downarrow 0} g^{(m+1)}(z) \leq \frac{dg^{(m)}(0_+)}{dz} = \alpha_{m+2}^m \leq \limsup_{z \downarrow 0} g^{(m+1)}(z)\,. \tag{29}$$

On the other hand, since $z^{m+2} g^{(m+1)}(z)$ has an asymptotic series the limit $\lim_{z \downarrow 0} g^{(m+1)}(z)$ always exists and equals either $\pm\infty$ or α_{m+2}^{m+1} depending on whether there exists an index $k < m+2$ s.t. $\alpha_k^{m+1} \neq 0$ or not. However the property (29) clearly excludes the first possibility. $\square$

4 Asymptotics of a solution $h(t)$ of the second order differential equation

Except of the last subsection, we still consider the particular case when $t_0 = 0$ and $h_1 > 0$ (see the remark at the end of Section 2). We shall proceed to the case of general initial condition only at the very end of the proof, in Subsection 4.5.

4.1 Reduction of the second order differential equation

Let us now complete some computations concerning the reduction of the second order differential equation (1), (2) to a first order differential equation. Let $g(z)$ be the left maximal solution of the first order differential equation

$$\left(1 - \frac{3}{4} z\, g(z) - z^2 g'(z)\right) g(z) = 1, \;\; g(z_0) = g_0,$$

where

$$z_0 = \frac{4}{h_0^4}, \;\; g_0 = h_0^3 h_1 \,.$$

¿From Section 3 we know that $g(z)$ is a positive function from the class $C^\infty([\,0, z_0\,])$ (Proposition 6 and Corollary 11). Consider the function

$$G(x) = \int_{h_0^4}^{x} \frac{ds}{g\left(\frac{4}{s}\right)}, \;\; h_0^4 \le x < \infty. \tag{30}$$

Then $G \in C^\infty([\,h_0^4, \infty[\,)$, G is strictly increasing, $G(h_0^4) = 0$, and, owing to (14), $\lim_{x\to\infty} G(x) = \infty$. So the inverse function satisfies $G^{-1} \in C^\infty([\,0, \infty[)$ with $G^{-1}(0) = h_0^4$. Set

$$h(t) = \left(G^{-1}\,(4t)\right)^{1/4}.$$

Then $h(t)$ solves the problem (1), (2).

Actually, $G(h(t)^4) = 4t$, $G'(h^4) = g(4/h^4)^{-1}$, and so

$$h^3 h' = \frac{1}{4G'(h^4)} \frac{d\,G(h^4)}{dt} = g\left(\frac{4}{h^4}\right). \tag{31}$$

Differentiating (31) once more gives

$$3h^2(h')^2 + h^3 h'' = -16\, h^{-5} h' g'\left(\frac{4}{h^4}\right).$$

Denote for brevity $z = 4/h^4$. Hence

$$\begin{aligned} h^3(h' + h'') &= g(z) - 3h^{-4}(h^3h')^2 - 16\, h^{-8}(h^3h')g'(z) \\ &= g(z) - \frac{3}{4} z\, g(z)^2 - z^2 g(z)\, g'(z) \\ &= 1\,. \end{aligned}$$

Furthermore, $h(0) = \left(G^{-1}(0)\right)^{1/4} = h_0$ and

$$\begin{aligned} h'(0) &= G^{-1}(0)^{-3/4}\,\frac{d}{ds}G^{-1}(0) \;=\; h_0^{-3}\left(\frac{d}{dx}G(h_0^4)\right)^{-1} \\ &= h_0^{-3}\,g\left(\frac{4}{h_0^4}\right) \;=\; h_1\,. \end{aligned}$$

4.2 Asymptotics of $G(x)$

First let us find, in $\mathbb{C}[[z]]$, the reciprocal element to the formal power series $\tilde{g}(z) = \sum_{k=0}^{\infty} \alpha_k z^k$ defined in (17), (18). Set

$$\tilde{g}(z)^{-1} = \sum_{n=0}^{\infty} \beta_n z^n\,.$$

The formal power series $\tilde{g}(z)$ solves the differential equation (12) and so an easy calculation shows that $\tilde{f}(z) = \tilde{g}(z)^{-1}$ solves the differential equation

$$f'(z) = -\frac{1}{z^2}\,f(z)^2(1 - f(z)) + \frac{3}{4}\frac{f(z)}{z}\,.$$

On the other hand, this differential equation implies a recursive rule on the coefficients β_n, namely rule (6) preceding the formulation of Theorem 1.

