SPECIAL VALUES OF THE SPECTRAL ZETA FUNCTION OF THE NON-COMMUTATIVE HARMONIC OSCILLATOR AND CONFLUENT HEUN EQUATIONS*

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Abstract. We study the special values at $s = 2$ and 3 of the spectral zeta function $\zeta_Q(s)$ of the non-commutative harmonic oscillator $Q(x, D_x)$ introduced in [PW1, 2]. It is shown that the series defining $\zeta_Q(s)$ converges absolutely for Re $s > 1$ and further the respective values $\zeta_Q(2)$ and $\zeta_Q(3)$ are represented essentially by contour integrals of the solutions, respectively, of a singly confluent Heun's ordinary differential equation and of exactly the same but an inhomogeneous equation. As a by-product of these results, we obtain integral representations of the solutions of these equations by rational functions.
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1. Introduction

The aim of the present paper is a study of some special values of the spectral zeta function of the non-commutative harmonic oscillator introduced in [PW1, 2] (We sometimes abbreviate it as the NCHO). This non-commutative harmonic oscillator $Q = Q(x, D_x)$ is the second order differential operator

$$Q(x, D_x) = A\Big(-\frac{\partial_x^2}{2} + \frac{x^2}{2}\Big) + J\Big(x\partial_x + \frac{1}{2}\Big), \quad x \in \mathbb{R}, \quad \partial_x := \frac{d}{dx}, \tag{1.1}$$

where $A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ and $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We assume that the parameters $\alpha, \beta \in \mathbb{R}$ are positive and satisfy $\alpha\beta > 1$. It is known [PW1, 2] that Q defines a positive, self-adjoint operator in the Hilbert space $L^2(\mathbb{R}) \otimes \mathbb{C}^2$, which has only a discrete spectrum $(0 <)\lambda_1 \le \lambda_2 \le \ldots \le \lambda_n \le \ldots \to \infty$. Here the eigenvalues are counted with multiplicity. It is also known [PW3] that the multiplicity of these eigenvalues are uniformly bounded, in fact, by 3. The spectral zeta function of the system is then defined as

$$\zeta_Q(s) = \zeta_{Q_{(\alpha,\beta)}}(s) = \sum_{j=1}^{\infty} \frac{1}{\lambda_j^s} \tag{1.2}$$

for sufficiently large Re $s > 0$. Indeed, it is seen in §3 that the series converges absolutely for Re $s > 1$ and hence it defines a holomorphic function in this half plane (see Theorem 3.3). In the proof, a crucial role is played by the Lieb–Thirring inequality (see [LT], also [Ar]). When $\alpha = \beta$, the non-commutative harmonic oscillator Q is unitarily equivalent to a couple of the usual quantum harmonic operators $\frac{1}{2}(-\partial_x^2 + x^2)I$ (up to the automorphisms of $L^2(\mathbb{R}) \otimes \mathbb{C}^2$ given by the dilation of the variable); thus in this case, the spectral zeta function $\zeta_Q(s)$ is equal to the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ times some elementary factor. Actually, when $\alpha = \beta = \sqrt{2}$, since it is shown [PW2] in this case that the spectrum of the system is given by $\{n+\frac{1}{2}\}$ $(n = 0, 1, 2, \ldots)$ with multiplicity two, we immediately conclude that $\zeta_{Q_{(\sqrt{2},\sqrt{2})}}(s) = 2\sum_{n=0}^{\infty} \frac{1}{(n+\frac{1}{2})^s} = 2(2^s - 1)\zeta(s)$. In this sense, our spectral zeta function $\zeta_Q(s)$ can be considered as a modification, or rather a "$q(:= \alpha/\beta)$"-analogue (for a fixed value $\alpha\beta$) of $\zeta(s)$ (see [IW]).

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In this paper, we treat only the first two special values $\zeta_Q(2)$ and $\zeta_Q(3)$. The main results are as follows:

(1) We show that the special value $\zeta_Q(2)$ is strongly connected with a solution of a *singly confluent* Heun's ordinary differential equation. Here the so-called Heun differential equation is a Fuchsian ordinary differential equation with four regular singular points in a complex domain. See, e.g. [WW] and also [SL] for a confluent case. Actually, the value $\zeta_Q(2)$ is represented essentially by a contour integral of a holomorphic solution (in the unit disk) of this equation (times $\zeta(2) = \frac{\pi^2}{6}$) when $\alpha\beta > 2$ (see Corollary 4.14).

(2) As a by-product of the investigation above, we find an integral representation of the solution of the singly confluent Heun's equation by a rational function (see Theorem 4.13).

(3) Similarly to the case of $\zeta_Q(2)$, we show that $\zeta_Q(3)$ can be essentially expressed as a sum of the value $\zeta(3)$ and a contour integral of a solution of the confluent Heun's equation mentioned above with an *inhomogeneous* term. Moreover, this solution of the inhomogeneous equation which is holomorphic in the unit disk has an integral expression analogous with the one obtained in the case (2) (see Theorem 6.5 and Corollary 6.6). Note also that this integral expression of the solution resembles the definite integral due to Beukers [B] which was employed to prove the irrationality of $\zeta(3)$ (a result first proved by Apéry [Ap] in 1978, see §7 in [AAR]).

However, for a general n, it seems difficult to establish such results about the special values $\zeta_Q(n)$ in connection with Heun's equations via the calculations we develop in this paper. Indeed, according to Ochiai [O], the spectral problem of the NCHO is equivalent to solving Heun's equation (not confluent), but it is still unclear why the values $\zeta_Q(2)$ and $\zeta_Q(3)$ can be described by the solutions of the confluent Heun equations. On the other hand, in [IW], we have studied analytic properties of $\zeta_Q(s)$ such as a meromorphic continuation to the entire complex plane $\mathbb{C}$, etc., based on the initial result obtained in the present paper which claims that $\zeta_Q(s)$ is holomorphic in $\mathrm{Re}\ s > 1$, and further we have shown that $\zeta_Q(s)$ has a zero at $s = -2m\,(m = (0), 1, 2, \dots)$ like the case of $\zeta(s)$. But still, it is hard to expect any functional equation for $\zeta_Q(s)$ like that for $\zeta(s)$. Therefore, in order to study the distribution of the non-trivial zeros of $\zeta_Q(s)$ it will be quite important, for lack of a functional equation, to explore a relation (if any) between the values of $\zeta_Q(s)$ at $s = -2m + 1$ and those at $s = 2m$ via Heun's differential equations.

The paper is organized as follows. In §2 we study the heat kernels of operators closely related to the non-commutative harmonic oscillator, which will lead us to get to its Green function. We show in §3 the existence of $\zeta_Q(s)$ for $\mathrm{Re}\, s > 1$ and obtain a certain integral expression (by a double integral) of the value $\zeta_Q(2)$. §4 pursues a further study of the value of $\zeta_Q(2)$ in connection with a confluent Heun's equation. The calculation developed here is based on the aforementioned integral expression. We seek a recurrence relation of the sequences to be defined by the double integral for the coefficients of the Taylor expansion of the function in the integrand. We find that these sequences provide a holomorphic solution of the Heun equation in the unit disk. In §5 and §6, we devote ourselves to calculating the value $\zeta_Q(3)$ to find its connection with an inhomogeneous Heun and confluent Heun equations. In the Appendix, we provide a determinant formula of a certain special symmetric matrix which can be closely related to the calculation of the special value $\zeta_Q(n)$ of our spectral zeta function for general $n \geq 2$ (see Remark 2 in §5).

2. Heat kernels and Green function

To get the special values of $\zeta_Q(s)$ at the positive integer points of s we need explicit expressions for the heat kernel of the non-commutative harmonic operator $Q(x, D_x)$ defined in (1.1) or the Green function to be obtained by the Laplace transform. However, it seems difficult to deal directly with the former. So we study in this section simpler operators than but closely related to the operator (1.1), and construct their heat kernels, through which we get the Green function for the operator $Q(x, D_x)$ in question by the Laplace transform.

First we consider an operator (see, in Section 3, p.28R in [PW1] and also [PW2, 3]) defined by

$$Q_h(x, D_x) = \frac{1}{2}(-\partial_x^2 + 2x^2)I + (x\partial_x + \frac{1}{2})J, \tag{2.1}$$

which can be also written as

$$Q_h(x, D_x) = -\frac{1}{2}(\partial_x - Jx)^2 + \frac{x^2}{2}. \tag{2.2}$$

It is known that the system defined by this operator is unitarily equivalent to a couple of the harmonic oscillators. (For an explicit construction of the creation and the annihilation operators with the map which gives the equivalence, see [PW1, 2].) The heat kernel $K_0(t,x,y)$ of this operator is then given by

$$K_0(t,x,y) = \exp\Big[\frac{x^2-y^2}{2}J\Big]p(t,x,y) = \Big[\cos\frac{x^2-y^2}{2} + J\sin\frac{x^2-y^2}{2}\Big]p(t,x,y), \tag{2.3a}$$

$$\begin{aligned} p(t,x,y) &= e^{-t/2}\pi^{-1/2}(1-e^{-2t})^{-1/2}\exp\Big[-\frac{x^2-y^2}{2} - \frac{(e^{-t}x-y)^2}{1-e^{-2t}}\Big] \\ &= \pi^{-1/2}e^{-t/2}(1-e^{-2t})^{-1/2}\exp\Big[-\frac{(1+e^{-2t})(x^2+y^2)-4e^{-t}xy}{2(1-e^{-2t})}\Big]. \end{aligned} \tag{2.3b}$$

Here $p(t,x,y)$ is the heat kernel for the harmonic oscillator $\frac{1}{2}(-\partial_x^2+x^2)$:

$$\partial_t p(t,x,y) = -\frac{1}{2}(-\partial_x^2+x^2)p(t,x,y), \quad t>0,\ (x,y)\in\mathbb{R}^2, \tag{2.4a}$$

$$p(t,x,y) \sim \frac{\pi^{-1/2}}{(2t)^{1/2}}e^{-\frac{(x-y)^2}{2t}} \to \delta(x-y), \quad t\downarrow 0. \tag{2.4b}$$

This explicit formula is known as Mehler's formula, and referred to e.g. [GJ]. Note that in [GJ] $p(t,x,y)$ denotes the heat kernel for the operator $\frac{1}{2}(-\partial_x^2+x^2-1)$. See also [EMOT], Vol. **3**, 19.12, Eq. (14), p.227.

To show (2.3a), we put $\widetilde{K}_0(t,x,y) = E(x,y)p(t,x,y)$, where $E(x,y) = \Big[\cos\frac{x^2-y^2}{2} + J\sin\frac{x^2-y^2}{2}\Big]$. Then, it is sufficient to check the equation

$$\partial_t\widetilde{K}_0(t,x,y) = -Q_h(x,D_x)\widetilde{K}_0(t,x,y), \quad t>0,\ (x,y)\in\mathbb{R}^2. \tag{2.5}$$

This can be done easily. In fact, since

$$\begin{aligned} \partial_x\widetilde{K}_0 &= \partial_x E(x,y)p + E(x,y)\partial_x p, \\ \partial_x^2\widetilde{K}_0 &= \partial_x^2 E(x,y)p + 2\partial_x E(x,y)\partial_x p + E(x,y)\partial_x^2 p, \end{aligned}$$

the equations

$$\partial_x E(x,y) = xJE(x,y), \quad \partial_x^2 E(x,y) = -x^2JE(x,y) + JE(x,y)$$

together with the formula (2.4a) yield the equation (2.5).

Let

$$Q := Q_{(\alpha,\beta)}(x,D_x) = A\frac{1}{2}(-\partial_x^2+x^2) + J(x\partial_x+\frac{1}{2}). \tag{2.6}$$

Throughout the paper we assume that

$$\gamma := (\alpha\beta)^{-1/2} < 1. \tag{2.7}$$

We then consider the operator

$$Q' := A^{-1/2}QA^{-1/2} = \frac{1}{2}(-\partial_x^2+x^2) + A^{-1/2}JA^{-1/2}(x\partial_x+\frac{1}{2}). \tag{2.8}$$

Since $A^{-1/2}JA^{-1/2} = \gamma J$, Q' can be expressed as

$$Q' = \frac{1}{2}(-\partial_x^2+x^2) + \gamma J(x\partial_x+\frac{1}{2}) = -\frac{1}{2}(\partial_x-\gamma Jx)^2 + \frac{1-\gamma^2}{2}x^2. \tag{2.9}$$

Using the equations (2.3ab), we obtain the following explicit expression of the heat kernel $K'(t,x,y)$ for the operator Q', though we cannot have one for the operator Q in (2.6).

Theorem 2.1. *The heat kernel $K'(t,x,y)$ for Q', i.e., the integral kernel of $e^{-tQ'}$, is given by*

$$K'(t,x,y) = (1-\gamma^2)^{1/4}\Big[\cos\frac{\gamma(x^2-y^2)}{2} + J\sin\frac{\gamma(x^2-y^2)}{2}\Big]p((1-\gamma^2)^{1/2}t,(1-\gamma^2)^{1/4}x,(1-\gamma^2)^{1/4}y), \quad (2.10a)$$

which satisfies

$$K'(t,x,y) \to \delta(x-y)I, \quad t\downarrow 0, \quad (2.10b)$$

where I is the 2×2-identity matrix.

Proof. Put, with $p(t,x,y)$ in (2.3b),

$$p_\gamma(t,x,y) = (1-\gamma^2)^{1/4}p((1-\gamma^2)^{1/2}t,(1-\gamma^2)^{1/4}x,(1-\gamma^2)^{1/4}y). \quad (2.11a)$$

Then it is easy to see that it satisfies the equation

$$\partial_t p_\gamma(t,x,y) = -\frac{1}{2}[\partial_x^2 + (1-\gamma^2)^{1/2}x^2]p_\gamma(t,x,y), \quad t>0,\ (x,y)\in\mathbb{R}^2. \quad (2.11b)$$

Since it holds that $p_\gamma(t,x,y)\to\delta(x-y)$ as $t\downarrow 0$, we have (2.10b).

Next we show that $K'(t,x,y)$ satisfies the heat equation for Q':

$$\partial_t K'(t,x,y) = -Q'K'(t,x,y), \quad t>0,\ (x,y)\in\mathbb{R}^2. \quad (2.12)$$

If we put

$$E_\gamma(x,y) = \Big[\cos\frac{\gamma(x^2-y^2)}{2} + J\sin\frac{\gamma(x^2-y^2)}{2}\Big] = \exp\Big[\frac{\gamma(x^2-y^2)}{2}J\Big], \quad (2.13)$$

then we have

$$K'(t,x,y) = E_\gamma(x,y)p_\gamma(t,x,y) \quad (2.14)$$

and

$$\partial_x E_\gamma(x,y) = -\gamma x\sin\frac{\gamma(x^2-y^2)}{2} + \gamma xJ\cos\frac{\gamma(x^2-y^2)}{2} = \gamma xJE_\gamma(x,y). \quad (2.15)$$

Write

$$Q' = -\frac{1}{2}(\partial_x - \gamma Jx)^2 + \frac{1-\gamma^2}{2}x^2 =: Q_1' + Q_2'. \quad (2.16)$$

To calculate $Q'K'(t,x,y)$, we use (2.4a) and (2.15). First we calculate $Q_1'K'(t,x,y)$. We have (I).

$$\partial_x K'(t,x,y) = \partial_x[E_\gamma(x,y)p_\gamma] = (\partial_x E_\gamma)p_\gamma + E_\gamma\partial_x p_\gamma = \gamma xJE_\gamma p_\gamma + E_\gamma\partial_x p_\gamma.$$

(II). By (I) above we see that

$$\begin{aligned}\partial_x^2 K'(t,x,y) &= \partial_x[\partial_x K'(t,x,y)]\\ &= [\gamma JE_\gamma + \gamma xJ\partial_x E_\gamma]p_\gamma + \gamma xJE_\gamma\partial_x p_\gamma + \partial_x E_\gamma\partial_x p_\gamma + E_\gamma\partial_x^2 p_\gamma\\ &= [\gamma JE_\gamma - (\gamma x)^2 f]p_\gamma + 2\gamma xJE_\gamma\partial_x p_\gamma + E_\gamma\partial_x^2 p_\gamma.\end{aligned}$$

(III). By (2.14) we have

$$-\gamma^2x^2K'(t,x,y) = -\gamma^2x^2E_\gamma p_\gamma.$$

(IV). Similarly

$$-\gamma JK'(t,x,y) = -\gamma JE_\gamma p_\gamma.$$

(V). By (I) again, we observe that

$$-2\gamma Jx\partial_x K'(t,x,y) = 2(\gamma x)^2E_\gamma p_\gamma - 2\gamma xJE_\gamma\partial_x p_\gamma.$$

Hence, by adding (II), (III), (IV) and (V), we have

$$\begin{aligned} Q_1' K'(t,x,y) &= -\frac{1}{2}(\partial_x - \gamma J x)^2 K'(t,x,y) \\ &= -\frac{1}{2}[\partial_x^2 - \gamma^2 x^2 - \gamma J - 2\gamma J x \partial_x] K'(t,x,y) \\ &= -\frac{1}{2} E_\gamma \partial_x^2 p_\gamma. \end{aligned}$$

Also for Q_2', we have

$$Q_2' K'(t,x,y) = \frac{1-\gamma^2}{2} x^2 K'(t,x,y) = \frac{1-\gamma^2}{2} x^2 E_\gamma p_\gamma.$$

Thus we obtain

$$Q' K'(t,x,y) = (Q_1' + Q_2')K' = E_\gamma \frac{1}{2}[-\partial_x^2 + (1-\gamma^2)x^2] p_\gamma.$$

On the other hand, since $\partial_t K'(t,x,y) = E_\gamma(x,y)\partial_t p_\gamma$, the equation (2.12) follows from (2.11b) immediately. This completes the proof of the theorem. □

Note that since

$$\int_0^\infty e^{-tQ'} dt = Q'^{-1} = A^{1/2} Q^{-1} A^{1/2}, \tag{2.17}$$

the integral kernel $Q^{-1}(x,y)$ of Q^{-1}, i.e. the Green function for Q, is given by

$$Q^{-1}(x,y) = \int_0^\infty A^{-1/2} e^{-tQ'} A^{-1/2} dt = \int_0^\infty A^{-1/2} K'(t,x,y) A^{-1/2} dt. \tag{2.18}$$

On the other hand, it is seen by Theorem 2.1 that the resolvent of the operator Q',

$$(Q' - \lambda)^{-1} = \int_0^\infty e^{-t(Q'-\lambda)} dt = (A^{-1/2} Q A^{-1/2} - \lambda)^{-1} = A^{1/2}(Q - \lambda A)^{-1} A^{1/2}, \tag{2.19}$$

has the integral kernel

$$\begin{aligned} (Q' - \lambda)^{-1}(x,y) &= \int_0^\infty e^{t\lambda} K'(t,x,y) dt \\ &= \Big[\cos\frac{\gamma(x^2-y^2)}{2} + J \sin\frac{\gamma(x^2-y^2)}{2}\Big] \\ &\quad \times \int_0^\infty e^{t\lambda} (1-\gamma^2)^{1/4} p((1-\gamma^2)^{1/2} t, (1-\gamma^2)^{1/4} x, (1-\gamma^2)^{1/4} y) dt. \end{aligned} \tag{2.20}$$

3. Existence of $\zeta_Q(s)$ for $\mathrm{Re}\, s > 1$ and $\zeta_Q(2)$

Let $\{\lambda_n\}_{n=1}^\infty$ and $\{\phi_n\}_{n=1}^\infty$ be the eigenvalues and eigenfunctions of the operator $Q = Q_{(\alpha,\beta)}(x, D_x)$ in (1.1) : $Q\phi_n = \lambda_n \phi_n$. Let $\zeta_Q(s) = \mathrm{Tr}\, Q^{-s}$ be the spectral zeta function (1.2) for the operator Q. Here and throughout this paper, Tr stands for the operator trace, while tr denotes the matrix trace. The main purpose of this section is to give an expression of $\zeta_Q(2)$ by a definite integral and to prove the series (1.2) converges absolutely and uniformly for Re $s \geq 1 + \epsilon$ for every $\epsilon > 0$.

