Entropic repulsion of an interface in an external field

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Abstract

We consider an interface above an attractive hard wall in the complete wetting regime, and submitted to the action of an external increasing, convex potential, and study its delocalization as the intensity of this potential vanishes. Our main motivation is the analysis of critical prewetting, which corresponds to the choice of a linear external potential.

We also present partial results on critical prewetting in the two dimensional Ising model, as well as a few (weak) results on pathwise estimates for the pure wetting problem for effective interface models.

1 Introduction and results

1.1 Effective interface models

There has been a lot of interest recently in the properties of effective interface models in the presence of an attractive hard wall, and the associated wetting phase transition [7, 12, 24], as well as the related problems of entropic repulsion [9, 6, 14, 5, 15, 16, 3, 2] and pinning by a local potential [20, 4, 17, 23, 8].

In the present work, we consider d-dimensional effective interface models with convex interactions. That is, the interface is described by a collection of non-negative real numbers X_i, $i \in \Lambda_N = \{-N, \dots, N\}^d$, representing the height of the interface above site i. The probability measure is given by

$$\mathrm{P}_{N,+,\lambda,\upsilon}(\mathrm{d}X) = \big(Z_{N,+,\lambda,\upsilon}\big)^{-1} \exp\Big\{- \sum_{\langle i,j\rangle \cap \Lambda_N \neq \emptyset} \mathsf{U}(X_i - X_j) - \sum_{i \in \Lambda_N} \lambda \, \mathsf{V}(X_i)\Big\} \times \prod_{i\in\Lambda_N} (\mathrm{d}X_i + \upsilon\delta_0(\mathrm{d}X_i)) \prod_{i\in\mathbb{Z}^d\setminus\Lambda_N} \delta_0(\mathrm{d}X_i)\,, \quad (1.1)$$

where δ_0 is the Dirac mass at 0, λ is a strictly positive real number, $\upsilon \geq 0$ and the function U satisfies $c < \mathsf{U}'' < 1/c$ for some $c > 0$. We are mostly interested in the case $\mathsf{V}(x) = |x|$, but we allow for general $\mathsf{V} : \mathbb{R}^+ \to \mathbb{R}^+$, which are convex, increasing, and satisfy $\mathsf{V}(0) = 0$ and the following growth condition: There exists $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that, for all $\alpha > 0$,

$$\limsup_{x\to\infty} \frac{V(\alpha x)}{V(x)} \leq f(\alpha) < \infty\,.$$

(Obviously any convex polynomial is fine.) The parameter υ corresponds to the strength of the pinning potential locally attracting the interface to the wall.

Remark 1 *Our results and proofs can be extended straightforwardly to the case of a square-well potential of the form $-b\mathbf{1}_{X_i\leq a}$, $b \geq 0$, $a > 0$.*

When the pinning potential is absent (i.e. $\upsilon = 0$), we simply omit the corresponding subscript. We denote by $\upsilon_{\rm c}$ the critical value of the pinning parameter,

$$\upsilon_{\rm c} = \sup\left\{\upsilon \,:\, \lim_{N\to\infty} |\Lambda_N|^{-1} \log \frac{Z_{N,+,0,\upsilon}}{Z_{N,+,0}} = 0\right\}. \tag{1.2}$$

This corresponds to the value of the pinning parameter at which the wetting transition takes place. In the case $\mathsf{U}(x) = x^2$, it is known that $\upsilon_{\rm c} > 0$ when $d \leq 2$ [12], while $\upsilon_{\rm c} = 0$ when $d \geq 3$ [7]. (The situation is different when one drops the assumption of strict convexity of U, see [12].) We write $\mathsf{CW} = \{\upsilon \geq 0 \,:\, \upsilon < \upsilon_c\} \cup \{0\}$ to denote the region of complete wetting; for these values of the pinning parameter, the interface is delocalized when $\lambda = 0$.

Let $\mathcal{H}_{\lambda,d}$ be defined as follows: In dimension $d = 2$, $\mathcal{H}(\lambda; 2) = |\log\lambda|$; in dimensions $d \geq 3$, $\mathcal{H}_{\lambda,d} = |\log\lambda|^{1/2}$. Then our main result can be stated as

Theorem 1 *Let $\upsilon \in \mathsf{CW}$. There exist dimension-dependent, strictly positive constants δ, λ_0, r, and c such that: For all $\lambda < \lambda_0$ and $N > \lambda^{-r}$,*

$$\mathrm{P}_{N,+,\lambda,\upsilon}\big(|\Lambda_N|^{-1} \sum_{i\in\Lambda_N} X_i \notin (\delta\mathcal{H}_{\lambda,d}, \delta^{-1}\mathcal{H}_{\lambda,d})\big) \leq \exp\left\{-c\,\lambda\,\mathsf{V}\left(\mathcal{H}_{\lambda,d}\right)\,|\Lambda_N|\right\}.$$

Actually, the upper bound can be strengthened as

$$\mathrm{P}_{N,+,\lambda,\upsilon}\big(|\Lambda_N|^{-1} \sum_{i\in\Lambda_N} X_i \notin (\delta\mathcal{H}_{\lambda,d}, \delta^{-1}\mathcal{H}_{\lambda,d})\big) \leq \exp\left\{-c\,\delta^{-1}\,\lambda\,\mathsf{V}\left(\mathcal{H}_{\lambda,d}\right)\,|\Lambda_N|\right\}.$$

Remark 2 *We do not state and prove here the corresponding results in dimension 1. The reason is that, in that case, much more detailed informations, valid for an immensely larger class of interactions U can be obtained. This is of interest, since one would like to understand the degree of universality in the behavior found here. These results will appear elsewhere [22]. Let us just mention that, when $d = 1$, the interface is repelled to a height of order $\mathcal{H}$, the unique solution of the equation*

$$\lambda\mathcal{H}^2\mathsf{V}(2\mathcal{H}) = 1\,.$$

The previous theorem only yields estimates on the height of the field averaged over the box. It is however very easy to derive from it the following local estimates.

Corollary 1 *Let $\upsilon \in \mathcal{CW}$ and fix $0 < \epsilon < 1$. There exist $\delta > 0$ and $\lambda_0 > 0$ such that, for all $\lambda < \lambda_0$ and $N > N_0(\lambda)$,*

$$\sup_{i\in\Lambda_N} \mathrm{E}_{N,+,\lambda,\upsilon}(X_i) \leq \delta^{-1} |\log\lambda|^{\nu(d)},$$

$$\inf_{i\in\Lambda_{\epsilon N}} \mathrm{E}_{N,+,\lambda,\upsilon}(X_i) \geq \delta\, |\log\lambda|^{\nu(d)}\,;$$

Our main motivation for studying this model comes from the physical phenomenon of *critical prewetting*. Consider some substance in the regime of phase coexistence; let us call the two equilibrium phases A and B. Suppose that phase A occupies the bulk of the vessel, while the boundary of the latter favors phase B. As a result, a film of B phase is generated along the walls. It is well-known that by increasing the temperature, the system may undergo a surface phase transition, the wetting transition, reflected in the behavior of this film: Below the wetting temperature the film has a microscopic width (partial wetting), while above it its width becomes macroscopic (complete wetting), in the sense that it diverges in the thermodynamic limit.

Suppose now that, starting from the complete wetting regime, the system is pulled away from phase coexistence and only phase A remains thermodynamically stable. The film of (now unstable) B phase, though still present, cannot occupy a macroscopic region and therefore its width stays microscopic (i.e. remains finite in the thermodynamic limit). Critical prewetting corresponds to the (continuous) divergence of the film thickness as the system nears the phase coexistence manifold. If we denote by λ the free energy difference between phase B and phase A, then one is interested in the behavior of the average film width as a function of λ.

The Gibbs measures (1.1) provide a natural modelization of this phenomenon. Indeed, when the external potential is chosen as $\mathsf{V}(x) = x$, these measures introduce a penalization of the form "$\lambda\times$ the volume of the wet layer", which correspond to the free energetic cost for creating such a layer. In that case, the averaged width of the wetting layer is of order $\lambda^{-1/3}$ when $d = 1$ [22].

In the Subsection 1.3, we introduce another, more realistic modelization of this phenomenon in a lattice gas, the two-dimensional Ising model, and prove some partial results indicating that this model has the same type of critical behavior, in particular that the critical exponent is also 1/3.

1.2 Some additional results about effective interface models

In this section, we present some new results about effective interface models, not directly related to the main problem investigated in the paper. Some turn out to be useful in the proof of Theorem 1, other are just of independent interest.

The first problem concerns the probability that an interface stays inside a fixed horizontal slab. This problem was first investigated in [9] in the Gaussian case. Our first result in an extension to the case of strictly convex interactions; our proof is also completely different. Let P_N denote the Gibbs measure with 0-b.c. without the positivity constraint and with $\lambda = \eta = 0$.

Theorem 2 *There exist strictly positive constants $c_-(d)$, $c_+(d)$, c and $\ell_0 < \infty$ such that, for any $\ell > \ell_0$ the following holds:*

- *If $d = 2$, for all $N > e^{c\ell}$,*

$$e^{-e^{-c_+\ell}\,|\Lambda_N|} \geq \mathrm{P}_N(|X_i| \leq \ell,\, \forall i \in \Lambda_N) \geq e^{-e^{-c_-\ell}\,|\Lambda_N|}\,.$$

- *If $d \geq 3$, for all $N > e^{c\ell^2}$,*

$$e^{-e^{-c_+\ell^2}\,|\Lambda_N|} \geq \mathrm{P}_N(|X_i| \leq \ell,\, \forall i \in \Lambda_N) \geq e^{-e^{-c_-\ell^2}\,|\Lambda_N|}\,.$$

We have not been able to extend the results on the variance and the mass proved in [9] in the Gaussian setting. The corresponding results in dimension 1, valid for a much larger class of models will appear in [22]

All other results concern the (pure) wetting transition in effective interface models. We start with results giving more informations on pathwise properties.

Theorem 3 *1. Let $\upsilon \in \mathsf{CW}$. For all $\epsilon > 0$, there exist $N_0(\upsilon, \epsilon)$ and $c(\upsilon) > 0$ such that, for all $N \geq N_0$,*

$$\mathrm{P}_{N,+,0,\upsilon}(|\Lambda_N|^{-1} \sum_{i \in \Lambda_N} 1_{\{X_i = 0\}} \geq \epsilon) \leq e^{-c\,\epsilon\,|\Lambda_N|}\,.$$

2. Let $d = 2$ and $\upsilon \in \mathsf{CW}$.

$$\lim_{N \to \infty} \mathrm{E}_{N,+,0,\upsilon}(|\Lambda_N|^{-1} \sum_{i \in \Lambda_N} X_i) = \infty\,.$$

Moreover, for all η small enough, there exists $C > 0$ such that

$$\mathrm{E}_{N,+,0,\upsilon}(|\Lambda_N|^{-1} \sum_{i \in \Lambda_N} X_i) \geq C \log N\,.$$

3. We consider the Gaussian case $\mathsf{U}(x) = x^2/2$. Let $d = 2$ and υ be large enough. Then there exists $C < \infty$ such that, for all $i \in \Lambda_N$,

$$\mathrm{E}_{N,+,0,\upsilon}(X_i) \leq C\,.$$

Moreover, there exists $C > 0$ such that, for all $x \in \mathbb{S}^1$,

$$\lim_{k \to \infty} -\frac{1}{k} \limsup_{N \to \infty} \log \mathrm{cov}_{N,+,0,\upsilon}(X_0, X_{[kx]}) > C\,,$$

where $[y] \in \mathbb{Z}^d$ denotes the componentwise integer part of $y \in \mathbb{R}^d$.

Finally, a long-term goal would be to control quantitatively the divergence of the interface as the wetting transition is approached from the partial wetting regime, i.e. the limit $\upsilon \searrow \upsilon_c$. This seems far from what can be achieved with today's techniques, but at least we can show (easily), in the $d \geq 3$ Gaussian case, that the transition is second order, in the sense that the average height does indeed diverge as $\upsilon \searrow \upsilon_c = 0$, and not stay finite and jump at the transition.

