THE DISCRETE SPECTRUM IN THE SPECTRAL GAPS OF SEMIBOUNDED OPERATORS WITH NON-SIGNDEFINITE PERTURBATIONS

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Abstract. Given two selfadjoint operators A and $V = V_+ - V_-$, we study the motion of the eigenvalues of the operator $A(t) = A - tV$ as t increases. Let $\alpha > 0$ and let λ be a regular point for A. We consider the quantities $N_+(V;\lambda,\alpha)$, $N_-(V;\lambda,\alpha)$, $N_0(V;\lambda,\alpha)$ defined as the number of the eigenvalues of the operator $A(t)$ that pass point λ from the right to the left, from the left to the right or change the direction of their motion exactly at point λ, respectively, as t increases from 0 to $\alpha > 0$. We study asymptotic characteristics of these quantities as $\alpha \to \infty$. In the present paper we extend the results obtained in [16] and give new applications to differential operators.

0. Introduction

1. In this paper we supplement the results obtained in [16]. We remind the setting of the problem. Let A be a selfadjoint, semibounded from below operator in a Hilbert space $\mathfrak{H}$, let V be a selfadjoint *non-signdefinite* perturbation. Assume that the quadratic form of the operator V is compact with respect to the form of A. Denote

$$A(\alpha) = A - \alpha V, \quad \alpha > 0.$$

Let the interval $\Lambda = (\lambda_-, \lambda_+)$ be a gap in the spectrum of A. Then the spectrum of $A(\alpha)$ is discrete in Λ. The eigenvalues of $A(\alpha)$ in Λ are continuous functions of α. If α grows, they may appear or disappear only at the endpoints λ_-, λ_+. Fix a number λ such that $\lambda_- < \lambda < \lambda_+$. The eigenvalues of $A(\alpha)$ may move in Λ crossing the point λ from the right to the left, from the left to the right, or "being reflected" by point λ, as α increases. Let us introduce the value $N_+(V;\lambda,\alpha)$ to be the number of the eigenvalues of the operator $A(t)$ that pass point λ from the right to the left as t increases from 0 to $\alpha > 0$. Let $N_-(V;\lambda,\alpha)$

be the same value corresponding to the motion of the eigenvalues from the left to the right. At last, denote by $N_0(V;\lambda,\alpha)$ the number of the eigenvalues of $A(t)$, which change the direction of their motion exactly at point λ as t increased from 0 to α. Our aim is to investigate the asymptotic behavior of the values $N_+(V;\lambda,\alpha), N_-(V;\lambda,\alpha), N_0(V;\lambda,\alpha)$ as $\alpha\to\infty$.

Put $N(V;\lambda,\alpha)=N_+(V;\lambda,\alpha)-N_-(V;\lambda,\alpha)$. This quantity has been studied in the paper [15] where asymptotics for $N(V;\lambda,\alpha)$ have been obtained in the case of a Schrödinger operator and its generalizations. In the present paper we obtain abstract conditions guaranteeing that

$$N_+(V;\lambda,\alpha)\sim N(V;\lambda,\alpha),\ \ \alpha\to\infty, \tag{1}$$

and

$$N_-(V;\lambda,\alpha)+N_0(V;\lambda,\alpha)=o(N(V;\lambda,\alpha)),\ \ \alpha\to\infty. \tag{2}$$

In applications these relations often allow one to apply already known results about asymptotics for $N(V;\lambda,\alpha)$. Finally note, that for asymptotics of order α^p with $p>1$ the relations (1), (2) were obtained in [16] and now we extend the results obtained in [16] to the case $0<p\le 1$.

2. Let us give a survey of the previous results connected with our investigations. First results on this problem (in a gap situation) are devoted to the case $V\ge 0$ and $V\le 0$. One should notice that this case is in a certain sense easier then the general one. We mention here the paper [11] (Klaus), where a Dirac operator has been considered. The papers [8] (Hempel) and [1] (Alama, Deift and Hempel) deal with Schrödinger operators. The paper [2] (Birman) substantially widens the results of [8]. Namely, in the case $d\ge 3$ for the operator

$$A=-\triangle+f(x),\quad f\in L_\infty(\mathbb{R}^d), \tag{3}$$

perturbed by a potential $V\ge 0$,

$$V\in L_{d/2}(\mathbb{R}^d), \tag{4}$$

the following asymptotics was obtained:

$$N(V;\lambda,\alpha)\sim(2\pi)^{-d}\omega_d\alpha^{d/2}\int V^{d/2}dx,$$

where $\omega_d=\mathrm{vol}\{x\in\mathbb{R}^d:|x|<1\}$. Similar asymptotics with slightly different conditions on V are valid for $d=1,2$.