Lemma 12. *The asymptotic series at infinity of the function $G(x)$ defined in (30) is given by*

$$G(x) \sim x - 3\,\ln(x) + c - 4\sum_{k=1}^{\infty}\frac{\beta_{k+1}}{k}\left(\frac{4}{x}\right)^k$$

where

$$c = \int_{h_0^4}^{\infty}\left(\frac{1}{g\left(\frac{4}{s}\right)} - 1 + \frac{3}{s}\right)ds - h_0^4 + 3\,\ln(h_0^4). \tag{32}$$

Proof. It holds

$$\begin{aligned} G(x) &= \int_{h_0^4}^{\infty}\left(\frac{1}{g\left(\frac{4}{s}\right)} - 1 + \frac{3}{s}\right)ds + \int_{h_0^4}^{x}\left(1 - \frac{3}{s}\right)ds \\ &\quad - \int_{x}^{\infty}\left(\frac{1}{g\left(\frac{4}{s}\right)} - 1 + \frac{3}{s}\right)ds\,. \end{aligned}$$

According to Proposition 10 we have the asymptotics at infinity,

$$\frac{1}{g\left(\frac{4}{s}\right)} - 1 + \frac{3}{s} \sim \sum_{k=2}^{\infty} \beta_k \left(\frac{4}{s}\right)^k .$$

The claim then follows straightforwardly. □

4.3 Asymptotics of $G^{-1}(x)$

Let us now focus on the inverse function G^{-1}.

Lemma 13. *There exists $x_\star$ such that for all $x > x_\star$ it holds true that*

$$0 \le G^{-1}(x) - x \le \frac{x}{x-4}\,(x - G(x)). \tag{33}$$

Proof. Choose $y_\star \ge G(h_0^4) > 0$ so that, for all $y \ge y_\star$,

$$0 \le 1 - \frac{1}{g\left(\frac{4}{y}\right)} \le \frac{4}{y} \ \text{ and } \ y - G(y) \ge 0. \tag{34}$$

This is possible owing to Proposition 10 and Lemma 12. Set $x_\star = \max\{4, y_\star\}$. For $x > x_\star$ fixed define a sequence $\{y_n\}_{n=0}^{\infty}$ by the recursive rule

$$y_0 = x,\ y_{n+1} = x + y_n - G(y_n).$$

Owing to our choice, $y_n \ge x \ge x_\star \ge y_\star$ for all n. Using (30) one finds that

$$y_{n+2} - y_{n+1} = y_{n+1} - y_n - \int_{y_n}^{y_{n+1}} \frac{ds}{g\left(\frac{4}{s}\right)} = \int_{y_n}^{y_{n+1}} \left(1 - \frac{1}{g\left(\frac{4}{s}\right)}\right) ds.$$

In virtue of (34), the sequence satisfies

$$0 \le y_{n+2} - y_{n+1} \le 4(\ln(y_{n+1}) - \ln(y_n)) \le \frac{4}{x}\,(y_{n+1} - y_n).$$

Since $y_1 - y_0 = x - G(x)$ we get

$$0 \le y_{n+1} - y_n \le \left(\frac{4}{x}\right)^n (x - G(x)),\ \forall n.$$

By the choice of $x_\star$ we have $4 < x$ and, consequently, the sequence $\{y_n\}$ is convergent. The limit $y = \lim y_n$ solves $0 = x - G(y)$ and so $y = G^{-1}(x)$. Moreover,

$$\begin{aligned} 0 \ \le\ y - y_n &= \sum_{k=n}^{\infty} (y_{k+1} - y_k) \\ &\le \sum_{k=n}^{\infty} \left(\frac{4}{x}\right)^k (x - G(x)) = \frac{x}{x-4}\,(x - G(x)) \left(\frac{4}{x}\right)^n . \end{aligned}$$

The particular case $n = 0$ in this relation is nothing but our claim. □

Combining Lemma 13 with Lemma 12 one immediately gets

Corollary 14. *It holds true that, as $x \to +\infty$,*

$$G^{-1}(x) = x + O(\ln(x)).$$

Recall that in (4), (5) we have introduced polynomials $\sigma_k^m(a_1, a_2, \ldots, a_k)$ labeled by indices $m \geq 0$ and $k \geq 1$.

