# Zero-energy resonances and the flux -across-surfaces theorem

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**Abstract**

The flux-across-surfaces conjecture represents a corner stone in quantum scattering theory because it is the key-assumption needed to prove the usual relation between differential cross section and scattering amplitude. We improve a recent result [TDMB] by proving the conjecture also in presence of zero-energy resonances or eigenvalues, both in point and potential scattering.

## 1 Introduction

In the framework of quantum scattering theory (QST), experimental data and theoretical predictions are usually compared by using the familiar formula which relates the differential cross section and the scattering amplitude.
Focusing on the case of potential scattering, we notice that this familiar formula follows rigorously [1] from the following basic assumption: the probability $P(\Sigma, \Psi_0)$ that a particle is detected by an apparatus with active surface $\Sigma$ when the beam is prepared in the state $\Psi_0$, is related to the asymptotic outgoing state $\Psi_{\text{out}}$ by the formula

$$P(\Sigma, \Psi_0) = \int_{C(\Sigma)} |\widehat{\Psi}_{\text{out}}(\mathrm{k})|^2 \, d^3\mathrm{k} \tag{1}$$

where $C(\Sigma) = \{\lambda \mathrm{x} \in \mathbb{R}^3 : \mathrm{x} \in \Sigma, \lambda \geq 0\}$ is the cone generated by $\Sigma$ and the symbol $\widehat{\ }$ denotes Fourier transform. The problem of deducing (1) from more basic principles, is then a corner-stone in the foundations of quantum scattering theory.

The first answer to this problem was *Dollard's theorem* [Do]. This theorem claims that, assuming existence and asymptotic completeness of the wave operators $W_{\pm} = \lim_{t \to \pm\infty} e^{iHt} e^{-iH_0 t}$ (where $H_0 = -\Delta$ with domain $H^2(\mathbb{R}^3)$ and $H = H_0 + V$), one has

$$\lim_{t \to \infty} \int_{C(\Sigma)} |\Psi_t(\mathrm{x})|^2 \, d^3\mathrm{x} = \int_{C(\Sigma)} |\widehat{\Psi}_{\text{out}}(\mathrm{k})|^2 \, d^3\mathrm{k}$$

where $\Psi_t = e^{-iHt}\Psi_0$ with $\Psi_0 = W_- \Psi_{\text{in}} = W_+^* \Psi_{\text{out}}$. Dollard's theorem gives a possible answer to the problem (1) if one assumes that $P(\Sigma, \Psi_0)$ can be identified with the probability that the particle will be (detected) in the cone $C(\Sigma)$ in the distant future.

[1] See for example [AJS], Ch. 7.

However, the experimental measurement is more closely related to the probability that the particle is detected by the detector surface $\Sigma$ at *any* time in the interval during which the detector is operating. This quantity can be computed by integrating the probability density current $\mathrm{j}^{\Psi_t} := \mathrm{Im}(\Psi_t^* \nabla \Psi_t)$ on the surface $\Sigma$ over the relevant time interval $(T_1, T_2)$. For the foundations of QST, it is then relevant to prove that

$$\lim_{R\to\infty} \lim_{T_2\to\infty} \int_{T_1}^{T_2} dt \int_{\Sigma_R} \mathrm{j}^{\Psi_t} \cdot \mathrm{n}\, d\sigma = \int_{C(\Sigma)} |\widehat{\Psi}_{\mathrm{out}}(\mathrm{k})|^2\, d^3\mathrm{k} \tag{FAS}$$

where $\Sigma_R = C(\Sigma) \cap S_R$ and $S_R = \{\mathrm{x} \in \mathbb{R}^3 : |\mathrm{x}| = R\}$.

This *Flux-Across-Surfaces conjecture* was formulated by Combes, Newton e Shtokhamer in 1975 [CNS]. It was proved by Daumer, Dürr, Goldstein, e Zanghì in [DDGZ$_1$] for the case $V = 0$ (see also [DT] for a simpler proof ). Then (FAS) has been proved for a wide class of *short range* potentials by Amrein and Zuleta and *long range* potentials by Amrein and Pearson. In [AP] and [AZ] it is assumed that $\widehat{\Psi}_{\mathrm{out}}$ has a compact support away from the origin; such condition – from a physical viewpoint – can be seen as an *energy cut-off* . In [TDMB] Teufel, Dürr and Münch-Berndl proved the FAS conjecture, for a different class of potentials, without assuming this energy cut-off .

Despite the generality of these results, none of them relates to the case in which there exists a zero-energy resonance or eigenvalue (for a rigorous definition of zero-energy resonance see Definition 3.4 and Sec. 4). Indeed, in [TDMB] this eventuality it is explicitly ruled out in the hypotheses. In [AP] and [AZ] it is assumed that $\widehat{\Psi}_{\mathrm{out}}$ has a compact support away from the origin and on this (dense) class of states a zero-energy resonance is harmless. However, a standard density argument cannot be applied, since the r.h.s of (FAS) is not continuous in $\Psi_0$ (nor in $\Psi_{\mathrm{in}}$) in the $L^2$ topology.

In this paper we focus precisely on the case in which there exist a zero-energy resonance for the pair $(H, H_0)$ or zero is an eigenvalue for the operator $H$ (we will refer to this situation as the *resonant case*). A zero-energy resonance affects the long-time behavior of the wavefunction, and therefore the proof of FAS theorem.
Notice that the usual mapping properties of the (inverse) wave operators – which, roughly speaking, guarantee that the outgoing state $\Psi_{\mathrm{out}}$ inherits the same smoothness properties of the initial state $\Psi_0$ – fails to hold in the resonant case (see [Ya]). In particular, we will show (see Proposition 3.8) that if there exists a zero-energy resonance the asymptotic outgoing state generally has a singularity in the origin of momentum space, in spite of the smoothness of the initial state. This shows that in the resonant case it is not natural at all to assume a smoothness condition on $\widehat{\Psi}_{\mathrm{out}}$, as done in [AP], [AZ] and [TDMB] in the regular case.
It is interesting to notice that the scattering operator $S = W_+^* W_-$ maintains some nice mapping properties also in the resonant case, as can be seen by using a stationary representation for $S$ (see, e.g. [Ag], Th. 7.2 ) and our Proposition 3.8.
However, we believe that the use of $\Psi_{\mathrm{in}}$ rather than $\Psi_0$ is somehow unphysical since the preparation procedure of the system is performed at some past but finite time. In contrast the limit $T_2 \to \infty$, which introduces the asymptotic state $\Psi_{\mathrm{out}}$, can be regarded as a reasonable approximation, convenient to make connection with scattering theory. Since the FAS problem is deeply related to its physical counterpart, we prefer to assume hypotheses only on the initial state $\Psi_0$. This reflects, as we have pointed out, in some additional mathematical troubles related to the singularity of $\widehat{\Psi}_{\mathrm{out}}$.

In Sec. 2 we prove (FAS) in the case of point interaction scattering. Although this solvable

model was already treated in [PT] we present an entirely different proof, using ideas which will be then applied to the general case of potential scattering. In Sec. 3. we will treat the case of potential scattering, by studying the behavior of the Lippman-Schwinger eigenfunctions in the resonant case (see Proposition 3.8) and using this result to prove the FAS theorem.
In Sec. 4 we summarize some previous results on zero-energy resonances and we prove that, under suitable assumption on the potential, some definitions of zero-energy resonance that can be found in the literature (and that we use in Sec. 3) are indeed equivalent. Although this is more or less common knowledge in scattering theory, we were not able to find in the literature a result analogous to Proposition 4.3.

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**Convention.** We denote with $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space of fast-decreasing smooth functions, with $\mathcal{S}'(\mathbb{R}^d)$ the space of tempered distributions and with $\langle \ldots, \ldots \rangle$ the *sesquilinear* pairing between them. For any $u \in \mathcal{S}'(\mathbb{R}^d)$ we denote its Fourier transform (resp. antitransform) as $\mathcal{F}u = \hat{u}$ (resp. $\mathcal{F}^{-1}u = \check{u}$). Derivatives and Fourier transforms will be always intended in the sense of tempered distributions.

**Convention.** Let be $\mathcal{E}$ a Banach space. We denote as $\mathcal{B}(\mathcal{E})$ the algebra of the bounded operators on $\mathcal{E}$ and as $\mathcal{B}_\infty(\mathcal{E})$ the ideal of the compact operators on $\mathcal{E}$.

## 2 A solvable example: point interaction scattering

Point interaction is a widely used model to describe physical situations in which a particle interacts with a potential whose range is negligible with respect to the de Broglie wavelength of the particle. The point interaction hamiltonian will be denoted as $H_{\gamma,\mathrm{y}}$ where $\mathrm{y} \in \mathbb{R}^3$ denotes the point in which the interaction is localized and $\gamma \in \mathbb{R}$ is a parameter related to the strength of the interaction (in particular $\gamma = +\infty$ corresponds to the free case).

From a mathematical point of view (see [AGHH] and references therein) $H_{\gamma,\mathrm{y}}$ can be rigorously defined by using the von Neumann-Kreĭn extension theory. It is known that the continuous spectrum is purely absolutely continuous and $\sigma_{\mathrm{ac}}(H_{\gamma,\mathrm{y}}) = [0, +\infty)$. The point spectrum is empty if $\gamma \leq 0$ and $\sigma_{\mathrm{p}}(H_{\gamma,\mathrm{y}}) = \left\{-(4\pi\gamma)^2\right\}$ for $\gamma > 0$. For $\gamma = 0$ the hamiltonian exhibits a zero-energy resonance.
By using the generalized eigenfunctions

$$\Phi_\pm(\mathrm{x}, \mathrm{k}) = e^{i\mathrm{k}\cdot\mathrm{x}} + \frac{e^{i\mathrm{k}\cdot\mathrm{y}}}{(4\pi\gamma \pm i|\mathrm{k}|)} \frac{e^{\mp i|\mathrm{k}||\mathrm{x}-\mathrm{y}|}}{|\mathrm{x}-\mathrm{y}|} \tag{2}$$

one can define two unitary maps $\mathcal{F}_\pm : \mathcal{H}_{ac}(H_{\gamma,\mathrm{y}}) \to L^2(\mathbb{R}^3)$ by posing

$$(\mathcal{F}_\pm f)(\mathrm{k}) = \operatorname*{s\,lim}_{R\to+\infty} \int_{B_R} \Phi^*_\pm(\mathrm{x}, \mathrm{k}) f(\mathrm{x}) \ (2\pi)^{-3/2} dx.$$

The operators $\mathcal{F}_\pm$ spectralize the operator $H_{\gamma,\mathrm{y}}$ and are related to the wave operators $W_\pm = \lim_{t\to\pm\infty} e^{H_{\gamma,\mathrm{y}}t} e^{-iH_0 t}$ (here $H_0 = -\frac{1}{2}\Delta$ ) by the intertwining properties

$$W_\pm^{-1} = \mathcal{F}^{-1}\mathcal{F}_\pm \qquad \text{and} \qquad W_\pm = \mathcal{F}_\pm^{-1}\mathcal{F} \tag{3}$$

where $\mathcal{F}$ is the usual Fourier transform.
Since the operators $H_{\gamma,\mathrm{y}}$ for different choice of $\mathrm{y} \in \mathbb{R}^3$ are unitarly equivalent, in the following we will consider only the case $\mathrm{y} = 0$.

As previously pointed out, when the hamiltonian exhibits a zero-energy resonance (i.e. for $\gamma = 0$), the asymptotic outgoing state $\Psi_{\text{out}}$ is singular in momentum space, in spite of the smoothness of the initial state $\Psi_0$. This can be immediately seen by noticing that

$$\begin{aligned}
\widehat{\Psi}_{\text{out}}(\mathrm{k}) &= (\mathcal{F}_+\Psi_0)(\mathrm{k}) = \int_{\mathbb{R}^3} \Phi_+(\mathrm{x},\mathrm{k})^* \Psi_0(\mathrm{x})\ (2\pi)^{-\frac{3}{2}} dx \\
&= \widehat{\Psi}_0(\mathrm{k}) + \frac{1}{4\pi\gamma - i|\mathrm{k}|} \int_{\mathbb{R}^3} \frac{e^{i|\mathrm{k}||\mathrm{x}|}}{|\mathrm{x}|} \Psi_0(\mathrm{x})\ (2\pi)^{-\frac{3}{2}} dx
\end{aligned} \tag{4}$$

and recalling that the zero-energy resonance corresponds to $\gamma = 0$.

However, in the solvable case of point interaction, it is possible to show that $\widehat{\Psi}_{\text{out}}$ has a smooth behavior outside the origin and some decreasing behavior at infinity. More precisely, we can prove the following lemma.