We continue some calculation initiated in the previous section. We first show the following claim by the results obtained in §2. (Using this we prove that $\zeta_Q(s)$ has a simple pole at $s = 1$. See [IW].)

Proposition 3.1.

$$\sum_{n=1}^\infty \frac{1}{\lambda_n} = \mathrm{Tr}\, Q^{-1} = +\infty. \tag{3.1}$$

Proof. We use (2.18) and (2.19), (2.20) with $\lambda = 0$. Then we have with (2.3b) that

$$\begin{aligned} Q^{-1}(x,x) &= (A^{-1/2}Q'^{-1}A^{-1/2})(x,x) \\ &= A^{-1}\int_0^\infty (1-\gamma^2)^{1/4} p((1-\gamma^2)^{1/2}t, (1-\gamma^2)^{1/4}x, (1-\gamma^2)^{1/4}x)dt. \end{aligned}$$

Therefore

$$\begin{aligned} \operatorname{Tr} Q^{-1} &= \operatorname{tr} A^{-1}\int dx \int_0^\infty (1-\gamma^2)^{1/4} p((1-\gamma^2)^{1/2}t, (1-\gamma^2)^{1/4}x, (1-\gamma^2)^{1/4}x)dt \\ &= (\alpha^{-1}+\beta^{-1})(1-\gamma^2)^{-1/2}\int_0^\infty dt' \int p(t', x', x')dx', \end{aligned}$$

where $t' = (1-\gamma^2)^{1/2}t$, $x' = (1-\gamma^2)^{1/4}x$. But this last dt'-integral is divergent, because

$$\begin{aligned} \int p(t,x,x)dx &= \pi^{-1/2}e^{-t/2}(1-e^{-2t})^{-1/2}\int \exp\Big[-\frac{(1-e^{-t})^2x^2}{(1-e^{-2t})}\Big]dx \\ &= \pi^{-1/2}e^{-t/2}(1-e^{-2t})^{-1/2}\sqrt{\frac{\pi(1-e^{-2t})}{(1-e^{-t})^2}} = \frac{e^{-t/2}}{1-e^{-t}}. \end{aligned}$$

This completes the proof. $\square$

We now prove the following expression of the value of the zeta function $\zeta_Q(s) = \operatorname{Tr} Q^{-s}$ at $s = 2$.

Theorem 3.2. *With $\gamma = (\alpha\beta)^{-1/2}$ in (2.7), it holds that*

$$\begin{aligned} \sum_{n=1}^\infty \frac{1}{\lambda_n^2} = \operatorname{Tr} Q^{-2} = \frac{1}{1-\gamma^2}\Big[&\frac{(\alpha^{-1}+\beta^{-1})^2}{2}(2^2-1)\zeta(2) \\ &+ \frac{(\alpha^{-1}-\beta^{-1})^2}{2}\int_0^\infty\int_0^\infty F(t,s;-\gamma(1-\gamma^2)^{-1/2}))dtds\Big], \end{aligned} \tag{3.2}$$

where the function $F(t,s;a)$ is given by

$$F(t,s;a) = \frac{e^{-(t+s)/2}}{\sqrt{(1-e^{-(t+s)})^2 + a^2(1-e^{-2t})(1-e^{-2s})}}. \tag{3.3}$$

Proof. We start with the expression connected with Mehler's formula (2.3b). We first note that $Q^{-2} = (A^{-1/2}Q'^{-1}A^{-1/2})^2$. Let $Q^{-2}(x,y)$ be the integral kernel of Q^{-2}. Then we see that

$$\begin{aligned} &Q^{-2}(x,x) \\ &= \int dy\Big[A^{-1}\cos\frac{\gamma(x^2-y^2)}{2} + \gamma J\sin\frac{\gamma(x^2-y^2)}{2}\Big]\Big[A^{-1}\cos\frac{\gamma(y^2-x^2)}{2} + \gamma J\sin\frac{\gamma(y^2-x^2)}{2}\Big] \\ &\qquad \times \int_0^\infty (1-\gamma^2)^{1/4}p((1-\gamma^2)^{1/2}t, (1-\gamma^2)^{1/4}x, (1-\gamma^2)^{1/4}y)dt \\ &\qquad \times \int_0^\infty (1-\gamma^2)^{1/4}p((1-\gamma^2)^{1/2}s, (1-\gamma^2)^{1/4}y, (1-\gamma^2)^{1/4}x)ds \\ &= \int dy\Big[A^{-2}\cos\frac{\gamma(x^2-y^2)}{2}\cos\frac{\gamma(y^2-x^2)}{2} - \gamma^2\sin\frac{\gamma(x^2-y^2)}{2}\sin\frac{\gamma(y^2-x^2)}{2} \\ &\quad + \gamma\Big(A^{-1}J\cos\frac{\gamma(x^2-y^2)}{2}\sin\frac{\gamma(y^2-x^2)}{2} + JA^{-1}\sin\frac{\gamma(x^2-y^2)}{2}\cos\frac{\gamma(y^2-x^2)}{2}\Big)\Big] \\ &\qquad \times \int_0^\infty (1-\gamma^2)^{1/4}p((1-\gamma^2)^{1/2}t, (1-\gamma^2)^{1/4}x, (1-\gamma^2)^{1/4}y)dt \\ &\qquad \times \int_0^\infty (1-\gamma^2)^{1/4}p((1-\gamma^2)^{1/2}s, (1-\gamma^2)^{1/4}y, (1-\gamma^2)^{1/4}x)ds. \end{aligned} \tag{3.4}$$

Therefore we obtain

$$\begin{aligned}\operatorname{Tr} Q^{-2} =&(1-\gamma^2)^{1/2}\int dx\int dy\Big[\operatorname{tr} A^{-2}\cos^2\frac{\gamma(x^2-y^2)}{2}+2\gamma^2\sin^2\frac{\gamma(x^2-y^2)}{2}\Big]\\ &\times\int_0^\infty p((1-\gamma^2)^{1/2}t,(1-\gamma^2)^{1/4}x,(1-\gamma^2)^{1/4}y)dt\\ &\qquad\times\int_0^\infty p((1-\gamma^2)^{1/2}s,(1-\gamma^2)^{1/4}y,1-\gamma^2)^{1/4}x)ds.\end{aligned}\tag{3.5}$$

For the first factor in the integrand on the right of (3.5) we have

$$\begin{aligned}&\operatorname{tr} A^{-2}\cos^2\frac{\gamma(x^2-y^2)}{2}+2\gamma^2\sin^2\frac{\gamma(x^2-y^2)}{2}\\ &=\frac{\alpha^{-2}+\beta^{-2}+2\gamma^2}{2}+\frac{\alpha^{-2}+\beta^{-2}-2\gamma^2}{2}\cos\gamma(x^2-y^2)\\ &=\frac{\alpha^{-2}+\beta^{-2}+2\gamma^2}{2}+\frac{\alpha^{-2}+\beta^{-2}-2\gamma^2}{2}\frac{e^{i\gamma(x^2-y^2)}+e^{-i\gamma(x^2-y^2)}}{2}\\ &=\frac{(\alpha^{-1}+\beta^{-1})^2}{2}+\frac{(\alpha^{-1}-\beta^{-1})^2}{2}\frac{e^{i\gamma(x^2-y^2)}+e^{-i\gamma(x^2-y^2)}}{2}.\end{aligned}$$

Therefore

$$\begin{aligned}\operatorname{Tr} Q^{-2} =&(1-\gamma^2)^{1/2}\int_0^\infty dt\int_0^\infty ds\\ &\times\int\int dxdy\Big[\frac{(\alpha^{-1}+\beta^{-1})^2}{2}+\frac{(\alpha^{-1}-\beta^{-1})^2}{2}\frac{e^{i\gamma(x^2-y^2)}+e^{-i\gamma(x^2-y^2)}}{2}\Big]\\ &\times p((1-\gamma^2)^{1/2}t,(1-\gamma^2)^{1/4}x,(1-\gamma^2)^{1/4}y)p((1-\gamma^2)^{1/2}s,(1-\gamma^2)^{1/4}y,(1-\gamma^2)^{1/4}x)\\ =&(1-\gamma^2)^{-1}\int_0^\infty dt\int_0^\infty ds\int\int dxdy\,p(t,x,y)p(s,y,x)\\ &\times\Big[\frac{(\alpha^{-1}+\beta^{-1})^2}{2}+\frac{(\alpha^{-1}-\beta^{-1})^2}{2}\cdot\frac{e^{i\gamma(1-\gamma^2)^{-1/2}(x^2-y^2)}+e^{-i\gamma(1-\gamma^2)^{-1/2}(x^2-y^2)}}{2}\Big],\end{aligned}\tag{3.6}$$

where in the last equality we have made the change of variables: $t'=(1-\gamma^2)^{1/2}t$, $s'=(1-\gamma^2)^{1/2}s$, $x'=(1-\gamma^2)^{1/4}x$, $y'=(1-\gamma^2)^{1/4}y$, and then rewritten t', s', x', y' as t, s, x, y again.

Put

$$F(t,s;a):=\int\int e^{ia(x^2-y^2)}p(t,x,y)p(s,x,y)dxdy.\tag{3.7}$$

We need to calculate the terms $F(t,s;a)$ in (3.6) for $a=0$, $\gamma(1-\gamma^2)^{-1/2}$ and $-\gamma(1-\gamma^2)^{-1/2}$. First we observe that

$$\begin{aligned}&p(t,x,y)p(s,y,x)\\ =&\pi^{-1}e^{-(t+s)/2}(1-e^{-2t})^{-1/2}(1-e^{-2s})^{-1/2}\\ &\times\exp\Big[-\frac{(1+e^{-2t})(x^2+y^2)-4e^{-t}xy}{2(1-e^{-2t})}\Big]\exp\Big[-\frac{(1+e^{-2s})(y^2+x^2)-4e^{-s}xy}{2(1-e^{-2s})}\Big]\\ =&\pi^{-1}e^{-(t+s)/2}(1-e^{-2t})^{-1/2}(1-e^{-2s})^{-1/2}\\ &\times\exp\Big[-\frac{(1-e^{-2(t+s)})(x^2+y^2)-2(e^{-t}+e^{-s})(1-e^{-(t+s)})xy}{(1-e^{-2t})(1-e^{-2s})}\Big].\end{aligned}\tag{3.8}$$

Hence we obtain

$$F(t,s;a)=\pi^{-1}e^{-(t+s)/2}(1-e^{-2t})^{-1/2}(1-e^{-2s})^{-1/2}\int\int F(t,s;x,y;a)dxdy,\tag{3.9a}$$

where

$$F(t,s;x,y;a) = e^{ia(x^2-y^2)} \exp\Big[-\frac{(1-e^{-2(t+s)})(x^2+y^2) - 2(e^{-t}+e^{-s})(1-e^{-(t+s)})xy}{(1-e^{-2t})(1-e^{-2s})}\Big]. \tag{3.9b}$$

We have hence

$$\begin{aligned}
&F(t,s;x,y;a)\\
=&\exp\Big[-\Big(\frac{(1-e^{-2(t+s)})}{(1-e^{-2t})(1-e^{-2s})}+ia\Big)y^2\Big]\\
&\times\exp\Bigg[-\Big(\frac{(1-e^{-2(t+s)})}{(1-e^{-2t})(1-e^{-2s})}-ia\Big)\Big(x-\frac{\frac{(e^{-t}+e^{-s})(1-e^{-(t+s)})}{(1-e^{-2t})(1-e^{-2s})}}{\Big(\frac{(1-e^{-2(t+s)})}{(1-e^{-2t})(1-e^{-2s})}-ia\Big)}y\Big)^2+\frac{\frac{(e^{-t}+e^{-s})^2(1-e^{-(t+s)})^2}{(1-e^{-2t})^2(1-e^{-2s})^2}}{\Big(\frac{(1-e^{-2(t+s)})}{(1-e^{-2t})(1-e^{-2s})}-ia\Big)}y^2\Bigg]\\
=&\exp\Big[-\Big(\frac{(1-e^{-2(t+s)})}{(1-e^{-2t})(1-e^{-2s})}-ia\Big)\Big(x-\frac{(e^{-t}+e^{-s})(1-e^{-(t+s)})}{(1-e^{-2(t+s)})-ia(1-e^{-2t})(1-e^{-2s})}y\Big)^2\Big]\\
&\quad\times\exp\Bigg[-\Big(\frac{(1-e^{-2(t+s)})}{(1-e^{-2t})(1-e^{-2s})}+ia-\frac{\frac{(e^{-t}+e^{-s})^2(1-e^{-(t+s)})^2}{(1-e^{-2t})^2(1-e^{-2s})^2}}{\Big(\frac{(1-e^{-2(t+s)})}{(1-e^{-2t})(1-e^{-2s})}-ia\Big)}\Big)y^2\Bigg]\\
=&\exp\Big[-\Big(\frac{(1-e^{-2(t+s)})}{(1-e^{-2t})(1-e^{-2s})}-ia\Big)\Big(x-\frac{(e^{-t}+e^{-s})(1-e^{-(t+s)})}{(1-e^{-2(t+s)})-ia(1-e^{-2t})(1-e^{-2s})}y\Big)^2\Big]\\
&\quad\times\exp\Big[-\frac{(1-e^{-2(t+s)})^2+a^2(1-e^{-2t})^2(1-e^{-2s})^2-(e^{-t}+e^{-s})^2(1-e^{-(t+s)})^2}{\Big((1-e^{-(t+s)})-ia(1-e^{-2t})(1-e^{-2s})\Big)(1-e^{-2t})(1-e^{-2s})}y^2\Big].
\end{aligned} \tag{3.10}$$

Here we see that the power of the second exponential of the last member of (3.10) is equal to

$$\begin{aligned}
&\frac{(1-e^{-(t+s)})^2(1-e^{-2t})(1-e^{-2s})+a^2(1-e^{-2t})^2(1-e^{-2s})^2}{\Big((1-e^{-(t+s)})-ia(1-e^{-2t})(1-e^{-2s})\Big)(1-e^{-2t})(1-e^{-2s})}\\
=&\frac{(1-e^{-(t+s)})^2+a^2(1-e^{-2t})(1-e^{-2s})}{(1-e^{-(t+s)})-ia(1-e^{-2t})(1-e^{-2s})},
\end{aligned} \tag{3.11}$$

because for part of its nominator we have

$$\begin{aligned}
&(1-e^{-2(t+s)})^2-(e^{-t}+e^{-s})^2(1-e^{-(t+s)})^2\\
=&\Big((1-e^{-2(t+s)})+(e^{-t}+e^{-s})(1-e^{-(t+s)})\Big)\Big((1-e^{-2(t+s)})-(e^{-t}+e^{-s})(1-e^{-(t+s)})\Big)\\
=&(1-e^{-(t+s)})^2(1-e^{-2t})(1-e^{-2s})(\geq 0).
\end{aligned}$$

Now we calculate the Gaussian integration ($\int_{\mathbf{R}} e^{-ax^2}dx = \sqrt{\pi/a}$) to get

$$\begin{aligned}
&\int\int F(t,s;x,y;a)dxdy\\
=&\pi\sqrt{\frac{(1-e^{-2t})(1-e^{-2s})}{1-e^{-2(t+s)}-ia(1-e^{-2t})(1-e^{-2s})}}\sqrt{\frac{(1-e^{-(t+s)})-ia(1-e^{-2t})(1-e^{-2s})}{(1-e^{-(t+s)})^2+a^2(1-e^{-2t})(1-e^{-2s})}}\\
=&\pi\sqrt{\frac{(1-e^{-2t})(1-e^{-2s})}{(1-e^{-(t+s)})^2+a^2(1-e^{-2t})(1-e^{-2s})}}.
\end{aligned}$$

Putting all the above together, by (3.9ab) we have actually (3.3) as

$$\begin{aligned}
F(t,s;a) &= \pi^{-1}e^{-(t+s)/2}(1-e^{-2t})^{-1/2}(1-e^{-2s})^{-1/2}\int\int F(t,s;x,y;a)dxdy\\
&= e^{-(t+s)/2}\Big((1-e^{-(t+s)})^2+a^2(1-e^{-2t})(1-e^{-2s})\Big)^{-1/2}.
\end{aligned}$$

Thus we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{\lambda_n^2} &= \operatorname{Tr} Q^{-2}\\ &= \frac{1}{1-\gamma^2}\Big[\frac{(\alpha^{-1}+\beta^{-1})^2}{2}\int_0^\infty\int_0^\infty F(t,s;0)dtds\\ &+\frac{(\alpha^{-1}-\beta^{-1})^2}{2}\int_0^\infty\int_0^\infty \frac{1}{2}\big(F(t,s;\gamma(1-\gamma^2)^{-1/2})+F(t,s;-\gamma(1-\gamma^2)^{-1/2})\big)dtds\Big].\end{aligned}$$

Note that $F(t,s,;\gamma(1-\gamma^2)^{-1})=F(t,s;-\gamma(1-\gamma^2)^{-1})$. Also, when $a=0$ we have

$$F(t,s;0)=\frac{e^{-(t+s)/2}}{1-e^{-(t+s)}}.$$

It follows that

$$\begin{aligned}\int_0^\infty\int_0^\infty F(t,s;0)dtds &= \sum_{k=0}^{\infty}\int_0^\infty\int_0^\infty e^{-(k+1/2)(t+s)}dtds\\ &=\sum_{k=0}^{\infty}\frac{1}{(k+1/2)^2}=\pi^2/2=3\zeta(2).\end{aligned}\tag{3.12}$$

This yields the expression (3.2). Hence the theorem follows. □

In the same way, we may calculate, in principle, $\sum_{n=1}^{\infty}1/\lambda_n^k=\operatorname{Tr}\ Q^{-k}$ for a general $k>2$. For instance, for $k=3$, we will encounter with a Gaussian integral by the three variables x, y and z in §5. However, it is practically very complicated in general.

We will show through the following three steps that Tr Q^{-s} is finite and holomorphic in the region Re $s>1$. It is uniformly bounded in Re $s\geq 1+\varepsilon$, for each fixed $\varepsilon>0$.

(I). For every real $s>1$, Tr Q'^{-s} is finite.
(II). For every real $s>1$, Tr Q^{-s} is finite.
(III). For every complex number s with Re $s>1$, $\operatorname{Tr} Q^{-s}$ is finite and holomorphic there. It is uniformly bounded in Re $s\geq 1+\varepsilon$, for each fixed $\varepsilon>0$.

Proof of (I), (II) *and* (III).

(I). For every real $s>1$, Tr Q'^{-s} is finite. In fact, by the Mellin transform, we have

$$Q'^{-s}=\Gamma(s)^{-1}\int_0^\infty t^{s-1}e^{-tQ'}dt.\tag{3.13}$$

Therefore by use of the integral kernel $K'(t,x,y)$ of $e^{-tQ'}$ in (2.10a), the integral kernel $Q'^{-s}(x,y)$ of Q'^{-s} is given by

$$\begin{aligned}Q'^{-s}(x,y)=\ &\Gamma(s)^{-1}\int_0^\infty t^{s-1}\Big[\cos\frac{\gamma(x^2-y^2)}{2}+J\sin\frac{\gamma(x^2-y^2)}{2}\Big]\\ &\times(1-\gamma^2)^{1/4}p((1-\gamma^2)^{1/2}t,(1-\gamma^2)^{1/4}x,(1-\gamma^2)^{1/4}y)dt.\end{aligned}\tag{3.14}$$

Hence we see that

$$\begin{aligned}\operatorname{Tr} Q'^{-s} &=\frac{2}{\Gamma(s)}\int_0^\infty t^{s-1}dt\int(1-\gamma^2)^{1/4}p((1-\gamma^2)^{1/2}t,(1-\gamma^2)^{1/4}x,(1-\gamma^2)^{1/4}x)dx\\ &=\frac{2(1-\gamma^2)^{-s/2}}{\Gamma(s)}\int_0^\infty t^{s-1}dt\int p(t,x,x)dx,\end{aligned}\tag{3.15}$$

where in the last equality we have made the change of variables $t' = (1-\gamma^2)^{1/2}t$, $x' = (1-\gamma^2)^{1/4}x$ and rewritten t' as t and x' as x again.