If V is a non-signdefinite perturbation, the problem becomes much more difficult. The motion of eigenvalues of the operator $A(\alpha)$ is no longer monotone and it is not clear what is the proper object to be investigated. In the papers [9] and [14] the quantity $N_-(V;\lambda,\alpha)+N_+(V;\lambda,\alpha)+N_0(V;\lambda,\alpha)$ was studied. For this function some lower asymptotic estimates have been obtained. The fine structure of the motion of the eigenvalues ("trapping and cascading") has been considered in [6].

There were some reasons to believe that under quite broad conditions on V the relations (1), (2) hold true. Firstly it was confirmed in the paper [10] where it was shown that for a compactly supported potential V the eigenvalues of the Schrödinger operator pass the point $\lambda \in \Lambda$ from the right to the left beginning from a certain value of α. Exceptions are the points λ which are the eigenvalues of the operator $-\triangle + f(x)$ in $L_2(\mathbb{R}^d \setminus G)$ with Dirichlet boundary conditions (see [10]), where G is open and coincides with the support of V up to a set of measure zero . Secondly for asymptotics of order α^p, $p > 1$ the relations (1), (2) were proved in [16].

Note that the formulae (1), (2) allow one to apply the results of [15] where the difference $N(\lambda, \alpha) = N_+(V; \lambda, \alpha) - N_-(V; \lambda, \alpha)$ was studied. In particular, in dimension $d \geq 3$ for the case of the Schrödinger operator (3) perturbed by a non-signdefinite potential (4) the asymptotic formula

$$N(\lambda, \alpha) \sim (2\pi)^{-d} \omega_d \alpha^{d/2} \int V_+^{d/2} dx, \quad 2V_+ = |V| + V,$$

was obtained. In dimensions $d = 1, 2$ similar asymptotics are valid under some additional conditions on V. Thus (1), (2) give an asymptotics for $N_+(\lambda, \alpha)$ and asymptotic estimates for $N_0(\lambda, \alpha)$ and $N_-(\lambda, \alpha)$.

It is a pleasure for the author to express his deep gratitude to Professor M. Sh. Birman for suggesting the problem. The author is grateful to M. Sh. Birman and A. A. Laptev for their attention to the work. The author wants to thank Professor M. Klein for reading the manuscript.

1. Statement of the main result

1. Let $\mathfrak{H}$ be a separable Hilbert space. Below $(\cdot, \cdot)$, $\|\cdot\|$, I are the scalar product, the norm and the identity operator in $\mathfrak{H}$. We denote by $\mathfrak{R}$ the space of continuous linear operators and by $\mathfrak{S}_\infty$ the space of compact operators acting from $\mathfrak{H}$ to $\mathfrak{H}$. For a densely defined linear operator M we denote its domain, range, kernel, adjoint operator, resolvent set and spectrum by the symbols $\mathcal{D}(M), \mathrm{Ran} M, \mathrm{Ker}\, M, M^*, \rho(M), \sigma(M)$ respectively. If $M = M^*$, we put $2M_\pm = |M| \pm M$. For $T \in \mathfrak{S}_\infty$ we denote by $s_k(T)$, $k \in \mathbb{N}$, the singular numbers of T, i.e., the consecutive eigenvalues of the operator $(T^*T)^{1/2}$. We introduce the counting function

$$n(s, T) = \mathrm{card}\{\ k : \ s_k(T) > s\ \}, \quad s > 0, \tag{5}$$

of the s-numbers. If $T = T^* \in \mathfrak{S}_\infty(\mathfrak{H})$, we denote

$$n_\pm(\cdot, T) = n(\cdot, T_\pm). \tag{6}$$

For $0 < p < \infty$ we consider the class (ideal) $\Sigma_p \subset \mathfrak{S}_\infty$ defined by the condition

$$\Sigma_p = \{T \in \mathfrak{S}_\infty : \sup_{s>0} s^p n(s, T) < \infty\}.$$

We introduce the following functionals on Σ_p:

$$\begin{cases} \Delta_p(T) = \limsup_{s\to 0} s^p n(s, T), \\ \delta_p(T) = \liminf_{s\to 0} s^p n(s, T). \end{cases} \tag{7}$$