**Lemma 2.1 (Asymptotic decrease of the outgoing state)** *Let be $\Psi_0 \in \mathcal{S}(\mathbb{R}^3)$. Let us define $\Psi_{\text{out}} := W_+^{-1}\Psi_0$ where $W_\pm$ are the wave operators with respect to the pair $(H_{\gamma,\mathrm{y}}, H_0)$ for $\gamma = 0$. Then $\widehat{\Psi}_{\text{out}} \in C^\infty(\mathbb{R}^3\backslash\{0\})$ and for every $m \in \mathbb{N}$ there exist positive constants $C_m$ and $K_m$ such that*

$$\left| \frac{\partial^m}{\partial |\mathrm{k}|^m} \widehat{\Psi}_{\text{out}}(\mathrm{k}) \right| \le \frac{C_m}{|\mathrm{k}|^{3+m}} \qquad \textit{for} \qquad |\mathrm{k}| \ge K_m. \tag{5}$$

**Proof.** Since $\widehat{\Psi}_0 \in \mathcal{S}(\mathbb{R}^3)$, we can consider only the second term appearing in the second line of equation (4) and we will denote it as $\zeta(k)$ where $k = |\mathrm{k}|$. By posing $\psi(x) = \int_{S_1} \Psi_0(x\omega)\ d\omega$, we get

$$\zeta(k) = \frac{i}{k} \int_0^{+\infty} e^{ikx} x\psi(x)\ dx \tag{6}$$

A dominated regularity argument immediately shows that $\zeta$ is $C^\infty$ for $k \in (0, +\infty)$. As for the behavior at infinity, we notice that

$$\begin{aligned}
\zeta(k) &= \frac{i}{k} \int_0^{+\infty} e^{ikx} \left( \psi(x) - \psi(0)\ e^{-x^2} - \psi'(0)\ xe^{-x^2} \right) x\ dx + \\
&\quad + \frac{i}{k}\psi(0) \int_0^{+\infty} e^{ikx}\ e^{-x^2} x\ dx + \frac{i}{k}\psi'(0) \int_0^{+\infty} e^{ikx}\ e^{-x^2} x^2\ dx \\
&= \zeta_1(k) + \zeta_2(k) + \zeta_3(k)
\end{aligned}$$

where $\psi'$ is the right derivative of $\psi$ with respect to $x$. The second and third term can be computed exactly, getting that $\zeta_2(k) \asymp k^{-3}$ and $\zeta_2(k) \asymp k^{-4}$ as $k \to +\infty$. For the first term, we notice that

$$k^3 \zeta_1(k) = -i \int_0^{+\infty} \left( \frac{d^2}{dx^2} e^{ikx} \right) \left( \psi(x) - \psi(0)\ e^{-x^2} - \psi'(0)\ xe^{-x^2} \right)\ dx.$$

By integrating by parts twice it follows that $|\zeta_1(k)| \le Ck^{-3}$. This proves (5) for $m = 0$. By differentiating explicitly (6) and using similar arguments, one obtains (5) for any $m \in \mathbb{N}$. ■

Using this lemma, we can prove the FAS theorem for the point-interaction scattering without making *ad hoc* assumptions on the asymptotic outgoing state. Although the statement is identical to [PT, Theorem 1] we present a completely different proof, which can be generalized to the case of potential scattering. With the previous notation, our result is the following.

**Theorem 2.2** *Let us fix* $\Psi_0 \in \mathcal{S}(\mathbb{R}^3) \cap \mathcal{H}_{ac}(H_{\gamma,\mathrm{y}})$. *Then* $\Psi_t := e^{-iH_{\gamma,\mathrm{y}}t}\Psi_0$ *is continuously differentiable in* $\mathbb{R}^3 \backslash \{\mathrm{y}\}$ *and relation (FAS) holds true, for every* $T_1 \in \mathbb{R}$.

We focus on the proof in the resonant case $\gamma = 0$ (see [PT] for $\gamma \neq 0$). To clarify the structure of the proof, we will decompose it in some steps. To streamline the exposition, it is convenient to introduce the following notation.

**Definition 2.3** *Fix* $\nu \in \mathbb{R}_+$ *(We are interested in cases* $\nu = \frac{1}{2}$ *and* $\nu = 1$*). Consider an interval* $[a,b] \subseteq \mathbb{R}$. *We say that* $F : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{C}$ *is of type* $\mathcal{O}_{[a,b]}(\frac{|\mathrm{x}|}{t^\nu})$ *if there exist* $T_0 > 0$, $R_0 > 0$ *such that*

$$\operatorname*{Sup}_{|\mathrm{x}| \geq R_0, t \geq T_0} \left( \frac{|\mathrm{x}|}{t^\nu} \right)^\tau |F(\mathrm{x},t)| \leq C_{\nu,\tau} \tag{7}$$

*for each* $\tau \in [a,b]$. *In this case we write* $F = \mathcal{O}_{[a,b]}(\frac{|\mathrm{x}|}{t^\nu})$.

If $[a,b] = [0,n]$ (with $n \in \mathbb{N}$) we write, with a harmless abuse of notation, $F = \mathcal{O}_n(\frac{|\mathrm{x}|}{t^\nu})$. The previous definitions are trivially extended to the case of $\mathbb{C}^d$-valued functions.

**Preliminaries.** Since $\mathcal{F}_+$ spectralize the hamiltonian $H_{\gamma,\mathrm{y}}$ and $\mathcal{F}_+\Psi_0 = \widehat{\Psi}_{\mathrm{out}}$ we get

$$\begin{aligned}
\Psi_t(\mathrm{x}) &= \int_{\mathbb{R}^3} e^{-i\mathrm{k}^2 t} \widehat{\Psi}_{\mathrm{out}}(\mathrm{k}) \Phi_+(\mathrm{x},\mathrm{k}) \, (2\pi)^{-3/2} dk \\
&= \int_{\mathbb{R}^3} e^{-i\mathrm{k}^2 t} \widehat{\Psi}_{\mathrm{out}}(\mathrm{k}) e^{i\mathrm{k}\cdot\mathrm{x}} \, (2\pi)^{-3/2} dk + \int_{\mathbb{R}^3} e^{-i\mathrm{k}^2 t} \widehat{\Psi}_{\mathrm{out}}(\mathrm{k}) \frac{1}{i|\mathrm{k}|} \frac{e^{-i|\mathrm{k}||\mathrm{x}|}}{|\mathrm{x}|} (2\pi)^{-3/2} dk \\
&\equiv \alpha(\mathrm{x},t) + \beta(\mathrm{x},t) \, .
\end{aligned}$$

Then the probability density current is

$$\mathrm{j}^{\Psi_t} = \mathrm{Im}(\alpha^* \nabla \alpha + \alpha^* \nabla \beta + \beta^* \nabla \alpha + \beta^* \nabla \alpha). \tag{8}$$

The first term $\mathrm{j}_0 = \mathrm{Im}(\alpha^* \nabla \alpha)$ corresponds to the free evolution of $\Psi_{\mathrm{out}}$, so using the free flux-across-surfaces theorem [DDGZ$_1$] one has

$$\lim_{R \to \infty} \int_T^{+\infty} dt \int_{\Sigma_R} \mathrm{j}_0(\mathrm{x},t) \cdot \mathrm{n} \, d\sigma = \int_{C(\Sigma)} |\widehat{\Psi}_{\mathrm{out}}(\mathrm{k})|^2 \, dk \, .$$

Therefore, to prove Theorem 2.2 what remains to show is that

$$\lim_{R \to \infty} \int_T^{+\infty} dt \int_{\Sigma_R} |\mathrm{j}_1(\mathrm{x},t) \cdot \mathrm{n}| \, d\sigma = 0 \tag{9}$$

where $\mathrm{j}_1 := \mathrm{Im}(\alpha^* \nabla \beta + \beta^* \nabla \alpha + \beta^* \nabla \beta)$. In order to prove (9) we need estimates on $\alpha, \beta$ and their gradients.

**Estimates on $\alpha$ and $\nabla\alpha$.** First of all, we decompose $\alpha$ as $\alpha_{\text{reg}} + \alpha_{\text{sing}}$ by extracting the singular part of $\widehat{\Psi}_{\text{out}}$, which can be read from (4). More precisely, we pose

$$f_1(\mathrm{k}) := \widehat{\Psi}_{\text{out}}(\mathrm{k}) - \frac{r}{|\mathrm{k}|}e^{-|\mathrm{k}|^2} \qquad \text{where} \qquad r := i\int_{\mathbb{R}^3} \frac{1}{|\mathrm{x}|}\Psi_0(\mathrm{x})\ dx \tag{10}$$

and then we define

$$\begin{aligned} \alpha_{\text{reg}}(\mathrm{x},t) &= \int_{\mathbb{R}^3} e^{i\mathrm{k}\cdot\mathrm{x}}e^{-ik^2t}\, f_1(\mathrm{k})(2\pi)^{-3/2}dk \\ \alpha_{\text{sing}}(\mathrm{x},t) &= -\int_{\mathbb{R}^3} e^{i\mathrm{k}\cdot\mathrm{x}}e^{-ik^2t}\frac{r}{|\mathrm{k}|}e^{-|\mathrm{k}|^2}\,(2\pi)^{-3/2}dk. \end{aligned}$$

Moreover,

$$\nabla\alpha(\mathrm{x},t) = i\int_{\mathbb{R}^3} e^{ik\cdot\mathrm{x}}e^{-ik^2t}\mathrm{k}\widehat{\Psi}_{\text{out}}(\mathrm{k})\,(2\pi)^{-3/2}dk.$$

The properties of $\alpha_{\text{reg}}$ and $\nabla\alpha$ are given in the following lemma, which will be useful also in the general case of potential scattering.

**Lemma 2.4 (Free evolution of a slow-decreasing state)** *Let us suppose that $f \in C^5(\mathbb{R}^3\setminus\{0\})$ satisfies the following assumptions:*

*a)* ***regularity in a neighborhood of the origin:*** *there exists a suitable punctured neighborhood $U_0$ of the origin such that*

$$\partial^\mu f \in L^\infty(U_0) \tag{11}$$

*for every multi-index $\mu \in \mathbb{N}^3$ with $1 \le |\mu| \le 5$;*

*b)* ***decrease at infinity:*** *for every multi-index $\mu \in \mathbb{N}^3$ with $1 \le |\mu| \le 5$ there exists positive $C_\mu$ and $K_\mu$ such that*

$$|\partial^\mu f(\mathrm{k})| \le \frac{C_\mu}{|\mathrm{k}|^{3+|\mu|}} \qquad \text{for } |\mathrm{k}| \ge K_\mu. \tag{12}$$

*Then*

$$\alpha_f(\mathrm{x},t) \equiv \int_{\mathbb{R}^3} e^{ik\cdot\mathrm{x}}e^{-ik^2t}f(\mathrm{k})\ dk = \frac{1}{t^{3/2}}\mathcal{O}_1\left(\frac{|\mathrm{x}|}{t}\right). \tag{13}$$

**Proof of lemma.** Since $f \in L^2(\mathbb{R}^3)$ as a consequence of hypotheses $(a)_{\mu=0}$ and $(b)_{\mu=0}$, we obtain that

$$\alpha_f(\mathrm{x},t) = (e^{-iH_0t}\check{f})(\mathrm{x}) = \frac{e^{i\frac{\mathrm{x}^2}{2t}}}{(2it)^{3/2}}\int_{\mathbb{R}^3} e^{-i\frac{\mathrm{x}\cdot\mathrm{y}}{t}}e^{i\frac{\mathrm{y}^2}{2t}}\check{f}(\mathrm{y})\ dy \tag{14}$$

where $\check{f}$ denotes the Fourier antitransform of $f$.