Theorem 4 *Let $d \geq 3$ and suppose that $\upsilon_c = 0$ (this is the case, e.g., when $\mathsf{U}(x) = x^2/2$). Then, for any $i \in \mathbb{Z}^d$,*

$$\lim_{\upsilon \searrow \upsilon_c} \lim_{N \to \infty} \mathrm{E}_{N,+,\lambda,\upsilon}(X_i) = \infty \,.$$

1.3 The two-dimensional Ising model

It would of course be very interesting to obtain results similar to those of Theorem 1 also in the case of lattice gases. However, such systems do not always display critical prewetting. In particular, below the roughening temperature, e.g. in the low-temperature 3-dimensional Ising model, or in subcritical Ising models in dimensions $d \geq 4$, it is expected that the continuous divergence of the width of the wet layer is replaced by an infinite sequence of first order phase transitions, the *layering transitions*, at which the interface jumps up one microscopic unit. This has never been established for the Ising model, though that should follow from a rather straightforward, but technically involved, Pirogov-Sinai analysis. This problem was studied in detail in the series of papers [18, 13, 25] in the case of the discrete SOS model.

On the other hand, critical prewetting is expected to occur in the rough phase of such models. In particular, it should occur for any subcritical temperatures in the case of the two-dimensional Ising model. This is what we analyze in this section.

Let N be even and set $\Lambda_N = \{-N/2+1, \ldots, N/2\} \times \{0, \ldots, N-1\}$, $\partial^+\Lambda_N = \{x \in \Lambda_N \,:\, \exists y \notin \Lambda_N,\, |x-y| = 1,\, y_2 \geq 0\}$ and $\partial^-\Lambda_N = \{x \in \Lambda_N \,:\, \exists y \notin \Lambda_N,\, |x-y| = 1,\, y_2 = -1\}$.

We define the following Hamiltonian acting on configurations $\sigma \in \{-1,1\}^{\Lambda_N}$,

$$H_{\lambda,h,\Lambda_N}(\sigma) = -\sum_{\langle x,y\rangle \subset \Lambda_N} \sigma_x \sigma_y - \lambda \sum_{x \in \Lambda_N} \sigma_x - \sum_{x \in \partial^+\Lambda_N} \sigma_x + h \sum_{x \in \partial^-\Lambda_N} \sigma_x \,.$$

$\lambda > 0$ is the bulk magnetic field, while the boundary field $h > 0$ serves to model the preference of the bottom wall toward $-$ spins. The Gibbs measure in Λ_N is the probability measure on $\sigma \in \{-1,1\}^{\Lambda_N}$ given by

$$\mu_{\beta,\lambda,h,\Lambda_N}(\sigma) = (Z_{\beta,\lambda,h,\Lambda_N})^{-1}\, e^{-\beta\, H_{\lambda,h,\Lambda_N}(\sigma)} \,.$$

Let $\beta > \beta_c$ and $\lambda = 0$. Because of our choice of boundary conditions, the bulk of the system is occupied, with probability asymptotically 1, by the $+$ phase, while the introduction of the boundary field h results in the creation of

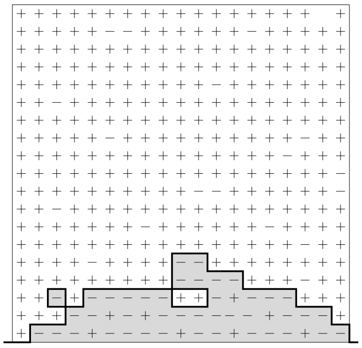

Figure 1: Part of the open contour $\gamma(\sigma)$ (bold line), and the set $\Lambda^-(\sigma)$ (shaded).

a layer of $-$ phase along the bottom wall. It is well-known [1, 21] that there is a critical value $h_w(\beta) > 0$ of h at which a wetting transition occurs. For $h < h_w(\beta)$, the layer has a finite thickness (see [26]), while for $h \geq h_w(\beta)$, the width of the film of $-$ phase becomes macroscopic (of order $N^{1/2}$) (this can be proved, for example, along the line of the argument leading to (2.8) below). It is actually expected that in this regime, once suitably rescaled, the microscopic interface converges to Brownian excursion; the sketch proof of such a result in the case $h = 1$ can be found in [19].

Let us now choose some $h \geq h_w(\beta)$ and let $\lambda > 0$. Our second result describes the behavior of the film of $-$ phase as $\lambda \to 0^+$. There is no canonical way of measuring the width of the film of $-$ phase in the Ising model, or, for that matter, even to define the film itself. We proceed as follows.

Given a configuration σ in the box Λ_N, construct its extension $\bar\sigma$ to $\mathbb{Z}^2$ by setting, for $x \notin \Lambda_N$, $\bar\sigma_x = -1$ if $x_2 < 0$ and $\bar\sigma_x = 1$ otherwise. Then in the configuration $\bar\sigma$ there is a single (infinite) open Peierls contour. This Peierls contour can intersect itself, and we use the deformation rules given in Fig. 5 to turn it into a simple curve. We call the resulting open contour $\gamma(\sigma)$. This contour splits $\mathbb{Z}^2$ into two components. We denote by $\Lambda^-(\sigma)$ the set of all sites of Λ_N lying "below" $\gamma(\sigma)$, see Fig. 1. The volume of $\Lambda^-(\sigma)$ is one of the measures of the size of the layer we use. The other one is $|C^-(\sigma)|$, where $C^-(\sigma)$ is the set of all sites in $\Lambda^-(\sigma)$ connected to the bottom wall by a path of $-$ spins in σ.

Our result is the following

Theorem 5 *Let $\beta > \beta_c$ and $h \geq h_w(\beta)$. There exist $\lambda_0 > 0$, $K < \infty$, and $C_2 > 0$ such that, for all $0 < \lambda < \lambda_0$ and $N > K\lambda^{-2/3}|\log\lambda|^3$,*

$$\mu_{\beta,\lambda,h,\Lambda_N}(|\Lambda^-| < |\log\lambda|^{-3}\,\lambda^{-1/3}\,N) \leq e^{-C_2\,|\log\lambda|^2\lambda^{2/3}\,N}\,, \tag{1.3}$$

and

$$\mu_{\beta,\lambda,h,\Lambda_N}(|C^-| > K\,|\log\lambda|^2\,\lambda^{-1/3}\,N) \leq e^{-C_2\,|\log\lambda|^2\lambda^{2/3}\,N}\,. \tag{1.4}$$

We expect that the two quantities $|\Lambda^-|$ and $|C^-(\sigma)|$ are typically the same, up to a multiplicative constant:

Conjecture 1 *There exists a constant $c > 0$ such that*

$$\lim_{N\to\infty} \mu_{\beta,\lambda,h,\Lambda_N}(|C^-| > c|\Lambda^-|) = 1\,.$$

(Of course $|\Lambda^-(\sigma)| \geq |C^-(\sigma)|$ for all σ.) We were unable to prove this conjecture. The difficulty is that the system can create big droplets of $+$ phase inside the layer of $-$ phase, and it seems that a rather complicated metastability analysis is required. Such an analysis is well understood in the case of regular macroscopic boxes, see e.g. [28]. However, we are here in a situation where the box is random, and very anisotropic.

Remark 3 *The logarithmic corrections present in Theorem 5 are artificial and should be removed. The reason of their appearance is the lack of local CLT type results for Ising random-lines in the vicinity of the boundary of the system. Such estimates have been proved in [11] in the case of random-lines in the bulk but their extension seems delicate.*

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2 Proofs

2.1 Proof of Theorem 1 and its corollary

2.1.1 Proof of Theorem 1

Let $\nu(2) = 1$ and $\nu(d) = 1/2$ when $d \geq 3$. Theorem 1 is a rather simple corollary of Theorem 2 and the following result.

Lemma 1 ([16], Lemma 2.9) *There exists $K > 0$ such that, for all $\epsilon > 0$,*

$$\lim_{N\to\infty} \mathrm{P}_{N,+,0}\Big(\Big|\Big\{i \in \Lambda_{[N/2]}\,:\, X_i \leq K(\log N)^{\nu(d)}\Big\}\Big| \geq \epsilon|\Lambda_{[N/2]}|\Big) = 0\,. \tag{2.5}$$

We first prove a lower bound on the partition function.

Lemma 2 *There exist $\lambda_0 > 0$, $r > 0$ and $C > 0$ such that, for all $\lambda < \lambda_0$ and $N > \lambda^{-r}$, and any nonnegative υ,*

$$Z_{N,+,\lambda,\upsilon} \geq e^{-C\,\lambda \vee (|\log\lambda|^{\nu(d)})\,|\Lambda_N|}\, Z_{N,+,0,\upsilon}\,.$$

Proof of Lemma 2. First observe that it is enough to prove the corresponding claim without pinning. Indeed, by FKG,

$$\begin{aligned}\frac{Z_{N,+,\lambda,\upsilon}}{Z_{N,+,0,\upsilon}} &= \langle e^{-\lambda\sum_i \mathsf{V}(X_i)}\rangle_{N,+,\upsilon} \\ &\geq \langle e^{-\lambda\sum_i \mathsf{V}(X_i)}\rangle_{N,+,0} = \frac{Z_{N,+,\lambda}}{Z_{N,+,0}}\,.\end{aligned}$$

Let now

$$\mathcal{H} = \left|\log\left(\lambda\mathsf{V}(|\log\lambda|^{\nu(d)})\right)/c_-(d)\right|^{\nu(d)}\,,$$

where $c_-(d)$ has been introduced in Theorem 2 and $\nu(d)$ is defined above.

$$Z_{N,+,\lambda} \geq e^{-\lambda|\Lambda_N|\mathsf{V}(2\mathcal{H})}\, Z_{N,0}\, \mathrm{P}_{N,0}(0 \leq X_i \leq 2\mathcal{H},\, \forall i\in\Lambda_N)\,,$$

Obviously, shifting the boundary condition from 0 to $\mathcal{H}$ and changing variables, we can write, thanks to the convexity of U,

$$\mathrm{P}_{N,0}(0 \leq X_i \leq 2\mathcal{H},\, \forall i\in\Lambda_N) \geq e^{-C\mathcal{H}^2 N^{d-1}}\mathrm{P}_{N,0}(|X_i| \leq \mathcal{H},\, \forall i\in\Lambda_N)\,.$$

Applying Theorem 2, we get that

$$Z_{N,+,\lambda} \geq e^{-(\lambda\mathsf{V}(2\mathcal{H})+e^{-c_-\mathcal{H}^{-\nu(d)}}+c\mathcal{H}^2N^{-1})|\Lambda_N|}\, Z_{N,0}\,.$$

Making use of the growth condition on V, the claim follows. ■

Using the convexity of U, the lemma immediately implies that

$$\begin{aligned}&\mathrm{P}_{N,+,\lambda,\upsilon}\left(\sum_{i\in\Lambda_N} X_i \geq \delta^{-1}|\log\lambda|^{\nu(d)}\,|\Lambda_N|\right)\\ &\quad\leq \mathrm{P}_{N,+,\lambda,\upsilon}\left(\sum_{i\in\Lambda_N}\mathsf{V}(X_i) \geq \mathsf{V}\left(\delta^{-1}|\log\lambda|^{\nu(d)}\right)\,|\Lambda_N|\right)\\ &\quad\leq \exp\left\{-\lambda\mathsf{V}\left(\delta^{-1}|\log\lambda|^{\nu(d)}\right)\,|\Lambda_N|\right\}\,\frac{Z_{N,+,0}}{Z_{N,+,\lambda}}\\ &\quad\leq \exp\left\{-\lambda\left(\mathsf{V}\left(\delta^{-1}|\log\lambda|^{\nu(d)}\right) - C\,\mathsf{V}\left(|\log\lambda|^{\nu(d)}\right)\right)\,|\Lambda_N|\right\}\\ &\quad\leq \exp\left\{-\lambda\mathsf{V}(|\log\lambda|^{\nu(d)})\left(\delta^{-1}-C\right))\,|\Lambda_N|\right\}\,,\end{aligned}$$

and one half of Theorem 1 is proved.