2. Let $a[\cdot, \cdot]$ be a semibounded from below closed sesquilinear form in $\mathfrak{H}$. We assume that its domain $d[a]$ is a dense set in $\mathfrak{H}$. The form a induces in $\mathfrak{H}$ the self-adjoint operator A. Fix a value of $\gamma \in \mathbb{R}$, such that $a_\gamma := a + \gamma \geq 1$, i.e.

$$a_\gamma[x, x] = a[x, x] + \gamma\|x\|^2 \geq \|x\|^2, \quad x \in d[a].$$

Denote by $H_\gamma[a]$ the (complete) Hilbert space $d[a]$ with the metric form

$$a_\gamma[x, x] = \|(A + \gamma I)^{1/2} x\|^2, \quad x \in d[a].$$

Let V be a self-adjoint operator in $\mathfrak{H}$, satisfying $\mathcal{D}(|V|^{1/2}) \supset d[a]$. Put $W_\pm = (\frac{1}{2}(|V| \pm V))^{1/2}$. Obviously, $\mathcal{D}(W_\pm) \supset d[a]$ and

$$W_\pm(A + \gamma I)^{-1/2} \in \mathfrak{R}.$$

We impose on $W_\pm$ the more restrictive conditions: there exists $p \in (0, \infty)$, such that

$$W_\pm(A + \gamma I)^{-1/2} \in \Sigma_{2p}. \tag{8}$$

Let us introduce the form

$$v[x, y] = (W_+ x, W_+ y) - (W_- x, W_- y). \tag{9}$$

The condition (8) implies that v is compact on $d[a]$. This means that the form v is continuous on $H_\gamma[a]$ and the corresponding operator Q, determined by the equation $a_\gamma[Qx, y] = v[x, y]$, $x, y \in d[a]$, is compact on $H_\gamma[a]$. Hence, the form v satisfies the estimate

$$v[x, x] \leq \varepsilon a_\gamma[x, x] + C(\varepsilon)\|x\|^2, \quad x \in d[a], \ \forall\varepsilon > 0.$$

Therefore, we can introduce a family of semibounded from below, closed forms $a(\alpha)$ on $d[a]$:

$$a(\alpha) = a - \alpha v, \quad \alpha > 0. \tag{10}$$

Denote by $A(\alpha)$ the self-adjoint operator in $\mathfrak{H}$, which corresponds to the form (10). Since the difference of the resolvents of A and $A(\alpha)$ is compact, the spectrum of $A(\alpha)$ in the gaps of $\sigma(A)$ is discrete. Let the interval $\Lambda = (\lambda_-, \lambda_+)$ be a gap in the spectrum $\sigma(A)$. We fix an "observation point" λ,

$$\lambda_- < \lambda < \lambda_+.$$

Assume that λ is an eigenvalue of multiplicity k of the operator $A(t)$ for some $t > 0$. There is a neighborhood of t where we can choose real-analytic functions $\lambda_j(s),\ j = 1, \ldots, k$, whose values are eigenvalues of $A(s)$ and

$$\lambda_j(t) = \lambda, \quad j = 1, \ldots, k.$$

We count the eigenvalues $\{\lambda_j(s)\}_{j=1}^k$ with their multiplicities.

Let us choose a suitable neighborhood of point t. Since none of the functions λ_j is constant, the zeros of the derivatives $d\lambda_j/ds$ are isolated. Hence we can choose such a neighborhood of t that

$$d\lambda_j(s)/ds \neq 0 \quad \text{for } s \neq t, \quad j = 1, \ldots, k.$$

Let $k_+(t),\ k_-(t),\ k_0(t)$ be the number of the functions λ_j which are monotone decreasing, monotone increasing and non-monotone (respectively) in the chosen neighborhood. If $\lambda \in \rho(A(t))$, then we assume that $k_+(t) = k_-(t) = k_0(t) = 0$. Introduce the following values

$$\begin{cases} N_\pm(V; \lambda, \alpha) = \sum\limits_{0<t<\alpha} k_\pm(t), \\ N_0(V; \lambda, \alpha) = \sum\limits_{0<t<\alpha} k_0(t), \\ N(V; \lambda, \alpha) = N_+(V; \lambda, \alpha) - N_-(V; \lambda, \alpha). \end{cases} \tag{11}$$

Our goal is to investigate the asymptotic behavior of the values (11) as $\alpha \to \infty$.