Proposition 15. *For all $n \in \mathbb{Z}_+$ it holds true that, as $x \to \infty$,*

$$G^{-1}(x) = x + p_0(c; \ln(x)) + \sum_{k=1}^{n} \frac{p_k(c; \ln(x))}{x^k} + O\Big(\left(\frac{\ln(x)}{x} \right)^{n+1} \Big) \tag{35}$$

where the polynomials $p_n(c; z)$ have been defined in (7), (8) and the constant c is given by equality (32).

Proof. Corollary 14 implies

$$\ln\left(G^{-1}(x)\right) = \ln(x) + O\left(\frac{\ln(x)}{x}\right), \ \frac{1}{G^{-1}(x)} = \frac{1}{x} + O\left(\frac{\ln(x)}{x^2}\right). \tag{36}$$

Combining (36) with Lemma 12 one derives the relation

$$x = G^{-1}(x) - 3\,\ln(G^{-1}(x)) + c - 4 \sum_{k=1}^{n} \frac{\beta_{k+1}}{k} \left(\frac{4}{G^{-1}(x)}\right)^k + O\left(\frac{1}{x^{n+1}}\right), \tag{37}$$

valid for all $n \geq 0$. Setting $n = 0$ in (37) one arrives at the case $n = 0$ in (35). To finish the proof one can proceed, in the obvious way, by induction in n when repeatedly using relation (37). $\square$

4.4 Asymptotics of $h(t)$ for particular initial data

We already know that $h(t) = \left(G^{-1}(4t)\right)^{1/4}$ solves the problem $h(t)^3(h''(t) + h'(t)) = 1$, $h(0) = h_0$, $h'(0) = h_1$. Using the known asymptotics of $G^{-1}(x)$ we get

$$h(t) = (4t)^{1/4} \left(1 + \sum_{k=1}^{n} \frac{p_{k-1}(c; \ln(4t))}{4^k t^k} + O\left(\frac{\ln(t)^n}{t^{n+1}}\right) \right)^{1/4}$$

and consequently

$$h(t) = (4t)^{1/4} \left(1 + \sum_{k=1}^{n} \frac{q_{k-1}(c; \ln(4t))}{t^k} + O\Big(\left(\frac{\ln(t)}{t}\right)^{n+1} \Big) \right)$$

where

$$q_k = \sum_{m=1}^{k} \frac{1}{4^k} \binom{\frac{1}{4}}{m} s_{m,k}(p_0, p_1, \ldots, p_{k-1})\,.$$

So q_k are exactly the polynomials introduced in Theorem 1. It is also easy to see that the degree of $q_k(c; z)$ is less than or equal to k since the same is true for the polynomials p_n with $n \geq 1$ and $\deg p_0 = 1$. This observation in fact completes the proof of Theorem 1 in the case when $t_0 = 0$ and $h_1 > 0$.

4.5 General initial conditions

Consider first a solution $h(t)$ of (1) with the initial conditions $h(0) = h_0$, $h'(0) = h_1$, assuming that h_1 is positive. Then, as we already know, the asymptotic behaviour of $h(t)$ is described by Theorem 1, i.e. equality (9) holds true with $c = c(0, h_0, h_1)$. Choose $s \in \mathbb{R}$ and set $\tilde{h}(t) = h(t+s)$. Then $\tilde{h}(t)$ solves equation (1) and satisfies the initial conditions $\tilde{h}(0) = \tilde{h}_0 = h(s)$, $\tilde{h}'(0) = \tilde{h}_1 = h'(s)$. But $h'(s)$ is positive for s sufficiently small and so equality (1) applies to $\tilde{h}(t)$ as well, with c being replaced by $\tilde{c} = c(0, \tilde{h}_0, \tilde{h}_1)$. Equating the asymptotics of $h(t+s)$ to the asymptotics of $\tilde{h}(t)$ one arrives at the equality

$$\begin{aligned}\big(4(t+s)\big)^{1/4} &\left(1 + \sum_{k=1}^{n} \frac{q_k\big(c; \ln\big(4(t+s)\big)\big)}{(t+s)^k}\right) \\ &= (4t)^{1/4} \left(1 + \sum_{k=1}^{n} \frac{q_k(\tilde{c}; \ln(4t))}{t^k} + O\Big(\Big(\frac{\ln(t)}{t}\Big)^{n+1}\Big)\right)\end{aligned} \tag{38}$$

valid for $t \to +\infty$ and every $n \in \mathbb{Z}_+$. ¿From (38) it is not difficult to derive the relation between c and $\tilde{c}$, it reads

$$\tilde{c} = c - 4s. \tag{39}$$

Thus the invariance of the differential equation (1) is reflected in an invariance of the asymptotic expansion of its solutions, as expressed by relations (38), (39). It is also clear that these relations must hold true not only for s small but even for all $s \in \mathbb{R}$.