Let us consider the second half. We first apply Lemma 2 to remove the external field,

$$\begin{aligned}&\mathrm{P}_{N,+,\lambda,\upsilon}(\sum_{i\in\Lambda_N} X_i \leq \delta|\log\lambda|^{\nu(d)})\\ &\qquad\leq e^{C\lambda|\log\lambda|^{\nu(d)}N^d}\,\mathrm{P}_{N,+,0,\upsilon}(\sum_{i\in\Lambda_N} X_i \leq \delta|\log\lambda|^{\nu(d)})\,.\end{aligned}$$

Let $R = R_0 \cdot \big(\lambda \mathsf{V}(|\log\lambda|^{\nu(d)})\big)^{-1/d}$, with $R_0 > 0$ some small enough real number to be chosen below. Let also

$$\Delta_R = \{x \in \mathbb{Z}^d \,:\, x_i \equiv 0 \mod 2(R+1),\ \text{for some } i \in \{1,\dots,d\}\}\,.$$

Using the usual expansion on the pinned sites (see e.g. [8]) and FKG inequalities, we can write

$$\begin{aligned}
\mathrm{P}_{N,+,0,\upsilon}\big(&\sum_{i\in\Lambda_N} X_i \leq \delta|\log\lambda|^{\nu(d)}|\Lambda_N|\big) \\
&= \sum_{A\subset\Lambda_N} \nu_{N,+,\upsilon}(A)\,\mathrm{P}_{A^c,+,0}\big(\sum_{i\in\Lambda_N} X_i \leq \delta|\log\lambda|^{\nu(d)}|\Lambda_N|\big) \\
&\leq \sum_{A\subset\Lambda_N} \nu_{N,+,\upsilon}(A)\,\mathrm{P}_{A^c,+,0}\big(\sum_{i\in\Lambda_N} X_i \leq \delta|\log\lambda|^{\nu(d)}|\Lambda_N| \;\big|\; X\equiv 0 \text{ on } \Delta_R\big)\,,
\end{aligned}$$

where $\nu_{N,+,\upsilon}$ is the induced probability measure on the subsets of Λ_N.

The grid Δ_R splits Λ_N into $M = |\Lambda_N|/(2R+2)^d$ cubic cells of sidelength $2R+1$. (To simplify the notation, we suppose that this splitting can be done exactly.) Given a realization A of the random set of the pinned sites, we declare a cell *clean* if it does not contain any site of A; otherwise it is *dirty*. We need to ensure that, for all $\upsilon \in \mathsf{CW}$, at least half of the cells of the grid are clean. This follows easily from the first claim of Theorem 3 with $\epsilon = \frac{1}{2}M/|\Lambda_N|$, which implies that at least one half of the cells are clean once N is large enough, up to an event of probability e^{-cM}. In the rest of this proof, we suppose N is at least that large.

Let us number the clean cells as $\mathcal{C}_1,\dots,\mathcal{C}_K$, $K \geq \frac{1}{2}M$, and let Y_k, $k = 1,\dots,K$, be the indicator function of the event

$$\sum_{i\in\mathcal{C}_k} X_i \leq 4\delta|\log\lambda|^{\nu(d)}\,|\mathcal{C}_k|\,.$$

We clearly have the inclusion (for all λ small enough)

$$\big\{\sum_{i\in\Lambda_N} X_i \leq \delta|\log\lambda|^{\nu(d)}|\Lambda_N|\big\} \subset \big\{\sum_{k=1}^{K} Y_i \geq \tfrac{1}{3}K\big\}\,.$$

Now, under $\mathrm{P}_{A^c,+,0}(\,\cdot\,|\,X\equiv 0 \text{ on } \Delta_R)$, the Y_i are i.i.d. Bernoulli random variables. Let us denote their common law by Q_R. Using (2.5), we see that

$$\begin{aligned}
\mathrm{Q}_R(Y_i = 1) &\leq \mathrm{P}_{R,+,0}\Big(\Big|\Big\{i\in\Lambda_{[R/2]} \,:\, X_i \leq 2^{d+3}\delta|\log\lambda|^{\nu(d)}\Big\}\Big| \geq \tfrac{1}{2}|\Lambda_{[R/2]}|\Big) \\
&\leq 1/10\,,
\end{aligned}$$

provided δ, and then λ, are chosen small enough. Consequently,

$$\otimes_{k=1}^{M}\mathrm{Q}_R\big(\sum_{k=1}^{M} Y_i \geq \tfrac{1}{6}M\big) \leq e^{-CM}\,,$$

and the conclusion follows, provided we take R_0 small enough.

2.1.2 Proof of Corollary 1

We only give the proof for $d = 2$ since the argument for $d \geq 3$ is identical. Notice first that it is enough to prove the result for X_0. Indeed, by FKG,

$$\mathrm{E}_{N,+,\lambda,\upsilon}(X_i) \leq \mathrm{E}_{2N,+,\lambda,\upsilon}(X_0)\,,$$

for all $i \in \Lambda_N$, while for $i \in \Lambda_{\epsilon N}$ $(0 < \epsilon < 1)$ we have that

$$\mathrm{E}_{N,+,\lambda,\upsilon}(X_i) \geq \mathrm{E}_{\epsilon N,+,\lambda,\upsilon}(X_0)\,,$$

Theorem 1 implies that

$$\mathrm{E}_{N,+,\lambda,\upsilon}(|\Lambda_N|^{-1} \sum_{i\in\Lambda_N} X_i) \in (a_1, a_2)\mathcal{H}_{\lambda,d}\,,$$

for some constants $0 < a_1 < a_2 < \infty$ provided λ is small enough and N large enough. Using FKG inequalities, we deduce that

$$|\Lambda_N|^{-1} \sum_{i\in\Lambda_N} \mathrm{E}_{N,+,\lambda,\upsilon}(X_i) \leq |\Lambda_N|^{-1} \sum_{i\in\Lambda_N} \mathrm{E}_{2N,+,\lambda,\upsilon}(X_0) = \mathrm{E}_{2N,+,\lambda,\upsilon}(X_0)\,,$$

from which it follows that $\mathrm{E}_{N,+,\lambda,\upsilon}(X_0) \geq a_1\mathcal{H}_{\lambda,d}$ for N large enough. The complementary bound is also a trivial consequence of FKG:

$$\begin{aligned}
|\Lambda_N|^{-1} \sum_{i\in\Lambda_N} \mathrm{E}_{N,+,\lambda,\upsilon}(X_i) &\geq |\Lambda_N|^{-1} \sum_{i\in\Lambda_{[N/2]}} \mathrm{E}_{N,+,\lambda,\upsilon}(X_i) \\
&\geq |\Lambda_N|^{-1} \sum_{i\in\Lambda_{[N/2]}} \mathrm{E}_{[N/2],+,\lambda,\upsilon}(X_0) \\
&= \tfrac{1}{4}\, \mathrm{E}_{[N/2],+,\lambda,\upsilon}(X_0)\,.
\end{aligned}$$

This shows that $\mathrm{E}_{N,+,\lambda,\upsilon}(X_0) \leq 4a_2\, \mathcal{H}_{\lambda,d}$.

2.2 Proof of Theorems 2, 3 and 4

Proof of Theorem 2. We first establish the lower bound. We write

$$\mathrm{P}_N(|X_i| \leq \ell,\, \forall i \in \Lambda_N) \geq \mathrm{P}_{N,+}(X_i \leq \ell,\, \forall i \in \Lambda_N)\, \mathrm{P}_N(X_i \geq 0,\, \forall i \in \Lambda_N)\,.$$

By [16, Theorem 3.1], $\mathrm{P}_N(X_i \geq 0,\, \forall i \in \Lambda_N) \geq e^{-c\,N^{d-1}}$ for some $c > 0$ if $d \geq 2$, and can thus be neglected. Let us consider the other one. We introduce $R = e^{\kappa\,\ell}$ if $d = 2$, and $R = e^{\kappa\,\ell^2}$ if $d \geq 3$, where κ is a small number to be chosen later, and the grid

$$\Delta_R = \{x \in \mathbb{Z}^d \,:\, x_i \equiv 0 \mod 2(R+1),\ \text{for some } i \in \{1,\ldots,d\}\}\,.$$

This grid splits Λ_N into $M = |\Lambda_N|/(2R+2)^d$ cubic cells of sidelength $2R+1$. (To simplify the notation, we suppose that this splitting can be done exactly.)

Denoting by $\mathrm{P}^t_{N,+}$ the measure with t boundary conditions, FKG inequalities then imply that

$$\begin{aligned}\mathrm{P}_{N,+}(X_i \leq \ell,\, \forall i \in \Lambda_N) &\geq \mathrm{P}^\ell_{N,+}(X_i \leq \ell,\, \forall i \in \Lambda_N \,|\, X \equiv \ell \text{ on } \Delta_R)\\ &= \left(\mathrm{P}_R(X_i \geq 0,\, \forall i \in \Lambda_R \,|\, X_i \leq \ell,\, \forall i \in \Lambda_R)\right)^M .\end{aligned}$$

Now,

$$\begin{aligned}&\mathrm{P}_R(X_i \geq 0,\, \forall i \in \Lambda_R \,|\, X_i \leq \ell,\, \forall i \in \Lambda_R)\\ &\qquad\geq \mathrm{P}_{R,+}(X_i \leq \ell,\, \forall i \in \Lambda_R)\, \mathrm{P}_R(X_i \geq 0,\, \forall i \in \Lambda_R)\,.\end{aligned}$$

As above,

$$\mathrm{P}_R(X_i \geq 0,\, \forall i \in \Lambda_R) \geq e^{-c\,R^{d-1}}\,,$$

while, using [16, Corollary 2.6],

$$\mathrm{P}_{R,+}(X_i \leq \ell,\, \forall i \in \Lambda_R) \geq 1 - |\Lambda_R| \sup_{i\in\Lambda_R} \mathrm{P}_{R,+}(X_i > \ell) \geq \tfrac{1}{2}\,,$$

provided that κ is chosen small enough, and then ℓ large enough (actually the above reference does not give explicitly a rate of convergence, but a glance at the proof shows that $\sup_{i\in\Lambda_R} \mathrm{P}_{R,+}(X_i > \ell) \leq R^{-c\,\ell}$).

Collecting all the estimates proves the lower bound.

Let us now turn to the upper bound. Δ_R is the same grid as above (for an R to be chosen later). Then it follows from convexity of U and FKG inequalities that

$$\begin{aligned}\mathrm{P}_N(|X_i| \leq \ell,\, \forall i \in \Lambda_N) &\leq \mathrm{P}_{N,+}(X_i \leq 2\ell,\, \forall i \in \Lambda_N)\, e^{c\ell^2\,|\partial\Lambda_N|}\\ &\leq \mathrm{P}_{N,+}(X_i \leq 2\ell,\, \forall i \in \Lambda_N \,|\, X \equiv 0 \text{ on } \Delta_R\ )\, e^{c\ell^2\,|\partial\Lambda_N|}\\ &= \left(\mathrm{P}_{R,+}(X_i \leq 2\ell,\, \forall i \in \Lambda_R)\right)^M\ e^{c\ell^2\,|\partial\Lambda_N|}\,.\end{aligned}$$

We choose R as follows: If $d = 2$, then $R = e^{\rho\ell}$, while in the case $d \geq 3$, we choose $R = e^{\rho\ell^2}$, with ρ large enough. Then

$$\mathrm{P}_{R,+}(X_i \leq 2\ell,\, \forall i \in \Lambda_R) \leq 1/2\,,$$

by the results of [16] (since the typical height of the repelled field is proportional to $(\log R)^{\nu(d)} = \rho^{\nu(d)}\ell$). This completes the proof. ■

Proof of Theorem 3.
First statement. Let $\mathcal{A}$ be the (random) set of pinned sites. The crucial observation is that

$$\begin{aligned}\mathrm{E}_{N,+,0,\upsilon}\left(e^{t|\mathcal{A}|}\right) &= \sum_{A\subset\Lambda_N} e^{t|A|}\,\upsilon^{|A|}\mathrm{Z}_{A^c,+,0,0}/\mathrm{Z}_{N,+,0,\upsilon}\\ &= \frac{\mathrm{Z}_{N,+,0,e^t\upsilon}}{\mathrm{Z}_{N,+,0,\upsilon}}\,.\end{aligned}$$

Therefore an application of the exponential Chebyshev inequality yields

$$\mathrm{P}_{N,+,0,\upsilon}\left(|\mathcal{A}| > \zeta|\Lambda_N|\right) \le e^{-t\zeta|\Lambda_N| + \log(\mathrm{Z}_{N,+,0,e^t\upsilon}/\mathrm{Z}_{N,+,0}) - \log(\mathrm{Z}_{N,+,0,\upsilon}/\mathrm{Z}_{N,+,0})}\,.$$