First of all, we show that $\check{f}$ satisfies the following properties:

$$\left|\check{f}(\mathrm{y})\right| \le \frac{C_m}{|\mathrm{y}|^m} \qquad \text{for } m = 2,\dots,5 \text{ and } \mathrm{y} \in \mathbb{R}^3, \tag{15}$$

$$\partial_j\ \check{f} \in L^1(\mathbb{R}^3) \qquad \text{for every } j = 1,\dots,3. \tag{16}$$

The bound (15) can be proved by observing that

$$e^{i\mathrm{k}\cdot\mathrm{y}} = (-i)^m \frac{1}{|\mathrm{y}|^m}\left(\frac{\mathrm{y}}{|\mathrm{y}|}\cdot\nabla_{\mathrm{k}}\right)^m e^{i\mathrm{k}\cdot\mathrm{y}} \tag{17}$$

so that (by integration by parts and a standard density argument) we get

$$\check{f}(\mathrm{y}) = i^m \frac{1}{|\mathrm{y}|^m}\int_{\mathbb{R}^3} e^{i\mathrm{k}\cdot\mathrm{y}}\left(\frac{\mathrm{y}}{|\mathrm{y}|}\cdot\nabla_{\mathrm{k}}\right)^m f(\mathrm{k})\; dk \tag{18}$$

Our assumptions imply that $\left(\frac{\mathrm{y}}{|\mathrm{y}|}\cdot\nabla_{\mathrm{k}}\right)^m f \in L^1(\mathbb{R}^3)$ for $m = 2,\dots,5$, so that (15) follows from the Riemann-Lebesgue lemma.

A similar argument shows that $\left|\partial_j\ \check{f}(\mathrm{y})\right| \leq \frac{C}{|\mathrm{y}|^4}$ for every $\mathrm{y}\in\mathbb{R}^3$. As for the local behavior the fact that $k_j f \in L^2$ implies that $\partial_j \check{f} \in L^2 \subseteq L^1_{\mathrm{loc}}$ . This completes the proof of (16).

The claim (13) means that for every $\tau\in[0,1]$ there exists $C_\tau > 0$ such that

$$\left(\frac{|\mathrm{x}|}{t}\right)^\tau |\alpha_f(\mathrm{x},t)| \leq \frac{C_\tau}{t^{3/2}}. \tag{19}$$

In the case $\tau = 0$, by using the representation (14) and the bound (15), we get

$$\begin{aligned} |\alpha_f(\mathrm{x},t)| &\leq \frac{C}{t^{3/2}}\int_{\mathbb{R}^3}\left|\check{f}(\mathrm{y})\right|\; dy \\ &\leq \frac{C}{t^{3/2}}\left\{\int_{B_1(0)}\frac{C_2}{|\mathrm{y}|^2}\; dy + \int_{\mathbb{R}^3\setminus B_1(0)}\frac{C_4}{|\mathrm{y}|^4}\; dy\right\} \leq \frac{C'}{t^{3/2}} \end{aligned}$$

In the case $\tau = 1$, by using (14) and (17) we obtain

$$\frac{|\mathrm{x}|}{t}\alpha_f(\mathrm{x},t) = \frac{-ie^{i\frac{\mathrm{x}^2}{2t}}}{(4\pi i t)^{3/2}}\int_{\mathbb{R}^3} e^{-i\frac{\mathrm{x}\cdot\mathrm{y}}{t}}\left(\frac{\mathrm{x}}{|\mathrm{x}|}\cdot\nabla\right) e^{i\frac{\mathrm{y}^2}{2t}}\check{f}(\mathrm{y})\; dy$$

and then

$$\frac{|\mathrm{x}|}{t}|\alpha_f(\mathrm{x},t)| \leq \frac{C}{t^{3/2}}\sum_{r=1}^{3}\int_{\mathbb{R}^3}\left|\frac{1}{t}y_r e^{i\frac{\mathrm{y}^2}{2t}}\check{f}(\mathrm{y}) + e^{i\frac{\mathrm{y}^2}{2t}}\partial_r\check{f}(\mathrm{y})\right|\; dy \tag{20}$$

As a consequence of (15) and (16) we have that $y_r\check{f}$ and $\partial_r\check{f}$ belong to $L^1(\mathbb{R}^3)$, so (20) implies (19) in the case $\tau = 1$. The general case $\tau\in(0,1)$ follows by an interpolation argument. ■

The previous lemma applies to $f_1(\mathrm{k})$ given by (10) and $f_2(\mathrm{k}) := \mathrm{k}\widehat{\Psi}_{\mathrm{out}}(\mathrm{k})$, so that we get

$$\alpha_{\mathrm{reg}} = \frac{1}{t^{3/2}}\mathcal{O}_1\left(\frac{|\mathrm{x}|}{t}\right) \qquad \text{and} \qquad \nabla\alpha = \frac{1}{t^{3/2}}\mathcal{O}_1\left(\frac{|\mathrm{x}|}{t}\right). \tag{21}$$

Moreover,

$$\alpha_{\mathrm{sing}} = \frac{1}{|\mathrm{x}|}\frac{1}{t^{1/2}}\mathcal{O}_{[-1,1]}\left(\frac{|\mathrm{x}|}{\sqrt{t}}\right) \tag{22}$$

Indeed, a straightforward computation gives

$$\alpha_{\mathrm{sing}}(\mathrm{x},t) = \frac{i\pi^{3/2}}{|\mathrm{x}|\sqrt{1+it}}\varphi\left(\frac{i|\mathrm{x}|}{2\sqrt{1+it}}\right)$$

where $\varphi(z) := e^{z^2}\left(\mathrm{erfc}(z) - \mathrm{erfc}(-z)\right)$ for $z \in \mathbb{C}$. To prove (22) it is then sufficient to show that

$$\operatorname*{Sup}_{\arg(z)<\frac{3}{4}\pi} \left(z^{\tau}\varphi(z)\right) \leq C_{\tau} \tag{23}$$

for every $\tau \in [-1,1]$. [2] For $\tau \in [-1,0]$ the bound (23) is trivially true, since $\varphi$ has a first order zero in $z = 0$ and is bounded at infinity in the specified region. For $\tau \in (0,1]$, one notices that the asymptotic expansion of the error function (in the specified region) assure that $z\varphi(z)$ is bounded at infinity.

**Estimates on $\beta$ and $\nabla\beta$.** We turn now to the estimates on $\beta$ and $\nabla\beta$. As before, it is convenient to extract the singular part of $\widehat{\Psi}_{\mathrm{out}}$. We pose

$$f_3(\mathrm{k}) := \frac{1}{|\mathrm{k}|}\widehat{\Psi}_{\mathrm{out}}(\mathrm{k}) - \frac{r}{|\mathrm{k}|^2}e^{-k^2} - \frac{c}{|\mathrm{k}|}e^{-k^2} \tag{24}$$

where $r \in \mathbb{C}$ has been defined in (10), and $c \in \mathbb{C}$ is the zeroth order term in the Laurent expansion of $\widehat{\Psi}_{\mathrm{out}}$; then we decompose $\beta$ as $\beta_{\mathrm{sing},2} + \beta_{\mathrm{sing},1} + \beta_{\mathrm{reg}}$ where

$$\beta_{\mathrm{sing},2}(\mathrm{x},t) \equiv \frac{-i}{|\mathrm{x}|}\int_{\mathbb{R}^3} e^{-i\mathrm{k}^2 t}\frac{r}{|\mathrm{k}|^2}e^{-\mathrm{k}^2}e^{-i|\mathrm{k}||\mathrm{x}|}(2\pi)^{-3/2}dk \tag{25}$$

$$\beta_{\mathrm{sing},1}(\mathrm{x},t) \equiv \frac{-i}{|\mathrm{x}|}\int_{\mathbb{R}^3} e^{-i\mathrm{k}^2 t}\frac{c}{|\mathrm{k}|}e^{-\mathrm{k}^2}e^{-i|\mathrm{k}||\mathrm{x}|}(2\pi)^{-3/2}dk \tag{26}$$

$$\beta_{\mathrm{reg}}(\mathrm{x},t) \equiv \frac{-i}{|\mathrm{x}|}\int_{\mathbb{R}^3} e^{-i\mathrm{k}^2 t}f_2(\mathrm{k})e^{-i|\mathrm{k}||\mathrm{x}|}(2\pi)^{-3/2}dk \tag{27}$$

The first two terms can be computed exactly by gaussian integration, obtaining

$$\beta_{\mathrm{sing},2} = \frac{1}{|\mathrm{x}|}\frac{1}{\sqrt{t}}\mathcal{O}_1\left(\frac{|\mathrm{x}|}{\sqrt{t}}\right) \qquad \text{and} \qquad \beta_{\mathrm{sing},1} = \frac{1}{|\mathrm{x}|}\frac{1}{t}\mathcal{O}_2\left(\frac{|\mathrm{x}|}{\sqrt{t}}\right). \tag{28}$$

As for the third term, we will show that

$$\beta_{\mathrm{reg}}(\mathrm{x},t) \leq \frac{1}{|\mathrm{x}|}\frac{C}{|\mathrm{x}|+t} \qquad \text{for } |\mathrm{x}| > R_0, t > T_0 \tag{29}$$

for suitable $R_0$ and $T_0$. First of all, we pose $k = |\mathrm{k}|$ and $\tilde{f}(k) := \int_{\mathbb{S}^2} f_3(k\omega)\, d\omega$ getting

$$\beta_{\mathrm{reg}}(\mathrm{x},t) = \frac{C}{|\mathrm{x}|}\int_0^{+\infty} e^{-i\left(k^2 t + k|\mathrm{x}|\right)}\tilde{f}(k)\; k^2 dk. \tag{30}$$

In order to apply a stationary phase method, we define

$$\eta := |\mathrm{x}| + t \qquad \text{and} \qquad \chi(k) := \frac{k^2 t + k|\mathrm{x}|}{|\mathrm{x}|+t} \tag{31}$$

observing moreover that

$$\frac{1}{\chi'(k)} \leq \mathrm{Max}(1, k^{-1}) \qquad \text{and} \qquad \frac{\chi''(k)}{\chi'(k)^2} \leq \frac{1}{k} \tag{32}$$

[2] Notice that, for $z = \frac{i|\mathrm{x}|}{2\sqrt{1+it}}$, one has $\arg(z) < \frac{3}{4}\pi$ for every value of x and $t$.

where $\chi'$ indicate the derivative of $\chi$ with respect to $k$. From (30) and definition (31) it follows that

$$\beta_{\mathrm{reg}}(\mathrm{x},t)=\frac{C'}{|\mathrm{x}|\left(|\mathrm{x}|+t\right)}\int_0^{+\infty}\left(\frac{d}{dk}e^{-i\eta\chi(k)}\right)\frac{1}{\chi'(k)}\tilde{f}(k)\ k^2dk$$

By recalling definition (24) and using (4) and Lemma 2.1 it is easy to show that $\tilde{f}$ and $\frac{d\tilde{f}}{dk}$ belong to $C^1(0,+\infty)$, are bounded in a neighborhood of zero and satisfy the bound

$$\tilde{f}(k)\leq Ck^{-3}\qquad\text{and}\qquad\frac{d\tilde{f}}{dk}(k)\leq Ck^{-4}\tag{33}$$

for $k\to+\infty$. These facts imply that integration by part is possible and that the boundary term is zero, so we get

$$\left|\beta_{\mathrm{reg}}(\mathrm{x},t)\right|\leq\frac{C'}{|\mathrm{x}|\left(|\mathrm{x}|+t\right)}\int_0^{+\infty}\left|\frac{d}{dk}\left(\frac{1}{\chi'(k)}\tilde{f}(k)\ k^2\right)\right|dk.\tag{34}$$

From (33) and (32) it follows that the integral appearing on the right-hand side of (34) is finite. This proves (29).