Note that the number "2" at the top of the second and third members of (3.15) is arising from the trace of the 2×2 identity matrix. Using the integral formula described at the end of the proof of Proposition 3.1, we have

$$\begin{aligned}
\zeta_{Q'}(s) = \operatorname{Tr} Q'^{-s} &= \frac{2(1-\gamma^2)^{-s/2}}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{e^{-t/2}}{1-e^{-t}} dt \\
&= \frac{2(1-\gamma^2)^{-s/2}}{\Gamma(s)} \sum_{k=0}^\infty \int_0^\infty t^{s-1} e^{-(k+1/2)} dt \\
&= \frac{2(1-\gamma^2)^{-s/2}}{\Gamma(s)} \sum_{k=0}^\infty \frac{\Gamma(s)}{(k+1/2)^s} \\
&= 2(1-\gamma^2)^{-s/2}(2^s-1)\zeta(s),
\end{aligned} \tag{3.16}$$

where $\zeta(s)$ is the Riemann zeta function. Hence $\zeta_{Q'}(s) = \operatorname{Tr} Q'^{-s}$ is obviously finite for Re $s > 1$.

(II). For every real $s > 1$, Tr Q^{-s} is finite: To see this, note that $Q^{-1} = A^{-1/2}Q'^{-1}A^{-1/2}$ and recall the inequality of Lieb and Thirring; for two positive self-adjoint operators A and B, we have $\operatorname{Tr}(B^{1/2}AB^{1/2})^{kq} \le \operatorname{Tr}(B^{q/2}A^qB^{q/2})^k$, for $q \ge 1$ and $k > 0$. (See [LT] and for the general case, [Ar].) Let $s > 1$. Then, putting $q = s$ and $k = 1$ in the inequality we have

$$\begin{aligned}
\operatorname{Tr} Q^{-s} &= \operatorname{Tr}(A^{-1/2}Q'^{-1}A^{-1/2})^s \\
&\le \operatorname{Tr} A^{-s/2}Q'^{-s}A^{-s/2} \\
&= \operatorname{Tr} A^{-s}Q'^{-s} = \operatorname{Tr} Q'^{-s/2}A^{-s}Q'^{-s/2} \\
&\le (\min\{\alpha,\beta\})^{-s}\operatorname{Tr} Q'^{-s},
\end{aligned} \tag{3.17}$$

which is finite by the assertion (I) above.

(III). For every complex number s with Re $s > 1$, $\operatorname{Tr} Q^{-s}$ is finite and holomorphic there: In fact, since Q^{-1} is a positive (definite) bounded self-adjoint operator, by the spectral theorem we first note that Q^{-s} is holomorphic in s with $\sigma := \operatorname{Re} s > 1$. Since

$$|\operatorname{Tr} Q^{-s}| = |\sum_{n=1}^\infty \frac{1}{{\lambda_n}^s}| \le \sum_{n=1}^\infty \frac{1}{{\lambda_n}^\sigma},$$

$\operatorname{Tr} Q^{-s}$ is finite by (II). Hence it is clear that $\operatorname{Tr} Q^{-s}$ is holomorphic in Re $s > 1$ and uniformly bounded in Re $s \ge 1+\varepsilon$ for fixed $\varepsilon > 0$. This ends the proof of (I), (II) and (III). Thus we have the following basic result.

Theorem 3.3. *The spectral zeta function $\zeta_Q(s) = \operatorname{Tr} Q^{-s}$ is holomorphic in the half plane Re $s > 1$.* □

As a final remark of this section we try to express the integral $\int_0^\infty\int_0^\infty F(t,s,a)dtds$ in a different form using the elliptic integral (or a special type of the Gauss hypergeometric function). It is also interesting to find any relation between this integral and the singly confluent Heun equation (and its solution), which will be discussed in §5. We now express (3.3) as

$$F(t,s,a) = \frac{1}{2\sqrt{\sinh^2(\frac{t+s}{2}) + a^2\sinh t \sinh s}}. \tag{3.18}$$

Put $a = \sinh\mu$, so that $1 + a^2 = \cosh^2\mu$. Then one has

$$\begin{aligned}
F(t,s;a)^{-2} &= 4\cosh^2\mu \sinh^2(\frac{t+s}{2}) - \sinh^2\mu\sinh^2(\frac{t+s}{2}) \\
&= 4\big\{\cosh\mu\sinh(\frac{t+s}{2}) + \sinh\mu\sinh(\frac{t-s}{2})\big\}\big\{\cosh\mu\sinh(\frac{t+s}{2}) - \sinh\mu\sinh(\frac{t-s}{2})\big\}
\end{aligned}$$

Now we put

$$u = \cosh\mu \sinh(\frac{t+s}{2}), \quad v = \sinh\mu \sinh(\frac{t-s}{2}).$$

Notice that the Jacobian of this transform is given by

$$\left|\frac{\partial(u,v)}{\partial(t,s)}\right| = \frac{1}{2}\sqrt{\cosh^2\mu + u^2}\sqrt{\sinh^2\mu + v^2}.$$

We note that

$$\int_0^\infty\int_0^\infty F(t,s;a)dtds = 2\int\int_{\{(t,s);t\geq s\}} F(t,s;a)dtds = 2\int_0^\infty dt\int_0^t F(t,s;a)ds,$$

so that we have only to integrate over (u,v) with $u\geq 0,\ (\tanh\mu)u \geq v \geq 0$. Hence we have

$$\int_0^\infty\int_0^\infty F(t,s;a)dtds = 2\int_0^\infty du\int_0^{(\tanh\mu)u} dv\frac{1}{\sqrt{(u^2-v^2)(u^2+\cosh^2\mu)(v^2+\sinh^2\mu)}}.$$

Recall the elliptic integral of the first kind or the Jacobi elliptic function $\mathrm{cn}\, z$:

$$\mathrm{cn}^{-1}(x) = K(\arccos x, k) = \int_x^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2+k^2x^2)}}, \quad 0<x<1,$$

for $0<k^2<1$. Using this, if we put $v = ux\ (v\to x)$ and $k^2 = \frac{u^2}{\sinh^2\mu + u^2}$ we have

$$\begin{aligned}\int_0^{(\tanh\mu)u}\frac{dv}{\sqrt{(u^2-v^2)(v^2+\sinh^2\mu)}} &= \int_0^{(\tanh\mu)}\frac{dx}{\sqrt{(1-x^2)(1-k^2+k^2x^2)(u^2+\sinh^2\mu)}}\\ &=(\mathrm{cn}^{-1}(0)-\mathrm{cn}^{-1}((\tanh\mu)))(u^2+\sinh^2\mu)^{-1/2}.\end{aligned}$$

Here $\mathrm{cn}^{-1}(0)$ is the complete elliptic integral of the first kind and is given by the Gauss hypergeometric function ${}_2F_1$ (see e.g. §3.2 in [AAR]):

$$\mathrm{cn}^{-1}(0) = \frac{\pi}{2}{}_2F_1(\frac{1}{2},\frac{1}{2},1;k^2) = K(\frac{\pi}{2},k) = \int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}.$$

Thus, as a function of a^2, we obtain the expressions

$$\begin{aligned}\int_0^\infty\int_0^\infty F(t,s;a)dtds &= 2\int_0^\infty \frac{\mathrm{cn}^{-1}(0)-\mathrm{cn}^{-1}((\tanh\mu))}{\sqrt{(u^2+\sinh^2\mu)(u^2+\cosh^2\mu)}}du\\ &= 2\int_0^\infty \frac{K(\frac{\pi}{2},\frac{u}{(u^2+\sinh^2\mu)^{1/2}}) - K(\arccos\frac{a}{\sqrt{1+a^2}},\frac{u}{(u^2+\sinh^2\mu)^{1/2}})}{\sqrt{(u^2+\sinh^2\mu)(u^2+\cosh^2\mu)}}du \qquad (3.19)\\ &= 2\int_0^\infty \frac{K(\frac{\pi}{2},\frac{u}{(u^2+a^2)^{1/2}}) - K(\arccos\frac{a}{\sqrt{1+a^2}},\frac{u}{(u^2+a^2)^{1/2}})}{\sqrt{(u^2+a^2)(u^2+1+a^2)}}du.\end{aligned}$$

Using the definition of $K(\frac{\pi}{2},k)$ above, we have also (putting $u = a\tan\varphi$)

$$\begin{aligned}\int_0^\infty\int_0^\infty F(t,s;a)dtds &= 2\int_0^\infty du\int_{\arccos\frac{a}{\sqrt{1+a^2}}}^{\pi/2}\frac{d\theta}{\sqrt{1-\frac{u^2\sin^2\theta}{u^2+a^2}}\sqrt{(u^2+a^2)(u^2+a^2+1)}}\\ &= 2\int_0^{\pi/2}d\varphi\int_{\arcsin\frac{1}{\sqrt{1+a^2}}}^{\pi/2}\frac{d\theta}{\sqrt{(1-\sin^2\varphi\sin^2\theta)(a^2+\cos^2\varphi)}} \qquad (3.20)\\ &\Big(\text{or} = 2\int_0^{\pi/2}d\varphi\int_{\arcsin\frac{1}{\sqrt{1+a^2}}}^{\pi/2}\frac{d\theta}{\sqrt{(1-\cos^2\varphi\sin^2\theta)(a^2+\sin^2\varphi)}}\Big).\end{aligned}$$

4. $\zeta_Q(2)$ and a confluent Heun differential equation

We show in this section that the value $\zeta_Q(2)$ is closely related to a solution of some confluent Heun's differential equation (see e.g., [WW], [R], [SL]).

Recall the expression of $\zeta_Q(2)$ obtained in Theorem 3.2.

$$\begin{aligned}\zeta_Q(2) &= \frac{(\alpha^{-1}+\beta^{-1})^2}{2(1-\gamma^2)}(2^2-1)\zeta(2) + \frac{(\alpha^{-1}-\beta^{-1})^2}{2(1-\gamma^2)}\int_0^\infty\int_0^\infty F(t,s;a)dtds, \\ F(t,s;a) &= \frac{e^{-(t+s)/2}}{\sqrt{(1-e^{-(t+s)})^2 + a^2(1-e^{-2t})(1-e^{-2s})}},\end{aligned} \tag{4.1}$$

where $a = \gamma(1-\gamma^2)^{-1/2}$ and $\gamma = (\alpha\beta)^{-1/2}$. Expand $F(t,s;a)$ into the Taylor series with respect to the variable a^2:

$$F(t,s;a) = F(t,s;0) + F'(t,s;0)a^2 + \frac{1}{2!}F''(t,s;0)a^4 + \cdots + \frac{1}{n!}F^{(n)}(t,s;0)a^{2n} + \cdots. \tag{4.2}$$

Here we have

$$\begin{aligned}&F^{(n)}(t,s;a) := \Big(\frac{d}{d\,a^2}\Big)^n F(t,s;a) \\ =&(-1)^n\frac{(2n-1)\cdot(2n-3)\cdots 3\cdot 1}{2^n}\frac{e^{-(t+s)/2}(1-e^{-2t})^n(1-e^{-2s})^n}{[(1-e^{-(t+s)})^2 + a^2(1-e^{-2t})(1-e^{-2s})]^{n+1/2}},\end{aligned} \tag{4.3}$$

and hence, putting $a^2=0$, we have

$$F^{(n)}(t,s;0) = (-1)^n\frac{(2n-1)!!}{2^n}\frac{e^{-(t+s)/2}(1-e^{-2t})^n(1-e^{-2s})^n}{(1-e^{-(t+s)})^{2n+1}}. \tag{4.4}$$

Since $(1-e^{-2t})(1-e^{-2s}) < (1-e^{-(t+s)})$ for $s,t>0$, the Taylor expansion (4.2) above converges absolutely when $a^2<1$.

Therefore, by Theorem 3.2 we have

$$\zeta_Q(2) := Z_1(2) + Z_0'(2) + Z_1'(2) + \cdots + Z_n'(2) + \cdots, \tag{4.5a}$$

$$Z_1(2) = \frac{(\alpha^{-1}+\beta^{-1})^2}{2(1-\gamma^2)}(2^2-1)\zeta(2), \tag{4.5b}$$

$$Z_n'(2) = \frac{(\alpha^{-1}-\beta^{-1})^2}{2(1-\gamma^2)}\frac{a^{2n}}{n!}\int_0^\infty\int_0^\infty F^{(n)}(t,s;0)dtds, \quad n=0,1,2,\ldots. \tag{4.5c}$$

We now evaluate the first two of the $Z_n'(2)$.

Lemma 4.1. *One has*

$$\begin{aligned}Z_0'(2) &= \frac{(\alpha^{-1}-\beta^{-1})^2}{2(1-\gamma^2)}(2^2-1)\zeta(2), \\ Z_1'(2) &= -\frac{(\alpha^{-1}-\beta^{-1})^2}{2(1-\gamma^2)}\frac{1}{2!}\frac{1}{2}\frac{9}{4}\zeta(2)a^2, \qquad a^2 = \frac{\gamma^2}{1-\gamma^2} = \frac{1}{\alpha\beta-1}.\end{aligned}$$

Proof. For $Z_0'(2)$ we have from (3.12)

$$Z_0'(2) = \frac{(\alpha^{-1}-\beta^{-1})^2}{2(1-\gamma^2)}\int_0^\infty\int_0^\infty F(t,s;0)dtds = \frac{(\alpha^{-1}-\beta^{-1})^2}{2(1-\gamma^2)}(2^2-1)\zeta(2).$$

To evaluate the value $Z_1'(2)$, we need to calculate

$$\begin{aligned}\int_0^\infty\int_0^\infty F'(t,s;0)dtds &= -\frac{1}{2}\int_0^\infty\int_0^\infty \frac{e^{-(t+s)/2}(1-e^{-2t})(1-e^{-2s})}{(1-e^{-(t+s)})^3}dtds \\ &= -\frac{1}{2}\int_0^\infty\int_0^\infty e^{-(t+s)/2}(1-e^{-2t})(1-e^{-2s})\sum_{k,l,m=0}^\infty e^{-(k+l+m)(t+s)}dtds \\ &= -\frac{1}{2}\cdot 4\cdot \sum_{k,l,m=0}^\infty \frac{1}{(k+l+m+1/2)^2(k+l+m+5/2)^2} \\ &= -2S,\end{aligned} \tag{4.6}$$

where we put

$$S := \sum_{k,\ell,m=0}^\infty \frac{1}{(k+\ell+m+1/2)^2(k+\ell+m+5/2)^2}.$$

To evaluate the sum S, for $N \geq 0$, we consider

$$p_3(N) = \#\{(k,\ell,m)\in\mathbb{Z}\,;\ k,\ell,m\geq 0, k+\ell+m=N\}.$$

It is easy to see that

$$p_3(N) = \binom{N+2}{N} = \frac{(N+1)(N+2)}{2}.$$

Hence we have

$$S = \frac{1}{2}\sum_{N\geq 0}\frac{(N+1)(N+2)}{(N+1/2)^2(N+5/2)^2} = 8\sum_{N\geq 0}\frac{N^2+3N+2}{(2N+1)^2(2N+5)^2}.$$

Note here that

$$(2N+1)^2+(2N+5)^2 = 8N^2+24N+26 = 8(N^2+3N+2)+10.$$

Hence it follows that

$$\begin{aligned}S &= \sum_{N\geq 0}\frac{(2N+1)^2+(2N+5)^2-10}{(2N+1)^2(2N+5)^2} \\ &= \sum_{N\geq 0}\frac{1}{(2N+1)^2}+\sum_{N\geq 0}\frac{1}{(2N+5)^2}-\sum_{N\geq 0}\frac{10}{(2N+1)^2(2N+5)^2}.\end{aligned} \tag{4.7}$$

We now use the following elementary partial fractional expansion:

$$\frac{4}{(z+a)^2(z+a+2)^2} = \frac{1}{(z+a)^2}+\frac{1}{(z+a+2)^2}-\left\{\frac{1}{z+a}-\frac{1}{z+a+1}+\frac{1}{z+a+1}-\frac{1}{z+a+2}\right\}.$$

Then it is easy to see that

$$\sum_{a\geq 0}\frac{4}{(z+a)^2(z+a+2)^2} = \sum_{a\geq 0}\frac{1}{(z+a)^2}+\sum_{a\geq 0}\frac{1}{(z+a+2)^2}-\left(\frac{1}{z}+\frac{1}{z+1}\right).$$

Therefore

$$\begin{aligned}\sum_{N\geq 0}\frac{10}{(2N+1)^2(2N+5)^2} &= \frac{10}{16}\cdot\frac{1}{4}\left[\sum_{N\geq 0}\frac{1}{(N+1/2)^2}+\sum_{N\geq 0}\frac{1}{(N+5/2)^2}-\left(2+\frac{2}{3}\right)\right] \\ &= \frac{10}{16}\left[\sum_{N\geq 0}\frac{1}{(2N+1)^2}+\sum_{N\geq 0}\frac{1}{(2N+5)^2}-\frac{2}{3}\right].\end{aligned}$$

It follows from (4.7) that

$$S = \frac{6}{16}\left[\sum_{N\geq 0}\frac{1}{(2N+1)^2} + \sum_{N\geq 0}\frac{1}{(2N+5)^2} + \frac{10}{6}\cdot\frac{2}{3}\right].$$

Since

$$\sum_{N\geq 0}\frac{1}{(2N+1)^s} = (1-2^{-s})\zeta(s),$$

we have

$$S = \frac{6}{16}\left\{\left(1-\frac{1}{4}\right)\zeta(2)\times 2 - 1 - \frac{1}{3^2} + \frac{10}{9}\right\} = \frac{3^2}{4^2}\zeta(2).$$

Hence we get immediately the desired expression of $Z_1'(2)$ from (4.5c). This completes the proof of the lemma. $\square$

More generally, in order to know the values $Z_n'(2)$, we try to find recurrence relations satisfied by the integrals $\int_0^\infty\int_0^\infty F^{(n)}(t,s;0)dtds$.

Let

$$J_n := \int_0^\infty\int_0^\infty \frac{e^{-\frac{t+s}{2}}(1-e^{-2t})^n(1-e^{-2s})^n}{(1-e^{-(t+s)})^{2n+1}}dtds. \tag{4.8}$$

Then it is clear from (4.4) and (4.5c) that

$$\begin{aligned} Z_n'(2) &= (-1)^n\frac{(\alpha^{-1}-\beta^{-1})^2}{2(1-\gamma^2)}\frac{(2n-1)!!}{2^n}\frac{a^{2n}}{n!}J_n \\ &= (-1)^n\frac{(\alpha^{-1}-\beta^{-1})^2}{(1-\gamma^2)}\binom{2n-1}{n}\left(\frac{a}{2}\right)^{2n}J_n \end{aligned} \tag{4.9}$$

and

$$J_0 = 3\zeta(2) = \frac{\pi^2}{2}, \quad J_1 = \frac{9}{4}\zeta(2) = \frac{3\pi^2}{8}. \tag{4.10}$$

From the definition it is immediate to see that

$$J_n = \int_0^\infty \frac{e^{-\frac{u}{2}}}{(1-e^{-u})^{2n+1}}\left\{\int_0^u (1-e^{-2t})^n(1-e^{-2u+2t})^n dt\right\}du.$$

Define the function $I_{n,m}(u)$ by the integral

$$I_{n,m} = I_{n,m}(u) := \int_0^u (1-e^{-2t})^n(1-e^{-2u+2t})^m dt. \tag{4.11}$$

Then it is clear that

$$I_{n,m}(u) = I_{m,n}(u), \tag{4.12}$$

and

$$J_n = \int_0^\infty \frac{e^{-\frac{u}{2}}}{(1-e^{-u})^{2n+1}}I_{n,n}(u)du. \tag{4.13}$$

In order to evaluate the integral J_n, we establish a certain recursion formula among $I_{n,n}(u)$. As an intermediate result we have the following proposition.