Choosing t such that $e^t = \frac{1}{2}(\upsilon + \upsilon_c)$ and using (1.2), we get the conclusion, for all $N > N_0(\upsilon, \zeta)$.
Second statement. Since $\lambda = 0$ in all the proof, we omit it from the notations. It is proved in [12] that the condition $\upsilon \in \mathsf{CW}$ implies that

$$\lim_{N\to\infty} |\Lambda_N|^{-1}\, \mathbb{E}_{N,+,\upsilon}\left[\rho_N\right] = 0\,,$$

where $\rho_N(X) = \sum_{i\in\Lambda_N} 1_{\{X_i=0\}}$. So there exists a sequence c_N, with $\lim c_N = 0$, such that $\mathbb{E}_{N,+,\upsilon}\left[\rho_N\right] \le c_N\,|\Lambda_N|$. Let $\mathcal{E}_N = \{X\,:\,\rho_N(X) \le 2c_N\,|\Lambda_N|\}$. Using again the expansion on pinned sites, we have the following

$$\begin{aligned}
\mathbb{E}_{N,+,\upsilon}\left[\sum_{i\in\Lambda_N} X_i\right] &\ge \mathbb{E}_{N,+,\upsilon}\left[\sum_{i\in\Lambda_N} X_i 1_{\mathcal{E}_N}\right] \\
&= \sum_{\substack{A\subset\Lambda_N\\ |A|\le 2c_N|\Lambda_N|}} \nu_{N,+,\upsilon}(A)\; \mathbb{E}_{A^c,+,0}\big(\sum_{i\in\Lambda_N} X_i\big) \\
&\ge \sum_{\substack{A\subset\Lambda_N\\ |A|\le 2c_N|\Lambda_N|}} \nu_{N,+,\upsilon}(A) \sum_{\substack{i\in\Lambda_N\\ d(x,A)\ge c_N^{-1/4}}} \mathbb{E}_{B_{c_N^{-1/4}}(x),+,0}(X_i) \\
&\ge C\,|\log c_N|\,|\Lambda_N|\,,
\end{aligned}$$

where $B_r(x) = \{y\,:\,|y-x| \le r\}$, and we used positivity of the field and FKG inequalities; the last inequality follows from Markov inequality and the usual entropic repulsion estimate for a square box of radius $c_N^{-1/4}$ with 0 b.c., see [16].

Since, when υ is sufficiently small, we can even choose $c_N \sim N^{-1}$ [12], in that case one has the stronger result

$$\mathbb{E}_N^{+,\varepsilon}\left[\sum_{x\in\Lambda_N} \phi_x\right] \ge C\,\log N\,|\Lambda_N|\,.$$

Third statement. Since $\lambda = 0$ in all the proof, we omit it from the notations. Our proof is based on the following estimate: There exist η_0 and $c > 0$ such that, for all $B \subset \Lambda$ and all $\upsilon > \upsilon_0$,

$$\nu_{N,+,\upsilon}(\mathcal{A}\cap B = \emptyset) \le e^{-c|\log\upsilon||B|}\,. \tag{2.6}$$

Let us first assume that (2.6) holds and prove the two statements of the Theorem. The first statement is very easy: By the usual decomposition over pinned sites, and FKG inequalities,

$$
\begin{aligned}
\mathrm{E}_{N,+,0,v}(X_i) &= \sum_{R\geq 1} \sum_{\substack{A\subset\Lambda_N \\ d(A,x)=R}} \nu_{N,+,v}(A)\, \mathrm{E}_{A^c,+,0,0}(X_i) \\
&\leq \sum_{R\geq 1} \sum_{\substack{A\subset\Lambda_N \\ d(A,x)=R}} \nu_{N,+,v}(A) \sup_{|j-i|=R} \mathrm{E}_{\mathbb{Z}^2\setminus\{j\},+,0,0}(X_i)\,.
\end{aligned}
$$

Since $\mathrm{E}_{\mathbb{Z}^2\setminus\{j\},+,0,0}(X_i) \leq C\log R$ uniformly in j such that $|j-i| = R$ [20], the first bound is proved.

Let us now establish the positivity of the mass. Decomposing again according to the pinned sites, we get

$$
\begin{aligned}
\mathrm{E}_{N,+,0,v}(X_iX_j) = &\sum_{\substack{A\subset\Lambda_N \\ i \overset{A^c}{\not\leftrightarrow} j}} \nu_{N,+,v}(A)\, \mathrm{E}_{A^c,+,0,0}(X_iX_j)\,, \\
&+ \sum_{\substack{A\subset\Lambda_N \\ i \overset{A^c}{\leftrightarrow} j}} \nu_{N,+,v}(A)\, \mathrm{E}_{A^c,+,0,0}(X_iX_j)\,,
\end{aligned}
$$

where $i \overset{A^c}{\leftrightarrow} j$ means that i and j are connected by a path containing only sites of $\Lambda_N \setminus A$, and $i \overset{A^c}{\not\leftrightarrow} j$ is the complementary event.

Suppose $i \overset{A^c}{\not\leftrightarrow} j$ occurs. Let γ_i be the innermost closed path of pinned sites surrounding j (possibly using sites outside from Λ_N), and let γ_j be the analogous object for j. If γ_i does not surround j, then let $\gamma = \gamma_i$, otherwise let $\gamma = \gamma_j$. Observe that under $i \overset{A^c}{\not\leftrightarrow} j$, γ surrounds only one of the two sites, and is the innermost path with this property. Suppose, for definiteness that it surrounds i. We then bound the first term as follows, writing $\mathrm{int}(\gamma)$ for the set of site surrounded by γ or along γ,

$$
\begin{aligned}
\mathrm{E}_{N,+,0,v}(X_iX_j \,|\, \gamma) &\leq \mathrm{E}_{\Lambda_N\setminus\mathrm{int}(\gamma),+,0,v}(X_j)\; \mathrm{E}_{\mathrm{int}(\gamma)\setminus\gamma,+,0,0}(X_i) \\
&\leq \mathrm{E}_{\Lambda_N,+,0,v}(X_j)\; \mathrm{E}_{\mathrm{int}(\gamma)\setminus\gamma,+,0,0}(X_i)\,.
\end{aligned}
$$

Since averaging the last expectation over γ yields $\mathrm{E}_{N,+,0,v}(X_i, i \overset{A^c}{\not\leftrightarrow} j, \gamma=\gamma_i)$, we see that

$$
\sum_{\substack{A\subset\Lambda_N \\ i \overset{A^c}{\not\leftrightarrow} j}} \nu_{N,+,v}(A)\, \mathrm{E}_{A^c,+,0,0}(X_iX_j) - \mathrm{E}_{N,+,0,v}(X_i)\mathrm{E}_{N,+,0,v}(X_j) \leq 0\,.
$$

Therefore, it only remains to prove that

$$
\sum_{\substack{A\subset\Lambda_N \\ i \overset{A^c}{\leftrightarrow} j}} \nu_{N,+,v}(A)\; \mathrm{E}_{A^c,+,0,0}(X_iX_j)
$$

is exponentially small in $|i-j|$. This follows from (2.6). Indeed, this expression is bounded above by

$$\sum_{R_1\geq 0, R_2\geq 0} \sum_{\substack{A\subset\Lambda_N,\, i \overset{A^c}{\leftrightarrow} j \\ d(A,x)=R_1, d(A,y)=R_2}} \nu_{N,+,\upsilon}(A)\, \mathrm{E}_{A^c,+,0,0}(X_i X_j)\,.$$

Now, proceeding as above,

$$\mathrm{E}_{A^c,+,0,0}(X_i X_j) \leq \Big(\mathrm{E}_{A^c,+,0,0}(X_i^2)\mathrm{E}_{A^c,+,0,0}(X_j^2)\Big)^{1/2} \leq C\log R_1 \log R_2\,.$$

Moreover, it follows from (2.6) that

$$\sum_{\substack{A\subset\Lambda_N,\, i \overset{A^c}{\leftrightarrow} j \\ d(A,x)=R_1, d(A,y)=R_2}} \nu_{N,+,\upsilon}(A) \leq e^{-C|\log\upsilon|\,|i-j|} e^{-C|\log\upsilon|\,({R_1}^2+{R_2}^2)}\,,$$

provided υ is chosen large enough. The statement follows.

Let us turn now to the proof of (2.6). The control of the distribution of pinned sites is done similarly as in [17, 23, 8]. Let $B\subset\Lambda_N$. We set $B_0=B$, and define

$$B_{k+1} = B_k \cup \{x\in\mathbb{Z}^2 \,:\, d(x,B_k)=1\}\,.$$

We then have

$$\nu_{N,+,\upsilon}(\mathcal{A}\cap B=\emptyset) \leq \sum_{k\geq 0} \nu_{N,+,\upsilon}(\mathcal{A}\cap B_k=\emptyset \,|\, \bar{\mathcal{A}}\cap B_{k+1}\neq\emptyset)\,,$$

where $\bar{\mathcal{A}} = \mathcal{A}\cup(\Lambda_N)^c$. It is therefore enough to prove that

$$\nu_{N,+,\upsilon}(\mathcal{A}\cap B_k=\emptyset \,|\, \bar{\mathcal{A}}\cap B_{k+1}\neq\emptyset) \leq e^{-c|\log\upsilon|\,|B|}\,.$$

We then write, as in [17, 23, 8],

$$\begin{aligned}
\nu_{N,+,\upsilon}(\mathcal{A}\cap B_k=\emptyset \,|\, \bar{\mathcal{A}}\cap B_{k+1}\neq\emptyset) &\leq \left\{\sum_{C\subset B_k} \upsilon^{|C|} \inf_{\substack{A\cap B_k=\emptyset \\ \bar{A}\cap B_{k+1}\neq\emptyset}} \frac{Z_{(A\cup C)^c,+,0}}{Z_{A^c,+,0}}\right\}^{-1} \\
&\leq \left\{\upsilon^{|B_k|} \inf_{\substack{A\cap B_k=\emptyset \\ \bar{A}\cap B_{k+1}\neq\emptyset}} \frac{Z_{(A\cup B_k)^c,+,0}}{Z_{A^c,+,0}}\right\}^{-1}\,.
\end{aligned}$$

Let us number the sites of $B_k=\{t_1,\ldots,t_{|B_k|}\}$ in such a way that t_1 is a neighbor of a site of $\bar{A}\cap B_{k+1}$, and each t_k, $k>1$, is a neighbor of at least one t_j, $j<k$. We write the last fraction as a telescoping product:

$$\frac{Z_{(A\cup B_k)^c,+,0}}{Z_{A^c,+,0}} = \prod_{k=1}^{|B_k|} \frac{Z_{(A\cup\{t_1,\ldots,t_k\})^c,+,0}}{Z_{(A\cup\{t_1,\ldots,t_{k-1}\})^c,+,0}}\,.$$

The conclusion follows once we prove that there exists $c > 0$ such that

$$\frac{Z_{(C\cup\{i\})^c,+,0}}{Z_{C^c,+,0}} \geq c\,,$$

uniformly in $C \subset \Lambda_N$, and i neighboring C.

$$\frac{Z_{(C\cup\{i\})^c,+,0}}{Z_{C^c,+,0}} = \lim_{\epsilon\searrow 0}\frac{1}{\epsilon}\mathbb{P}_{(C\cup\{i\})^c,+,0}(|X_i|\leq\epsilon) \geq \lim_{\epsilon\searrow 0}\frac{1}{\epsilon}\mathbb{P}_{\mathbb{Z}^2,+,0}(|X_i|\leq\epsilon\,|\,X_j=0)\,,$$

where j in the last probability is a site of C which is nearest neighbor to i (the last inequality follows from FKG). Let us estimate this probability. By the Markov property, we have ($u \sim v$ meaning u and v nearest neighbors)

$$\begin{aligned}
&\mathbb{P}_{\mathbb{Z}^2,+,0}(|X_i|\leq\epsilon\,|\,X_j=0)\\
&\qquad\geq \int \mathbb{P}_{\mathbb{Z}^2,+,0}(\mathrm{d}X\,|\,X_j=0)\, 1_{\{\max_{k\sim i} X_k<h_1\}}\mathbb{P}_{\{i\},+,X}(X_i\leq\epsilon)\\
&\qquad\geq \int \mathbb{P}_{\mathbb{Z}^2,+,0}(\mathrm{d}X\,|\,X_j=0)\, 1_{\{\max_{k\sim i} X_k<h_1\}}\mathbb{P}^{h_1}_{\{i\},+,0}(X_i\leq\epsilon)\\
&\qquad= \mathbb{P}_{\mathbb{Z}^2,+,0}(\max_{k\sim i} X_k<h_1\,|\,X_j=0)\;\mathbb{P}^{h_1}_{\{i\},+,0}(X_i\leq\epsilon)\,.
\end{aligned}$$

Now, this last probability can easily be bounded from below:

$$\mathbb{P}^{h_1}_{\{i\},+,0}(X_i\leq\epsilon) \geq c_1\,\epsilon\,,$$

while the other probability can be bounded below by $1/2$ if h_1 is chosen large enough. Indeed, since

$$\mathbb{E}_{\mathbb{Z}^2,+,0}(X_k\,|\,X_j=0) \leq c_2\,,$$

when $k \sim i \sim j$, Markov inequality yields

$$\mathbb{P}_{\mathbb{Z}^2,+,0}(\max_{k\sim i} X_k \geq h_1\,|\,X_j=0) \leq 4\,c_2/h_1\,.$$

■

Proof of Theorem 4. The proof is trivial; by FKG inequalities,

$$\mathbb{E}_{\mathbb{Z}^2,+,0,\upsilon}(X_i) \geq \mathbb{E}_{\mathbb{Z}^2,0,\upsilon}(X_i 1_{\{X_i\geq 0\}}) = \tfrac{1}{2}\mathbb{E}_{\mathbb{Z}^2,0,\upsilon}(|X_i|) \geq C\sqrt{|\log\upsilon|}\,,$$

the last inequality being proved in [8]. ■

2.3 Proof of Theorem 5

2.3.1 Preliminaries

Let us first introduce our notations; note that, for reasons of convenience, the notations we use here differ slightly from those used in the introduction.