Remark 1.1. The imposed on V conditions are invariant with respect to multiplication of V by real constants. In particular, as a perturbation we can take the operator $(-1)V$. Thus, the quantities $N_+(-V; \lambda, \alpha), N_-(-V; \lambda, \alpha), N_0(-V; \lambda, \alpha)$ are also well-defined.

3. Let $W = |V|^{1/2},\ G = W(A + \gamma I)^{-1/2}$ and $S = \mathrm{sign} V$. For $\lambda \in \rho(A)$ we consider the operator

$$X(\lambda) := G(A + \gamma I)(A - \lambda I)^{-1} G^* S.$$

The condition (8) implies, that this operator is of the class Σ_p. Note that $X(\lambda)$ admits the following equivalent definition:

$$X(\lambda) g = W(A - \lambda I)^{-1} W S g, \quad g \in \mathcal{D}(W).$$

Note also that

$$n(s, X(\lambda)) = n_+(s, X(\lambda)S) + n_-(s, X(\lambda)S), \quad s > 0, \tag{12}$$

where

$$X(\lambda) S g = W(A - \lambda I)^{-1} W g, \quad g \in \mathcal{D}(W).$$

It is well known that the description of the discrete spectrum of $A(\alpha)$ can be reduced to the study of the spectrum of the compact operator $X(\lambda)$ (the Birman-Schwinger principle). We present here a version of this reduction suitable for perturbations of forms.

Proposition 1.1. *Let $\lambda = \overline{\lambda} \in \rho(A)$. The following two statements are equivalent:*

1) the point λ is an eigenvalue of multiplicity k for the operator $A(\alpha)$;

2) the point α^{-1} is an eigenvalue of geometric multiplicity k for the operator $X(\lambda)$.

Proposition 1.1 is a direct consequence of Lemma 1 from [13]. For additive operator perturbations a simple proof can be found in [12]. Note, that in the case $V \geq 0$ or $V \leq 0$ this statement was later proved in [2] and this proof can be also applied for non-signdefinite perturbations.

Now we are ready to formulate the main result of the paper.

Theorem 1.1. *Let the form v in* (10) *be defined by* (9) *and let the condition* (8) *be fulfilled for some $p > 0$. Assume that for $\lambda = \overline{\lambda} \in \rho(A)$*

$$\lim_{\alpha\to\infty} \alpha^{-p} N(V;\lambda,\alpha) = J_+, \tag{13}$$

$$\lim_{\alpha\to\infty} \alpha^{-p} N(-V;\lambda,\alpha) = J_- \tag{14}$$

and

$$\lim_{s\to 0} s^p n(s, X(\lambda)) =: J_- + J_+. \tag{15}$$

Then

$$\lim_{\alpha\to\infty} \alpha^{-p} N_+(V;\lambda,\alpha) = J_+,$$

$$\lim_{\alpha\to\infty} \alpha^{-p} N_+(-V;\lambda,\alpha) = J_-,$$

$$\lim_{\alpha\to\infty} \alpha^{-p} (N_0(V;\lambda,\alpha) + N_-(V;\lambda,\alpha)) = 0$$

and

$$\lim_{\alpha\to\infty} \alpha^{-p} (N_0(-V;\lambda,\alpha) + N_-(-V;\lambda,\alpha)) = 0.$$

2. Additional information concerning compact operators

1. Let $T \in \mathfrak{S}_\infty$ and let $\{\lambda_k(T)\}_1^\nu$ be the sequence of the eigenvalues of T, enumerated so that their absolute values are non-increasing, and counted with their *algebraic* multiplicities. Here ν is a positive integer or $\nu = \infty$. If the sequence $\{\lambda_k(T)\}_1^\nu$ is finite, we continue it by zeros. After this we can assume that $\nu = \infty$. Note that T may be a non-selfadjoint operator. We introduce the distribution function

$$n_0(s,T) = \mathrm{card}\{\ k :\ |\lambda_k(T)| > s\ \},\quad s > 0,$$

of the eigenvalues of T.