Finally, we give an estimate on $\nabla\beta$. By direct computation we obtain

$$\begin{aligned}\nabla\beta(\mathrm{x},t)&=-\frac{1}{|\mathrm{x}|^2}\int_{\mathbb{R}^3}e^{-i\mathrm{k}^2t}e^{-i|\mathrm{k}||\mathrm{x}|}\widehat{\Psi}_{\mathrm{out}}(\mathrm{k})\frac{1}{i|\mathrm{k}|}(2\pi)^{-3/2}d\mathrm{k}\\&\quad+\frac{1}{|\mathrm{x}|}\frac{\mathrm{x}}{|\mathrm{x}|}\int_{\mathbb{R}^3}e^{-i\mathrm{k}^2t}e^{-i|\mathrm{k}||\mathrm{x}|}\widehat{\Psi}_{\mathrm{out}}(\mathrm{k})(2\pi)^{-3/2}d\mathrm{k}\\&\equiv-\frac{1}{|\mathrm{x}|}\beta(\mathrm{x},t)+\left(\nabla\beta\right)_{\mathrm{r}}(\mathrm{x},t)\end{aligned}$$

The term $\left(\nabla\beta\right)_{\mathrm{r}}$ can be treated exactly as $\beta$; however, since the second order pole does not appear, we get the bound

$$\left(\nabla\beta\right)_{\mathrm{r}}\leq\frac{1}{|\mathrm{x}|}\frac{1}{t}\mathcal{O}_2\left(\frac{|\mathrm{x}|}{\sqrt{t}}\right)+\frac{1}{|\mathrm{x}|}\frac{C}{|\mathrm{x}|+t}.\tag{35}$$

**Proof of Theorem 2.2.** We remarked that in order to prove Theorem 2.2 it is sufficient to prove (9). To achieve the proof, we notice that the singular term

$$-\frac{1}{|\mathrm{x}|}\beta^*_{\mathrm{sing},2}(\mathrm{x},t)\beta_{\mathrm{sing},2}(\mathrm{x},t)$$

appearing in $\beta\nabla\beta$ is *real*, so it does not contribute to $\mathrm{Im}(\beta^*\nabla\beta)$. As for all the remaining terms, they can be shown to vanish as $R\to+\infty$ by using estimates (21), (22), (28), (29) and (35). As an example we show how to prove that

$$\lim_{R\to\infty}\int_T^{+\infty}dt\int_{\Sigma_R}\left|\,\mathrm{Im}(\beta^*\nabla\alpha)\cdot\mathrm{n}\right|d\sigma=0\,.\tag{36}$$

We observe that

$$\begin{aligned}\int_{\Sigma_R} |\operatorname{Im}(\beta^* \nabla \alpha)\cdot \mathrm{n}|\, d\sigma &\leq 4\pi R^2 \left|\beta_{\mathrm{sing},2} + \beta_{\mathrm{sing},1} + \beta_{\mathrm{reg}}\right| |\nabla \alpha| \\ &\leq 4\pi R \left\{ \frac{1}{\sqrt{t}} \mathcal{O}_1\left(\frac{R}{\sqrt{t}}\right) + \frac{1}{t}\mathcal{O}_2\left(\frac{R}{\sqrt{t}}\right) + \frac{C}{R+t} \right\} \left\{ \frac{1}{t^{3/2}} \mathcal{O}_1\left(\frac{R}{t}\right) \right\}\end{aligned}$$

where we used (21), (28) and (35) by identifying $R = |\mathrm{x}|$. Now one makes use of property (7) with suitable choices of $\tau$ and $\nu$ in order to control the previous expression. For example, the first term is

$$\frac{R}{\sqrt{t}} \mathcal{O}_1\left(\frac{R}{\sqrt{t}}\right) \frac{1}{R^{\varepsilon} t^{3/2-\varepsilon}} \frac{R^{\varepsilon}}{t^{\varepsilon}} \mathcal{O}_1\left(\frac{R}{t}\right) \leq \frac{C}{R^{\varepsilon} t^{3/2-\varepsilon}}$$

for every $\varepsilon \in (0, \frac{1}{2})$. By similar computations we get

$$\int_{\Sigma_R} |\operatorname{Im}(\beta^* \nabla \alpha)\cdot \mathrm{n}|\, d\sigma \leq \frac{C}{R^{\varepsilon} t^{3/2-\varepsilon}} + \frac{C'}{R t^{3/2}}$$

This bound is sufficient to prove the vanishing of the left-hand side of (36) by applying the dominated convergence theorem. This completes the proof of Theorem 2.2 ■

# 3 Zero energy resonances and the FAS theorem in potential scattering

In this section and the following one we will consider the scattering theory for the pair $(H, H_0)$ where $H_0 = -\Delta$ on the domain $\mathcal{D}(H_0) = H^2(\mathbb{R}^3)$ and $H = H_0 + V$. Later on, we will focus on potentials satisfying the following assumptions.

**Definition 3.1** *We say that a measurable function $V : \mathbb{R}^3 \to \mathbb{R}$ belongs to the* Ikebe class $(\mathrm{I})_n$ *(with $n \in \mathbb{N}$), if:*

(i) *$V$ is locally Hölder continuous except that in a finite number of points*

(ii) $V \in L^2(\mathbb{R}^3)$

(iii) *there exist $R_0 > 0$ and $\varepsilon > 0$ such that $|V(\mathrm{x})| \leq \frac{C_0}{|\mathrm{x}|^{n+\varepsilon}}$ for $|\mathrm{x}| \geq R_0$.*

*Moreover, we define $(\mathrm{I})_\infty = \bigcap_{n\in\mathbb{N}} (\mathrm{I})_n$.*

The terminology follows from the fact that, for $n = 2$, these are the hypotheses under which Ikebe's eigenfunction expansion theorem [Ik] has been proved. Under these assumptions, the operator $H$ is self-adjoint on $\mathcal{D}(H_0)$. Moreover $H$ has neither positive eigenvalues nor singular continuous spectrum and $\sigma_{\mathrm{ac}}(H) = [0, +\infty)$. Finally, the wave operators $W_\pm = \lim_{t\to\pm\infty} e^{iHt}e^{-iH_0t}$ exist and are asymptotically complete.

In what follows the Laplace operator $\Delta$ will be intended to act on the space of tempered distributions $\mathcal{S}'(\mathbb{R}^d)$. The operator $-\Delta + \kappa^2$ ($\kappa \in \mathbb{C}_+$), seen as an operator in $\mathcal{S}'(\mathbb{R}^d)$, has a *right* inverse $G_\kappa$ given explicitly by the convolution (in the sense of tempered distributions) with $\tilde{G}_\kappa(\mathrm{x}) = \frac{e^{i\kappa|\mathrm{x}|}}{|\mathrm{x}|}$.

## 3.1 Lippman-Schwinger eigenfunctions and zero energy resonances

The main tool of our analysis will be the fact that the operator $H$ can be "diagonalized" by means of the so-called Lippman-Schwinger eigenfunctions, or generalized eigenfunction. The classical results concerning this generalized eigenfunctions, proved in [Ik] and [Po], are summarized in [TDMB]. Here we point out few basic facts.

If $V \in (\mathrm{I})_2$, then for every $\mathrm{k} \in \mathbb{R}^3 \setminus \{0\}$ the generalized eigenfunction $\Phi_\pm(\cdot, \mathrm{k}) : \mathbb{R}^3 \to \mathbb{C}$ is defined as the unique *continuous* solution of the Lippman-Schwinger (LS) equation

$$\Phi_\pm(\mathrm{x}, \mathrm{k}) = e^{i\mathrm{k}\cdot\mathrm{x}} - \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{\mp i|\mathrm{k}||\mathrm{x}-\mathrm{y}|}}{|\mathrm{x}-\mathrm{y}|} V(\mathrm{y})\Phi_\pm(\mathrm{y}, \mathrm{k}) \; dy \tag{LS}$$

which satisfies the asymptotic condition $\lim_{|\mathrm{x}|\to\infty} \left(\Phi_\pm(\mathrm{x}, \mathrm{k}) - e^{i\mathrm{k}\cdot\mathrm{x}}\right) = 0$. In more abstract terms[3], the function $\eta_\mathrm{k}(\mathrm{x}) := \Phi_+(\mathrm{x}, \mathrm{k}) - e^{i\mathrm{k}\cdot\mathrm{x}}$ is the unique solution of the equation

$$(1 + G_{|\mathrm{k}|}V)\eta_\mathrm{k} = g_\mathrm{k} \tag{37}$$

[3] Here and in the following we will consider only the upper sign in $\Phi_\pm$, omitting pedices.

in $C_\infty(\mathbb{R}^3)$, the space of continuous functions vanishing at infinity, where $g_{\mathrm{k}}$ is given by

$$g_{\mathrm{k}}(\mathrm{x}) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{-i|\mathrm{k}||\mathrm{x}-\mathrm{y}|}}{|\mathrm{x}-\mathrm{y}|} V(\mathrm{y}) e^{i\mathrm{k}\cdot\mathrm{y}}\, dy.$$

However, for our purposes, the choice of $C_\infty(\mathbb{R}^3)$ is not the most suitable one. As pointed out by Agmon and Kuroda (see e.g. [Ag]) a convenient alternative topology for setting the LS equation is given by the weighted Sobolev spaces, defined by

$$H^{m,s}(\mathbb{R}^d) := \left\{ u \in \mathcal{S}'(\mathbb{R}^d) : \left\| \left(1+|x|^2\right)^{\frac{s}{2}} (1-\Delta)^{\frac{m}{2}} u \right\|_{L^2} < +\infty \right\}.$$

By varying the indexes $m, s \in \mathbb{R}$ one gets a net of spaces (in the picture $s, s' \geq 0$)

$$\begin{array}{ccccccccc}
\subseteq & H^{2,s} & \subseteq & H^{1,s} & \subseteq & L^2_s & \subseteq & H^{-1,s} & \subseteq \\
 & \cap & & \cap & & \cap & & \cap & \\
\subseteq & H^{2} & \subseteq & H^{1} & \subseteq & L^2 & \subseteq & H^{-1} & \subseteq \\
 & \cap & & \cap & & \cap & & \cap & \\
\subseteq & H^{2,-s'} & \subseteq & H^{1,-s'} & \subseteq & L^2_{-s'} & \subseteq & H^{-1,-s'} & \subseteq
\end{array}$$

where one recognizes the usual weighted $L^2$-spaces $L^2_s \equiv H^{0,s}$ and the usual Sobolev spaces $H^m \equiv H^{m,0}$. Moreover we define the spaces

$$H^{m,s-0} := \bigcap_{r<s} H^{m,r} \qquad \text{and} \qquad H^{m,s+0} := \bigcup_{r>s} H^{m,r}$$

regarded as linear spaces.

The main advantage of this approach is that, if $G_\kappa = (-\Delta + \kappa^2)^{-1}$ $(\kappa \in \mathbb{C}_+)$ is regarded as an operator between weighted Sobolev spaces, then the map $\kappa \mapsto G_\kappa$ can be continuously extended to the closed upper half-plane $\overline{\mathbb{C}}_+$ .
Notice that if $G_\kappa$ $(\kappa \in \mathbb{C}_+)$ is regarded as an element of $\mathcal{B}(L^2, H^2)$ it coincides with the free resolvent $R_0(\kappa) = (H_0 - \kappa^2)^{-1} \in \mathcal{B}(L^2)$ but – in such a case – the map $\kappa \mapsto R_0(\kappa)$ *cannot* be continuously extended to $\overline{\mathbb{C}}_+$, since $\kappa \in \mathbb{R}$ implies $\kappa^2 \in \sigma(H_0)$.
The following lemma – which has been proved in [JK] – makes precise the previous statements.

**Lemma 3.2 (Extension of the free resolvent)** *Assume $s, s' > \frac{1}{2}$ , $s + s' > 2$ and $m \in \mathbb{Z}$. Then $\kappa \mapsto G_\kappa$, considered as a $\mathcal{B}(H^{m-2,s}, H^{m,-s'})$-valued function, can be* continuously *extended to the region $\kappa \in \overline{\mathbb{C}}_+$.*

The previous lemma allow us to give a meaning to $1+G_\kappa V$, as an operator between weighted Sobolev spaces, also for $\kappa$ on the real axis. To this end, we consider potentials satisfying the following condition:

$$V \text{ is a compact operator from } H^{m,0} \text{ to } H^{m-2,\beta} \text{ for some } \beta > 2. \qquad (\mathrm{V}.m.\beta)$$

This condition implies that $V$ is a compact operator from $H^{m,s}$ to $H^{m-2,s+\beta}$ for every $s \in \mathbb{R}$ since $V$ commutes with multiplication by $\left(1+|x|^2\right)^{\frac{s}{2}}$. If $V \in (\mathrm{I})_n$ then condition (V.$m.\beta$) holds true for $m = 1, 2$ for every $\beta < n + \varepsilon$ (with $\varepsilon$ from Def. 3.1.iii) (for $m = 2$ this is a consequence of the fact that $\langle\cdot\rangle^{-\beta} V$ is an $H_0$-compact operator in $L^2$; for $m = 1$ see [JK]).

Assume condition (V.$m$.$\beta$) for $m = 1,2$ and $\beta > 2$. Then, for every $s > \frac{1}{2}$ and $m = 1,2$ one has a compact-operator-valued analytic map $\mathbb{C}_+ \to \mathcal{B}_\infty(H^{m,-s}), \kappa \mapsto G_\kappa V$ that, in virtue of Lemma 3.2, can be extended by continuity to the (positive) real axis. This allows us to formulate the LS equation in $H^{m,-s}$. The comparison with the previous approach is given in the following proposition.