Proposition 4.2. *Suppose $n\geq 2$. Then*

$$I_{n,n} = (e^{-2u}-1)I_{n-1,n-1} + 2I_{n,n-1}, \tag{4.14}$$

$$nI_{n,n-1} = (n-1)(e^{-2u}-1)I_{n-1,n-2} + (2n-1)I_{n-1,n-1}. \tag{4.15}$$

To prove the proposition we establish a series of lemmas. First we begin with the following recursion formula.

Lemma 4.3. *Suppose that $n, m \geq 1$. Then*

$$(n-m)I_{n,m} = nI_{n-1,m} - mI_{n,m-1}. \tag{4.16}$$

Proof. Integration by parts yields

$$\begin{aligned} I_{n,m} &= I_{n-1,m} - \int_0^u e^{-2t}(1-e^{-2t})^{n-1}(1-e^{-2u+2t})^m dt \\ &= I_{n-1,m} - \int_0^u \frac{1}{2n}\left(\frac{d}{dt}(1-e^{-2t})^n\right)(1-e^{-2u+2t})^m dt \\ &= I_{n-1,m} - \frac{m}{n}\int_0^u (1-e^{-2t})^n e^{-2u+2t}(1-e^{-2u+2t})^{m-1} dt. \end{aligned}$$

Hence we have

$$nI_{n,m} = nI_{n-1,m} - me^{-2u}\int_0^u (1-e^{-2t})^n e^{2t}(1-e^{-2u+2t})^{m-1} dt. \tag{4.17}$$

On the other hand, it is immediate to see that

$$I_{n,m} = I_{n,m-1} - e^{-2u}\int_0^u (1-e^{-2t})^n e^{2t}(1-e^{-2u+2t})^{m-1} dt. \tag{4.18}$$

The desired assertion follows by subtracting (4.18)$\times m$ from (4.17). $\square$

Lemma 4.4. *Suppose $n \geq 1, m \geq 0$. Then*

$$(m+1)e^{-2u}I_{n,m} = (n+1)I_{n+1,m+1} - (2n+1)I_{n,m+1} + nI_{n-1,m+1}. \tag{4.19}$$

Proof. Note first that

$$(1-e^{-2t})^n = -e^{2t}(1-e^{-2t})^{n+1} + e^{2t}(1-e^{-2t})^n.$$

It follows that

$$I_{n,m} = \int_0^u (1-e^{-2t})^n e^{2t}(1-e^{-2u+2t})^m dt - (1-e^{-2t})^{n+1}e^{2t}(1-e^{-2u+2t})^m dt.$$

Since

$$\frac{d}{dt}(1-e^{-2u+2t})^{m+1} = -2(m+1)e^{-2u}e^{2t}(1-e^{-2u+2t})^m,$$

integration by parts again yields

$$\begin{aligned} -2(m+1)e^{-2u}I_{n,m} &= -2n\int_0^u e^{-2t}(1-e^{-2t})^{n-1}(1-e^{-2u+2t})^{m+1} dt \\ &\quad + 2(n+1)\int_0^u e^{-2t}(1-e^{-2t})^n(1-e^{-2u+2t})^{m+1} dt. \end{aligned}$$

Since

$$I_{n,m} = I_{n-1,m} - \int_0^u e^{-2t}(1-e^{-2t})^n(1-e^{-2u+2t})^m dt$$

we have

$$-2(m+1)e^{-2u}I_{n,m} = -2n\{I_{n-1,m+1} - I_{n,m+1}\} + 2(n+1)\{I_{n,m+1} - I_{n+1,m+1}\}.$$

Hence we get the desired equation immediately. $\square$

Putting $n = m$ and replacing n by $n-1$ we have the following

Corollary 4.5. *For $n \geq 2$, we have*

$$ne^{-2u}I_{n-1,n-1} = nI_{n,n} - (2n-1)I_{n-1,n} + (n-1)I_{n-2,n}. \tag{4.20}$$

Furthermore, we have the following

Lemma 4.6. *Suppose $n \geq 2, m \geq 1$. Then*

$$nI_{n,m} = (2n-1)I_{n-1,m} - (n-1)I_{n-2,m} + e^{-2u} m I_{n-1,m-1}. \tag{4.21}$$

In particular, one has

$$nI_{n,n} = (2n-1)I_{n-1,n} - (n-1)I_{n-2,n} + e^{-2u} n I_{n-1,n-1}. \tag{4.22}$$

Proof. Using twice (4.17) we obtain

$$\begin{aligned}
-\frac{n}{m}(I_{n,m} - I_{n-1,m}) &= e^{-2u}\int_0^u (1-e^{-2t})^n e^{2t}(1-e^{-2u+2t})^{m-1}dt \\
&= e^{-2u}\int_0^u (1-e^{-2t})^{n-1} e^{2t}(1-e^{-2u+2t})^{m-1}dt - e^{-2u}I_{n-1,m-1} \\
&= -\frac{n-1}{m}(I_{n-1,m} - I_{n-2,m}) - e^{-2u}I_{n-1,m-1}.
\end{aligned}$$

Hence the formula (4.21) follows. This proves the lemma. □

Proof of Proposition 4.2. Since $I_{n,m} = I_{m,n}$, by putting $m = n-1$ in (4.21), the formula (4.15) immediately follows.

If we put $m = n-1$ in (4.16) then we have

$$I_{n,n-1} = nI_{n-1,n-1} - (n-1)I_{n,n-2}, \tag{4.23a}$$

in other words,

$$I_{n,n-1} + (n-1)I_{n,n-2} = nI_{n-1,n-1}. \tag{4.23b}$$

Plugging this (4.23b) into the right-hand side of (4.22) we see that

$$nI_{n,n} = 2nI_{n-1,n} - nI_{n-1,n-1} + e^{-2u}nI_{n-1,n-1}.$$

It follows hence that, for $n \geq 2$,

$$I_{n,n} = (e^{-2u}-1)I_{n-1,n-1} + 2I_{n,n-1}. \tag{4.24}$$

Using (4.23b), we can express (4.24) as

$$I_{n,n} = (e^{-2u}-1)I_{n-1,n-1} + 2nI_{n-1,n-1} - 2(n-1)I_{n,n-2}. \tag{4.25}$$

Summing up the expressions (4.24) and (4.25) and dividing by 2 we conclude that

$$I_{n,n} = (e^{-2u}-1+n)I_{n-1,n-1} + I_{n,n-1} - (n-1)I_{n,n-2}. \tag{4.26}$$

Notice now that (4.22) can be rewritten as

$$e^{-2u}nI_{n-1,n-1} = nI_{n,n} - (2n-1)I_{n-1,n} + (n-1)I_{n-2,n}. \tag{4.27}$$

Adding (4.26) and (4.27) to kill $I_{n-2,n}$ we have

$$I_{n,n} + e^{-2u}nI_{n-1,n-1} = nI_{n,n} + (e^{-2u}-1+n)I_{n-1,n-1} - 2(n-1)I_{n,n-1}.$$

Hence it follows that

$$(n-1)I_{n,n} = (n-1)(e^{-2u}-1)I_{n-1,n-1} + 2(n-1)I_{n,n-1}.$$

This proves (4.14) and completes the proof of the proposition. □

Lemma 4.7. *The functions $I_{n,m}(u)$ satisfy the difference-differential equations*

$$\frac{d}{du}I_{n,m} = 2n(I_{n-1,m} - I_{n,m}). \tag{4.28}$$

In particular, one has

$$\frac{d}{du}I_{n,n} = 2n(I_{n-1,n} - I_{n,n}). \tag{4.29}$$

Proof. Differentiating with respect to the variable u we easily see that

$$\frac{d}{du}I_{n,m} = 2m\int_0^u (1-e^{-2t})^n e^{-2u}e^{2t}(1-e^{-2u+2t})^{m-1}dt.$$

Hence by (4.17) we obtain

$$\frac{d}{du}I_{n,m} = 2(nI_{n-1,m} - nI_{n,m}),$$

whence (4.28) follows. This proves the lemma. □

For simplicity we put now

$$a_n(u) = I_{n,n}(u), \qquad b_n(u) = I_{n,n-1}(u).$$

Then the equations (4.14) and (4.15) are respectively written as follows:

$$\begin{cases} a_n(u) &= (e^{-2u}-1)a_{n-1} + 2b_n(u) \\ nb_n(u) &= (2n-1)a_{n-1}(u) + (n-1)(e^{-2u}-1)b_{n-1}(u). \end{cases}$$

From these relations we can eliminate $b_n(u)$'s. In fact, since

$$2b_n(u) = a_n(u) - (e^{-2u}-1)a_{n-1}, \tag{4.30}$$

we substitute the corresponding expressions for $2b_n(u)$ and $2b_{n-1}(u)$ into the second formula above to have the following relation.

Lemma 4.8. *Let $a_n(u) = I_{n,n}(u)$. Then the recurrence relation*

$$na_n(u) = (2n-1)(e^{-2u}+1)a_{n-1}(u) - (n-1)(e^{-2u}-1)^2 a_{n-2}(u) \quad (n \geq 2). \tag{4.31}$$

holds. □

It is also clear from (4.29) and (4.30) to see the following

Lemma 4.9. *The difference-differential equation*

$$a_n'(u) = -n\{a_n(u) + (e^{-2u}-1)a_{n-1}(u)\} \quad (n \geq 2). \tag{4.32}$$

holds. □

We now list the first few terms of $a_n(u)$:

$$\begin{aligned} a_0(u) &= u, \\ a_1(u) &= u(e^{-2u}+1) + (e^{-2u}-1), \\ a_2(u) &= u(e^{-4u}+4e^{-2u}+1) + \frac{3}{2}(e^{-4u}-1), \\ \cdots \quad & \cdots\cdots\cdots \end{aligned}$$

It is easy to check directly that the orders of the zero of $a_1(u)$ and $a_2(u)$ at $u=0$ are equal to 3 and 5 respectively. Therefore, by successive use of the equations (3.31) and (3.32) above we have the following assertion by induction on n.

Proposition 4.10. *Suppose $n \geq 0$. Then the function $a_n(u)$ has a zero of order $2n+1$ at $u=0$. In particular, the definite integral J_n converges.*

Proof. We show the first assertion by induction. Suppose it holds for k less than n. Then by (4.31) we see that $a_n(u)$ has a zero of order $2n-1$ at $u=0$; $a_n(0)=a_n'(u)=\cdots=a_n^{(2n-2)}(u)=0$. It follows that $a_n'(u)$ has a zero of order $2n-1$ by (4.32). Namely we have

$$a_n^{(2n-1)}(u)=\frac{d^{(2n-2)}}{du^{(2n-2)}}a_n'(u)=0.$$

Now differentiating (4.32) again we have

$$a_n''(u)=-n\{a_n'(u)+(e^{-2u}-1)a_{n-1}'(u)-2e^{-2u}a_{n-1}(u)\}.$$

This equation shows that $a_n''(u)$ has a zero of order $2n-1$ at 0, that is, one has $a_n^{(2n)}(0)=0$. This proves the first assertion of the proposition.

In particular then, the integrand in (4.13) is continuous at $u=0$. Further, either by induction using (4.31) or by the integral representation of $a_n(u)=I_{n,n}(u)$ using the binomial expansion, we easily verify that

$$a_n(u)\in\mathbb{R}[u,p^2] \qquad (p:=e^{-u})$$

and the degree of $a_n(u)$ as a polynomial in p (as an "independent variable" of u) is $2n$. Hence it is obvious that the integral J_n in (4.8) converges since the denominator of the integrand in (4.13) is of degree $2n+1$ with respect to p. Hence the second assertion follows. This completes the proof of the proposition. □

To perform the calculation further we introduce a new function $A_n(u)$ by

$$A_n(u)=e^{nu}a_n(u).$$

Then the equations (4.31) and (4.32) respectively read as follows:

$$nA_n(u)=2(2n-1)\cosh u\,A_{n-1}(u)-4(n-1)(\sinh u)^2A_{n-2}(u), \tag{4.31'}$$

$$A_n'(u)=2n\sinh u\,A_{n-1}(u). \tag{4.32'}$$

By the definition (4.8) of J_n, it is also easy to see that

$$J_n=\int_0^\infty\frac{e^{-\frac{u}{2}}}{(1-e^{-2u})^{2n+1}}a_n(u)du=\frac{1}{2^{2n+1}}\int_0^\infty\frac{A_n(u)}{(\sinh\frac{u}{2})^{2n+1}}du.$$

Hence, if we put

$$B_n(u)=\frac{A_n(u)}{(\sinh\frac{u}{2})^{2n+1}} \tag{4.33}$$

we have

$$J_n=\frac{1}{2^{2n+1}}\int_0^\infty B_n(u)du. \tag{4.34}$$

We establish now the following recurrence relation among J_n's which will be crucial in the discussion.

Proposition 4.11. *For $n\geq 2$ it holds that*

$$4n^2J_n=(8n^2-8n+3)J_{n-1}-4(n-1)^2J_{n-2}, \tag{4.35}$$

or equivalently

$$4n^2(J_n-J_{n-1})=4(n-1)^2(J_{n-1}-J_{n-2})-J_{n-1}. \tag{4.35'}$$

Note also that $4J_1=3J_0(=9\zeta(2))$.

Proof. We prove (4.35). By (4.32') with the duplication formula $\sinh 2x=2\sinh x\cosh x$ we observe that

$$B_n'(u)=\Big(\frac{A_n(u)}{(\sinh\frac{u}{2})^{2n+1}}\Big)'=4n\frac{\sinh\frac{u}{2}\cosh\frac{u}{2}\,A_{n-1}(u)}{(\sinh\frac{u}{2})^{2n+1}}-\frac{1}{2}(2n+1)\frac{\cosh\frac{u}{2}}{\sinh\frac{u}{2}}\frac{A_n(u)}{(\sinh\frac{u}{2})^{2n+1}}.$$

Thus we obtain

$$2(\tanh\frac{u}{2})B_n'(u) = 8nB_{n-1}(u) - (2n+1)B_n(u). \tag{4.36}$$

Similarly we have from (4.31') that

$$nB_n(u) = 2(2n-1)\Big(2(\coth\frac{u}{2})^2 - \frac{1}{(\sinh\frac{u}{2})^2}\Big)B_{n-1}(u) - 4(n-1)\cdot 4(\coth\frac{u}{2})^2B_{n-2}(u).$$

Hence it follows that

$$n(\tanh\frac{u}{2})^2B_n(u) = 4(2n-1)B_{n-1}(u) - \frac{2(2n-1)}{(\cosh\frac{u}{2})^2}B_{n-1}(u) - 16(n-1)B_{n-2}(u). \tag{4.37}$$

We introduce here an auxiliary integration K_n by

$$K_n = \frac{1}{2^{2n+1}}\int_0^\infty \frac{1}{(\cosh\frac{u}{2})^2}B_n(u)du. \tag{4.38}$$

Then by (4.36), integration by parts yields

$$\begin{aligned} &8n\cdot 2^{2n-1}J_{n-1} - (2n+1)2^{2n+1}J_n \\ =&2\Big\{\Big[\tanh\frac{u}{2}\,B_n(u)\Big]_0^\infty - \frac{1}{2}\int_0^\infty \frac{1}{(\cosh\frac{u}{2})^2}B_n(u)du\Big\} \\ =&-2^{2n+1}K_n. \end{aligned}$$

Hence we obtain

$$K_n = (2n+1)J_n - 2nJ_{n-1}. \tag{4.39}$$

Since $(\tanh\frac{u}{2})^2 = 1 - \frac{1}{(\cosh\frac{u}{2})^2}$, we have by (4.37)

$$n(1 - \frac{1}{(\cosh\frac{u}{2})^2})B_n(u) = 4(2n-1)B_{n-1}(u) - \frac{2(2n-1)}{(\cosh\frac{u}{2})^2}B_{n-1}(u) - 16(n-1)B_{n-2}(u).$$

Thus we see that

$$n\cdot 2^{2n+1}(J_n - K_n) = 4(2n-1)2^{2n-1}J_{n-1} - 2(2n-1)2^{2n-1}K_{n-1} - 16(n-1)2^{2n-3}J_{n-2},$$

that is, we have

$$2n(J_n - K_n) = (2n-1)(2J_{n-1} - K_{n-1}) - 2(n-1)J_{n-2}. \tag{4.40}$$

Plugging the expressions K_n and K_{n-1} of J_n's obtained by (4.39) into (4.40) we get the desired recursion formula (4.35). This proves the proposition. $\square$

Since the equation (4.35) is linear we immediately see from (4.10) the following

Corollary 4.12. *The number J_n is positive and*

$$J_n \in \mathbb{Q}\,\zeta(2) = \mathbb{Q}\,\pi^2.$$

For instance, $J_2 = \frac{123}{64}\zeta(2) = \frac{41}{128}\pi^2$.

Proof. Positivity follows from the definition (4.8) of J_n. The other assertions are clear from the recursion equation (4.35). $\square$

Using Proposition 4.11 we prove that the power series defined by the sequence $\{J_n\}$ as its coefficients gives a solution of a singly confluent Heun's differential equation (see [SL]).

Theorem 4.13. *Define the function $w(z)$ by the series*

$$w(z) = \sum_{n=0}^{\infty} J_n z^n.$$

Then the function $w(z)$ is holomorphic in $|z| < 1$ and is a (unique) solution of the singly confluent case of Heun's differential equation

$$z(1-z)^2 \frac{d^2 w}{dz^2} + (1-3z)(1-z)\frac{dw}{dz} + \big(z - \frac{3}{4}\big)w = 0 \tag{4.41}$$

satisfying $w(0) = 3\zeta(2)$ (and $w'(0) = \frac{9}{4}\zeta(2)$). Moreover, $w(z)$ has an integral representation

$$w(z) = 4\int_0^1 \int_0^1 \frac{1 - x^2y^2}{(1-x^2y^2)^2 - (1-x^4)(1-y^4)z} dxdy. \tag{4.42}$$

Proof. Since $J_n > 0$ it follows from (4.35') that $4n^2(J_n - J_{n-1}) < 4(n-1)^2(J_{n-1} - J_{n-2})$. Hence by induction we conclude that the sequence $\{J_n\}$ is monotone decreasing. Then the series in question converges in the region $|z| < 1$, whence $w(z)$ defines a holomorphic function in this region. The second assertion which claims the $w(z)$ satisfies the Heun equation (4.41) is straightforward (or, see §3.6.3 in [SL]). Indeed, if we define an ordinary differential operator D by

$$D := z(1-z)^2 \frac{d^2}{dz^2} + (1-3z)(1-z)\frac{d}{dz} + \big(z - \frac{3}{4}\big) \tag{4.43}$$

then we have

$$4Dw(z) = \sum_{n=0}^{\infty} \Big[4(n+1)^2 J_{n+1} - \{8(n+1)^2 - 8(n+1) + 3\}J_n + 4n^2 J_{n-1}\Big] z^n = 0. \tag{4.44}$$

For the third assertion, we note that if we change the variables as $x = e^{-\frac{t}{2}}, y = e^{-\frac{s}{2}}$ in the definition (4.8) of J_n then we have

$$J_n = 4\int_0^1 \int_0^1 \frac{1}{1-x^2y^2} \Big\{\frac{(1-x^4)(1-y^4)}{(1-x^2y^2)^2}\Big\}^n dxdy.$$

Since $(1-ab)^2 > (1-a^2)(1-b^2)$ for $0 < a, b < 1$, we may change the order of the integral and the summation. Thus we obtain

$$w(x) = 4\int_0^1 \int_0^1 \frac{1}{1-x^2y^2} \frac{1}{1 - \frac{(1-x^4)(1-y^4)}{(1-x^2y^2)^2} z} dxdy.$$

Hence the assertion follows. □

From this theorem we find that the value $\zeta_Q(2)$ can be represented as an integral transform of the solution above of the singly confluent Heun's equation (4.41) for sufficiently large $\alpha\beta > 1$.