There is a relation (discovered by H. Weyl) between the eigenvalues and the singular numbers of T

Proposition 2.1. *Let $T \in \mathfrak{S}_\infty$. Then*

$$\prod_{k=1}^{l} |\lambda_k(T)| \le \prod_{k=1}^{l} s_k(T), \quad l = 1, 2, \dots \tag{16}$$

The proof of this proposition can be found in [3]. Let n be defined in (5), then this proposition could be rewritten in the following form.

Proposition 2.2. *Let $T \in \mathfrak{S}_\infty$ and $\eta > 0$. Then*

$$\int_\eta^\infty n_0(s,T)\frac{ds}{s} \le \int_\eta^\infty n(s,T)\frac{ds}{s}. \tag{17}$$

Proof. Inequality (16) can be rewritten in the form

$$\sum_{k=1}^{l} \log|\lambda_k(T)| \le \sum_{k=1}^{l} \log s_k(T).$$

Hence, for every $\eta > 0$ we have

$$\sum_{k=1}^{l}(\log|\lambda_k(T)| - \log\eta) \le \sum_{k=1}^{l}(\log s_k(T) - \log\eta). \tag{18}$$

Let us take the number $l = n_0(\eta, T)$. Then we can substitute the sum on the left-hand side of (18) by

$$\sum_{|\lambda_k(T)|>\eta} (\log|\lambda_k(T)| - \log\eta).$$

Obviously,

$$\sum_{k=1}^{l}(\log s_k(T) - \log\eta) \le \sum_{s_k(T)>\eta} (\log s_k(T) - \log\eta).$$

Thus we have

$$\sum_{|\lambda_k(T)|>\eta} (\log|\lambda_k(T)| - \log\eta) \le \sum_{s_k(T)>\eta} (\log s_k(T) - \log\eta). \tag{19}$$

It remains to note that the left- and right-hand sides in (17) are the same as in (19) respectively. The proof is completed. □

2. Here we consider the functionals

$$\begin{cases} \Delta_p^{(0)}(T) = \limsup_{s\to 0} s^p n_0(s,T), \\ \delta_p^{(0)}(T) = \liminf_{s\to 0} s^p n_0(s,T), \end{cases}$$

which are finite on Σ_p (see [3]).

Assume that $T \in \Sigma_p,\ p > 0$, and for every $s > 0$

$$n_0(s,T) \geq n(s,T) - \phi(s), \tag{20}$$

where $\phi \in L_{\infty,loc}(0,\infty)$ is a real-valued function, such that

$$\lim_{s\to 0} s^p \phi(s) = 0. \tag{21}$$

Let the functionals $\Delta_p,\ \delta_p$ be defined in (7). As an application of the inequality (17) we present the following result.

Proposition 2.3. *Let $T \in \Sigma_p,\ p > 0$. Assume that the conditions* (20) *and* (21) *are fulfilled. Then*

$$\Delta_p^{(0)}(T) = \Delta_p(T), \tag{22}$$

$$\delta_p^{(0)}(T) = \delta_p(T). \tag{23}$$

Proof. To be definite we consider the functional $\Delta_p^{(0)}(T)$. The equality (23) can be checked similarly. It follows from (20), (21) that

$$\Delta_p^{(0)}(T) \geq \Delta_p(T).$$

Let us now prove the inequality $\leq$ in (22). There exists $s_0 > 0$, such that $n_0(s,T) = n(s,T) = 0$ for $s > s_0$. Therefore, multiplying ϕ by the indicator function of $(0, s_0]$ we can assume that

$$\phi(s) = 0,\ \forall s > s_0.$$

Then for every $t > 0$, according to (20), we have

$$\int_t^\infty n_0(s,T)\frac{ds}{s} \geq \int_t^\infty n(s,T)\frac{ds}{s} - \int_t^\infty \phi(s)\frac{ds}{s}. \tag{24}$$