**Proposition 3.3** *Let be $s > \frac{1}{2}$ . Assume that $V \in (\mathrm{I})_2$. Then, for every $\mathrm{k} \in \mathbb{R}^3 \setminus \{0\}$, there exists a unique solution $\tilde{\eta}_{\mathrm{k}}$ of the equation*

$$(1 + G_{|\mathrm{k}|}V)\tilde{\eta}_{\mathrm{k}} = g_{\mathrm{k}} \tag{38}$$

*in $H^{1,-s}(\mathbb{R}^3)$ (which in fact belongs to $H^{2,-s}(\mathbb{R}^3)$) and this solution can be identified with the unique solution $\eta_{\mathrm{k}}$ of the equation (37) in $C_\infty(\mathbb{R}^3)$.*

**Proof.** Recall that any $V \in (\mathrm{I})_2$ satisfies condition (V.$m$.$\beta$) for $m = 1,2$ and some $\beta > 2$. It follows that the map $\kappa \mapsto G_\kappa V$ is a $\mathcal{B}_\infty(H^{m,-s})$-valued function, analytic in the upper half-plane and continuous in $\overline{\mathbb{C}}_+$. Then a variant of Fredholm theory (analogous to [RS3], Proposition on page 101) shows that there exists $(1 + G_k V)^{-1} \in \mathcal{B}(H^{m,-s})$ for every $k \in [0, +\infty) \setminus N_m$, where $N_m$ is a closed set of Lebesgue measure zero. Moreover, $k$ belongs to $N_m$ if and only if there exists a non-zero solution $\psi \in H^{m,-s}$ of the homogenous equation $(1 + G_k V)\psi = 0$.
From the fact that $H$ has no positive eigenvalues, it follows that the previous homogenous equation can not have any non-zero solution in $H^{2,-s}$ if $k \neq 0$ (this follows from [Ag], Th. 3.3; notice that $\psi \in H^{2,-s}$ implies $\hat{\psi} \in L^1_{\mathrm{loc}}$). Then $N_2 \subseteq \{0\}$.
Moreover, an argument similar to the proof of Prop. 4.3 shows that any $\psi \in H^{1,-s}$ which solves the equation $(1 + G_k V)\psi = 0$ with $V \in (\mathrm{I})_2$ really belongs to $H^{2,-s}$. Then $N_1 \subseteq N_2 \subseteq \{0\}$.
Since $g_{\mathrm{k}} \in H^{2,-s}$ for every $\mathrm{k} \in \mathbb{R}^3$, the unique solution of equation (38) in $H^{2,-s}$ is explicitly given by $\tilde{\eta}_{\mathrm{k}} = (1 + G_{|\mathrm{k}|}V)^{-1} g_{\mathrm{k}}$ for every $\mathrm{k} \in \mathbb{R}^3 \setminus \{0\}$. This is also the unique solution of the equation in $H^{1,-s}$, since $H^{1,-s} \supseteq H^{2,-s}$ and $N_1 \subseteq N_2$.

Now we prove that $\tilde{\eta}_{\mathrm{k}} \in H^{2,-s}$ can be identified with $\eta_{\mathrm{k}} \in C_\infty(\mathbb{R}^3)$.
A straightforward argument, shows that the condition $f \in H^{2,-s}$ is equivalent to the condition $\langle x \rangle^{-s} f \in H^2$, where $\langle x \rangle^{-s}$ represent the multiplication times $(1 + |\mathrm{x}|^2)^{-\frac{s}{2}}$. Then, we notice that the function

$$\tilde{\xi}_{\mathrm{k}} := \langle x \rangle^{-s} \tilde{\eta}_{\mathrm{k}} \in H^2(\mathbb{R}^3) \subseteq C_\infty(\mathbb{R}^3)$$

satisfies the equation

$$(1 + \langle x \rangle^{-s} G_{|\mathrm{k}|} V \langle x \rangle^{-s})\tilde{\xi}_{\mathrm{k}} = \langle x \rangle^{-s} g_{\mathrm{k}} \tag{39}$$

in $C_\infty$. On the other hand, also $\xi_{\mathrm{k}} := \langle x \rangle^{-s} \eta_{\mathrm{k}}$ satisfies equation (39) in $C_\infty$, since $\eta_{\mathrm{k}}$ satisfies (37). An argument based on Fredholm theory shows that equation (39) has a unique solution in $C_\infty(\mathbb{R}^3)$; then $\tilde{\xi}_{\mathrm{k}} = \xi_{\mathrm{k}}$ and this proves our claim. ■

**Remark.** In the following we will not distinguish anymore between $\tilde{\eta}_{\mathrm{k}}$ and $\eta_{\mathrm{k}}$. The pointwise values of $\eta_{\mathrm{k}}$will be denoted as $\eta_{\mathrm{k}}(\mathrm{y}) \equiv \eta(\mathrm{y}, \mathrm{k})$ for every $\mathrm{y} \in \mathbb{R}^3$.

We turn now to study the behavior of the generalized eigenfunctions in presence of a zero-energy resonance. Although the following definition could appear quite *ad hoc*, we will show in the Appendix that it is completely equivalent to the most common ones (see Prop. 4.3).

**Definition 3.4** *We say that there is a* ***zero-energy resonance*** *for the pair* $(H, H_0)$ *if there exists a* $\psi \in H^{1,-\frac{1}{2}-0}(\mathbb{R}^3)$ *such that* $(1 + G_0 V)\,\psi = 0$ *but* $\psi \notin L^2(\mathbb{R}^3)$.

Such a $\psi$, if it exists, is unique up to a complex phase. It will be called the **resonance function** and denoted with $\psi_{\rm res}$.

Roughly speaking, if the pair $(H, H_0)$ admits a zero-energy resonance, then the map $\kappa \mapsto (1 + G_\kappa V)^{-1}$ has a singular behavior as $\kappa \to 0$. This behavior is described by the following theorem, which has been proved in [JK, Lemmas 4.2 and 4.4].

**Theorem 3.5 (Jensen-Kato, 1979)** *Assume that* $V$ *satisfies condition (V.1.$\beta$) with* $\beta > 7$. *Let* $s$ *satisfy* $7/2 < s < \beta - 7/2$.

*Assume that there is a zero-energy resonance for the pair* $(H, H_0)$ *and that* $0 \notin \sigma_{\rm p}(H)$. *Then for* $\kappa \to 0, \kappa \in \mathbb{R}_+$, *we have in* $\mathcal{B}(H^{1,-s})$ *the expansion*

$$(1 + G_\kappa V)^{-1} = -\frac{i}{\kappa}\,\langle \cdot, V\psi_{\rm res}\rangle\,\psi_{\rm res} + C_0 + \kappa C_1 + O(\kappa^2) \tag{40}$$

*where* $C_0$ *and* $C_1$ *are explicitly computable operators in* $\mathcal{B}(H^{1,-s})$.

An explicit expression for the operators $C_0$ and $C_1$ can be found in [JK, Lemma 4.3]. As pointed out in [JK, Remark 4.6], by assuming sufficiently large $\beta$ and $s$ it is possible to obtain an expansion of $(1 + G_\kappa V)^{-1}$ to any order in $\kappa$. However, the actual computation of coefficients becomes rather difficult.

**Remark 3.6** *If there is a zero-energy resonance for the pair* $(H, H_0)$ *and moreover* $0 \in \sigma_{\rm p}(H)$ *then expansion (40) is replaced by*

$$(1 + G_\kappa V)^{-1} = -\frac{1}{\kappa^2}P_0 V - \frac{i}{\kappa}\left(\langle \cdot, V\psi_{\rm res}\rangle\,\psi_{\rm res} - P_0 V C V P_0 V\right) + \tilde{C}_0 + O(1) \tag{41}$$

*where* $P_0$ *is the projector on the eigenspace relative to zero (naturally extended to* $H^{-1,-\frac{1}{2}+0}$ *by using the fact that the eigenfunctions belongs to* $H^{1,\frac{1}{2}-0}$*) and* $C$ *is the convolution operator with kernel* $C(\mathrm{x},\mathrm{y}) = \frac{1}{24\pi}|\mathrm{x}-\mathrm{y}|^3$.

**Remark 3.7** *According to Jensen and Kato (see [JK, Remark 6.7]) the asymptotic expansion (40) can be differentiated any number of times, in the sense that for every* $r \in \mathbb{N}$

$$\frac{d^r}{d\kappa^r}\left[(1 + G_\kappa V)^{-1} - \sum_{j=-1}^{n} \kappa^j C_j\right] = o(\kappa^{n-r}). \tag{42}$$

*However, these asymptotic expansions require larger values of* $s$ *and* $\beta$ *than for* $r = 0$. *To fix notation, for every* $n, r \in \mathbb{N}$ *with* $r < n$ *there exists a real number* $\bar{\beta} = \bar{\beta}(n, r)$ *such that (41) and (42) holds true, provided that* $\beta > \bar{\beta}$ *and* $s$ *satisfies* $\frac{\bar{\beta}}{2} < s < \beta - \frac{\bar{\beta}}{2}$.

It would be tempting to rephrase the previous result by saying that $(1+G_\kappa V)^{-1}$ has a "pole" in $\kappa = 0$, but this term is usually reserved for meromorphic functions (which are defined in an open neighborhood of the point $\kappa = 0$) while we are facing with a function with an asymptotic expansion only on $\Omega = \{\kappa \in \mathbb{C} : \kappa^2 \in \overline{\mathbb{C}}_+\}$. In such a case we prefer to use the term **polar**

**singularity**. The usual concept of complex analysis (simple pole, residue,...) extends trivially to this case.
We emphasize that, when zero is a resonance but not an eigenvalue, the expected polar singularity for $(1+G_\kappa V)^{-1}$ must be *simple* and the residue corresponding to the pole is a *rank-one operator*, projecting on the subspace generated by the resonance function $\psi_{\rm res}\in H^{1,-\frac{1}{2}-0}(\mathbb{R}^3)$.

By recalling that the solution $\eta_{\rm k}$ of the Lippman-Schwinger equation (37) is given by $\eta_{\rm k}=(1+G_\kappa V)^{-1}g_{\rm k}$ we can prove the following result.

**Proposition 3.8** *Assume that $V$ satisfies condition $(V.1.\beta)$ for $\beta>\bar\beta\equiv\bar\beta(6,5)$ and that the pair $(H,H_0)$ has a zero-energy resonance. Fix $s$ so that $\frac{\bar\beta}{2}<s<\beta-\frac{\bar\beta}{2}$.*
*If $0\notin\sigma_{\rm p}(H)$, then the solution $\eta_{\rm k}$ of the Lippman-Schwinger equation (38) can be decomposed as*

$$\eta(\mathrm{x},\mathrm{k})=\frac{r_0}{|\mathrm{k}|}\psi_{\rm res}(\mathrm{x})+\rho(\mathrm{x},\mathrm{k}) \tag{43}$$

*where $r_0=-i\,\langle g_0,V\psi_{\rm res}\rangle=i\,\langle V,\psi_{\rm res}\rangle$ and the map $\mathrm{k}\mapsto\rho_{\rm k}$ from $\mathbb{R}^3\setminus\{0\}$ to $H^{1,-s}(\mathbb{R}^3)$ is bounded (with all its derivatives until order $r=5$ at least) in a punctured neighborhood of the origin.*
*If $0\in\sigma_{\rm p}(H)$ and $\{\psi_j\}_{j=1}^p\subseteq H^{1,\frac{1}{2}-0}\subseteq L^2$ are the corresponding eigenfunctions, then $\eta_{\rm k}$ be decomposed as*

$$\eta(\mathrm{x},\mathrm{k})=\frac{r_0}{|\mathrm{k}|}\psi_{\rm res}(\mathrm{x})+\sum_{j=1}^{p}\frac{r_j}{|\mathrm{k}|}\psi_j(\mathrm{x})+\bar\rho(\mathrm{x},\mathrm{k})$$

*where the map $\mathrm{k}\mapsto\bar\rho_{\rm k}$ has the same properties stated above.*

**Proof.** First consider the case $0\notin\sigma_{\rm p}(H)$. From the expansion (40) (at higher order) it follows that $\rho_{\rm k}$ is given by

$$\rho_{\rm k}=-\frac{i}{|\mathrm{k}|}\langle g_{\rm k}-g_0,V\psi_{\rm res}\rangle\,\psi_{\rm res}+C_0g_{\rm k}+|\mathrm{k}|C_1g_{\rm k}+\cdots+O(|\mathrm{k}|^5).$$

Then the claim follows from the differentiability of the map $\mathrm{k}\mapsto\langle g_{\rm k},V\psi_{\rm res}\rangle$ together with Remark 3.7.