Corollary 4.14. *Let $u(z)$ be a unique holomorphic solution of the confluent Heun equation (4.41) in the region $|z| < 1$ satisfying $u(0) = 1$. Assume that $\alpha\beta > 2$. Then for any r satisfying $a^2 := \frac{1}{\alpha\beta - 1} < r < 1$ we have*

$$\zeta_Q(2) = \frac{3}{2}\zeta(2) \cdot \Big\{\frac{\alpha+\beta}{\sqrt{\alpha\beta(\alpha\beta-1)}}\Big\}^2 \cdot \left[1 + \frac{1}{2\pi\sqrt{-1}}\Big(\frac{\alpha-\beta}{\alpha+\beta}\Big)^2 \int_{|z|=r} z^{-1}(1+a^2z^{-1})^{-1/2} u(z) dz\right]. \tag{4.45}$$

Proof. Retain the notation of Theorem 4.13. Then it is clear that $w(x) = 3\zeta(2)u(x)$. Hence the Cauchy integral formula asserts that

$$J_n = 3\zeta(2) \cdot \frac{1}{2\pi\sqrt{-1}} \int_{|z|=r} z^{-n-1} u(z) dz$$

for any $r<1$. Namely, we have

$$\sum_{n=0}^{\infty} Z_n'(2) = 3\zeta(2)\frac{(\alpha^{-1}-\beta^{-1})^2}{(1-\gamma^2)}\sum_{n=0}^{\infty}\binom{2n-1}{n}(-1)^n\Big(\frac{a}{2}\Big)^{2n}\frac{1}{2\pi\sqrt{-1}}\int_{|z|=r} z^{-n-1}u(z)dz.$$

Recalling the binomial expansion

$$(1+x)^{-1/2} = \sum_{n=0}^{\infty}(-1)^n\frac{(2n-1)!!}{n!2^n}x^n \qquad (|x|<1), \tag{4.46}$$

we have

$$\begin{aligned}&\sum_{n=0}^{\infty}\binom{2n-1}{n}(-1)^n\Big(\frac{a}{2}\Big)^{2n}z^{-n-1}\\ =&z^{-1}\sum_{n=0}^{\infty}(-1)^n\frac{(2n-1)!!}{n!}2^{n-1}\Big(\frac{a}{2}\Big)^{2n}z^{-n} = \frac{1}{2z}\sum_{n=0}^{\infty}(-1)^n\frac{(2n-1)!!}{n!2^n}(a^2z^{-1})^n\\ =&\frac{1}{2z}(1+a^2z^{-1})^{-1/2},\end{aligned}$$

which is valid for $|z|>a^2=\frac{1}{\alpha\beta-1}$. Note that $a^2<1$ whenever $\alpha\beta>2$. Therefore, taking r as $a^2<r<1$ we see that

$$\sum_{n=0}^{\infty} Z_n'(2) = 3\zeta(2)\cdot\frac{(\alpha^{-1}-\beta^{-1})^2}{(1-\gamma^2)}\cdot\frac{1}{2}\cdot\frac{1}{2\pi\sqrt{-1}}\int_{|z|=r} z^{-1}(1+a^2z^{-1})^{-1/2}u(z)dz.$$

Hence, in view of the equations (4.5ab), a little manipulation gives the assertion of the corollary. □

Remark 1. It is shown in [O] that the equivalence between the spectral problem for the NCHO and the resolution of a Heun's equation is realized essentially by considering the modified Laplace transform $\mathcal{L}u(z) := \int_0^\infty u(yz)e^{-y^2/2}ydy$. Our calculation of $\zeta_Q(2)=\operatorname{Tr} Q^{-2}$ also involves this type of integrals. Moreover, note that the continued fractions [PW2, 3, 4] which determine (at least in principle) the spectrum of Q are defined through some three terms recurrence equations, which seem to have also a close connection with Heun equations. However the reason is completely unclear why $\zeta_Q(2)$ is controlled by a singly confluent Heun's equation (see [SL]) as we saw above.

5. $\zeta_Q(3)$ and an inhomogeneous Heun equation

The aim of this section is to calculate the value $\zeta_Q(3) = \operatorname{Tr} Q^{-3}$ in the same way as $\operatorname{Tr} Q^{-2}$ as in §3. In fact, the following expression holds.

Theorem 5.1. *We have the expression*

$$\begin{aligned}\zeta_Q(3) =&\operatorname{Tr} Q^{-3}\\ =&\frac{1}{(1-\gamma^2)^{3/2}}\Big[\frac{(\alpha^{-1}+\beta^{-1})^3}{4}(2^3-1)\zeta(3)\\ &+\frac{3(\alpha^{-1}+\beta^{-1})(\alpha^{-1}-\beta^{-1})^2}{4}\int_0^\infty\int_0^\infty\int_0^\infty F(t,s,u;a)dtdsdu\Big],\end{aligned} \tag{5.1}$$

where we put

$$F(t,s,u;a) = \frac{e^{-(t+s+u)/2}}{\sqrt{(1-e^{-(t+s+u)})^2+a^2(1-e^{-2t})(1-e^{-2(s+u)})}}. \tag{5.2}$$

Proof. Note first that the relation $Q^{-3} = (A^{-1/2}Q'^{-1}A^{-1/2})^3$. Denote the integral kernel of Q^{-3} by $Q^{-3}(x,y)$. Then, in an analogous way used to have obtained the formula (3.4) for $Q^{-2}(x,x)$ we can get

$$\begin{aligned}
Q^{-3}(x,x) = &\int\int dydz\Big[A^{-1}\cos\frac{\gamma(x^2-y^2)}{2} + \gamma J\sin\frac{\gamma(x^2-y^2)}{2}\Big]\\
&\times\Big[A^{-1}\cos\frac{\gamma(y^2-z^2)}{2} + \gamma J\sin\frac{\gamma(y^2-z^2)}{2}\Big]\\
&\times\Big[A^{-1}\cos\frac{\gamma(z^2-x^2)}{2} + \gamma J\sin\frac{\gamma(z^2-x^2)}{2}\Big]\\
&\times\int_0^\infty (1-\gamma^2)^{1/4}p((1-\gamma^2)^{1/2}t,(1-\gamma^2)^{1/4}x,(1-\gamma^2)^{1/4}y)dt\\
&\times\int_0^\infty (1-\gamma^2)^{1/4}p((1-\gamma^2)^{1/2}s,(1-\gamma^2)^{1/4}y,(1-\gamma^2)^{1/4}z)ds\\
&\times\int_0^\infty (1-\gamma^2)^{1/4}p((1-\gamma^2)^{1/2}u,(1-\gamma^2)^{1/4}z,(1-\gamma^2)^{1/4}x)du.
\end{aligned}\tag{5.3}$$

We note that the first three factors in the integrand of (5.3) above can be expressed as

$$\begin{aligned}
&\Big[A^{-1}\cos\frac{\gamma(x^2-y^2)}{2} + \gamma J\sin\frac{\gamma(x^2-y^2)}{2}\Big]\Big[A^{-1}\cos\frac{\gamma(y^2-z^2)}{2} + \gamma J\sin\frac{\gamma(y^2-z^2)}{2}\Big]\\
&\qquad\times\Big[A^{-1}\cos\frac{\gamma(z^2-x^2)}{2} + \gamma J\sin\frac{\gamma(z^2-x^2)}{2}\Big]\\
&=\Big[A^{-3}\cos\frac{\gamma(x^2-y^2)}{2}\cos\frac{\gamma(y^2-z^2)}{2}\cos\frac{\gamma(z^2-x^2)}{2} - \gamma^2A^{-1}\sin\frac{\gamma(x^2-y^2)}{2}\sin\frac{\gamma(y^2-z^2)}{2}\cos\frac{\gamma(z^2-x^2)}{2}\\
&-\gamma^2A^{-1}\cos\frac{\gamma(x^2-y^2)}{2}\sin\frac{\gamma(y^2-z^2)}{2}\sin\frac{\gamma(z^2-x^2)}{2} + \gamma^2JA^{-1}J\sin\frac{\gamma(x^2-y^2)}{2}\cos\frac{\gamma(y^2-z^2)}{2}\sin\frac{\gamma(z^2-x^2)}{2}\Big]\\
&+\gamma\Big[A^{-2}J\cos\frac{\gamma(x^2-y^2)}{2}\cos\frac{\gamma(y^2-z^2)}{2}\sin\frac{\gamma(z^2-x^2)}{2} - \gamma^2J\sin\frac{\gamma(x^2-y^2)}{2}\sin\frac{\gamma(y^2-z^2)}{2}\sin\frac{\gamma(z^2-x^2)}{2}\\
&+A^{-1}JA^{-1}\cos\frac{\gamma(x^2-y^2)}{2}\sin\frac{\gamma(y^2-z^2)}{2}\cos\frac{\gamma(z^2-x^2)}{2} + JA^{-2}\sin\frac{\gamma(x^2-y^2)}{2}\cos\frac{\gamma(y^2-z^2)}{2}\cos\frac{\gamma(z^2-x^2)}{2}\Big].
\end{aligned}\tag{5.4}$$

Here note that the second term on the right of (5.4) is traceless. Since $-\operatorname{tr} JA^{-1}J = \operatorname{tr} A^{-1} = \alpha^{-1}+\beta^{-1}$

it follows hence that

$$
\begin{aligned}
\operatorname{Tr} Q^{-3} = \int\int\int dxdydz\Big[&(\mathrm{tr}A^{-3})\cos\frac{\gamma(x^2-y^2)}{2}\cos\frac{\gamma(y^2-z^2)}{2}\cos\frac{\gamma(z^2-x^2)}{2}\\
&-\gamma^2(\mathrm{tr}A^{-1})\sin\frac{\gamma(x^2-y^2)}{2}\sin\frac{\gamma(y^2-z^2)}{2}\cos\frac{\gamma(z^2-x^2)}{2}\\
&-\gamma^2(\mathrm{tr}A^{-1})\cos\frac{\gamma(x^2-y^2)}{2}\sin\frac{\gamma(y^2-z^2)}{2}\sin\frac{\gamma(z^2-x^2)}{2}\\
&+\gamma^2(\mathrm{tr}JA^{-1}J)\sin\frac{\gamma(x^2-y^2)}{2}\cos\frac{\gamma(y^2-z^2)}{2}\sin\frac{\gamma(z^2-x^2)}{2}\Big]\\
&\times\int_0^\infty(1-\gamma^2)^{1/4}p((1-\gamma^2)^{1/2}t,(1-\gamma^2)^{1/4}x,(1-\gamma^2)^{1/4}y)dt\\
&\times\int_0^\infty(1-\gamma^2)^{1/4}p((1-\gamma^2)^{1/2}s,(1-\gamma^2)^{1/4}y,(1-\gamma^2)^{1/4}z)ds\\
&\times\int_0^\infty(1-\gamma^2)^{1/4}p((1-\gamma^2)^{1/2}u,(1-\gamma^2)^{1/4}z,(1-\gamma^2)^{1/4}x)du\\
=\int\int\int dx'dy'dz'(1-\gamma^2)^{-3/4}\Big[&(\alpha^{-3}+\beta^{-3})\cos\frac{\gamma(x'^2-y'^2)}{2(1-\gamma^2)^{1/2}}\cos\frac{\gamma(y'^2-z'^2)}{2(1-\gamma^2)^{1/2}}\cos\frac{\gamma(z'^2-x'^2)}{2(1-\gamma^2)^{1/2}}\\
&-\gamma^2(\alpha^{-1}+\beta^{-1})\sin\frac{\gamma(x'^2-y'^2)}{2(1-\gamma^2)^{1/2}}\sin\frac{\gamma(y'^2-z'^2)}{2(1-\gamma^2)^{1/2}}\cos\frac{\gamma(z'^2-x'^2)}{2(1-\gamma^2)^{1/2}}\\
&+\gamma^2(\alpha^{-1}+\beta^{-1})\sin^2\frac{\gamma(x'^2-z'^2)}{2(1-\gamma^2)^{1/2}}\Big]\\
&\times\int_0^\infty\int_0^\infty\int_0^\infty(1-\gamma^2)^{3/4-3/2}p(t',x',y')p(s',y',z')p(u',z',x')dt'ds'du',
\end{aligned}
\tag{5.5}
$$

where we have made the change of variables $t'=(1-\gamma^2)^{1/2}t$, etc. and $x'=(1-\gamma^2)^{1/4}x$, etc.

Recall the triple product formula of the cosine function

$$
\begin{aligned}
&\cos A\cos B\cos C\\
&\quad=\frac{1}{4}[\cos(A+B-C)+\cos(B+C-A)+\cos(C+A-B)+\cos(A+B+C)].
\end{aligned}
\tag{5.6}
$$

Using this formula we see that the part $[\cdots]$ in the integrand of the last member of (5.5) above turns to be

$$
\begin{aligned}
&\big((\alpha^{-3}+\beta^{-3})-\gamma^2(\alpha^{-1}+\beta^{-1})\big)\cos\frac{\gamma(x'^2-y'^2)}{2(1-\gamma^2)^{1/2}}\cos\frac{\gamma(y'^2-z'^2)}{2(1-\gamma^2)^{1/2}}\cos\frac{\gamma(z'^2-x'^2)}{2(1-\gamma^2)^{1/2}}\\
&+\gamma^2(\alpha^{-1}+\beta^{-1})\Big[\cos\frac{\gamma(x'^2-y'^2)}{2(1-\gamma^2)^{1/2}}\cos\frac{\gamma(y'^2-z'^2)}{2(1-\gamma^2)^{1/2}}\cos\frac{\gamma(z'^2-x'^2)}{2(1-\gamma^2)^{1/2}}\\
&-\sin\frac{\gamma(x'^2-y'^2)}{2(1-\gamma^2)^{1/2}}\sin\frac{\gamma(y'^2-z'^2)}{2(1-\gamma^2)^{1/2}}\cos\frac{\gamma(z'^2-x'^2)}{2(1-\gamma^2)^{1/2}}\Big]+\gamma^2(\alpha^{-1}+\beta^{-1})\sin^2\frac{\gamma(x'^2-z'^2)}{2(1-\gamma^2)^{1/2}}\\
=&\,\gamma^2(\alpha^{-1}+\beta^{-1})\\
&+(\alpha^{-1}+\beta^{-1})(\alpha^{-1}-\beta^{-1})^2\cos\frac{\gamma(x'^2-y'^2)}{2(1-\gamma^2)^{1/2}}\cos\frac{\gamma(y'^2-z'^2)}{2(1-\gamma^2)^{1/2}}\cos\frac{\gamma(z'^2-x'^2)}{2(1-\gamma^2)^{1/2}}.
\end{aligned}
\tag{5.7}
$$

Therefore we put $t=t'$, etc. and $x=x'$, etc. to get

$$
\begin{aligned}
\mathrm{Tr}Q^{-3}=&\,\frac{1}{(1-\gamma^2)^{3/2}}\int\int\int dxdydz\Big[\gamma^2(\alpha^{-1}+\beta^{-1})+(\frac{1}{\alpha}+\frac{1}{\beta})(\frac{1}{\alpha}-\frac{1}{\beta})^2\\
&\times\cos\frac{\gamma(x^2-y^2)}{2(1-\gamma^2)^{1/2}}\cos\frac{\gamma(y^2-z^2)}{2(1-\gamma^2)^{1/2}}\cos\frac{\gamma(z^2-x^2)}{2(1-\gamma^2)^{1/2}}\Big]\\
&\times\int_0^\infty\int_0^\infty\int_0^\infty p(t,x,y)p(s,y,z)p(u,z,x)dtdsdu\\
=:&\;I_1+I_2.
\end{aligned}
\tag{5.8}
$$

We now study I_1 and I_2. Since $\int p(t,x,y)p(s,y,z)dy = p(t+s,x,z)$, it is easy to see that

$$\begin{aligned} I_1 &= \frac{\gamma^2}{(1-\gamma^2)^{3/2}}(\alpha^{-1}+\beta^{-1}) \\ &\quad\times \int_0^\infty\int_0^\infty\int_0^\infty dtdsdu \int\int\int p(t,x,y)p(s,y,z)p(u,z,x)dxdydz \\ &= \frac{\gamma^2}{(1-\gamma^2)^{3/2}}(\alpha^{-1}+\beta^{-1})\int_0^\infty\int_0^\infty\int_0^\infty dtdsdu\int p(t+s+u,x,x)dx. \end{aligned}$$

Hence it follows from (3.12) that

$$\begin{aligned} I_1 &= \frac{\gamma^2}{(1-\gamma^2)^{3/2}}(\alpha^{-1}+\beta^{-1})\int_0^\infty\int_0^\infty\int_0^\infty \frac{e^{-(t+s+u)/2}}{1-e^{-(t+s+u)}}dtdsdu \\ &= \frac{\gamma^2}{(1-\gamma^2)^{3/2}}(\alpha^{-1}+\beta^{-1})\sum_{k=0}^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-(t+s+u)(k+1/2)}dtdsdu \\ &= \frac{\gamma^2}{(1-\gamma^2)^{3/2}}(\alpha^{-1}+\beta^{-1})\sum_{k=0}^\infty(k+1/2)^{-3} \\ &= \frac{\gamma^2}{(1-\gamma^2)^{3/2}}(\alpha^{-1}+\beta^{-1})(2^3-1)\zeta(3). \end{aligned} \tag{5.9}$$

Next, to study the integral

$$\begin{aligned} I_2 &= \frac{1}{(1-\gamma^2)^{3/2}}(\alpha^{-1}+\beta^{-1})(\alpha^{-1}-\beta^{-1})^2\int\int\int dxdydz \\ &\quad\times \Big[\cos\frac{\gamma(x^2-y^2)}{2(1-\gamma^2)^{1/2}}\cos\frac{\gamma(y^2-z^2)}{2(1-\gamma^2)^{1/2}}\cos\frac{\gamma(z^2-x^2)}{2(1-\gamma^2)^{1/2}}\Big] \\ &\quad\times \int_0^\infty\int_0^\infty\int_0^\infty p(t,x,y)p(s,y,z)p(u,z,x)dtdsdu, \end{aligned} \tag{5.10}$$

we begin with observing the following claims (a), (b), (c) and (d).
(a) Note the part $[\cdots]$ of the integrand of I_2 equals

$$\begin{aligned} &\cos\frac{\gamma(x^2-y^2)}{2(1-\gamma^2)^{1/2}}\cos\frac{\gamma(y^2-z^2)}{2(1-\gamma^2)^{1/2}}\cos\frac{\gamma(z^2-x^2)}{2(1-\gamma^2)^{1/2}} \\ &= \frac14\Big[\cos\frac{\gamma(x^2-z^2)}{(1-\gamma^2)^{1/2}}+\cos\frac{\gamma(y^2-x^2)}{(1-\gamma^2)^{1/2}}+\cos\frac{\gamma(z^2-y^2)}{(1-\gamma^2)^{1/2}}+1\Big], \end{aligned} \tag{5.11}$$

where we used the formula (5.6) again.
(b) Note by (2.3b) that

$$\begin{aligned} &p(t,x,y)p(s,y,z)p(u,z,x) \\ =&\pi^{-3/2}e^{-(t+s+u)/2}(1-e^{-2t})^{-1/2}(1-e^{-2s})^{-1/2}(1-e^{-2u})^{-1/2} \\ &\quad\times\exp\Big[-\frac{(1+e^{-2t})(x^2+y^2)-4e^{-t}xy}{2(1-e^{-2t})}\Big]\exp\Big[-\frac{(1+e^{-2s})(y^2+z^2)-4e^{-s}yz}{2(1-e^{-2s})}\Big] \\ &\quad\times\exp\Big[-\frac{(1+e^{-2u})(z^2+x^2)-4e^{-u}zx}{2(1-e^{-2u})}\Big] \\ =&\pi^{-3/2}e^{-(t+s+u)/2}(1-e^{-2t})^{-1/2}(1-e^{-2s})^{-1/2}(1-e^{-2u})^{-1/2} \\ &\times\exp\Big[-\frac{(1-e^{-2s})(1-e^{-2(u+t)})x^2+(1-e^{-2u})(1-e^{-2(t+s)})y^2+(1-e^{-2t})(1-e^{-2(s+u)})z^2}{(1-e^{-2t})(1-e^{-2s})(1-e^{-2u})}\Big] \\ &\times\exp\Big[-\frac{-2e^{-t}(1-e^{-2s})(1-e^{-2u})xy-2e^{-s}(1-e^{-2u})(1-e^{-2t})yz-2e^{-u}(1-e^{-2t})(1-e^{-2s})zx}{(1-e^{-2t})(1-e^{-2s})(1-e^{-2u})}\Big] \end{aligned} \tag{5.12}$$