Let $\Lambda \Subset \mathbb{Z}^2$, $h \in \mathbb{R}$, and let, for each bond $e = \langle x, y\rangle$, $x, y \in \mathbb{Z}^2$,

$$J(e) = \begin{cases} h & \text{if } \min(x_2, y_2) = 0 \text{ and } \max(x_2, y_2) = 1; \\ 1 & \text{otherwise.} \end{cases}$$

We define the following Hamiltonian acting on configurations $\sigma \in \{-1, 1\}^\Lambda$,

$$H_{\lambda,h,\Lambda}(\sigma) = -\sum_{\langle x,y\rangle \subset \Lambda} J(\langle x, y\rangle)\,(\sigma_x\sigma_y - 1) - \lambda \sum_{x\in\Lambda} \sigma_x\,.$$

λ and h are respectively the bulk and boundary magnetic fields. Let $\partial\Lambda = \{x \in \Lambda\,:\, \exists y \notin \Lambda,\, |x - y| = 1\}$. We define three Gibbs measures in Λ; all are probability measures on $\sigma \in \{-1, 1\}^\Lambda$. Let $s = \pm 1$;

$$\mu_{s,\beta,\lambda,h,\Lambda} = \begin{cases} (Z_{s,\beta,\lambda,h,\Lambda})^{-1}\, e^{-\beta\, H_{\lambda,h,\Lambda}(\sigma)} & \text{if } \sigma_x = s,\ \forall x \in \partial\Lambda\,; \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\mu_{\pm,\beta,\lambda,h,\Lambda} = \begin{cases} (Z_{\pm,\beta,\lambda,h,\Lambda})^{-1}\, e^{-\beta\, H_{\lambda,h,\Lambda}(\sigma)} & \text{if } \sigma_x = \mathrm{sign}(x_2),\ \forall x \in \partial\Lambda\,; \\ 0 & \text{otherwise,} \end{cases}$$

where we set $\mathrm{sign}(0) = -1$.

Let $\beta > \beta_c$. We denote by $h_w(\beta) \ge 0$ the value of the boundary field at which wetting takes place when $\lambda = 0$. Since $\beta > \beta_c$ and $h \ge h_w(\beta)$ are kept fixed, we often lighten the notations by omitting the corresponding subscripts.

For N even, consider the square box

$$\Lambda_N = \{-N/2 + 1, \ldots, N/2\} \times \{0, \ldots, N - 1\}.$$

In the complete wetting regime $h \ge h_w(\beta)$, when $\lambda = 0$ the wall free energy is given by the surface tension in the horizontal direction. In the presence of a bulk field $\lambda > 0$, the latter does not make sense anymore since the $-$ phase isn't stable, however the former remains meaningful since in the vicinity of the wall the $-$ phase is stabilized by the boundary conditions. We therefore define the (finite-volume) wall free energy by the usual formula,

$$\tau_{\mathrm{bd}}(\beta, h, \lambda, N) = -\frac{1}{N} \log \frac{Z_{\pm,\beta,\lambda,h,\Lambda_N}}{Z_{+,\beta,\lambda,h,\Lambda_N}}\,.$$

The next lemma states that the effect of the bulk field λ on the surface tension is to increase the latter by an amount of order at most $\lambda^{2/3+o(1)}$.

Lemma 3 *For all $\beta > \beta_c$ and $h > h_w(\beta)$, there exist $C > 0$, $\lambda_0 > 0$ and $K > 0$ such that, for all $0 < \lambda < \lambda_0$ and $N > K\lambda^{-2/3}|\log\lambda|^3$,*

$$\tau_{\mathrm{bd}}(\beta, h, 0, N) \le \tau_{\mathrm{bd}}(\beta, h, \lambda, N) \le \tau_{\mathrm{bd}}(\beta, h, 0, N) + C\lambda^{2/3}|\log\lambda|^2\,.$$

Proof of Lemma 3.

Let $F_\lambda(\sigma) = e^{\lambda \sum_{x\in\Lambda_N} \sigma_x}$. The statement of the lemma can be rewritten in the following form,

$$\langle F_\lambda \rangle_{+,\beta,0,h,\Lambda_N} \geq \langle F_\lambda \rangle_{\pm,\beta,0,h,\Lambda_N} \geq e^{-C_4\,\lambda^{2/3}\,|\log\lambda|^2\,N} \langle F_\lambda \rangle_{+,\beta,0,h,\Lambda_N} \,.$$

Since the first inequality is a direct consequence of FKG inequalities, we only have to prove the second one.

Let $\mathcal{H} = \lambda^{-1/3}|\log\lambda|^2$ and $\mathcal{S}_{\mathcal{H}} = \{x \in \Lambda_N \,:\, x_2 \leq \mathcal{H}\}$ be the strip of width $\mathcal{H}$ along the bottom wall.

According to the discussion in Appendix A,

$$Z_{\pm,\beta,\lambda,h,\Lambda_N} \geq \sum_{\gamma\subset\mathcal{S}_{\mathcal{H}}} w(\gamma)\, Z_{+,\beta,\lambda,\Lambda^+(\gamma)}\, Z_{-,\beta,\lambda,h,\Lambda^-(\gamma)} \,.$$

To simplify somewhat the notations we omit, in the rest of the proof, the subscripts β and h, and write simply Λ^+ and Λ^-.

Our first task is to remove the λ-dependence in the last equation. Indeed our main tool, the random-line representation briefly described in Appendix A, only applies in the absence of bulk field.

$$\begin{aligned} Z_{+,\lambda,\Lambda^+}\, Z_{-,\lambda,\Lambda^-} &\geq Z_{+,\lambda,\Lambda^+}\, Z_{-,-\lambda,\Lambda^-}\; e^{-2\lambda\,|\Lambda^-(\gamma)|} \\ &\geq \frac{Z_{+,\lambda,\Lambda^+}\, Z_{-,-\lambda,\Lambda^-}}{Z_{+,0,\Lambda^+}\, Z_{+,0,\Lambda^-}}\; Z_{+,0,\Lambda^+}\, Z_{+,0,\Lambda^-}\; e^{-2\lambda\,\mathcal{H}\,N} \,. \end{aligned}$$

Writing $x \sim \gamma$ if x is one the sites whose value is completely determined by γ, we can estimate the ratio

$$\begin{aligned} \frac{Z_{+,\lambda,\Lambda^+}\, Z_{-,-\lambda,\Lambda^-}}{Z_{+,0,\Lambda^+}\, Z_{+,0,\Lambda^-}} &= \frac{\mu_{+,\lambda,\Lambda_N}(\sigma_x = 1,\, \forall x\sim\gamma)}{\mu_{+,0,\Lambda_N}(\sigma_x = 1,\, \forall x\sim\gamma)}\; \frac{Z_{+,\lambda,\Lambda_N}}{Z_{+,0,\Lambda_N}} \\ &\geq \frac{Z_{+,\lambda,\Lambda_N}}{Z_{+,0,\Lambda_N}} \,. \end{aligned}$$

Here, the identity follows from the $+/-$ symmetry of the model, and the last inequality is a consequence of FKG inequalities. Collecting all these estimates together, we get, see (A.12),

$$Z_{\pm,\lambda,\Lambda_N} \geq e^{-2\lambda\,\mathcal{H}\,N}\, Z_{+,\lambda,\Lambda_N} \sum_{\gamma\subset\mathcal{S}_{\mathcal{H}}} q_N(\gamma) \,,$$

which is precisely what we were after. At this stage, it is possible to use the tools discussed in Appendix A in order to control the last sum.

Let K be a sufficiently large integer. We split the slab $\mathcal{S}_{\mathcal{H}}$ into M disjoint rectangles, $\mathcal{R}_1, \ldots, \mathcal{R}_M$, with height $\mathcal{H}$ and basis of length $[\mathcal{H}^2/(K\log\mathcal{H})]$, except possibly for the rightmost one which may have a shorter basis. We also introduce the dual sites a_k, $k = 0, \ldots, M$, defined by

$$a_k = (-(N-1)/2 + k[\mathcal{H}^2/(K\log\mathcal{H})] \wedge (N-1)/2, 1/2) \,.$$

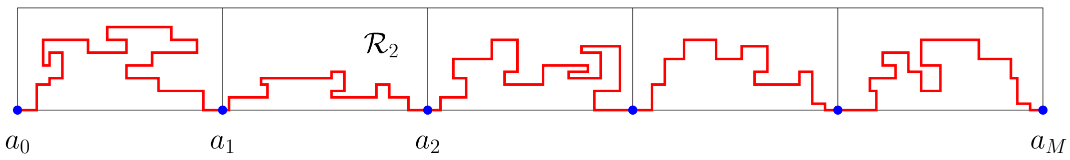

Figure 2: The restricted family of open contours.

(a_{k-1} and a_k are thus the bottom corners of the rectangle $\mathcal{R}_k$).

We further restrict the summation to the set of open contours γ satisfying the following conditions (see Fig. 2):

- $\gamma = \eta_1 \amalg \eta_2 \amalg \cdots \amalg \eta_M$.
- The piece η_k, $k = 1, \ldots, M$, use only inner edges of the kth rectangle, and connects a_{k-1} and a_k.

By (A.15) and (A.14),

$$q_N(\gamma) \geq \prod_{k=1}^{M} q_{\text{s.i.}}(\eta_k)\,.$$

Therefore

$$\sum_{\gamma \subset \mathcal{S}_{\mathcal{H}}} q_N(\gamma) \geq \prod_{k=1}^{M} \sum_{\eta_k \subset \mathcal{R}_k} q_{\text{s.i.}}(\eta_k)\,;$$

the last summation runs over contours satisfying the condition above. It is controlled thanks to (A.17), which implies that

$$\sum_{\eta_k \subset \mathcal{R}_k} q_{\text{s.i.}}(\eta_k) \geq \mathcal{H}^{-C} \sum_{\zeta :\, a_{k-1} \to a_k} q_{\text{s.i.}}(\zeta)\,,$$

provided K is big enough. We then combine (A.13) and (A.23) to obtain

$$\begin{aligned}\sum_{\zeta :\, a_{k-1} \to a_k} q_{\text{s.i.}}(\zeta) &\geq C\,|a_{k-1} - a_k|^{-3/2}\, e^{-\tau_\beta(\mathbf{e}_1)|a_{k-1}-a_k|}\\ &\geq C\,\mathcal{H}^{-3}\, e^{-\tau_\beta(\mathbf{e}_1)\,|a_{k-1}-a_k|}\,,\end{aligned}$$

where $\tau_\beta(\mathbf{e}_1)$ is the surface tension in the horizontal direction. Since, by (A.11) (with $\lambda = 0$) and (A.12) , (A.13) and (A.23),

$$e^{-\tau_\beta(\mathbf{e}_1)\,N} \geq \frac{Z_{\pm,0,\Lambda_N}}{Z_{+,0,\Lambda_N}}\,,$$

the conclusion follows. ■

2.3.2 Proof of the lower bound

We turn now to the proof of (1.3). As before, we omit reference to β and h in the notations, write simply Λ^+ and Λ^-, and also set $C_{\lambda,N} = \lambda^{-1/3} |\log\lambda|^{-3} N$. We also denote by $\mathcal{A}$ the set of open contours such that $|\Lambda^-| < C_{\lambda,N}$. Applying Lemma 3 yields

$$\mu_{\pm,\lambda,\Lambda_N}(\mathcal{A}) \leq e^{C\,\lambda^{2/3}|\log\lambda|^2\,N}\, \frac{Z_{+,0,\Lambda_N}}{Z_{\pm,0,\Lambda_N}} \sum_{\gamma\in\mathcal{A}} w(\gamma)\, \frac{Z_{+,\lambda,\Lambda^+}\,Z_{-,\lambda,\Lambda^-}}{Z_{+,\lambda,\Lambda_N}}\,.$$