Now we turn to the case in which zero is an eigenvalue for $H$. By using the expansion (41) one gets that the coefficient of the second-order pole is given by

$$P_0Vg_0=\sum_{j=1}^{p}\left\langle\psi_j,Vg_0\right\rangle\psi_j$$

However, this term is identically zero since (we use the fact that $(1+G_0V)\psi_j=0$ since $\psi_j$ is an eigenfunction, see the Appendix) one has

$$\left\langle\psi_j,Vg_0\right\rangle=\left\langle\psi_j,VG_0V\cdot1\right\rangle=\left\langle G_0V\psi_j,V\cdot1\right\rangle=-\left\langle\psi_j,V\cdot1\right\rangle=0$$

where the last equality comes from the fact that $V\psi_j$ is orthogonal (in the sense of dual pairing) to 1 since $\psi_j$ is an *eigenfunction* (see Lemma 3).

Taking into account the fact that $P_0 V g_0 = 0$ one gets

$$\eta_{\mathrm{k}} = -\frac{1}{|\mathrm{k}|^2} \sum_{j=1}^{p} \langle \psi_j, V(g_{\mathrm{k}} - g_0) \rangle \psi_j - \frac{i}{|\mathrm{k}|} \langle V\psi_{\mathrm{res}}, g_{\mathrm{k}} \rangle \psi_{\mathrm{res}} + \tilde{C}_0 g_{\mathrm{k}} + \cdots + O(|\mathrm{k}|^5)$$

and the claim follows as in the previous case. ■

The previous lemma gives relevant information about the behavior of the generalized eigenfunctions in a neighborhood of the point $\mathrm{k} = 0$. What about the behavior away from the origin in momentum space? The regularity in k of the generalized eigenfunctions has been studied in depth in [TDMB]. The following result follows from the proof of Th. 3.1 in the cited paper.

**Proposition 3.9** *Let be $V \in (\mathrm{I})_n$ for some $n \geq 3$. Then*

(i) *for every fixed $\mathrm{x} \in \mathbb{R}^3$, the function $\Phi_\pm(\mathrm{x}, \cdot)$ belongs to $C^{n-2}(\mathbb{R}^3 \setminus \{0\})$ and the partial derivatives $\partial_k^\alpha \Phi_\pm(\mathrm{x}, \mathrm{k})$ (for every multindex $\alpha \in \mathbb{N}^3$ with $|\alpha| \leq n-2$) are continuous with respect to $\mathrm{x} \in \mathbb{R}^3$ and $\mathrm{k} \in \mathbb{R}^3 \setminus \{0\}$.*

(ii) *for every compact set $K \subseteq \mathbb{R}^3$ containing the origin*

$$\operatorname*{Sup}_{\mathrm{k} \in \mathbb{R}^3 \setminus K, \mathrm{x} \in \mathbb{R}^3} |\Phi_\pm(\mathrm{x}, \mathrm{k})| \leq c_K \tag{44}$$

$$\operatorname*{Sup}_{\mathrm{k} \in \mathbb{R}^3 \setminus K, \mathrm{x} \in \mathbb{R}^3} |\partial_k^\alpha \Phi_\pm(\mathrm{x}, \mathrm{k})| \leq c_{K,\alpha} (1 + |x|)^{|\alpha|} \tag{45}$$

*for every $\alpha \in \mathbb{N}^3$ with $|\alpha| \leq n-2$.*

### 3.2 The flux-across-surfaces theorem

We are now in position to study the FAS problem for an hamiltonian $H = H_0 + V$ with a zero-energy resonance. We treat first the case in which zero is not an eigenvalue.

First of all, let us focus on the properties of the asymptotic outgoing state $\Psi_{\mathrm{out}}$ corresponding to a given $\Psi_0 \in \mathcal{S}(\mathbb{R}^3)$. From the eigenfunction expansion theorem we get

$$\begin{aligned}
\widehat{\Psi}_{\mathrm{out}}(\mathrm{k}) &= (\mathcal{F}_+ \Psi_0)(\mathrm{k}) = \int_{\mathbb{R}^3} \Phi_+(\mathrm{x}, \mathrm{k})^* \Psi_0(\mathrm{x}) \; (2\pi)^{-\frac{3}{2}} dx \\
&= \widehat{\Psi}_0(\mathrm{k}) + \int_{\mathbb{R}^3} \eta(\mathrm{x}, \mathrm{k})^* \Psi_0(\mathrm{x}) \; (2\pi)^{-\frac{3}{2}} dx.
\end{aligned} \tag{46}$$

The local behavior of $\widehat{\Psi}_{\mathrm{out}}$ can be analyzed using essentially Proposition 3.8. Indeed, by using the decomposition (43) we get

$$\widehat{\Psi}_{\mathrm{out}}(\mathrm{k}) = \widehat{\Psi}_0(\mathrm{k}) + \frac{r}{|\mathrm{k}|} + \int_{\mathbb{R}^3} \rho(\mathrm{x}, \mathrm{k})^* \Psi_0(\mathrm{x}) \; dx \tag{47}$$

where $r = i \langle g_0, V\psi_{\mathrm{res}} \rangle^* \langle \psi_{\mathrm{res}}, \Psi_0 \rangle$.
Decomposition (47) shows the typical *singular behavior* of the asymptotic outgoing state when the hamiltonian exhibits a zero energy resonance, and Proposition 3.8 assure that the remaining part is *regular* in some suitable sense.

On the contrary, we have little control on the asymptotic decrease of $\widehat{\Psi}_{\rm out}(\mathrm{k})$ as $|\mathrm{k}| \to +\infty$. Indeed, the mapping properties of the wave operators, which usually assure that $\Psi_{\rm out}$ inherits the properties of $\Psi_0$, fail to hold when the hamiltonian has a zero-energy resonance (see [Ya]). For this reason we are forced to assume *a priori* a suitable decrease of $\widehat{\Psi}_{\rm out}$ as $|\mathrm{k}| \to +\infty$. We will assume the following assumption (DA): $\widehat{\Psi}_{\rm out} \in C^5(\mathbb{R}^3 \backslash \{0\})$ and for every $m = 0, ..., 5$

$$\left| \frac{\partial^m}{\partial |\mathrm{k}|^m} \widehat{\Psi}_{\rm out}(\mathrm{k}) \right| \leq \frac{C_m}{|\mathrm{k}|^{3+\varepsilon+m}} \qquad \text{for} \qquad |\mathrm{k}| \geq K_m \tag{DA}$$

where $C_m$, $K_m$ and $\varepsilon$ are suitable positive constants.
Although this condition is stronger than what we proved in the case of point interaction (see Lemma 2.1) the estimate (47) can be proved to hold for some solvable smooth potential. For example, the *Bargmann potential*

$$V_b(\mathrm{x}) = -\frac{2b^2}{\cosh^2(b|\mathrm{x}|)} \qquad (b > 0)$$

admits a zero-energy resonance and, by using the explicit form of the Lippman-Schwinger radial wavefunctions (see, for example, [CS])[(4)] it is possible to prove that for any $\Psi_0 \in \mathcal{S}(\mathbb{R}^3)$ the corresponding $\widehat{\Psi}_{\rm out}$ decreases at infinity, with its derivatives, faster than the inverse of any polynomial.

**Theorem 3.10** *Assume $V \in (\mathrm{I})_\infty$ and that the hamiltonian $H = H_0 + V$ has a zero-energy resonance or/and eigenvalue. Let be $\Psi_0 \in \mathcal{H}_{ac}(H) \cap \mathcal{S}(\mathbb{R}^3)$ such that the corresponding asymptotic outgoing state $\Psi_{\rm out} = W_+^{-1}\Psi_0$ satisfies the assumption (DA). Then relation (FAS) holds, for every $T_1 \in \mathbb{R}$.*

**Remark.** We emphasize that the hypothesis $V \in (\mathrm{I})_\infty$ has been assumed only for sake of simplicity. The proof works assuming that $V \in (\mathrm{I})_n$ for $n$ sufficiently large.

We keep the notation as close as possible to the notation used in Section 2 and in [TDMB].

**Proof.** The first part of the proof follows closely the proof of Theorem 2.2. Using the properties of $\mathcal{F}_+$ and (3) we obtain

$$\begin{aligned}
\Psi_t(\mathrm{x}) &= \int_{\mathbb{R}^3} e^{-i\mathrm{k}^2 t} \widehat{\Psi}_{\rm out}(\mathrm{k}) \Phi_+(\mathrm{x},\mathrm{k})\, (2\pi)^{-3/2} dk \\
&= \int_{\mathbb{R}^3} e^{-i\mathrm{k}^2 t} \widehat{\Psi}_{\rm out}(\mathrm{k}) e^{i\mathrm{k}\cdot\mathrm{x}}\, (2\pi)^{-3/2} dk + \int_{\mathbb{R}^3} e^{-i\mathrm{k}^2 t} \widehat{\Psi}_{\rm out}(\mathrm{k}) \eta(\mathrm{x},\mathrm{k}) (2\pi)^{-3/2} dk \\
&\equiv \alpha(\mathrm{x},t) + \beta(\mathrm{x},t)\,.
\end{aligned}$$

As in the case of point interaction, the claim follows if one can prove that

$$\lim_{R\to\infty} \int_T^{+\infty} dt \int_{\Sigma_R} |\mathrm{j}_1(\mathrm{x},t)\cdot\mathrm{n}|\, d\sigma = 0 \tag{48}$$

where $\mathrm{j}_1 \equiv \mathrm{Im}(\alpha^* \nabla\beta + \beta^*\nabla\alpha + \beta^*\nabla\alpha)$. In order to prove (48) we need good estimates on $\alpha, \beta$ and their gradients. We focus first on the case in which zero is not an eigenvalue.

[4]It is well-known that, for a spherically simmetric potential, a zero-energy resonance affects only the $s$-wave component of the scattering operator (see, e.g. [Ba]). Then one needs only the $s$-wave component of the Lippman-Schwinger wavefunction.

**Estimates on $\alpha$ and $\nabla\alpha$** One decomposes $\alpha$ as $\alpha_{\rm reg}+\alpha_{\rm sing}$ where

$$\begin{aligned}
\alpha_{\rm reg}(\mathrm{x},t) &= \int_{\mathbb{R}^3} e^{i\mathrm{k}\cdot\mathrm{x}}e^{-ik^2t}\left(\widehat{\Psi}_{\rm out}(\mathrm{k})-\frac{r}{|\mathrm{k}|}e^{-|\mathrm{k}|^2}\right)(2\pi)^{-3/2}dk\\
\alpha_{\rm sing}(\mathrm{x},t) &= \int_{\mathbb{R}^3} e^{i\mathrm{k}\cdot\mathrm{x}}e^{-ik^2t}\frac{r}{|\mathrm{k}|}e^{-|\mathrm{k}|^2}\,(2\pi)^{-3/2}dk
\end{aligned}$$

and from the properties of distributional Fourier transform we get

$$\nabla\alpha(\mathrm{x},t)=i\int_{\mathbb{R}^3} e^{i\mathrm{k}\cdot\mathrm{x}}e^{-ik^2t}\mathrm{k}\widehat{\Psi}_{\rm out}(\mathrm{k})\,(2\pi)^{-3/2}dk.$$

Decomposition (47) and Proposition 3.8 show that the functions

$$f_1(\mathrm{k}) = \widehat{\Psi}_{\rm out}(\mathrm{k})-\frac{r}{|\mathrm{k}|}e^{-|\mathrm{k}|^2} \tag{49}$$

$$f_2(\mathrm{k}) = \mathrm{k}\widehat{\Psi}_{\rm out}(\mathrm{k}) \tag{50}$$

satisfy the condition (11). Moreover,assumption (DA) implies that they satisfy also (12). So from Lemma 2.4 it follows that

$$\alpha_{\rm reg}=\frac{1}{t^{3/2}}\mathcal{O}_1\left(\frac{|\mathrm{x}|}{t}\right)\qquad\text{and}\qquad\nabla\alpha=\frac{1}{t^{3/2}}\mathcal{O}_1\left(\frac{|\mathrm{x}|}{t}\right) \tag{51}$$

Finally, as in the proof of Theorem 2.2 one proves that

$$\alpha_{\rm sing}=\frac{1}{|\mathrm{x}|}\frac{1}{t^{1/2}}\mathcal{O}_{[-1,1]}\left(\frac{|\mathrm{x}|}{\sqrt{t}}\right). \tag{52}$$