(c) Define the two Gaussian integrals L and L_{cs} by

$$\begin{aligned}
L &= L(A,B,C,E,F,G) \\
&:= \int\int\int e^{ia(x^2-y^2)} e^{-(Ax^2+By^2+Cz^2-2Exy-2Fyz-2Gzx)} dxdydz, \\
L_{cs} &= L_{cs}(A,B,C,E,F,G) \\
&:= \int\int\int (\cos a(x^2-y^2)) e^{-(Ax^2+By^2+Cz^2-2Exy-2Fyz-2Gzx)} dxdydz.
\end{aligned}$$

Lemma 5.2. *We have*

$$L(A,B,C,E,F,G) = \frac{\pi^{3/2}}{\sqrt{A'+iaB'}}, \tag{5.13a}$$

$$L_{cs}(A,B,C,E,F,G) = \frac{\pi^{3/2}}{2^{1/2}}\sqrt{\frac{C'}{D'}}, \tag{5.13b}$$

where

$$\begin{aligned}
&A' = ABC - CE^2 - AF^2 - BG^2 - 2EFG + a^2C, \quad B' = AC - BC + F^2 - G^2, \\
&C' = ({A'}^2 + a^2{B'}^2)^{1/2} + A', \quad D' = {A'}^2 + a^2{B'}^2.
\end{aligned}$$

Moreover, if $B'=0$, then it turns out that

$$L = L_{cs} = \frac{\pi^{3/2}}{\sqrt{A'}} = \frac{\pi^{3/2}}{\sqrt{ABC - CE^2 - AF^2 - BG^2 - 2EFG + a^2C}}. \tag{5.13c}$$

Proof. The integrand of L is equal to

$$\begin{aligned}
&e^{ia(x^2-y^2)} e^{-(Ax^2+By^2+Cz^2-2Exy-2Fyz-2Gzx)} \\
=&e^{-[(A-ia)x^2-2(Ey+Gz)x+(B+ia)y^2+Cz^2-2Fyz]} \\
=&e^{-[(A-ia)(x-\frac{Ey+Gz}{A-ia})^2-\frac{(Ey+Gz)^2}{A-ia}+(B+ia)y^2+Cz^2-2Fyz]} \\
=&e^{-[(A-ia)(x-\frac{Ey+Gz}{A-ia})^2+(B+ia-\frac{E^2}{A-ia})\left(y-\frac{EG+(A-ia)F}{(A-ia)(B+ia)-E^2}z\right)^2]} e^{-[\frac{(A-ia)(B+ia)C-CE^2-(B+ia)G^2-(A-ia)F^2-2EFG}{(A-ia)(B+ia)-E^2}]z^2}
\end{aligned}$$

Now we perform a Gaussian integration to get

$$\begin{aligned}
L =&\sqrt{\frac{\pi}{A-ia}}\sqrt{\frac{\pi}{B+ia-\frac{E^2}{A-ia}}}\sqrt{\frac{\pi((A-ia)(B+ia)-E^2)}{(A-ia)(B+ia)C-CE^2-(A-ia)F^2-(B+ia)G^2-2EFG}} \\
=&\frac{\pi^{3/2}}{\sqrt{(A-ia)(B+ia)C-CE^2-(A-ia)F^2-(B+ia)G^2-2EFG}} \\
=&\frac{\pi^{3/2}}{\sqrt{ABC-CE^2-AF^2-BG^2-2EFG+a^2C+ia(AC-BC+F^2-G^2)}}.
\end{aligned}$$

This shows (5.13a).

To get L_{cs} in (5.13b), we have only to note that

$$\frac{1}{2}\left(\frac{1}{\sqrt{a+ib}}+\frac{1}{\sqrt{a-ib}}\right) = \sqrt{\frac{(a^2+b^2)^{1/2}+a}{2(a^2+b^2)}}.$$

The formula (5.13c) immediately follows from the relation

$$C' = {D'}^{1/2} = A' = ABC - CE^2 - AF^2 - BG^2 - 2EFG + a^2C$$

when $B' = 0$. This completes the proof of the lemma. □

(d) We have with L in (5.13a) and L_{cs} in (5.13b) respectively that

$$\begin{aligned} &\int\int\int e^{ia(x^2-y^2)}p(t,x,y)p(s,y,z)p(u,z,x)dxdydz \\ =&\pi^{-3/2}e^{-(t+s+u)/2}(1-e^{-2t})^{-1/2}(1-e^{-2s})^{-1/2}(1-e^{-2u})^{-1/2}L(A,B,C,E,F,G), \end{aligned} \tag{5.14a}$$

and

$$\begin{aligned} F(t,s,u;a) :=& \int\int\int (\cos(a(x^2-y^2)))p(t,x,y)p(s,y,z)p(u,z,x)dxdydz \\ =&\pi^{-3/2}e^{-(t+s+u)/2}(1-e^{-2t})^{-1/2}(1-e^{-2s})^{-1/2}(1-e^{-2u})^{-1/2}L_{cs}(A,B,C,E,F,G). \end{aligned} \tag{5.14b}$$

Here we put

$$\begin{aligned} A &= \frac{(1-e^{-2s})(1-e^{-2(u+t)})}{(1-e^{-2t})(1-e^{-2s})(1-e^{-2u})} = \frac{(1-e^{-2(u+t)})}{(1-e^{-2t})(1-e^{-2u})}, \\ B &= \frac{(1-e^{-2u})(1-e^{-2(t+s)})}{(1-e^{-2t})(1-e^{-2s})(1-e^{-2u})} = \frac{(1-e^{-2(t+s)})}{(1-e^{-2t})(1-e^{-2s})}, \\ C &= \frac{(1-e^{-2t})(1-e^{-2(s+u)})}{(1-e^{-2t})(1-e^{-2s})(1-e^{-2u})} = \frac{(1-e^{-2(s+u)})}{(1-e^{-2s})(1-e^{-2u})}, \\ E &= \frac{e^{-t}(1-e^{-2s})(1-e^{-2u})}{(1-e^{-2t})(1-e^{-2s})(1-e^{-2u})} = \frac{e^{-t}}{(1-e^{-2t})}, \\ F &= \frac{e^{-s}(1-e^{-2u})(1-e^{-2t})}{(1-e^{-2t})(1-e^{-2s})(1-e^{-2u})} = \frac{e^{-s}}{(1-e^{-2s})}, \\ G &= \frac{e^{-u}(1-e^{-2t})(1-e^{-2s})}{(1-e^{-2t})(1-e^{-2s})(1-e^{-2u})} = \frac{e^{-u}}{(1-e^{-2u})}. \end{aligned} \tag{5.15}$$

With these A, B, C, E, F, G in (5.15), we see that

$$\begin{aligned} B' &= (A-B)C + (E^2 - G^2) \\ &= \frac{(e^{-2u}-e^{-2s})(1-e^{-2(s+u)})}{(1-e^{-2s})^2(1-e^{-2u})^2} + \frac{(e^{-2s}-e^{-2u})(1-e^{-2(s+u)})}{(1-e^{-2s})^2(1-e^{-2u})^2} = 0. \end{aligned} \tag{5.16}$$

On the other hand, we get

$$\begin{aligned} A' &= ABC - CE^2 - AF^2 - BG^2 - 2EFG + a^2C \\ &= \frac{(1-e^{-2(s+u)})[(1-e^{-2(u+t)})(1-e^{-2(t+s)}) - e^{-2t}(1-e^{-2s})(1-e^{-2u})]}{(1-e^{-2t})^2(1-e^{-2s})^2(1-e^{-2u})^2} \\ &\quad - AF^2 - BG^2 - 2EFG + a^2C \\ &= \frac{(1-e^{-2(s+u)})(1-e^{-2(t+s+u)}) - e^{-2s}(1-e^{-(u+t)}(1-e^{-2u})}{(1-e^{-2t})(1-e^{-2s})^2(1-e^{-2u})^2} - BG^2 - 2EFG + a^2C \\ &= \frac{(1-e^{-2(t+s+u)}) - e^{-2u}(1-e^{-(t+s)})}{(1-e^{-2t})(1-e^{-2s})(1-e^{-2u})^2} - 2EFG + a^2C \\ &= \frac{(1-e^{-(t+s+u)})^2}{(1-e^{-2t})(1-e^{-2s})(1-e^{-2u})} + a^2C \\ &= \frac{(1-e^{-(t+s+u)})^2 + a^2(1-e^{-2t})(1-e^{-2(s+u)})}{(1-e^{-2t})(1-e^{-2s})(1-e^{-2u})}. \end{aligned} \tag{5.17}$$

Hence we have by (5.13c) (noting (5.16) and (5.17))

$$\begin{aligned} F(t,s,u;a) &= \pi^{-3/2}e^{-(t+s+u)/2}(1-e^{-2t})^{-1/2}(1-e^{-2s})^{-1/2}(1-e^{-2u})^{-1/2}L_{cs} \\ &= e^{-(t+s+u)/2}(1-e^{-2t})^{-1/2}(1-e^{-2s})^{-1/2}(1-e^{-2u})^{-1/2}A'^{-1/2} \\ &= \frac{e^{-(t+s+u)/2}}{\sqrt{(1-e^{-(t+s+u)})^2 + a^2(1-e^{-2t})(1-e^{-2(s+u)})}}. \end{aligned} \tag{5.18}$$

Therefore, noting the symmetry of the variables x, y, z and using the equation (5.11) in (a) we have

$$
\begin{aligned}
I_2 = & \frac{1}{(1-\gamma^2)^{3/2}} \frac{(\alpha^{-1}+\beta^{-1})(\alpha^{-1}-\beta^{-1})^2}{4} \Big(3 \int_0^\infty \int_0^\infty \int_0^\infty F(t,s,u;a) dtdsdu \\
& + \int_0^\infty \int_0^\infty \int_0^\infty dtdsdu \int\int\int p(t,x,y)p(s,y,z)p(u,z,x)dxdydz\Big) \qquad (5.19) \\
= & \frac{1}{(1-\gamma^2)^{3/2}} \frac{(\alpha^{-1}+\beta^{-1})(\alpha^{-1}-\beta^{-1})^2}{4} \Big(3 \int_0^\infty \int_0^\infty \int_0^\infty F(t,s,u;a) dtdsdu + (2^3-1)\zeta(3)\Big)
\end{aligned}
$$

Thus, by (5.8), (5.9), (5.10) and (5.16) with $F(t,s,u;a)$ in (5.18) ($a = \gamma(1-\gamma^2)^{-1}$) we conclude that

$$
\begin{aligned}
\zeta_Q(3) = & \operatorname{Tr} Q^{-3} = I_1 + I_2 \\
= & \frac{1}{(1-\gamma^2)^{3/2}} (\alpha^{-1}+\beta^{-1}) \\
& \times \Big[7\zeta(3)\gamma^2 + \frac{(\alpha^{-1}-\beta^{-1})^2}{4} \Big(3 \int_0^\infty \int_0^\infty \int_0^\infty F(t,s,u;a) dtdsdu + 7\zeta(3)\Big)\Big] \\
= & \frac{1}{(1-\gamma^2)^{3/2}} \Big[\frac{(\alpha^{-1}+\beta^{-1})^3}{4} (2^3-1)\zeta(3) \\
& + \frac{3(\alpha^{-1}+\beta^{-1})(\alpha^{-1}-\beta^{-1})^2}{4} \int_0^\infty \int_0^\infty \int_0^\infty F(t,s,u;a) dtdsdu\Big].
\end{aligned}
$$

This proves the theorem. □

Remark 1. There seems to be the following relation between the function $F(t,s,u;a)$ in (5.18) and the previous $F(t,s;a)$ in (3.3):

$$
\int_0^\infty \int_0^\infty \int_0^\infty F(t,s,u;a) dtdsdu = \int_0^\infty \int_0^\infty sF(t,s;a) dtds. \qquad \square \tag{5.21}
$$

Remark 2. In order to get an expression of Tr Q^{-3}, we had to calculate Gaussian integrals like (5.13a). It should be noticed that the $A' + iaB'$ inside the root of its denominator in the last member of (5.13a) is nothing but the determinant of the coefficient matrix associated with *that* quadratic form on the power of the exponential in the integrand of (5.13a), *which*, of course, has to include the term like $ia(x^2-y^2)$. Thus, to go further to the cases $s = k = 4, 5, \ldots$, we may use this observation to calculate the determinant of the quadratic form, which we will have from the product of four, five, ... , of the heat kernel $p(t,x,y)$ of the usual harmonic oscillator; we have to calculate an expression like in (5.12). The coefficients of this quadratic form are functions of $t, s, u, w, \ldots$, like (5.15). We actually treat this problem in the Appendix. □

6. $\zeta_Q(3)$ and some inhomogeneous confluent Heun equation

The aim of this section is to develop a study of $\zeta_Q(3)$ similar to the one of $\zeta_Q(2)$ in §4.

Recall the function $F(t,s,u;a)$ and make a Taylor expansion of it with respect to the variable a^2:

$$
\begin{aligned}
F(t,s,u;a) = & \frac{e^{-(t+s+u)/2}}{\sqrt{(1-e^{-(t+s+u)})^2 + a^2(1-e^{-2t})(1-e^{-2(s+u)})}} \\
= & \sum_{n=0}^\infty (-1)^n \frac{(2n-1)!!}{2^n n!} \frac{e^{-(t+s+u)/2}(1-e^{-2t})^n(1-e^{-2(s+u)})^n}{(1-e^{-(t+s+u)})^{2n+1}} a^{2n} \qquad (a^2<1),
\end{aligned}
\tag{6.1}
$$

because we have

$$
\begin{aligned}
F^{(n)}(t,s,u;a) = & \Big(\frac{d}{da^2}\Big)^n F(t,s,u;a) \\
= & (-1)^n \frac{(2n-1)!!}{2^n} \frac{e^{-(t+s+u)/2}(1-e^{-2t})^n(1-e^{-2(s+u)})^n}{\{(1-e^{-(t+s+u)})^2 + a^2(1-e^{-2t})(1-e^{-2(s+u)})\}^{\frac{2n+1}{2}}}.
\end{aligned}
$$

As in the case of the integral J_n we define J_n^1 by

$$J_n^1 = \int_0^\infty \int_0^\infty \int_0^\infty \frac{e^{-(t+s+u)/2}(1-e^{-2t})^n(1-e^{-2(s+u)})^n}{(1-e^{-(t+s+u)})^{2n+1}} dtdsdu \tag{6.2}$$

$$= 2^n \int_0^1 \int_0^1 \int_0^1 \frac{1}{(1-p^2q^2r^2)} \left\{ \frac{(1-p^4)(1-q^4r^4)}{(1-p^2q^2r^2)^2} \right\}^n dpdqdr. \tag{6.2'}$$

We will show a recursion formula of J_n^1 similar to the one for J_n. Using the integral $I_{n,m}(u)$ in (4.11) we have first the following

Lemma 6.1. *For $n \geq 0$ we have*

$$J_n^1 = \frac{1}{2} \int_0^\infty \frac{we^{-w/2}}{(1-e^{-w})^{2n+1}} I_{n,n}(w) dw. \tag{6.3}$$

Proof. Let

$$I_{n,m}^1(w) = \int_0^w p(1-e^{-2w+2p})^n(1-e^{-2p})^m dp.$$

Then, it is not hard to see that

$$J_n^1 = \int_0^\infty \frac{e^{-w/2}}{(1-e^{-w})^{2n+1}} I_{n,n}^1(w) dw. \tag{6.4}$$

Putting $p = w - x$ in the integral of $I_{n,m}^1(w)$ we have

$$I_{n,m}^1(w) = \int_0^w (w-x)(1-e^{-2x})^n(1-e^{-2w+2x})^m dx = wI_{n,m}(w) - I_{m,n}^1(w),$$

whence

$$I_{n,m}^1(w) + I_{m,n}^1(w) = wI_{n,m}(w). \tag{6.5}$$

Thus, putting $n = m$ we have

$$I_{n,n}^1(w) = \frac{1}{2} wI_{n,n}(w). \tag{6.6}$$

The claim immediately follows from (6.4) and (6.6). $\square$

Recall the function $A_n(u) = e^{nu} I_{n,n}(u)$ and $B_n(u) = \frac{A_n(u)}{(\sinh \frac{u}{2})^{2n+1}}$ in §4. Put

$$C_n(u) = \frac{u}{2} B_n(u) = \frac{uA_n(u)}{2(\sinh \frac{u}{2})^{2n+1}}. \tag{6.7}$$

Then by the lemma above it is easy to see that

$$J_n^1 = \frac{1}{2^{2n+1}} \int_0^\infty C_n(u) du. \tag{6.8}$$

Let

$$K_n^1 = \frac{1}{2^{2n+1}} \int_0^\infty \frac{1}{(\cosh \frac{u}{2})^2} C_n(u) du, \tag{6.9}$$

$$M_n = \frac{1}{2^{2n+1}} \int_0^\infty \tanh \frac{u}{2} \, B_n(u) du. \tag{6.10}$$

From (4.36) we have

$$\frac{u}{2} \cdot 2 \tanh \frac{u}{2} \Big(\frac{2}{u} C_n(u) \Big)' = 8nC_{n-1}(u) - (2n+1)C_n(u).$$

Hence, integration by parts yields

$$\begin{aligned}
&8n\cdot 2^{2n-1}J^1_{n-1}-(2n+1)\cdot 2^{2n+1}J^1_n=\int_0^\infty u\tanh\frac{u}{2}\Big(\frac{2}{u}C_n(u)\Big)'du\\
=&\Big[u\tanh\frac{u}{2}\frac{2}{u}C_n(u)\Big]_0^\infty-\int_0^\infty\Big(\tanh\frac{u}{2}+\frac{u}{2}\frac{1}{(\cosh\frac{u}{2})^2}\Big)\frac{2}{u}C_n(u)du\\
=&-\int_0^\infty\tanh\frac{u}{2}\,B_n(u)du-\int_0^\infty\frac{1}{(\cosh\frac{u}{2})^2}C_n(u)du\\
=&-2^{2n+1}M_n-2^{2n+1}K^1_n.
\end{aligned}$$

It follows that

$$2nJ^1_{n-1}-(2n+1)J^1_n=-M_n-K^1_n. \tag{6.11}$$

Moreover, by (4.37) we obtain

$$n\Big(1-\frac{1}{(\cosh\frac{u}{2})^2}\Big)C_n(u)=4(2n-1)C_{n-1}(u)-2(2n-1)\frac{1}{(\cosh\frac{u}{2})^2}C_{n-1}(u)-16(n-1)C_{n-2}(u).$$

Integrating this equation over $[0,\infty)$, it is easy to see that

$$2n(J^1_n-K^1_n)=2(2n-1)J^1_{n-1}-(2n-1)K^1_{n-1}-2(n-1)J^1_{n-2}. \tag{6.12}$$

Here we have the following lemma.