Our first task is again to get rid of the bulk field. This is the content of the following

Lemma 4 *For all $\beta > \beta_c$, there exists $C_1 < \infty$ and $C_2 > 0$ such that, for any set $\mathcal{C}$ of open contours,*

$$\frac{Z_{+,0,\Lambda_N}}{Z_{\pm,0,\Lambda_N}} \sum_{\gamma\in\mathcal{C}} w(\gamma)\, \frac{Z_{+,\lambda,\Lambda^+}\,Z_{-,\lambda,\Lambda^-}}{Z_{+,\lambda,\Lambda_N}} \leq e^{C_1\,\lambda\,N}\, \frac{Z_{+,0,\Lambda_N}}{Z_{\pm,0,\Lambda_N}} \sum_{\gamma\in\mathcal{C}} q_N(\gamma) \;+\; e^{-C_2\,N}\,. \quad (2.7)$$

Proof of Lemma 4. By symmetry,

$$\begin{aligned}\frac{Z_{+,\lambda,\Lambda^+}\,Z_{-,\lambda,\Lambda^-}}{Z_{+,\lambda,\Lambda_N}} &= \frac{Z_{+,-\lambda,\Lambda^-}}{Z_{+,\lambda,\Lambda^-}}\,\mu_{+,\lambda,\Lambda_N}(\sigma_x = 1,\,\forall x\sim\gamma)\\ &\leq \mu_{+,\lambda,\Lambda_N}(\sigma_x = 1,\,\forall x\sim\gamma)\,,\end{aligned}$$

where as before $x \sim \gamma$ if its value is completely determined by γ, and we used

$$\frac{Z_{+,-\lambda,\Lambda^-}}{Z_{+,\lambda,\Lambda^-}} = \exp\left\{\sum_{x\in\Lambda^-}\int_0^\lambda \big(\langle\sigma_x\rangle_{+,-s,\Lambda^-} - \langle\sigma_x\rangle_{+,s,\Lambda^-}\big)\,\mathrm{d}s\right\} \leq 1\,.$$

Now,

$$\begin{aligned}\mu_{+,\lambda,\Lambda_N}(\sigma_x = 1,\,\forall x\sim\gamma) &= \mu_{+,0,\Lambda_N}(\sigma_x = 1,\,\forall x\sim\gamma)\\ &\quad\times\exp\left\{\sum_{x\in\Lambda_N}\int_0^\lambda \big(\langle\sigma_x\,|\,\sigma_y = 1,\,\forall y\sim\gamma\rangle_{+,s,\Lambda_N} - \langle\sigma_x\rangle_{+,s,\Lambda_N}\big)\,\mathrm{d}s\right\}\,.\end{aligned}$$

The difference in the exponent can easily be bounded using (A.26): There exists $C < \infty$ such that, uniformly in $s \geq 0$,

$$\langle\sigma_x\,|\,\sigma_y = 1,\,\forall y\sim\gamma\rangle_{+,s,\Lambda_N} - \langle\sigma_x\rangle_{+,s,\Lambda_N} \leq C\sum_{z\sim\gamma} e^{-|x-z|/C}\,,$$

and therefore

$$\sum_{x\in\Lambda}\int_0^\lambda \big(\langle\sigma_x\,|\,\sigma_y = 1,\,\forall y\sim\gamma\rangle_{+,s,\Lambda_N} - \langle\sigma_x\rangle_{+,s,\Lambda_N}\big)\,\mathrm{d}s \leq C'\,\lambda\,\#\{z\,:\,z\sim\gamma\}\,.$$

We are almost done. Observe now that $\#\{z : z \sim \gamma\} < 4|\gamma|$. Let K be a large number, which will be chosen below. In the case $|\gamma| < K\,N$, we get

$$\mu_{+,\lambda,\Lambda_N}(\sigma_x = 1,\, \forall x \sim \gamma) \leq e^{4C'K\,\lambda\,N}\, \mu_{+,0,\Lambda_N}(\sigma_x = 1,\, \forall x \sim \gamma)\,.$$

This yields the first term in (2.7). The second term takes care of the remaining open contours. Indeed,

$$\begin{aligned}\sum_{\substack{\gamma:\\ |\gamma|\geq KN}} q_N(\gamma)\, e^{CC'\,\lambda\,|\gamma|} &\leq \sum_{k\geq 0} e^{CC'\,\lambda\,(K+k+1)\,N} \sum_{l=(K+k)\,N}^{(K+k+1)\,N-1} \sum_{\substack{\gamma:\\ |\gamma|=l}} q_N(\gamma)\\ &\leq \sum_{k\geq 0} e^{-C''(K+k)\,N}\\ &\leq e^{-C'''K\,N}\,,\end{aligned}$$

where we used (A.22). Consequently, since

$$\frac{Z_{+,0,\Lambda_N}}{Z_{\pm,0,\Lambda_N}} \leq C\,N^{3/2}\, e^{\tau_\beta(\mathbf{e}_1)\,N}\,,$$

we see that

$$\frac{Z_{+,0,\Lambda_N}}{Z_{\pm,0,\Lambda_N}} \sum_{\gamma:\,|\gamma|\geq K\,N} w(\gamma)\, \frac{Z_{+,\lambda,\Lambda^+}\, Z_{-,\lambda,\Lambda^-}}{Z_{+,\lambda,\Lambda_N}} \leq e^{-C\,N}\,,$$

provided K is chosen large enough. ■

Applying the lemma, we have

$$\begin{aligned}\mu_{\pm,\lambda,\Lambda_N}(\mathcal{A}) &\leq e^{C\,\lambda^{2/3}|\log\lambda|^2\,N}\, \frac{Z_{+,0,\Lambda_N}}{Z_{\pm,0,\Lambda_N}} \sum_{\substack{\gamma:\\ |\Lambda^-|<C_{\lambda,N}}} q_N(\gamma) + e^{-a\,N}\\ &= e^{C\,\lambda^{2/3}|\log\lambda|^2\,N}\, \mu_{\pm,0,\Lambda_N}(|\Lambda^-| < C_{\lambda,N}) + e^{-a\,N}\,,\end{aligned}$$

for some $a > 0$, and so we are left with estimating the probability that $|\Lambda^-| < C_{\lambda,N}$ in the absence of bulk magnetic field, a purely entropic problem.

The next step is to observe that the event $\{|\Lambda^-| < C_{\lambda,N}\}$ is increasing. So, by FKG inequalities,

$$\mu_{\pm,0,\Lambda_N}(|\Lambda^-| < C_{\lambda,N}) \leq \mu_{\pm,0,\Omega}(|\Lambda^-| < C_{\lambda,N})$$

where Ω is the box (see Fig. 3)

$$\Omega = \Lambda_N \setminus \{z :\, z_1 = -N/2 + 1 + k[A\lambda^{-2/3}|\log\lambda|^{-2}],\, k \in \mathbb{N}\}\,,$$

A being a small number to be chosen later. Let us denote by Λ_k^- the component of Λ^- contained in the kth slice Ω_k of Ω. Let $K_\lambda = \lambda^{-1}|\log\lambda|^{-9/2}$. We are going to show that

$$\mu_{\pm,0,\Omega_1}(|\Lambda_1^-| < K_\lambda) < 7/8\,. \tag{2.8}$$

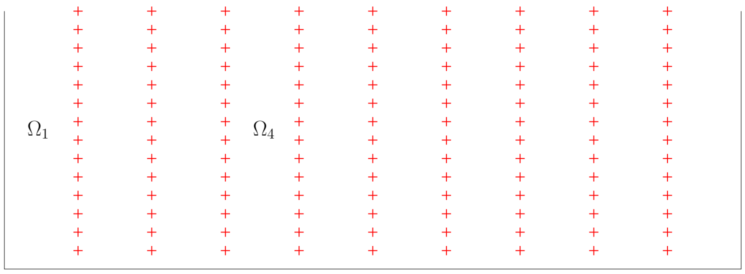

Figure 3: The box Ω is obtained by splitting Λ_N into strips of width $[A\lambda^{-2/3}|\log\lambda|^{-2}]$ by forcing all the spins on the corresponding vertical half-lines to take the value $+1$.

From this and a standard large deviations estimate, we get that

$$\mu_{\pm,0,\Omega}\left(\#\{k : |\Lambda_k^-| < K_\lambda\} \geq \frac{15}{16}\,\frac{N}{[A\lambda^{-2/3}|\log\lambda|^{-2}]}\right) \leq e^{-CA^{-1}\lambda^{2/3}|\log\lambda|^2 N} .$$

(Notice that the events $\{|\Lambda_k^-| < K_\lambda\}$ are independent under $\mu_{\pm,0,\Omega}$). The proposition easily follows from this, provided λ is chosen small enough (for a fixed small A). Indeed, on the complementary event, i.e.

$$\#\{k : |\Lambda_k^-| > K_\lambda\} > \frac{1}{16}\,\frac{N}{[A\lambda^{-2/3}|\log\lambda|^{-2}]} ,$$

we have

$$|\Lambda^-| > \tfrac{1}{16}N/[A\lambda^{-2/3}|\log\lambda|^{-2}]\,K_\lambda > C_{\lambda,N} ,$$

for small λ. Let us now prove (2.8).

Denote by x^l and x^r the dual sites which are the bottom left and bottom right corners of Ω_1 (so these are the endpoints of the open path in Ω_1, see Fig. 4). Let $a_l = x_1^l + [(\frac{1}{2} - A^2|\log\lambda|^{-1})A\lambda^{-2/3}|\log\lambda|^{-2}]$ and $a_r = x_r - [(\frac{1}{2} - A^2|\log\lambda|^{-1})A\lambda^{-2/3}|\log\lambda|^{-2}]$. We introduce the following sets of dual sites:

$$\begin{aligned}
\Delta_l &= \{x \in \mathbb{Z}^{2,*} : x_1 = a_l,\ \tfrac{1}{2} \leq x_2 < A\lambda^{-1/3}|\log\lambda|^{-1}\} \\
\Delta_r &= \{x \in \mathbb{Z}^{2,*} : x_1 = a_r,\ \tfrac{1}{2} \leq x_2 < A\lambda^{-1/3}|\log\lambda|^{-1}\} \\
\Delta &= \Delta_l \cup \Delta_r \\
B &= \{x \in \mathbb{Z}^{2,*} : a_l < x_1 < a_r,\ \tfrac{1}{2} \leq x_2 < \tfrac{1}{2}A\lambda^{-1/3}|\log\lambda|^{-1}\} .
\end{aligned}$$

We are going to show that

$$\sum_{\substack{\gamma : x^l \to x^r \\ \gamma\cap B=\emptyset}} q_{\Omega_1}(\gamma) \geq \tfrac{1}{8}\,\langle\sigma_{x^l}\sigma_{x^r}\rangle_{\Omega_1} , \tag{2.9}$$

which readily implies (2.8) since $|B| > K_\lambda$ (when λ is small). To prove (2.9), we first show that the open contour typically does not hit Δ. This is equivalent to saying that Λ_1^- contains the set Δ. But this is a decreasing event. Therefore, if we introduce the new box

$$\widetilde{\Omega}_1 = \{x = (x_1, x_2) : (x_1, |x_2|) \in \Omega_1\},$$

and the new set

$$\widetilde{\Delta} = \{x = (x_1, x_2) : (x_1, |x_2|) \in \Delta\},$$

then by FKG inequalities,

$$\mu_{\pm,0,\Omega_1}(\Lambda_1^- \supset \Delta) \geq \mu_{\pm,0,\widetilde{\Omega}_1}(\Lambda_1^- \supset \Delta) \geq \mu_{\pm,0,\widetilde{\Omega}_1}(\Lambda_1^- \supset \widetilde{\Delta}).$$