**Estimates on $\beta$ and $\nabla\beta$.** We use the Lippman-Schwinger equation and, by Fubini's theorem, we get

$$\begin{aligned}
\beta(\mathrm{x},t) &= \int_{\mathbb{R}^3} e^{-ik^2t}\widehat{\Psi}_{\rm out}(\mathrm{k})\int_{\mathbb{R}^3}\frac{e^{-i|\mathrm{k}||\mathrm{x}-\mathrm{y}|}}{|\mathrm{x}-\mathrm{y}|}\,V(\mathrm{y})\;\Phi_+(\mathrm{y},\mathrm{k})(2\pi)^{-3/2}\,dy\;dk\\
&= \int_{\mathbb{R}^3}\frac{V(\mathrm{y})}{|\mathrm{x}-\mathrm{y}|}\int_{\mathbb{R}^3} e^{-i\left(\mathrm{k}^2t+|\mathrm{k}||\mathrm{x}-\mathrm{y}|\right)}\;\widehat{\Psi}_{\rm out}(\mathrm{k})\;\Phi_+(\mathrm{y},\mathrm{k})\;(2\pi)^{-3/2}\,dy\;dk\\
&\equiv \int_{\mathbb{R}^3}\frac{V(\mathrm{y})}{|\mathrm{x}-\mathrm{y}|}\Upsilon(\mathrm{x},\mathrm{y},t)\;dy.
\end{aligned}$$

We extract from $\Upsilon$ the singular contributions. By standard arguments we show that the function

$$\tilde\rho(\mathrm{x},\mathrm{k})=\eta(\mathrm{x},\mathrm{k})-\frac{r_0}{|\mathrm{k}|}e^{-|\mathrm{k}|^2}\psi_{\rm res}(\mathrm{x}) \tag{53}$$

has all the regularity properties claimed in Proposition 3.8. By using definition (49) and decomposition (53) we get

$$\Upsilon(\mathrm{x},\mathrm{y},t)=\int_{\mathbb{R}^3} e^{-i\left(\mathrm{k}^2t+|\mathrm{k}||\mathrm{x}-\mathrm{y}|\right)}\left(\frac{r}{|\mathrm{k}|}e^{-|\mathrm{k}|^2}+f_1(\mathrm{k})\right)\left(e^{i\mathrm{k}\cdot\mathrm{y}}+\frac{r_0}{|\mathrm{k}|}e^{-|\mathrm{k}|^2}\psi_{\rm res}(\mathrm{y})+\tilde\rho(\mathrm{y},\mathrm{k})\right)\frac{dk}{(2\pi)^{3/2}}$$

so we can decompose $\Upsilon$ as $\Upsilon_{\mathrm{sing},2} + \Upsilon_{\mathrm{sing},1} + \Upsilon_{\mathrm{reg}}$ where

$$\begin{aligned}
\Upsilon_{\mathrm{sing},2}(\mathrm{x},\mathrm{y},t) &= \psi_{\mathrm{res}}(\mathrm{y}) \int_{\mathbb{R}^3} e^{-i\left(\mathrm{k}^2 t + |\mathrm{k}||\mathrm{x}-\mathrm{y}|\right)} e^{-2|\mathrm{k}|^2} \frac{r r_0}{|\mathrm{k}|^2} (2\pi)^{-3/2}\, dk, \\
\Upsilon_{\mathrm{sing},1}(\mathrm{x},\mathrm{y},t) &= \psi_{\mathrm{res}}(\mathrm{y}) \int_{\mathbb{R}^3} e^{-i\left(\mathrm{k}^2 t + |\mathrm{k}||\mathrm{x}-\mathrm{y}|\right)} e^{-|\mathrm{k}|^2} f_1(\mathrm{k}) \frac{r_0}{|\mathrm{k}|} (2\pi)^{-3/2}\, dk, \\
\Upsilon_{\mathrm{reg}}(\mathrm{x},\mathrm{y},t) &= \int_{\mathbb{R}^3} e^{-i\left(\mathrm{k}^2 t + |\mathrm{k}||\mathrm{x}-\mathrm{y}|\right)} f_1(\mathrm{k}) \left(e^{i\mathrm{k}\cdot\mathrm{y}} + \tilde{\rho}(\mathrm{y},\mathrm{k})\right) (2\pi)^{-3/2}\, dk.
\end{aligned}$$

An explicit gaussian integration gives

$$\Upsilon_{\mathrm{sing},2}(\mathrm{x},\mathrm{y},t) = \frac{1}{\sqrt{t}} \mathcal{O}_1\left(\frac{|\mathrm{x}-\mathrm{y}|}{\sqrt{t}}\right) \psi_{\mathrm{res}}(\mathrm{y}). \tag{54}$$

As for $\Upsilon_{\mathrm{sing},1}$, we perform explicit integration of the singular part and we use a stationary phase method (see Estimates on $\beta$ in Sec. 2) on the regular part getting

$$\Upsilon_{\mathrm{sing},1}(\mathrm{x},\mathrm{y},t) = \frac{1}{t} \mathcal{O}_2\left(\frac{|\mathrm{x}-\mathrm{y}|}{\sqrt{t}}\right) \psi_{\mathrm{res}}(\mathrm{y}). \tag{55}$$

¿From (54) and (55) it follows then that the functions

$$\beta_{\mathrm{sing},j}(\mathrm{x},t) \equiv \int_{\mathbb{R}^3} \frac{V(\mathrm{y})}{|\mathrm{x}-\mathrm{y}|} \Upsilon_{\mathrm{sing},j}(\mathrm{x},\mathrm{y},t)\, dy \qquad (j=1,2)$$

satisfy the bound (28).

The estimate on $\Upsilon_{\mathrm{reg}}$ is not so simple, since we have only an indirect control on the behavior of $\tilde{\rho}$ . From [TDMB, Th. 3.1] (see Proposition 3.9) it follows that $\eta(\mathrm{y},\cdot) \in C^1(\mathbb{R}^3 \setminus \{0\})$ for every $\mathrm{y} \in \mathbb{R}^3$. We can then employ the stationary phase technique of Section 2 in order to obtain

$$\Upsilon_{\mathrm{reg}}(\mathrm{x},\mathrm{y},t) \leq \frac{C}{|\mathrm{x}-\mathrm{y}|+t} \int_{\mathbb{R}^3} \left| \frac{d}{dk} \left( \frac{1}{\chi'(k)}\ f_1(\mathrm{k}) \left(e^{i\mathrm{k}\cdot\mathrm{y}} + \tilde{\rho}(\mathrm{y},\mathrm{k})\right) k^2 \right) \right| (2\pi)^{-3/2}\, dk\, d\omega \tag{56}$$

where we posed $\mathrm{k} = k\omega$ and $d\omega$ is the Lebesgue measure on the sphere. Now we prove that

$$\beta_{\mathrm{reg}}(\mathrm{x},t) \equiv \int_{\mathbb{R}^3} \frac{V(\mathrm{y})}{|\mathrm{x}-\mathrm{y}|} \Upsilon_{\mathrm{reg}}(\mathrm{x},\mathrm{y},t)\, dy \leq \frac{C}{|\mathrm{x}|\,(|\mathrm{x}|+t)}. \tag{57}$$

By computing explicitly the derivative in (56) we get the expression

$$\int_{\mathbb{R}^3} \frac{V(\mathrm{y})}{|\mathrm{x}-\mathrm{y}|} \frac{C}{|\mathrm{x}-\mathrm{y}|+t} \int_{\mathbb{R}^3} \xi_1(\mathrm{k})\tilde{\rho}(\mathrm{y},\mathrm{k}) + \xi_2(\mathrm{k}) \frac{d\tilde{\rho}}{dk}(\mathrm{y},\mathrm{k})\, dk\, dy$$

where $\xi_1$ and $\xi_2$ are given explicitly by computing the derivative. Now we observe that

$$\Theta_1 \equiv \int_{B_1(0)} \left( \xi_1(\mathrm{k})\tilde{\rho}_{\mathrm{k}} + \xi_2(\mathrm{k}) \frac{d\tilde{\rho}_{\mathrm{k}}}{dk} \right) dk \in H^{1,-s}(\mathbb{R}^3) \tag{58}$$

since $\xi_1$, $\xi_2$, $\tilde{\rho}_k$ and $\frac{d\tilde{\rho}}{dk}$ are bounded in every neighborhood of the origin (see Prop. 3.8). The condition $\Theta_1 \in H^{1,-s}(\mathbb{R}^3)$ it is sufficient to prove that

$$\int_{\mathbb{R}^3} \frac{|V(\mathrm{y})|}{|\mathrm{x}-\mathrm{y}|} \frac{|\Theta_1(\mathrm{y})|}{|\mathrm{x}-\mathrm{y}|+t} dy \leq \frac{C}{|\mathrm{x}|\,(|\mathrm{x}|+t)}.$$

As for the remaining part, given by

$$\Theta_2(\mathrm{y}) \equiv \int_{\mathbb{R}^3\setminus B_1(0)} \left( \xi_1(\mathrm{k})\tilde{\rho}_{\mathrm{k}} + \xi_2(\mathrm{k})\frac{d\tilde{\rho}_{\mathrm{k}}}{dk} \right) dk,$$

we obtain that

$$|\Theta_2(\mathrm{y})| \leq \int_{\mathbb{R}^3\setminus B_1(0)} |\xi_1(\mathrm{k})| \, \|\tilde{\rho}_{\mathrm{k}}\|_\infty + |\xi_2(\mathrm{k})| \left\| \frac{d\tilde{\rho}_{\mathrm{k}}}{dk} \right\|_\infty dk.$$

The decrease of $\xi_j$ $(j = 1, 2)$ and the uniform bounds on $\tilde{\rho}_{\mathrm{k}}$ and $\frac{d\tilde{\rho}_{\mathrm{k}}}{dk}$ are sufficient to show that the last integral is finite. This proves (57).
As far as $\nabla\beta$ is concerned, a similar argument give the estimate (35).

Finally, we use estimates on $\alpha$, $\nabla\alpha$, $\beta$ and $\nabla\beta$ in order to prove (48). Notice that, in the present case, the most singular part of $\beta^*\nabla\beta$ is *not real* as was in the case of point interaction. The most singular part (the only one that cannot be controlled by using the previous estimates) is explicitly given by

$$\begin{aligned} \mathrm{j}_{\mathrm{cr}}(\mathrm{x}, t) &= \mathrm{Im}\left( \left( \int_{\mathbb{R}^3} \frac{V(\mathrm{y})}{|\mathrm{x}-\mathrm{y}|} \Upsilon_{\mathrm{sing},2}(\mathrm{x},\mathrm{y},t)\, dy \right)^* \left( -\frac{\mathrm{x}}{|\mathrm{x}|} \int_{\mathbb{R}^3} \frac{V(\mathrm{y}')}{|\mathrm{x}-\mathrm{y}'|^2} \Upsilon_{\mathrm{sing},2}(\mathrm{x},\mathrm{y}',t)\, dy' \right) \right) \\ &= C\frac{\mathrm{x}}{|\mathrm{x}|} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{V(\mathrm{y})\psi_{\mathrm{res}}(\mathrm{y})}{|\mathrm{x}-\mathrm{y}|} \frac{V(\mathrm{y}')\psi_{\mathrm{res}}(\mathrm{y}')}{|\mathrm{x}-\mathrm{y}'|^2} \mathrm{Im}\left( \varphi^*_{\mathrm{cr}}(\mathrm{x}-\mathrm{y},t)\varphi_{\mathrm{cr}}(\mathrm{x}-\mathrm{y}',t) \right) dy\, dy' \end{aligned}$$

where we used the reality of $\psi_{\mathrm{res}}$ (see Sec. 4) and we defined

$$\varphi_{\mathrm{cr}}(\mathrm{x}-\mathrm{y},t) \equiv \int_{\mathbb{R}^3} e^{-i\left(\mathrm{k}^2 t + |\mathrm{k}||\mathrm{x}-\mathrm{y}|\right)} e^{-|\mathrm{k}|^2} \frac{1}{|\mathrm{k}|^2}\, dk = \frac{\sqrt{\pi}}{2}\frac{1}{\sqrt{1+it}} \left\{ e^{z^2}\mathrm{erfc}(z) \right\}_{z=\frac{i|\mathrm{x}-\mathrm{y}|}{\sqrt{1+it}}}.$$

Now we observe that

$$\varphi_{\mathrm{cr}}(\mathrm{x}-\mathrm{y},t) = \frac{c}{\sqrt{t}} \left\{ e^{z^2} - e^{z^2}\mathrm{erf}(z) \right\}_{z=\frac{i|\mathrm{x}-\mathrm{y}|}{\sqrt{1+it}}}$$

Since the second term has a first order zero in $z = 0$ we can show that

$$\varphi_{\mathrm{cr}}(\mathrm{x}-\mathrm{y},t) = \frac{c}{\sqrt{t}} \left\{ e^{z^2} \right\}_{z=\frac{i|\mathrm{x}-\mathrm{y}|}{\sqrt{1+it}}} + \frac{1}{\sqrt{t}}\mathcal{O}_{[-1,0]}\left( \frac{|\mathrm{x}|}{\sqrt{t}} \right)$$