Note that

$$\int_t^\infty \phi(s)\frac{ds}{s} = o(t^{-p}), \quad t \to 0. \tag{25}$$

Subtracting (24) from (17), we obtain

$$\int_\eta^t n_0(s,T)\frac{ds}{s} \leq \int_\eta^t n(s,T)\frac{ds}{s} + \int_t^\infty \phi(s)\frac{ds}{s}.$$

Putting $\eta = (1-\varepsilon)t,\ 0 < \varepsilon < 1$, and estimating the left- and right-hand side, we find that

$$\varepsilon n_0(t,T) \leq \frac{\varepsilon}{1-\varepsilon} n((1-\varepsilon)t,T) + \int_t^\infty \phi(s)\frac{ds}{s}, \quad t > 0.$$

Multiplying the last inequality by t^p/ε and taking upper limits as $t \to 0$, in view of (25) we obtain

$$\Delta_p^{(0)}(T) \leq (1-\varepsilon)^{-1-p}\Delta_p(T).$$

The passage to the limit as $\varepsilon \to 0$ gives the estimate

$$\Delta_p^{(0)}(T) \leq \Delta_p(T).$$

The proof is completed. □

3. The proof of the main result (Theorem 1.1)

It is sufficient to prove that

$$\begin{gathered} N_-(V;\lambda,\alpha) + N_0(V;\lambda,\alpha) + N_-(-V;\lambda,\alpha) \\ +N_0(-V;\lambda,\alpha) = o(\alpha^p), \quad \alpha \to \infty. \end{gathered} \tag{26}$$

It follows from the Proposition 1.1 that

$$\begin{gathered} N_+(V;\lambda,\alpha) + N_-(V;\lambda,\alpha) + N_0(V;\lambda,\alpha) = \\ \textstyle\sum_{0<t<\alpha} \dim \operatorname{Ker}(X(\lambda) - t^{-1}I), \end{gathered}$$

$$\begin{gathered} N_+(-V;\lambda,\alpha) + N_-(-V;\lambda,\alpha) + N_0(-V;\lambda,\alpha) = \\ \textstyle\sum_{0<t<\alpha} \dim \operatorname{Ker}(X(\lambda) + t^{-1}I). \end{gathered}$$

Hence,

$$\begin{gathered} N_+(V;\lambda,\alpha) + N_-(V;\lambda,\alpha) + N_0(V;\lambda,\alpha) + N_+(-V;\lambda,\alpha) \\ +N_-(-V;\lambda,\alpha) + N_0(-V;\lambda,\alpha) \leq n_0(\alpha^{-1}, X(\lambda)). \end{gathered} \tag{27}$$

According to (13), (14) we have

$$\begin{gathered} N_+(V;\lambda,\alpha) - N_-(V;\lambda,\alpha) + N_+(-V;\lambda,\alpha) - \\ N_-(-V;\lambda,\alpha) \sim \alpha^p(J_+ + J_-), \quad \alpha \to \infty. \end{gathered} \tag{28}$$

By the comparison (27) and (28), we see that to get (26) it is sufficient to prove that

$$\lim_{s\to 0} s^p n_0(s, X(\lambda)) = J_+ + J_-. \tag{29}$$

We apply Proposition 2.3 putting $T = X(\lambda)$ and $\phi(s) = n(s, X(\lambda)) - N(V;\lambda,s^{-1}) - N(-V;\lambda,s^{-1})$. Then the condition (20) follows from the inequality

$$n_0(\alpha^{-1}, X(\lambda)) \geq N(V;\lambda,\alpha) + N(-V;\lambda,\alpha).$$

The equalities (13),(14),(15) imply the assumption (21). Thus we obtain (29). □

4. Application to differential operators

We give here an example of how the Theorem 1.1 can be applied to the study of spectral properties of differential operators.