The latter probability is easily bounded. Indeed, by symmetry it is larger than

$$\tfrac{1}{2} \langle \sigma_{x^l} \sigma_{x^r} \rangle_{\widetilde{\Omega}_1}^{-1} \sum_{\substack{\gamma : x^l \to x^r \\ \gamma \cap \widetilde{\Delta} = \emptyset}} q_{\widetilde{\Omega}_1}(\gamma),$$

and the latter expression is larger than 1/4 since

$$\begin{aligned}
\sum_{\substack{\gamma : x^l \to x^r \\ \gamma \cap \widetilde{\Delta} \neq \emptyset}} q_{\widetilde{\Omega}_1}(\gamma) &\leq \sum_{u \in \widetilde{\Delta}} \langle \sigma_{x^l} \sigma_u \rangle_{\widetilde{\Omega}_1} \langle \sigma_u \sigma_{x^r} \rangle_{\widetilde{\Omega}_1} \\
&\leq C \sum_{u \in \widetilde{\Delta}} \frac{e^{-\tau(u - x^l)}}{|u - x_l|^{1/2}} \frac{e^{-\tau(x^r - u)}}{|x_r - u|^{1/2}} \\
&\leq 3C \, \frac{e^{-\tau(x^r - x^l)}}{|x_r - x_l|^{1/2}} \; \frac{|\widetilde{\Delta}|}{|x_r - x_l|^{1/2}} \\
&\leq \tfrac{1}{2} \langle \sigma_{x^l} \sigma_{x^r} \rangle_{\widetilde{\Omega}_1},
\end{aligned}$$

provided A is chosen small enough. We have thus shown that

$$\mu_{\pm,0,\Omega_1}(\Lambda_1^- \supset \Delta) \geq 1/4. \tag{2.10}$$

We now restrict our attention to open paths γ in Ω_1 not intersecting Δ, but intersecting B. To each such paths we associate three dual sites: the first visit to the set $\{x : x_1 = a_l, x \notin \Delta_l\}$, the last visit to $\{x : x_1 = a_r, x \notin \Delta_r\}$, and the first visit to B. We denote these dual sites by b^l, b^r and b^m respectively. Using (A.16), we then have

$$\sum_{\substack{\gamma : x^l \to x^r \\ \gamma \cap \Delta = \emptyset \\ \gamma \cap B \neq \emptyset}} q_{\Omega_1}(\gamma) \leq \sum_{b^l, b^m, b^r} \langle \sigma_{x^l} \sigma_{b^l} \rangle_{\Omega_1} \langle \sigma_{b^l} \sigma_{b^m} \rangle_{\Omega_1} \langle \sigma_{b^m} \sigma_{b^r} \rangle_{\Omega_1} \langle \sigma_{b^r} \sigma_{x^r} \rangle_{\Omega_1}$$

We first observe that

$$\sum_{b^l : b_2^l > \lambda^{-1}} \langle \sigma_{x^l} \sigma_{b^l} \rangle_{\Omega_1} \leq e^{-C\lambda^{-1}} \ll \langle \sigma_{x^l} \sigma_{x^r} \rangle_{\Omega_1},$$

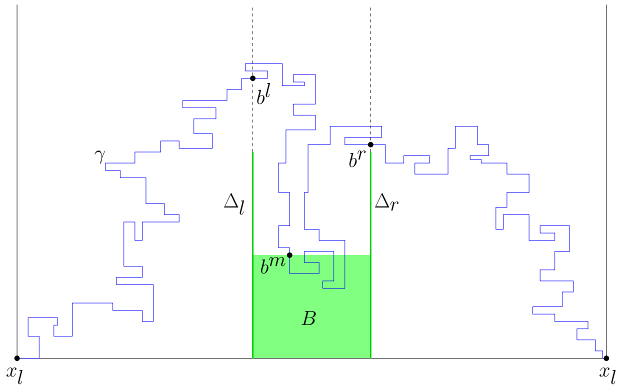

Figure 4: The settings for the proof of (2.8).

and similarly for b^r, so that we can restrict the sum over these points to those with second component smaller than λ^{-1}. We now use the sharp triangle inequality (A.25) to get

$$\begin{aligned}\tau(b^m - b^l) + \tau(b^r - b^m) &\geq \tau(b^r - b^l) + \kappa\left(|b^m - b^l| + |b^r - b^m| - |b^r - b^l|\right)\\ &\geq \tau(b^r - b^l) + CA^{-1}\,|\log\lambda|\,,\end{aligned}$$

and therefore, using (A.24),

$$\begin{aligned}\langle\sigma_{b^l}\sigma_{b^m}\rangle_{\Omega_1}\ \langle\sigma_{b^m}\sigma_{b^r}\rangle_{\Omega_1} &\leq \langle\sigma_{b^l}\sigma_{b^m}\rangle\ \langle\sigma_{b^m}\sigma_{b^r}\rangle\\ &\leq e^{-C'A^{-1}\,|\log\lambda|}\ \langle\sigma_{b^l}\sigma_{b^r}\rangle\,|b^r - b^l|\\ &\leq e^{-C''A^{-1}\,|\log\lambda|}\ \langle\sigma_{b^l}\sigma_{b^r}\rangle_{\Omega_1}\,.\end{aligned}$$

Using this and

$$\langle\sigma_{x^l}\sigma_{b^l}\rangle_{\Omega_1}\ \langle\sigma_{b^l}\sigma_{b^r}\rangle_{\Omega_1}\ \langle\sigma_{b^r}\sigma_{x^r}\rangle_{\Omega_1} \leq \langle\sigma_{x^l}\sigma_{x^r}\rangle_{\Omega_1}\,,$$

which follows from GKS inequalities, we finally get

$$\sum_{\substack{\gamma:x^l\to x^r\\ \gamma\cap\Delta=\emptyset\\ \gamma\cap B\neq\emptyset}} q_{\Omega_1}(\gamma) \leq e^{-C'''A^{-1}\,|\log\lambda|}\ \langle\sigma_{x^l}\sigma_{x^r}\rangle_{\Omega_1} \leq \tfrac{1}{8}\,\langle\sigma_{x^l}\sigma_{x^r}\rangle_{\Omega_1}\,.$$

We deduce from the latter bound and (2.10) that

$$\sum_{\substack{\gamma:x^l\to x^r\\ \gamma\cap B\neq\emptyset}} q_{\Omega_1}(\gamma) \leq \left(\tfrac{1}{8} + \tfrac{3}{4}\right)\langle\sigma_{x^l}\sigma_{x^r}\rangle_{\Omega_1}\,,$$

and (2.9) is proved.

2.3.3 Proof of the upper bound

Let us write $V = K|\log\lambda|^2\lambda^{-1/3}N$, where K will be chosen sufficiently large later on. We want to bound from above the following probability:

$$\mu_{\pm,\lambda,\Lambda_N}\left(|C^-(\,\cdot\,)| \geq V\right) ,$$

where $C^-(\sigma)$ is the set of all sites in Λ_N that are connected to the bottom wall by a path of $-$ spins in σ. Our aim is to show that this probability goes to zero with N, provided K has been chosen large enough.

We start with the following obvious upper bound:

$$\frac{Z_{\pm,\lambda,\Lambda_N}\left(|C^-(\,\cdot\,)| \geq V\right)}{Z_{\pm,\lambda,\Lambda_N}} \leq \frac{Z_{\pm,\lambda,\Lambda_N}\left(|C^-(\,\cdot\,)| \geq V\right)}{Z_{\pm,0,\Lambda_N}\left(|C^-(\,\cdot\,)| \geq V\right)}\,\frac{Z_{\pm,0,\Lambda_N}}{Z_{\pm,\lambda,\Lambda_N}}$$

By Lemma 3,

$$\frac{Z_{\pm,0,\Lambda_N}}{Z_{\pm,\lambda,\Lambda_N}} \leq e^{C|\log\lambda|^2\lambda^{2/3}N}\,\frac{Z_{+,0,\Lambda_N}}{Z_{+,\lambda,\Lambda_N}} .$$

Now observe that we can write

$$\frac{Z_{+,0,\Lambda_N}}{Z_{+,\lambda,\Lambda_N}} = \exp\left\{-\int_0^\lambda \mathrm{d}\lambda' \sum_{x\in\Lambda_N} \langle\sigma_x\rangle_{+,\lambda',\Lambda_N}\right\} ,$$

and, similarly,

$$\frac{Z_{\pm,\lambda,\Lambda_N}\left(|C^-(\,\cdot\,)| \geq V\right)}{Z_{\pm,0,\Lambda_N}\left(|C^-(\,\cdot\,)| \geq V\right)} = \exp\left\{\int_0^\lambda \mathrm{d}\lambda' \sum_{x\in\Lambda_N} \langle\sigma_x\,|\,|C^-(\,\cdot\,)| \geq V\rangle_{\pm,\lambda',\Lambda_N}\right\} .$$

We estimate the product of these two expressions using a coupling argument. Since, by FKG inequalities and the fact that $\{|C^-(\,\cdot\,)| \geq V\}$ is non-increasing,

$$\mu_{+,\lambda',\Lambda_N} \succeq \mu_{\pm,\lambda',\Lambda_N}(\,\cdot\,|\,|C^-(\,\cdot\,)| \geq V) ,$$

there exists a coupling $\nu_{\lambda'}$ of $\mu_{+,0,\Lambda_N}$ and $\mu_{\pm,\lambda',\Lambda_N}(\,\cdot\,|\,|C^-(\,\cdot\,)| \geq V)$ such that

$$\nu(\{\sigma \geq \sigma'\}) = 1 .$$

In fact, it is well-known that one can construct this coupling explicitly. We quickly sketch this, because we'll need some non-degeneracy property below. First, we order all the sites in Λ_N, say $i_1,\ldots,i_{|\Lambda_N|}$. Then we introduce a family $(X_{i_k})_{k=1}^{|\Lambda_N|}$ of iid random variables, with uniform distribution over $[0,1]$. Now we set $\sigma_{i_1} = 1$ (resp. $\sigma'_{i_1} = 1$) if $\mu_{+,\lambda',\Lambda_N}(\sigma_{i_1} = 1) \geq X_{i_1}$ (resp. if $\mu_{\pm,\lambda',\Lambda_N}(\sigma_{i_1} = 1\,|\,|C^-(\,\cdot\,)| \geq V) \geq X_{i_1}$); otherwise it is set to -1. Suppose we have already constructed the first $k-1$ spins. Then we set $\sigma_{i_k} = 1$ if $\mu_{+,\lambda',\Lambda_N}(\sigma_{i_k} = 1\,|\,\sigma_{i_l},\, l<k) \geq X_{i_k}$, and similarly for σ'_{i_k}. Just observe that, by FKG inequalities,

$$\mu_{+,\lambda',\Lambda_N}(\sigma_{i_k} = 1\,|\,\sigma_{i_l},\, l<k) \geq \mu_{\pm,\lambda',\Lambda_N}(\sigma_{i_k} = 1\,|\,|C^-(\,\cdot\,)| \geq V,\, \sigma_{i_l},\, l<k) .$$

Using this notation, we are left with estimating $\sum_{x\in\Lambda_N}\left(\nu_{\lambda'}(\sigma'_x-\sigma_x)\right)$. Since $\nu_{\lambda'}(\sigma\geq\sigma')=1$, we can write

$$\nu_{\lambda'}\big(\sum_{x\in\Lambda_N}(\sigma'_x-\sigma_x)\big)\leq\nu_{\lambda'}\big(\sum_{x\in C^-(\sigma')}(\sigma'_x-\sigma_x)\big)\,.$$

Now observe that $\sum_{x\in C^-(\sigma')}\sigma'_x=-|C^-(\sigma')|$. Moreover, from the above construction, we see that the coupling satisfies

$$\inf_{x\in\Lambda_N}\nu_{\lambda'}\big(\sigma_x\;\big|\;\sigma_y,\,\forall y\in\Lambda_N\setminus\{x\},\sigma'_z,\,\forall z\in\Lambda_N\big)>-1-\epsilon_\beta\,,$$

with ϵ_β depending only on the parameter β, for all λ small enough. Therefore, putting everything together, we obtain

$$\mu_{\pm,\lambda,\Lambda_N}\left(|C^-(\,\cdot\,)|\geq V\right)\leq e^{-(K\epsilon_\beta-c)|\log\lambda|^2\lambda^{2/3}N}\,,$$

and the conclusion follows once K is taken large enough.

A Some tools

In this section we collect several results from [26, 29, 27] which we are using in the paper. This is not supposed to be an introduction to the random-line representation, and we refer the reader to the latter works for detailed explanations.