Lemma 6.2. *For $n\geq 2$, it holds that*

$$4n^2J^1_n-(8n^2-8n+3)J^1_{n-1}+4(n-1)^2J^1_{n-2}=2nM_n-(2n-1)M_{n-1}. \tag{6.13}$$

Proof. The proof is done by straightforward calculations using (6.11) and (6.12) to kill K^1_n and K^1_{n-1}. □

This lemma shows that the sequence $\{J^1_n\}$ satisfies a recurrence relation (4.35) quite similar to that of J_n's but is different, in fact, in being an inhomogeneous one of (4.35). We thus next calculate the sequence $\{P_n\}$ defined by

$$P_n:=2nM_n-(2n-1)M_{n-1}. \tag{6.14}$$

By (4.32') we see that

$$\begin{aligned}
\tanh\frac{u}{2}\,B_n(u)=&\tanh\frac{u}{2}\frac{A_n(u)}{(\sinh\frac{u}{2})^{2n+1}}=\frac{A_n(u)}{(\cosh\frac{u}{2})(\sinh\frac{u}{2})^{2n}}\\
=&\frac{A'_{n+1}(u)}{2(n+1)\sinh u}\frac{1}{(\cosh\frac{u}{2})(\sinh\frac{u}{2})^{2n}}=\frac{1}{4(n+1)}\frac{A'_{n+1}(u)}{(\cosh\frac{u}{2})^2(\sinh\frac{u}{2})^{2n+1}}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
&2^{2n+1}M_n\\
=&\int_0^\infty\tanh\frac{u}{2}\,B_n(u)du=\frac{1}{4(n+1)}\int_0^\infty\frac{A'_{n+1}(u)}{(\cosh\frac{u}{2})^2(\sinh\frac{u}{2})^{2n+1}}\\
=&\frac{1}{4(n+1)}\Big\{\Big[\frac{A_{n+1}(u)}{(\cosh\frac{u}{2})^2(\sinh\frac{u}{2})^{2n+1}}\Big]_0^\infty-\int_0^\infty\Big(\frac{1}{(\cosh\frac{u}{2})^2(\sinh\frac{u}{2})^{2n+1}}\Big)'A_{n+1}(u)du\Big\}.
\end{aligned}$$

Since $A_{n+1}(u)$ has a zero of order $2n+3$ and is of polynomial growth at infinity, we have

$$2^{2n+3}(n+1)M_n=\int_0^\infty\frac{A_{n+1}(u)}{(\cosh\frac{u}{2})^3(\sinh\frac{u}{2})^{2n}}du+\frac{2n+1}{2}\cdot 2^{2n+3}M_{n+1}. \tag{6.15}$$

By (4.31'), since we have

$$\begin{aligned}(n+1)A_{n+1}(u) =&2(2n+1)\cosh u\, A_n(u) - 4n(\sinh u)^2 A_{n-1}(u)\\ =&2(2n+1)\Big\{2(\cosh\frac{u}{2})^2 - 1\Big\}\, A_n(u) - 16n(\cosh\frac{u}{2})^2(\sinh\frac{u}{2})^2 A_{n-1}(u),\end{aligned}$$

we see from (6.15) that

$$\begin{aligned}&2^{2n+3}(n+1)M_n - 2^{2n+2}(2n+1)M_{n+1}\\ =&\frac{1}{n+1}\left[2(2n+1)\int_0^\infty \frac{2(\cosh\frac{u}{2})^2-1}{(\cosh\frac{u}{2})^3(\sinh\frac{u}{2})^{2n}}A_n(u)du - 16n\int_0^\infty \frac{A_{n-1}(u)}{(\cosh\frac{u}{2})(\sinh\frac{u}{2})^{2n-2}}du\right]\\ =&\frac{1}{n+1}\left[4(2n+1)\cdot 2^{2n+1}M_n - 2(2n+1)\cdot 2^{2n+1}N_n - 16n\cdot 2^{2n-1}M_{n-1}\right],\end{aligned}$$

where we put

$$\begin{aligned}N_n :=&\frac{1}{2^{2n+1}}\int_0^\infty \frac{A_n(u)}{(\cosh\frac{u}{2})^3(\sinh\frac{u}{2})^{2n}}du\\ =&\frac{1}{2^{2n+1}}\int_0^\infty \frac{1}{(\cosh\frac{u}{2})^2}\tanh\frac{u}{2}\, B_n(u)du.\end{aligned}$$

Hence we have

$$\begin{aligned}&(n+1)\Big[2^{2n+3}(n+1)M_n - 2^{2n+2}(2n+1)M_{n+1}\Big]\\ &\quad = (2n+1)2^{2n+3}M_n - (2n+1)2^{2n+2}N_n - n2^{2n+3}M_{n-1}.\end{aligned}$$

This shows that

$$(2n+1)N_n = (2n+1)(n+1)M_{n+1} - 2n^2M_n - 2nM_{n-1}. \tag{6.16}$$

Since

$$\begin{aligned}&n\Big(1-\frac{1}{(\cosh\frac{u}{2})^2}\Big)\tanh\frac{u}{2}\, B_n(u)\\ =&4(2n-1)\tanh\frac{u}{2}\, B_{n-1}(u) - 2(2n-1)\frac{1}{(\cosh\frac{u}{2})^2}\tanh\frac{u}{2}\, B_{n-1}(u) - 16(n-1)\tanh\frac{u}{2}\, B_{n-2}(u),\end{aligned}$$

we integrate this equation to have again from (4.37) that

$$2n(M_n - N_n) = 2(2n-1)M_{n-1} - (2n-1)N_{n-1} - 2(n-1)M_{n-2}. \tag{6.17}$$

Multiplying (6.17) by $(2n+1)$ yields

$$\begin{aligned}&2n(2n+1)M_n - 2(2n+1)(2n-1)M_{n-1} + 2(n-1)(2n+1)M_{n-2}\\ =&2n\cdot(2n+1)N_n - (2n+1)\cdot(2n-1)N_{n-1}\\ =&2n\big[(2n+1)(n+1)M_{n+1} - 2n^2M_n - 2nM_{n-1}\big]\\ &\quad - (2n+1)\big[(2n-1)M_n - 2(n-1)^2M_{n-1} - 2(n-1)M_{n-2}\big],\end{aligned}$$

where we used (6.16) to obtain the second equality. Hence we have

$$2(2n+1)(n+1)M_{n+1} - (8n^2+4n+1)M_n + 2n(2n-1)M_{n-1} = 0. \tag{6.18}$$

It follows from (6.18) that

$$\begin{aligned}(2n+1)P_{n+1} &= (2n+1)\big[2(n+1)M_{n+1} - (2n+1)M_n\big]\\ &= (8n^2+4n+1)M_n - 2n(2n-1)M_{n-1} - (2n+1)^2M_n\\ &= 2n\big[2nM_n - (2n-1)M_{n-1}\big] = 2nP_n.\end{aligned}$$

This shows that $\frac{P_n}{P_{n-1}} = \frac{2(n-1)}{2n-1}$, whence we have the the following explicit formula of P_n.

Lemma 6.3.

$$P_n = \frac{2^{n-1}(n-1)!}{(2n-1)!!} P_1 \quad \text{with} \quad P_1 = 2. \tag{6.19}$$

Proof. It suffices to show that $P_1 = 2$. Recall that

$$\begin{aligned}
&A_0(u) = u,\\
&A_1(u) = 2u\cosh u - 2\sinh u = 2u\big\{2(\cosh\frac{u}{2})^2 - 1\big\} - 4\cosh\frac{u}{2}\sinh\frac{u}{2}.
\end{aligned}$$

Hence

$$\begin{aligned}
P_1 = 2M_1 - M_0 &= \frac{1}{4}\int_0^\infty \tanh\frac{u}{2}\cdot\frac{A_1(u)}{(\sinh\frac{u}{2})^3}du - \frac{1}{2}\int_0^\infty \tanh\frac{u}{2}\cdot\frac{A_0(u)}{\sinh\frac{u}{2}}du\\
&= \frac{1}{2}\int_0^\infty \tanh\frac{u}{2}\cdot\frac{u\cosh u - \sinh u}{(\sinh\frac{u}{2})^3}du - \frac{1}{2}\int_0^\infty \tanh\frac{u}{2}\cdot\frac{u}{\sinh\frac{u}{2}}du\\
&= \frac{1}{2}\int_0^\infty u\tanh\frac{u}{2}\Big(\frac{\cosh u}{(\sinh\frac{u}{2})^3} - \frac{1}{\sinh\frac{u}{2}}\Big)du - \frac{1}{2}\int_0^\infty \tanh\frac{u}{2}\cdot\frac{\sinh u}{(\sinh\frac{u}{2})^3}\\
&= \frac{1}{2}\int_0^\infty \frac{u\cosh\frac{u}{2}}{(\sinh\frac{u}{2})^2}du - \int_0^\infty \frac{du}{\sinh\frac{u}{2}}\\
&= 2\int_0^\infty \frac{t\cosh t - \sinh t}{(\sinh t)^2}dt = 2\Big[-\frac{t}{\sinh t}\Big]_0^\infty = 2,
\end{aligned}$$

where in the last second equality we have put $u = 2t$. This completes the proof. $\square$

By Lemma 6.2 and Lemma 6.3 we immediately have the recurrence relation satisfied by J_n^1.

Proposition 6.4. *It holds that*

$$4n^2 J_n^1 - (8n^2 - 8n + 3)J_{n-1}^1 + 4(n-1)^2 J_{n-2}^1 = \frac{2^n(n-1)!}{(2n-1)!!}, \tag{6.20}$$

with $J_0^1 = 7\zeta(3)$ *and* $J_1^1 = \frac{21}{4}\zeta(3) + \frac{1}{2}$. *In particular, we have*

$$J_n^1 \in \mathbb{Q}\zeta(3) + \mathbb{Q}. \tag{6.21}$$

Proof. The value J_0^1 is obtained from (5.9). Therefore we have now only to calculate the value of J_1^1. Similarly to the calculation of J_1 in §4, we can get from (6.2) that

$$\begin{aligned}
J_1^1 = \sum_{k,\ell,m=0}^\infty \Bigg[&\frac{1}{(k+\ell+m+\frac{1}{2})^3} - \frac{1}{(k+\ell+m+\frac{1}{2})^2(k+\ell+m+\frac{5}{2})}\\
&- \frac{1}{(k+\ell+m+\frac{1}{2})(k+\ell+m+\frac{5}{2})^2} + \frac{1}{(k+\ell+m+\frac{5}{2})^3}\Bigg].
\end{aligned}$$

Since $p_3(N) = \#\{(k,\ell,m) \in \mathbb{Z}\,;\, k,\ell,m \geq 0, k+\ell+m = N\} = \frac{(N+1)(N+2)}{2}$, we have

$$\begin{aligned}
2J_1^1 &= \sum_{N=0}^\infty (N+1)(N+2)\Bigg[\frac{1}{(N+\frac{1}{2})^3} - \frac{1}{(N+\frac{1}{2})^2(N+\frac{5}{2})} - \frac{1}{(N+\frac{1}{2})(N+\frac{5}{2})^2} + \frac{1}{(N+\frac{5}{2})^3}\Bigg]\\
&= \sum_{N=0}^\infty \Bigg[\frac{3}{4}\frac{1}{(N+\frac{1}{2})^3} + \frac{3}{4}\frac{1}{(N+\frac{5}{2})^3} + \frac{13}{8}\Big\{\frac{1}{(N+\frac{1}{2})^2} - \frac{1}{(N+\frac{5}{2})^2}\Big\}\Bigg]\\
&= \frac{3}{4}\cdot 7\zeta(3) + \frac{3}{4}\Big\{7\zeta(3) - 2^3 - \big(\frac{2}{3}\big)^3\Big\} + \frac{13}{8}\Big\{2^3 + \big(\frac{2}{3}\big)^3\Big\}\\
&= \frac{21}{2}\zeta(3) + 1.
\end{aligned}$$

Hence the assertion follows. □

Remark 1. Since $\zeta(3) \notin \mathbb{Q}$ (see [Ap]), we may write $J_n^1 = A_n\zeta(3) + B_n$ $(A_n, B_n \in \mathbb{Q})$ from the proposition above. Then, it is immediate to see that the A_n's satisfy exactly the same recurrence equation (4.35) as the integrals J_n possess, and the B_n's satisfy (6.20) because the coefficients of (6.20) are all rational numbers and $B_0 = 0, B_1 = \frac{1}{2}$. Hence, the sequence $\{A_n\}_{n=0,1,\dots}$ is proportional to the sequence $\{J_n\}_{n=0,1,\dots}$. Since $A_0 = 7$ (note also, resp. $A_1 = \frac{21}{4}$) and $J_0 = 3\zeta(2)$ (resp. $J_1 = \frac{9}{4}\zeta(2)$) we conclude that $A_n = \frac{14}{\pi^2}J_n$. Namely, we obtain $J_n^1 = \frac{14}{\pi^2}J_n\zeta(3) + B_n$. □

Theorem 6.5. *Let D be the (singly confluent) Heun differential operator defined by (4.43):*

$$D = z(1-z)^2\frac{d^2}{dz^2} + (1-3z)(1-z)\frac{d}{dz} + \big(z - \frac{3}{4}\big).$$

Let $v(z) = \sum_{n=0}^{\infty} J_n^1 z^n$. Then the function $v(z)$ is holomorphic in $|z| < 1$ and is a (unique) solution of the singly confluent Heun's differential equation with an inhomogeneous term

$$Dv(z) = \frac{1}{2}\cdot {}_2F_1(1,1,\frac{3}{2};z) \tag{6.22}$$

satisfying $v(0) = 7\zeta(3)$ (and $v'(0) = \frac{1}{4}(21\zeta(3)+2)$). Here ${}_2F_1$ is the hypergeometric function ${}_2F_1(a,b,c;z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}z^n$. In the present case, one has ${}_2F_1(1,1,\frac{3}{2};\sin^2 x) = \frac{2x}{\sin 2x}$.

Moreover, the solution $v(z)$ has an integral representation

$$v(z) = 8\int_0^1\int_0^1\int_0^1 \frac{1-p^2q^2r^2}{(1-p^2q^2r^2)^2 - (1-p^4)(1-q^4r^4)z}dpdqdr. \tag{6.23}$$

Proof. First we show that the sequence of positive numbers J_n^1 is bounded. In fact, put $C_n = \frac{(1-a^2)^n(1-b^2)^n}{(1-ab)^{2n+1}}$. Since $(1-ab)^2 > (1-a^2)(1-b^2)$ for $0 < a, b < 1$, it is easy to show that $C_{n+1} < C_n$. Thus, in view of the definition (6.2) (the definite integral expression) of J_n^1 it can be shown that $J_{n+1}^1 < J_n^1$, in particular, J_n^1 is bounded by $J_0^1 = 7\zeta(3)$. Hence, the series $v(z) = \sum_{n=0}^{\infty} J_n^1 z^n$ converges absolutely in $|z| < 1$ and defines a holomorphic function.

Further, we have

$$\begin{aligned} 4Dv(z) &= \sum_{n=0}^{\infty}\Big[4(n+1)^2J_{n+1}^1 - \{8(n+1)^2 - 8(n+1) + 3\}J_n^1 + 4n^2J_{n-1}^1\Big]z^n \\ &= \sum_{n=0}^{\infty}\frac{2^{n+1}(n)!}{(2n+1)!!}z^n = 2\sum_{n=0}^{\infty}\frac{(1)_n(1)_n}{(\frac{3}{2})_n n!}z^n \\ &= 2\,{}_2F_1(1,1,\frac{3}{2};z). \end{aligned} \tag{6.24}$$

The proof of the integral representation is exactly the same as the one in Theorem 4.13. This completes the proof of the theorem. □

Remark 2. Note that the differential operator D in (4.43) does not depend on the choice of the parameters α and β. □

Note from Theorem 5.1 and (6.1) that

$$\zeta_Q(3) = \frac{1}{\big\{\alpha\beta(\alpha\beta-1)\big\}^{\frac{3}{2}}}\left[\frac{7(\alpha+\beta)^3}{4}\zeta(3) + \frac{3(\alpha+\beta)(\alpha-\beta)^2}{4}\sum_{n=0}^{\infty}(-1)^n\frac{(2n-1)!!}{2^n n!}J_n^1 a^{2n}\right], \tag{6.25}$$

where $a^2 = 1/(\alpha\beta - 1)$. Since $v(z)$ is a holomorphic solution of the inhomogeneous Heun equation (6.22) satisfying $v(0) = 7\zeta(3)$ and $v'(0) = \frac{1}{4}(21\zeta(3)+2)$ at $z = 0$ in Theorem 6.5, it is clear that

$$J_n^1 = \frac{1}{2\pi\sqrt{-1}}\int_{|z|=r} z^{-n-1}v(z)dz.$$

for $0 < r < 1$. Hence we may obtain an integral representation of the value $\zeta_Q(3)$ in terms of the this solution as in Corollary 4.14. In fact, using the binomial expansion (4.46) again we have

$$\sum_{n=0}^{\infty}(-1)^n \frac{(2n-1)!!}{2^n n!} J_n^1 a^{2n} = \frac{1}{2\pi\sqrt{-1}} \int_{|z|=r} z^{-1}(1+a^2 z^{-1})^{-1/2} v(z) dz$$

for $a^2 < r < 1$ provided $\alpha\beta > 2$. By analogous arguments used in the proof of Corollary 4.4, we hence obtain the following expression of $\zeta_Q(3)$.

Corollary 6.6. *Suppose that $\alpha\beta > 2$. Then, for $a^2 = \frac{1}{\alpha\beta-1} < r < 1$,*

$$\zeta_Q(3) = \frac{1}{4} \cdot \Big\{ \frac{\alpha+\beta}{\sqrt{\alpha\beta(\alpha\beta-1)}} \Big\}^3 \cdot \left[7\zeta(3) + \frac{3}{2\pi\sqrt{-1}} \Big(\frac{\alpha-\beta}{\alpha+\beta}\Big)^2 \int_{|z|=r} z^{-1}(1+a^2 z^{-1})^{-1/2} v(z) dz \right]. \quad \square \tag{6.26}$$

Remark 3. The integral expressions (4.42) and (6.23) naturally leads us to consider the function defined by the integral

$$w_n(z) = 2^n \int_0^1 \cdots \int_0^1 \frac{1 - \prod_{j=1}^n x_j^2}{\big(1-\prod_{j=1}^n x_j^2\big)^2 - (1-x_1^4)\big(1-\prod_{j=2}^n x_j^4\big) z} dx_1 \cdots dx_n. \tag{6.27}$$

It would be interesting if the integral (6.27) can give a (holomorphic) solution of the singly confluent type, inhomogeneous Heun equation of the form

$$D w_n(z) = D_n \times {}_2F_1(a_n, b_n, c_n; z)$$

in the unit disk $|z| < 1$, where D is the Heun operator defined by (4.43), and D_n, a_n, b_n and c_n are some constants determined by n. It should be noted that one has (see (3.12), (5.9))

$$w_n(0) = 2^n \int_0^1 \cdots \int_0^1 \frac{1}{1-\prod_{j=1}^n x_j^2} dx_1 \cdots dx_n = (2^n - 1)\zeta(n). \tag{6.28}$$

Besides the investigation about special values $\zeta_Q(n)$ relative to the values $\zeta(n)$, it is also interesting to study the integrals like (6.27) from the Fuchsian differential equation points of view. We shall treat these problems elsewhere. $\square$

Remark 4. Put $B(z) = \sum_{n=0}^{\infty} B_n z^n$, where $\{B_n\}_{n=0,1,\ldots}$ is the sequence defined in Remark 1 above. Then by the relation $J_n^1 = \frac{14}{\pi^2} J_n \zeta(3) + B_n$ we have

$$w_3(z) = \frac{14}{\pi^2}\,\zeta(3)\, w_2(z) + B(z). \quad \square \tag{6.29}$$

Appendix

The aim of this appendix is to establish some determinant formula which seems intimately connected with calculations of the special values of the spectral zeta function $\zeta_Q(s)$ at the positive integer $n > 1$. See Remark 2 (and also (5.16)) in §5.