Below we use the following notations $\int = \int_{\mathbb{R}^d}$, $D_j = -i\partial/\partial x_j$, $D = (D_1, \dots, D_d)$ and $\omega_d = \mathrm{vol}\{x \in \mathbb{R}^d : |x| < 1\}$. By $H^\ell(\mathbb{R}^d)$ we denote the Sobolev classes on $\mathbb{R}^d$ of order $\ell > 0$. The cubes $\mathbb{Q}_n = \mathbb{Q} + n$, $n \in \mathbb{Z}^d$, form a partition of $\mathbb{R}^d$. With this partition we associate the classes $\ell_r(\mathbb{Z}^d; L_q(\mathbb{Q}))$, $0 < r, q < \infty$, of all functions on $\mathbb{R}^d$, for which the sequence of $L_q(\mathbb{Q}_n)$-norms belongs to $\ell_r(\mathbb{Z}^d)$. Let Φ be the Fourier transform on $\mathbb{R}^d$ defined on the Schwartz class by the formula

$$\Phi u(\xi) = (2\pi)^{-d/2} \int e^{-i\langle \xi, x\rangle} u(x) dx.$$

We write $\hat{u} := \Phi u$.

In our example $\mathfrak{H} = L_2(\mathbb{R}^d)$, $d[a] = H^\ell(\mathbb{R}^d)$, $\ell \in \mathbb{N}$ and

$$a[u, u] = \int |\xi|^{2\ell} |\hat{u}(\xi)|^2 d\xi + \sum_{|\beta| \le m} \int f_\beta(x) |D^\beta u(x)|^2 dx, \qquad (30)$$

$$|\beta| \le m \in \mathbb{N}, \ 0 \le m < \ell.$$

As V we take an operator of multiplication by a measurable function $V : \mathbb{R}^d \longrightarrow \mathbb{R}$. Let us put

$$p = d/2\ell.$$

We borrow from [2] the following assumptions:

$$f_\beta : \mathbb{R}^d \longrightarrow \mathbb{R}, \quad f_\beta \in L_\infty(\mathbb{R}^d), \quad |\beta| \le m, \qquad (31)$$

and

$$\left.\begin{array}{c} V \in L_p(\mathbb{R}^d) \quad \text{for } p > 1, \\ V \in \ell_1(\mathbb{Z}^d; L_q(\mathbb{Q})), \ q > 1, \quad \text{for } p = 1, \\ V \in \ell_p(\mathbb{Z}^d; L_1(\mathbb{Q})) \quad \text{for } p < 1. \end{array}\right\} \qquad (32)$$

The form a is semibounded from below and closed on $d[a]$ (see, for example, [2]). Obviously, $W_\pm$ are the operators of multiplication by the functions $V_\pm^{1/2}$, $2V_\pm = |V| \pm V$, and

$$v[u, u] = \int V(x) |u|^2 dx. \qquad (33)$$

The condition (32) ensures (8) (see [2]). At last, for the corresponding to the form (30) operator A and for V from (32) we have (see [15], §6)

$$\lim_{\alpha \to \infty} \alpha^{-p} N(V; \lambda, \alpha) = (2\pi)^{-d} \omega_d \int V_+^p dx, \qquad (34)$$

$$\lim_{\alpha \to \infty} \alpha^{-p} N(-V; \lambda, \alpha) = (2\pi)^{-d} \omega_d \int V_-^p dx. \qquad (35)$$

Theorem 4.1. *Let a, v be the forms defined as in* (30) *and* (33), *and let the conditions* (31) *and* (32) *be fulfilled. Then for all $\lambda = \overline{\lambda} \in \rho(A)$ the following formulae hold*

$$\lim_{\alpha\to\infty} \alpha^{-p} N_+(V;\lambda,\alpha) = (2\pi)^{-d}\omega_d \int V_+^p dx,$$

$$\lim_{\alpha\to\infty} \alpha^{-p} (N_-(V;\lambda,\alpha) + N_0(V;\lambda,\alpha)) = 0.$$

Proof. It is sufficient to verify the conditions of Theorem 1.1 with $J_+ = (2\pi)^{-d}\omega_d \int V_+^p dx$ and $J_- = (2\pi)^{-d}\omega_d \int V_-^p dx$. As it was noted, the condition (32) implies (8). The asymptotics (34),(35) mean $N(V;\lambda,\alpha) \sim \alpha^p J_+, \alpha \to \infty$, and $N(-V;\lambda,\alpha) \sim \alpha^p J_-, \alpha \to \infty$. Taking into account (12), it remains to note that (see [2])

$$\lim_{s\to 0} s^p n_+(s, X(\lambda)S) = J_+ + J_-,$$

$$\lim_{s\to 0} s^p n_-(s, X(\lambda)S) = 0.$$

The theorem is proved. $\square$

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