Let $\mathcal{E}^*$ be the set of all dual bonds of the infinite lattice. One of the basic objects in this paper is the partition function $Z_{\pm,\beta,\lambda,h,\Lambda_N}$. To any configuration σ with non-zero weight under $\mu_{\pm,\beta,\lambda,h,\Lambda_N}$, we can associate a subset $n(\sigma)$ of

$$\mathcal{E}^*_N=\{e^*\in\mathcal{E}^*\,:\,e^*\text{ dual to }e\subset\Lambda_N,\,e\not\subset\partial\Lambda_N\}$$

composed of all dual edges e^* such that $e=\langle x,y\rangle\not\subset\partial\Lambda_N$ with $\sigma_x\neq\sigma_y$. We then split this subset into a collection of contours, by applying the rules given in Fig. 5 each time more than two bonds are incident on a given vertex; contours are the families of edges remaining connected after these operations. Among all the contours, there is a unique open contour connecting the dual sites $(-(N-1)/2,1/2)$ and $((N+1)/2,1/2)$; we call it γ.

We want to expand the partition functions in terms of the open contour γ. Notice that once γ is fixed, some spins in the box Λ_N become frozen (i.e. in any configuration with positive weight having γ as open contour, these spins take the same values). We denote by $\Lambda^+(\gamma)$ the set of all non-frozen spins located inside a component surrounded by + spins. Similarly, we write $\Lambda^-(\gamma)$ for the set of all non-frozen spins located inside a component surrounded by − spins.

We then have

$$Z_{\pm,\beta,\lambda,h,\Lambda_N}=\sum_\gamma w(\gamma)\;Z_{+,\beta,\lambda,\Lambda^+(\gamma)}\;Z_{-,\beta,\lambda,h,\Lambda^-(\gamma)}\,,\tag{A.11}$$

where $Z_{+,\beta,\lambda,\Lambda^+(\gamma)}$ and $Z_{-,\beta,\lambda,h,\Lambda^-(\gamma)}$ are the two partition functions associated to the restriction of our system to $\Lambda^+(\gamma)$ and $\Lambda^-(\gamma)$, and $w(\gamma)=e^{-2\beta|\gamma|}$, where $|\gamma|=\sum_{e^*\subset\gamma}J(e)$.

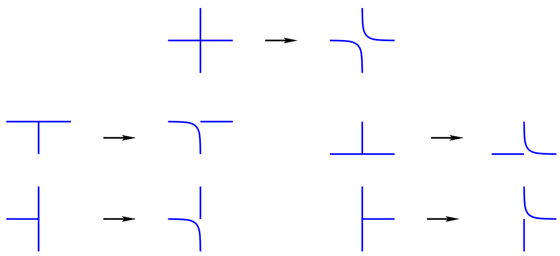

Figure 5: Deformation rules.

We introduce the weight (notice that it is computed at zero bulk field)

$$q_{\beta,h,N}(\gamma) = w(\gamma)\, \frac{Z_{+,\beta,0,\Lambda^+(\gamma)}\, Z_{-,\beta,0,h,\Lambda^-(\gamma)}}{Z_{+,\beta,0,h,\Lambda_N}}\,. \tag{A.12}$$

Let β^*, h^* and $J^*(e^*)$ be such that $\tanh(\beta^*) = \exp(-2\beta)$, $\tanh(\beta^* h^*) = \exp(-2\beta h)$ and $\tanh(\beta^* J^*(e^*)) = \exp(-2\beta J(e))$.

The Gibbs measure in $\mathcal{E}_N^*$ with free b.c. is defined by

$$\mu_{\beta^*,h^*,N}(\sigma) = (Z_{\beta^*,h^*,N})^{-1} \prod_{e^*=\langle x,y\rangle \in \mathcal{E}_N^*} e^{-\beta^* J^*(e^*)\,\sigma_x \sigma_y}\,.$$

A basic duality argument, see e.g. Lemma 6.2 in [27], shows that

$$\sum_\gamma q_{\beta,h,N}(\gamma) = \langle \sigma_{(-(N-1)/2,1/2)} \sigma_{((N+1)/2,1/2)} \rangle_{\beta^*,h^*,N}\,. \tag{A.13}$$

One has in fact a deeper relation between high and low temperature models. To each subset of $\mathcal{E}_N^*$ which remains connected after applying the rules of Fig. 5, we can associate a weight $q^*_{\beta^*,h^*,N}$ (see (6.8) of [27] for a definition) such that

$$q^*_{\beta^*,h^*,N}(\gamma) = q_{\beta,h,N}(\gamma)\,.$$

However, $q^*_{\beta^*,h^*,N}$ being defined for much more general subsets of $\mathcal{E}_N^*$ than $q_{\beta,h,N}$, it is a much more useful quantity. We'll state its properties for arbitrary $E \Subset \mathcal{E}^*$, and will denote the corresponding weight by $q^*_{\beta^*,h^*,E}$. We list now the properties of these weights that we are using in the proof. Let us denote by $\mathcal{E}^*_{\text{s.i.}} = \bigcup_{N\to\infty} \mathcal{E}_N^*$.

- ([27], Lemma 6.3) The limiting weights $q^*_{\beta^*,h^*} = \lim_{E\uparrow\mathcal{E}^*} q^*_{\beta^*,h^*,E}$ and $q^*_{\beta^*,h^*,\text{s.i.}} = \lim_{E\uparrow\mathcal{E}^*_{\text{s.i.}}} q^*_{\beta^*,h^*,E}$ are well-defined quantities. Moreover, for any $\gamma \subset E$,
$$q^*_{\beta^*,h^*,E}(\gamma) \geq q^*_{\beta^*,h^*,\text{s.i.}}(\gamma) \geq q^*_{\beta^*,h^*}(\gamma)\,. \tag{A.14}$$
- ([27], Lemma 6.4) If $\gamma = \gamma_1 \amalg \gamma_2$ ($\amalg$ is a concatenation operation, see [27] for a definition), then
$$q^*_{\beta^*,h^*,E}(\gamma) \geq q^*_{\beta^*,h^*,E}(\gamma_1)\, q^*_{\beta^*,h^*,E}(\gamma_2)\,. \tag{A.15}$$

- ([27], Lemma 6.5) Let $t_1, \ldots, t_n$ be distinct dual sites in E. Then

$$\sum_{\gamma:\, t_1 \to \cdots \to t_n} q^*_{\beta^*,h^*,E}(\gamma) \le \prod_{k=1}^{n-1} \sum_{\gamma:\, t_k \to \cdots \to t_{k+1}} q^*_{\beta^*,h^*,E}(\gamma)\,. \tag{A.16}$$

- ([29], Lemma 4.4.6, and [27], Lemma 6.10) Let $\mathcal{R}$ be a rectangular subset of $\mathcal{E}^*$ having length $R^2/(K \log R)$ and height R, with basis contained inside $\{e^* = \langle x, y\rangle \in \mathcal{E}^* \,:\, x_2 = y = 2 = 1/2\}$. We denote by u and v the dual sites at the bottom left and bottom right corners of $\mathcal{R}$. Then, for K large enough, there exists $C < \infty$ such that

$$\sum_{\substack{\gamma:\, u \to v\\ \gamma \subset \mathcal{R}}} q^*_{\beta^*,h^*}(\gamma) \ge R^{-C} \sum_{\gamma:\, u \to v} q^*_{\beta^*,h^*}(\gamma)\,. \tag{A.17}$$

- ([27], (6.9) and Lemma 6.9) For any vertices $x, y \in E$,

$$\sum_{\gamma:\, x \to y} q^*_{\beta^*,h^*,E}(\gamma) = \langle \sigma_x \sigma_y \rangle_{\beta^*,h^*,E}\,. \tag{A.18}$$

  Moreover

$$\sum_{\gamma:\, x \to y} q^*_{\beta^*,h^*,\mathrm{s.i.}}(\gamma) = \langle \sigma_x \sigma_y \rangle_{\beta^*,h^*,\mathrm{s.i.}}\,, \tag{A.19}$$

  and

$$\sum_{\gamma:\, x \to y} q^*_{\beta^*,h^*}(\gamma) = \langle \sigma_x \sigma_y \rangle_{\beta^*,h^*}\,, \tag{A.20}$$

$$\tag{A.21}$$

  where $\langle\,\cdot\,\rangle_{\beta^*,h^*}$ and $\langle\,\cdot\,\rangle_{\beta^*,h^*,\mathrm{s.i.}}$ denote expectation w.r.t. the infinite and semi-infinite volume Gibbs measures.

- There exists $c > 0$ and $l_0 < \infty$ such that, for all l,

$$\sum_{\substack{\gamma:\\ |\gamma| \ge l}} q_{\beta,h,N}(\gamma) \le e^{-c(l-l_0)}\,. \tag{A.22}$$

Proof of (A.22)**.** The proof is similar to that of Lemma 5.6 in [26]; we only sketch it. Let R be some big positive number. We make a coarse-graining of the path γ on the scale R, i.e. we set t_1 to be the left endpoint of γ, t_k to be the first point of γ after t_{k-1} which is outside the square of sidelength R centered on t_{k-1}, and the procedure stops when one reaches the other endpoint of γ. One then first sum over these sites $t_0, t_1, \ldots, t_L$ (observing that, given t_k there are at most CR choices for t_{k+1}, and that the total length of the piece of γ between two consecutive points is at most R^2, so that $L \ge l/R^2$) and uses (A.14), (A.18), and the upper bound in (A.24) below. The conclusion follows since τ_β is uniformly strictly positive when $\beta > \beta_c$. ■

We also use the following results about the asymptotic behavior of the boundary 2-point function $\langle\sigma_x\sigma_y\rangle_{\beta^*,h^*,\text{s.i.}}$: Suppose that $h > h_w(\beta)$; then there exist constants K_1 and K_2 such that, for any x, y with $x_2 = y_2 = 1/2$,

$$K_1\,|x-y|^{-3/2}e^{-\tau_\beta(\mathbf{e}_1)\,|x-y|} \geq \langle\sigma_x\sigma_y\rangle_{\beta^*,h^*,\text{s.i.}} \geq K_2\,|x-y|^{-3/2}e^{-\tau_\beta(\mathbf{e}_1)\,|x-y|}\,, \tag{A.23}$$

where $\tau_\beta(\mathbf{e}_1)$ is the surface tension in the horizontal direction.

We also need the corresponding result for the bulk 2-point. There exists $K < \infty$ such that, for any $h > h_w(\beta)$, and any pair of dual sites x, y in E such that the set

$$\mathcal{S}_K(x,y) = \{u \in \mathbb{Z}^{2,*} \,:\, \|x-u\|_2 + \|y-u\|_2 \leq \|x-y\|_2 + K\log\|x-y\|_2\}\,,$$

satisfies $\mathcal{S}_K(x,y) \subset \{x \in E \,:\, x_2 \geq 3/2\}$, the following holds: There exist constants K_1 and K_2 such that ([29], Proposition 4.6.1.)

$$K_1\,|x-y|^{-1/2}e^{-\tau_\beta(\frac{x-y}{|x-y|})\,|x-y|} \geq \langle\sigma_x\sigma_y\rangle_{\beta^*,h^*} \geq K_2\,|x-y|^{-1/2}e^{-\tau_\beta(\mathbf{e}_1)\,|x-y|}\,. \tag{A.24}$$

An important property of surface tension is that it satisfies the following *sharp triangle inequality* ([27], Theorem 2.1): There exists $\kappa > 0$ such that, for any x, y,

$$\tau_\beta(x) + \tau_\beta(y) \geq \tau_\beta(x+y) + \kappa(|x| + |y| - |x+y|)\,. \tag{A.25}$$

Finally, we also need the following estimate on the relaxation of correlation functions ([10]): Suppose that h and λ are non-negative; then there exists $C(\beta) > 0$ and $K < \infty$ such that, for any $\Lambda_1, \Lambda_2 \subset \mathbb{Z}^2$ and $A \subset \Lambda_1 \cup \Lambda_2$,

$$|\langle\sigma_A\rangle_{+,\beta,\lambda,h,\Lambda_1} - \langle\sigma_A\rangle_{+,\beta,\lambda,h,\Lambda_2}| \leq K \sum_{\substack{t\in A\\ t'\in\Lambda_1\triangle\Lambda_2}} e^{-C(\beta)\,|t'-t|}\,, \tag{A.26}$$

where $\sigma_A = \prod_{x\in A}\sigma_x$.

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