Put $A(t) = -\frac{1}{2}\coth t$, $B(t) = \frac{1}{2}(\sinh t)^{-1}$. Then one remarks

$$A(t)^2 - B(t)^2 = \frac{1}{4}, \tag{A.1}$$

$$A'(t) = 2B(t)^2, \tag{A.2}$$

$$B'(t) = 2B(t)A(t). \tag{A.3}$$

Recall the definition of $p(t,x,y)$ in (2.3b): It is of the form

$$p(t,x,y) = C(t)\exp[A(t)(x^2+y^2) + 2B(t)xy],$$

whence

$$
\begin{aligned}
&p(t_1, x_1, x_2)p(t_2, x_2, x_3)\cdots p(t_n, x_n, x_1)\\
&=\prod_{i=1}^{n} C(t_i)\exp\left[\sum_{i=1}^{n} A(t_i)(x_i^2 + x_{i+1}^2) + 2\sum_{i=1}^{n} B(t_i)x_i x_{i+1}\right].
\end{aligned}
$$

Here we assume the periodicity $x_{n+1} = x_1$.

Define the symmetric matrix (see Remark 2 in §5)

$$
\begin{aligned}
&\mathbb{A} = \mathbb{A}(t)\\
&= \begin{pmatrix}
A(t_n)+A(t_1) & B(t_1) & 0 & 0 & \cdots & 0 & B(t_n)\\
B(t_1) & A(t_1)+A(t_2) & B(t_2) & 0 & \cdots & \cdots & 0\\
0 & B(t_2) & A(t_2)+A(t_3) & B(t_3) & \cdots & \cdots & 0\\
0 & 0 & B(t_3) & \cdots & \cdots & \cdots & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0 & \cdots & \cdots & \cdots & \cdots & B(t_{n-2}) & 0\\
0 & 0 & \cdots & \cdots & \cdots & A(t_{n-2})+A(t_{n-1}) & B(t_{n-1})\\
B(t_n) & 0 & \cdots & \cdots & 0 & B(t_{n-1}) & A(t_{n-1})+A(t_n)
\end{pmatrix}.
\end{aligned}
$$

Put

$$
\mathbb{A}(t,a) = \mathbb{A}_k(t,a) := \mathbb{A}(t) + \mathrm{diag}(-ia, ia, -ia, ia, \ldots, -ia, ia, 0, 0, \ldots, 0),
$$

where k denotes the number of the pairs $(-a, a)$ which appeared in the diagonal subtraction. If we need to specify the size of k we use $\mathbb{A}_k(t,a)$; otherwise we write it simply as $\mathbb{A}(t,a)$.

Lemma A.1. *The determinant* $\det\mathbb{A}(t,a)$ *of the matrix* $\mathbb{A}(t,a)$ *is even as a polynomial in* a*. In particular,* $\det\mathbb{A}(t,a)$ *is real-valued for* $a \in \mathbb{R}$*.*

Proof. Put $f(t,a) = \det\mathbb{A}(t,a)$. Then by the definition of the matrix $\mathbb{A}(t,a)$, it is easy to see that

$$
f(-t,a)(= f(-t_1, -t_2, \ldots, -t_n, a)) = (-1)^n f(t,-a). \tag{A.4}
$$

Hence

$$
\frac{\partial}{\partial a} f(-t,a) = (-1)^{n+1}\frac{\partial}{\partial a} f(t,-a), \tag{A.5}
$$

whence putting $a = 0$ we have

$$
\frac{\partial}{\partial a} f(-t,0) = (-1)^{n+1}\frac{\partial}{\partial a} f(t,0). \tag{A.6}
$$

On the other hand, if we differentiate $\det\mathbb{A}(t,a)$ (with respect to a) by column by column, then we have

$$
\begin{aligned}
&\frac{\partial}{\partial a} f(t,a)\\
&= \begin{vmatrix}
-i & B(t_1) & 0 & \cdots & \cdots & \cdots\\
0 & A(t_1)+A(t_2)+ia & B(t_2) & \cdots & \cdots & \cdots\\
0 & B(t_2) & A(t_2)+A(t_3)-ia & \cdots & \cdots & \cdots\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0 & \cdots & \cdots & \cdots & \cdots & \cdots
\end{vmatrix}\\
&+ \begin{vmatrix}
A(t_n)+A(t_1)-ia & 0 & 0 & \cdots & \cdots & \cdots\\
B(t_1) & i & B(t_2) & \cdots & \cdots & \cdots\\
0 & 0 & A(t_2)+A(t_3)-ia & \cdots & \cdots & \cdots\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0 & \cdots & \cdots & \cdots & \cdots & \cdots
\end{vmatrix} + \text{etc.}
\end{aligned}
$$

By this expression we see that

$$
\frac{\partial}{\partial a} f(-t,a) = (-1)(-1)^{n-1}\frac{\partial}{\partial a} f(t,-a) = (-1)^n\frac{\partial}{\partial a} f(t,-a). \tag{A.7}
$$

If we put $a=0$ in (A.7) and compare with (A.5) then we get $\frac{\partial}{\partial a}f(t,0)=0$.

Continuing the procedure above successively (by taking many derivatives with respect to a) we conclude that the coefficients of the odd degree vanish. $\square$

Remark 1. The lemma above agrees with the fact that $B'=0$ in (5.16). $\square$

The main calculation of the present appendix is as follows:

Theorem A.2.

$$\det \mathbb{A}(t) = (-1)^n \frac{(1-\prod_{j=1}^n e^{-t_j})^2}{\prod_{j=1}^n (1-e^{-2t_j})} = (-1)^n \frac{(1-e^{-\sum_{j=1}^n t_j})^2}{\prod_{j=1}^n (1-e^{-2t_j})} = (-1)^n \frac{(\sinh \sum_{j=1}^n t_j)^2}{\prod_{j=1}^n \sinh t_j}$$

Before going to the actual proof of the assertion we first show the following lemma.

Lemma A.3. *Consider the $n-1\times n-1$ matrix $P(t)$ for $n\geq 2$ given by*

$$P(t) = \begin{pmatrix} A(t_1)+A(t_2) & B(t_2) & 0 & \cdots & \cdots & 0 \\ B(t_2) & A(t_2)+A(t_3) & B(t_3) & \cdots & \cdots & 0 \\ 0 & B(t_3) & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & A(t_{n-2})+A(t_{n-1}) & B(t_{n-1}) \\ 0 & \cdots & \cdots & 0 & B(t_{n-1}) & A(t_{n-1})+A(t_n) \end{pmatrix}.$$

Then one has

$$\det P(t) = \det P(t_1,t_2,\dots,t_n) = (-1)^{n-1} \frac{1-\prod_{j=1}^n e^{-2t_j}}{\prod_{j=1}^n (1-e^{-2t_j})}. \tag{A.8}$$

Proof. We prove the formula by induction. For $n=2$ we have

$$A(t_1)+A(t_2) = -\frac{1}{2}\left(\frac{1+e^{-2t_1}}{1-e^{-2t_1}} + \frac{1+e^{-2t_2}}{1-e^{-2t_2}}\right) = -\frac{1-e^{-2t_1-2t_2}}{(1-e^{-2t_1})(1-e^{-2t_2})} \left(= \frac{B(t_1)B(t_2)}{B(T_1+t_2)}\right).$$

Assume the formula holds for k less than n. First note that if one expands the determinant by the first column we have

$$\det P(t_1,t_2,\dots,t_n) = (A(t_1)+A(t_2))\det P(t_2,t_3,\dots,t_n) - B(t_2)^2 \det P(t_3,t_4,\dots,t_n).$$

Hence the induction hypothesis implies

$$\begin{aligned}
&\det P(t_1,t_2,\dots,t_n) \\
&= -(-1)^{n-2}\frac{1-e^{-2t_1-2t_2}}{(1-e^{-2t_1})(1-e^{-2t_2})}\frac{1-\prod_{j=2}^n e^{-2t_j}}{\prod_{j=2}^n(1-e^{-2t_j})} - (-1)^{n-3}\frac{e^{-2t_2}}{(1-e^{-2t_2})^2}\frac{1-\prod_{j=3}^n e^{-2t_j}}{\prod_{j=3}^n (1-e^{-2t_j})} \\
&= (-1)^{n-1}\frac{(1-e^{-2t_1-2t_2})(1-\prod_{j=2}^n e^{-2t_j}) - e^{-2t_2}(1-e^{-2t_1})(1-\prod_{j=3}^n e^{-2t_j})}{(1-e^{-2t_2})\prod_{j=1}^n (1-e^{-2t_j})} \\
&= (-1)^{n-1}\frac{1-\prod_{j=1}^n e^{-2t_j}}{\prod_{j=1}^n(1-e^{-2t_j})}.
\end{aligned}$$

This completes the proof. $\square$

Proof of Proposition A.2. The strategy we now employ is to compare the first order derivative of the both sides in question with respect to the first variable. Thus we put

$$f(t_0) = \frac{(1-\prod_{j=0}^n e^{-t_j})^2}{\prod_{j=0}^n (1-e^{-2t_j})} = \frac{(1-e^{-t_0}\prod_{j=1}^n e^{-t_j})^2}{(1-e^{-2t_0})\prod_{j=1}^n (1-e^{-2t_j})} \tag{A.9}$$

and

$$a(t_0) = \det A(t_0, t_1, t_2, \dots, t_n). \tag{A.10}$$

An elementary manipulation yields

$$f'(t) = \frac{2}{D(1-e^{-2t})^2}\left[Ce^{-t} - (C^2+1)e^{-2t} + Ce^{-3t}\right], \tag{A.11}$$

where we put

$$C := \prod_{j=1}^{n} e^{-t_j}, \quad D := \prod_{j=1}^{n}(1-e^{-2t_j}).$$

By taking the derivative and using the relations (A.2) and (A.3) we observe that

$$\begin{aligned}
a'(t) =& 2B(t)\begin{vmatrix} B(t) & B(t) & 0 & \cdots & \cdots & B(t_n) \\ A(t) & A(t)+A(t_1) & B(t_1) & 0 & \cdots & 0 \\ 0 & B(t_1) & A(t_1)+A(t_2) & B(t_2) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} \\
&+2B(t)\begin{vmatrix} A(t_n)+A(t) & A(t) & 0 & \cdots & \cdots & B(t_n) \\ B(t) & B(t) & B(t_1) & 0 & \cdots & 0 \\ 0 & 0 & A(t_1)+A(t_2) & B(t_2) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ B(t_n) & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} \\
=& 2B(t)\begin{vmatrix} B(t) & 0 & 0 & \cdots & \cdots & B(t_n) \\ A(t) & A(t_1) & B(t_1) & 0 & \cdots & 0 \\ 0 & B(t_1) & A(t_1)+A(t_2) & B(t_2) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} \\
&-2B(t)\begin{vmatrix} A(t) & A(t_n) & 0 & \cdots & \cdots & B(t_n) \\ B(t) & 0 & B(t_1) & 0 & \cdots & 0 \\ 0 & 0 & A(t_1)+A(t_2) & B(t_2) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & B(t_n) & \cdots & \cdots & \cdots & \cdots \end{vmatrix}.
\end{aligned}$$

Here we omit the detailed description of the down-right corner since the point we have to remark here is those coefficients do not involve the variable "t".

Let us now call the first determinant and the second one in the last expression above by Q and R respectively. By noting the fact $A(\frac{\pi}{2}i) = 0$, we see that

$$\begin{aligned}
Q =& 2B(t)^2 \det P(\frac{\pi}{2}i, t_1, t_2, \dots, t_n) \\
&-2B(t)A(t)\begin{vmatrix} 0 & 0 & \cdots & \cdots & \cdots & \cdots & B(t_n) \\ B(t_1) & A(t_1)+A(t_2) & B(t_2) & \cdots & \cdots & \cdots & 0 \\ 0 & B(t_2) & A(t_2)+A(t_3) & B(t_3) & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & B(t_{n-1}) \\ 0 & \cdots & \cdots & \cdots & \cdots & B(t_{n-1}) & A(t_{n-1})+A(t_n) \end{vmatrix} \\
=& 2B(t)^2 \det P(\frac{\pi}{2}i, t_1, t_2, \dots, t_n) - (-1)^{n+1}2B(t)A(t)\cdot B(t_n)\cdot B(t_1)\cdots B(t_{n-1}) \\
=& 2B(t)^2 \det P(\frac{\pi}{2}i, t_1, t_2, \dots, t_n) + (-1)^n 2B(t)A(t)\prod_{j=1}^{n} B(t_j).
\end{aligned}$$

Using Lemma A.2 we calculate

$$Q = 2(-1)^n B(t)^2 \cdot \frac{1+\prod_{j=1}^n e^{-2t_j}}{2\prod_{j=1}^n (1-e^{-2t_j})} + 2(-1)^n B(t)A(t) \cdot \frac{\prod_{j=1}^n e^{-t_j}}{\prod_{j=1}^n (1-e^{-2t_j})}. \tag{A.12}$$

We next evaluate the determinant R as follows.

$$\begin{aligned}
R = -2B(t)A(t) &\begin{vmatrix} 0 & B(t_1) & 0 & \cdots & \cdots & 0 \\ 0 & A(t_1)+A(t_2) & B(t_2) & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & B(t_{n-1}) \\ B(t_n) & 0 & \cdots & \cdots & \cdots & A(t_{n-1})+A(t_n) \end{vmatrix} \\
+ 2B(t)^2 &\begin{vmatrix} A(t_n) & 0 & \cdots & \cdots & \cdots & B(t_n) \\ 0 & A(t_1)+A(t_2) & B(t_2) & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & B(t_{n-1}) \\ B(t_n) & 0 & \cdots & \cdots & \cdots & A(t_{n-1})+A(t_n) \end{vmatrix} \\
&= -2B(t)A(t)(-1)^{n+1} B(t_n)\cdot B(t_1)\cdots B(t_{n-1}) \\
&\quad + 2B(t)^2 \left\{ A(t_n) \det P(t_1,t_2,\dots,t_n) + (-1)^{n+1}B(t_n)\cdot(-1)^n B(t_n)\cdot \det P(t_1,t_2,\dots,t_{n-1}) \right\} \\
&= (-1)^n 2B(t)A(t)\prod_{j=1}^n B(t_j) + 2B(t)^2 \left\{ A(t_n)\det P(t_1,t_2,\dots,t_n) - B(t_n)^2 \det P(t_1,t_2,\dots,t_{n-1}) \right\}
\end{aligned}$$

Hence one has

$$\begin{aligned}
R =& (-1)^n 2B(t)A(t) \cdot \frac{\prod_{j=1}^n e^{-t_j}}{\prod_{j=1}^n (1-e^{-2t_j})} \\
&+ 2B(t)^2 \Bigg[-\frac{1}{2}\cdot \frac{1+e^{-2t_n}}{1-e^{-2t_n}} (-1)^{n-1} \frac{1-\prod_{j=1}^n e^{-2t_j}}{\prod_{j=1}^n (1-e^{-2t_j})} \\
&\qquad - (-1)^{n-2} \frac{e^{-2t_n}}{(1-e^{-2t_n})^2} \frac{(1-\prod_{j=1}^{n-1} e^{-2t_j})}{\prod_{j=1}^{n-1}(1-e^{-2t_j})} \Bigg].
\end{aligned} \tag{A.13}$$

Combining (A.12), (A.13) with (A.8) we have

$$\begin{aligned}
a'(t) &= Q + R \\
=& 4(-1)^n B(t)A(t) \cdot \frac{\prod_{j=1}^n e^{-t_j}}{\prod_{j=1}^n (1-e^{-2t_j})} \\
&+ 2(-1)^n B(t)^2 \left(\frac{1+\prod_{j=1}^n e^{-2t_j}}{2\prod_{j=1}^n (1-e^{-2t_j})} + \frac{1+e^{-2t_n}}{1-e^{-2t_n}} \frac{1+\prod_{j=1}^n e^{-2t_j}}{2\prod_{j=1}^n (1-e^{-2t_j})} - \frac{e^{-t_n}}{1-e^{-2t_n}} \frac{(e^{-t_n} - \prod_{j=1}^n e^{-2t_j})}{\prod_{j=1}^n (1-e^{-2t_j})} \right) \\
=& 4(-1)^{n+1} \frac{e^{-t}(1+e^{-2t})}{2(1-e^{-2t})^2} \cdot \frac{\prod_{j=1}^n e^{-t_j}}{\prod_{j=1}^n (1-e^{-2t_j})} + 2(-1)^n \frac{e^{-2t}}{(1-e^{-2t})^2} \cdot \frac{1+\prod_{j=1}^n e^{-2t_j}}{\prod_{j=1}^n (1-e^{-2t_j})} \\
=& \frac{2(-1)^{n+1}}{(1-e^{-2t})^2} \left(\frac{C}{D}(e^{-t}+e^{-3t}) - \frac{1+C^2}{D} e^{-2t} \right) \\
=& \frac{2(-1)^{n+1}}{D(1-e^{-2t})^2} \left(C(e^{-t}+e^{-3t}) - (1+C^2)e^{-2t} \right).
\end{aligned}$$

Comparing (A.11) with the expression above we conclude that

$$a'(t) = (-1)^{n+1} f'(t), \tag{A.14}$$

whence we may write

$$a(t) - (-1)^{n+1} f(t) = K \tag{A.15}$$

with K being a constant independent of the variable t but dependent on $t_1, \dots, t_n$. In order to determine the constant K we consider the limit for $t \to +\infty$. We have immediately from the definition of $f(t)$ that

$$\lim_{t\to+\infty} f(t) = \frac{1}{D}. \tag{A.16}$$

If $t \to +\infty$, the definition of $a(t) = \det A(t, t_1, t_2, \dots, t_n)$ tells

$$\lim_{t\to+\infty} a(t) = \begin{vmatrix} A(t_n)+A(\infty) & 0 & 0 & \cdots & \cdots & B(t_n) \\ 0 & A(\infty)+A(t_1) & B(t_1) & \cdots & \cdots & 0 \\ 0 & B(t_1) & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & A(t_{n-2})+A(t_{n-1}) & B(t_{n-1}) \\ B(t_n) & \cdots & \cdots & 0 & B(t_{n-1}) & A(t_{n-1})+A(t_n) \end{vmatrix}$$
$$=(A(t_n)+A(\infty)) \det P(\infty, t_1, t_2, \dots, t_n) + (-1)^{n+2}(-1)^{n+1} B(t_n)^2 \det P(\infty, t_1, t_2, \dots, t_{n-1}).$$

Since it is easy to see that $\det P(\infty, t_1, t_2, \dots, t_n) = (-1)^n D^{-1}$, we have

$$\begin{aligned}\lim_{t\to+\infty} a(t) &= -\frac{1}{2} \cdot \frac{2}{1-e^{-2t_n}} \frac{(-1)^n}{D} - \frac{e^{-2t_n}}{(1-e^{-2t_n})^2} \frac{(-1)^n}{\prod_{j=1}^{n-1}(1-e^{-2t_j})} \\ &= \frac{(-1)^{n+1}}{D(1-e^{-2t_n})} + \frac{(-1)^n e^{-2t_n}}{D(1-e^{-2t_n})} = \frac{(-1)^{n+1}}{D}.\end{aligned}$$

By (A.16) this shows that

$$K = \lim_{t\to+\infty} \big(a(t) - (-1)^{n+1} f(t)\big) = 0,$$

whence we have $a(t) = (-1)^{n+1} f(t)$. This completes the proof of the theorem. □

Remark 2. It is interesting to calculate $\det \mathbb{A}_k(t, a)$ for general k. For instance, the coefficient of a^2 in $\det \mathbb{A}_2(t, a)$ for $n = 4$ is calculated as

$$\begin{aligned}&1 + (A(t_1)+A(t_3))(A(t_2)+A(t_4)) \\ =&1 + \frac{(1-e^{-2(s+u)})(1-e^{-2(t+v)})}{(1-e^{-2s})(1-e^{-2t})(1-e^{-2u})(1-e^{-2v})}.\end{aligned}$$

Apart from the present study of the special values of $\zeta_Q(s)$ at the positive integers, it is also interesting to establish special formulas similar to the one in this appendix in some general context such as Dynkin's diagram for the root system of a simple Lie algebra, etc. □

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