Choose now arbitrary initial data $(t_0, h_0, h_1) \in \mathbb{R} \times]0, \infty[\, \times \mathbb{R}$ and let $h(t)$ be the corresponding solution. Then, as we know from Corollary 4, $h'(t) > 0$ for all sufficiently large t. Fix $s > t_0$ such that $h'(s) > 0$ and set $\tilde{h}(t) = h(t+s)$. We use once more the already proven fact that $\tilde{h}(t)$ satisfies equality

(9), with c being replaced by $\tilde{c} = c(0, h(s), h'(s))$. This implies that the asymptotic behaviour of $h(t) = \tilde{h}(t-s)$ is given by

$$h(t) = \big(4(t-s)\big)^{1/4}\left(1 + \sum_{k=1}^{n} \frac{q_k(\tilde{c}; \ln\big(4(t-s)\big))}{(t-s)^k} + O\Big(\Big(\frac{\ln(t)}{t}\Big)^{n+1}\Big)\right),$$

$n \in \mathbb{Z}_+$. But in that case one deduces from (38), (39) that $h(t)$ satisfies equality (9) as well, with $c = \tilde{c} + 4s$. Theorem 1 is proven.

5 Additional remark: comparison with the asymptotics of $-W_{-1}(-e^{-x})$

This is a digression whose aim is to emphasize a rather close similarity of the asymptotic behaviour of the function $G^{-1}(x)$ with that of Lambert function. The Lambert function $W(z)$ gives the principal solution for w in $z = we^w$ and $W_k(z)$ gives the k^{th} solution. Surprisingly, it is not documented in some standard text books and reference books on special functions though we may have missed some sources. On the other hand, the Lambert function seems to have attracted even in a rather recent period some attention, particularly from the computational and combinatorial point of view (see [3] for a summary). It is also implemented in some standard computer algebra systems like Maple and Mathematica where it is called LambertW and ProductLog, respectively. Let us just briefly recall that $W(z)$ is analytic in a neighbourhood of $z = 0$ with the convergence radius equal to e^{-1},

$$W(z) = \sum_{k=1}^{\infty} \frac{(-1)^k k^{k-1}}{k!}\, z^k\,.$$

The coefficients have a combinatorial interpretation when counting distinct oriented trees.

Consider now the equation

$$y - \ln(y) = x,$$

or, equivalently,

$$ye^{-y} = e^{-x}\,.$$

It is elementary to see that for $x \in]1, +\infty[$ there are exactly two real solutions, $y_1(x)$ and $y_2(x)$, with $y_1(x) \in]0, 1[$ and $y_2(x) \in]1, +\infty[$. The both solutions can be expressed with the aid of the Lambert function, namely

$$y_1(x) = -W(-e^{-x}), \quad y_2(x) = -W_{-1}(-e^{-x})\,.$$

The aim of this remark is to point out that the asymptotics of the second solution, i.e. $-W_{-1}(-e^{-x})$, as $x \to +\infty$, can be derived in a way quite similar to what we have done in Subsection 4.3 when treating the function $G^{-1}(x)$. To this end let us recursively define polynomials $\tilde{p}_k(z)$,

$$\tilde{p}_0(z) = z,\ \tilde{p}_{k+1}(z) = \sigma^0_{k+1}(\tilde{p}_0(z), \tilde{p}_1(z), \ldots, \tilde{p}_k(z)).$$

For $k \geq 1$, the degree of the polynomial $\tilde{p}_k(z)$ is k. Here are several first polynomials:

$$\tilde{p}_1(z) = z,\ \tilde{p}_2(z) = z - \frac{1}{2}\,z^2,\ \tilde{p}_3(z) = z - \frac{3}{2}\,z^2 + \frac{1}{3}\,z^3,\ \ldots .$$

Proposition 16. *It holds, as $x \to +\infty$,*

$$-W_{-1}(-e^{-x}) = x + O(\ln x)$$

and, for $n \geq 0$,

$$-W_{-1}(-e^{-x}) = x + \sum_{k=0}^{n} \frac{\tilde{p}_k(\ln x)}{x^k} + O\left(\left(\frac{\ln x}{x}\right)^{n+1}\right).$$

The proposition can be proven using a similar approach as the one used in the proof of Lemma 13. In fact, this asymptotic expansion is well known and is in agreement with what has been published in [4, 5] and [3] though the derivation and presentation here is somewhat different.

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