The property $\mathcal{O}_{[-1,0]}$ of the second term can be now used to improve the decrease in time of the corresponding terms, leading to the usual vanishing argument. The only difficult term is then

$$\mathrm{Im}(\exp(z^2 + (z')^2)) = e^{-\frac{1}{1+t^2}\left(|\mathrm{x}-\mathrm{y}'|^2 + |\mathrm{x}-\mathrm{y}|^2\right)} \sin\left( \frac{t}{1+t^2}\left( |\mathrm{x}-\mathrm{y}'|^2 - |\mathrm{x}-\mathrm{y}|^2 \right) \right)$$

and a convenient bound is given by

$$\left| \mathrm{Im}(\exp(z^2 + (z')^2)) \right| \leq \frac{C}{t^{1/4}} \left( |\mathrm{x}-\mathrm{y}'|^2 + |\mathrm{x}-\mathrm{y}|^2 \right)^{1/4}.$$

One proves that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\left( |\mathrm{x}-\mathrm{y}'|^2 + |\mathrm{x}-\mathrm{y}|^2 \right)^{1/4}}{|\mathrm{x}-\mathrm{y}||\mathrm{x}-\mathrm{y}'|^2} V(\mathrm{y})\psi_{\mathrm{res}}(\mathrm{y})V(\mathrm{y}')\psi_{\mathrm{res}}(\mathrm{y}')\, dy\, dy' \leq \frac{C}{|\mathrm{x}|^{5/2}}$$

and then it follows that

$$|\mathrm{j}_{\mathrm{cr}}(\mathrm{x},t)| \leq \frac{C_1}{|\mathrm{x}|^{5/2} t^{5/4}} + \frac{C_2}{|\mathrm{x}|^3 t} \mathcal{O}_{[-1,0]}\left(\frac{|\mathrm{x}|}{\sqrt{t}}\right)^2 .$$

The time decreasing of the first term and the property $\mathcal{O}_{[-1,0]}$ of the second term are sufficient to employ the dominated convergence theorem and prove (48). This concludes the proof of Theorem 3.10 in the case in which zero is not an eigenvalue of $H$.

If zero is an eigenvalue, the proof follows a similar line, the main ingredient being again Prop. 3.8 ■

# 4 Appendix: some remarks on zero-energy resonances

From a phenomenological point of view, the idea of *quantum resonance* is related to a "bump" of an observable quantity as a function of some experimental parameter, like – for example – the scattering cross section as a function of the energy of the incoming particles.[5]
The appearing of these "bumps" in the scattering cross-section can be related, from a theoretical point of view, to some mathematical properties of the pair $(H, H_0)$, as, for example:

(i) the existence of almost-$L^2$ solutions of the stationary Schrödinger equation

(ii) the poles of a suitable meromorphic extension of the *resolvent*

(iii) the poles of the analytically continued *scattering operator* in the momentum decomposition.

In many situations, as for example in the case of exponentially decaying potentials, it can be shown that the previous properties are equivalent.

## 4.1 Resonances as complex poles

Roughly speaking, the mathematical theory of resonances is based upon the idea that, under suitable assumptions on the potential, there exists – in some suitable topology – a convenient *meromorphic continuation* of the resolvent

$$R(\kappa) = (H - \kappa^2)^{-1} \qquad (\kappa \in \mathbb{C}_+, \kappa^2 \in \rho(H))$$

to a region larger than $\mathbb{C}_+$. It is clear that the resolvent does not admit a meromorphic continuation as a $\mathcal{B}(L^2)$-valued function, since for $\kappa \in \mathbb{R}$ we have $\kappa^2 \in \sigma(H)$. However, one introduces suitable Banach spaces $X$, $Y$ (sometimes with the additional conditions $X \subseteq L^2$ and $H^2 \subseteq Y$) such that the resolvent admits a meromorphic continuation as a $\mathcal{B}(X, Y)$-valued function.[6]
One notices that, under general assumptions on the potential, the resolvent $R(\kappa) \in \mathcal{B}(L^2)$, satisfy the *resolvent equations*

$$R(\kappa) = (1 + R_0(\kappa)V)^{-1} R_0(\kappa) \tag{59}$$

$$R(\kappa) = R_0(\kappa)(1 + V R_0(\kappa))^{-1} \tag{60}$$

and is meromorphic in $\mathbb{C}_+$ with poles at the discrete eigenvalues of $H$. Decompositions (59) and (60) suggest defining the meromorphic continuation of the resolvent by an analytic continuation of the *free* resolvent and the previous factorization. [7]

[5] For a general overview of the relationship between the phenomenology and the mathematical theory of resonance see [AFS] or [RS4], Sec. XII.6.

[6] The choice of the spaces $X, Y$ – although sometime suggested by the properties of the potential – is not natural and there is no *a priori* warranty that the position of the poles of the meromorphic continuation do not depend on this choice. This ambiguity leads to the problem of the *equivalence of definitions* of resonances (see Prop. 4.3 for zero-energy resonances).

[7] Sometime it is convenient to notice that, under the assumption that $V$ belongs to the Rollnik class, the resolvent can be expressed as (see [RS3], Chapter XI, Problem 61 or [Ge])

$$R(\kappa) = R_0(\kappa) + \left(R_0(\kappa)|V|^{1/2}\right)(1 + K_\kappa)^{-1}\left(V^{1/2} R_0(\kappa)\right)$$

In order to illustrate this method, we focus first on the class of *exponentially decaying* potentials and we follow an exposition dual to the one given by [Ba].

To be precise, we require that

$$V = e^{-ar} W e^{-ar} \qquad (r = |x|) \tag{61}$$

where $W$ is an $H_0$-compact, symmetric operator. Under this assumption the resolvent equations (59) and (60) hold true for every $\kappa \in \mathbb{C}_+$. It is convenient to define the spaces (for $\sigma \geq 0$)

$$H_\sigma(\mathbb{R}^d) := \left\{ u \in \mathcal{S}'(\mathbb{R}^d) : \left\| e^{\sigma|x|} u \right\|_{L^2} < +\infty \right\}$$

$$H^2_{\pm\sigma}(\mathbb{R}^d) := \left\{ u \in \mathcal{S}'(\mathbb{R}^d) : \left\| e^{\pm\sigma|x|} u \right\|_{H^2} < +\infty \right\}$$

since the free resolvent, regarded as a $\mathcal{B}(H_\sigma, H^2_{-\sigma})$-valued function, admits a meromorphic continuation $\widetilde{R}_0(\kappa)$, to the region

$$\Omega_\sigma = \{\kappa \in \mathbb{C} : \mathrm{Im}(\kappa) > -\sigma\}.$$

Condition (61) implies that $V \in \mathcal{B}_\infty(H^2_{-a}, H_a)$, where $a$ is given in (61), so that the map $\kappa \mapsto 1 + \widetilde{R}_0(\kappa)V$ is a compact operator valued analytic function on $\Omega_a$. Then the Fredholm theory implies that the r.h.s of (59) is a meromorphic function on $\Omega_a$(with a discrete set of singularities $\Sigma$) that can be seen as a meromorphic continuation of the resolvent. Moreover a point $\kappa_0 \in \Omega_a$ belongs to $\Sigma$ if and only if the equation

$$(1 + \tilde{R}_0(\kappa_0)V)\psi = 0 \tag{62}$$

admits a non trivial solution $\psi \in H^2_{-a}$.

It is possible to prove that a point $\kappa_0 \in \Sigma \cap \mathbb{C}_+$ must lie on the imaginary axis and corresponds to a discrete negative eigenvalue of $H$ with eigenfunction $\psi$. A point $\kappa_0 \in \Sigma \cap (\mathbb{R} \setminus \{0\})$ corresponds to a positive eigenvalue of $H$ (embedded in the continuous spectrum). So we are lead to the following definition.

**Definition 4.1** *A singular point $\kappa_0 \in \Sigma$ in the lower half-plane $\mathbb{C}_-$ is called a* ***resonance point*** *for the pair $(H, H_0)$. A non trivial solution $\psi \in H^2_{-a}$ of the equation (62) is called the* ***resonance function*** *corresponding to $\kappa_0$.*

We emphasize that the concept of resonance is related to the pair $(H, H_0)$ and not only to the operator $H$, since decompositions (59) and (60) are based upon the decomposition of $H$ as $H_0 + V$.

A singularity of the resolvent in the point $\kappa_0 = 0$ corresponds to the fact that the equation

$$\left(1 + \tilde{R}_0(0)V\right)\psi = 0 \tag{63}$$

has non trivial solutions in $H^2_{-a}$. If there exists a solution $\psi \in H^2_{-a} \cap L^2$ then zero is an eigenvalue for $H$. If there exists a solution $\psi \in H^2_{-a}$ such that $\psi \notin L^2$ then we say that there

where

$$K_\kappa := V^{1/2} R_0(\kappa) |V|^{1/2} \qquad (\kappa \in \mathbb{C}_+).$$

This symmetric decomposition can be useful in order to define a convenient meromorphic extension of $R(\kappa) - R_0(\kappa)$ to a region $\Omega$ larger than $\mathbb{C}_+$.

is a *zero-energy resonance* and $\psi$ is called *resonance function.* Notice that $\kappa_0 = 0$ can be both (and simultaneously) an eigenvalue for $H$ and a resonance for the pair $(H, H_0)$ (see Lemma 3).

For exponentially decaying potentials, it is possible to find a relationship between to poles of the resolvent and the poles of the analytically continued scattering matrix and then a connection with observable quantities is possible (see again [Ba] for details).

### 4.2 General theory of zero-energy resonances

In the previous analysis, we introduced the *exponentially weighted* Sobolev spaces $H^m_{\pm\sigma}$ in order to obtain an *analytic* continuation of the free resolvent to the open region $\Omega_a \supseteq \mathbb{C}_+$. This forced us to consider exponentially decreasing potentials, to make $G_0 V$ a well-defined compact operator on $H^2_{-a}$.

However, we are only interested in *zero-energy resonances*, since only the poles of the resolvent on the *real axis* (in momentum complex plane) can affect the behavior of the Lippman-Schwinger generalized eigenfunctions. To define zero-energy resonances, we need only to *continuously* extend the free resolvent to the closed upper half-plane, and this extension is provided – in the weighted Sobolev space topology – by Lemma 3.2. Then formula (63) suggests to define – for a wide class of potential – a zero energy resonance function as an element of the null space of $1 + G_0 V \in \mathcal{B}(H^{2,-s})$ which does not belong to $L^2$. The following analysis make this idea precise and gives the connection with other usual approaches.

Assume condition (V.1.$\beta$) for $\beta > 2$. Then Lemma 3.2 assures that $G_0 V \in \mathcal{B}_\infty(H^{1,-s})$ and $V G_0 \in \mathcal{B}_\infty(H^{-1,s})$, where $s > \frac{1}{2}$ and $\mathcal{B}_\infty$ denotes the space of compact operators. We denote with $\mathfrak{M}_s$ the kernel of $1 + G_0 V$ in $H^{1,-s}$ and with $\mathfrak{N}_s$ the kernel of $1 + V G_0$ in $H^{-1,s}$. These spaces may depend on $s$, but $\mathfrak{M}_s$ is monotone increasing and $\mathfrak{N}_s$ is monotone decreasing in $s$ and – by duality – we have $\dim \mathfrak{M}_s = \dim \mathfrak{N}_s = d < +\infty$. Thus they are independent of $s$ and we denote them simply as $\mathfrak{M}$ and $\mathfrak{N}$; moreover

$$\mathfrak{M} \subseteq H^{1,-\frac{1}{2}-0} \qquad \mathfrak{N} \subseteq H^{-1,\beta-\frac{1}{2}-0}$$

We summarize the properties of this spaces, proved by Jensen and Kato in [JK], in the following lemma.

**Lemma 4.2** *Fix $s \in (\frac{1}{2}, \frac{3}{2}]$. Then:*

(i) *the kernel of $-\Delta + V$ in $H^{1,-s}$ coincides with $\mathfrak{M}$*

(ii) *both $-\Delta$ and $V$ are injective from $\mathfrak{M}$ onto $\mathfrak{N}$ and $G_0$ is injective from $\mathfrak{N}$ onto $\mathfrak{M}$ with inverse $-\Delta$.* [8]

(iii) *the eigenspace $P_0\mathcal{H}$ relative to the self-adjoint operator $H$ in $L^2(\mathbb{R}^3)$ is included in $\mathfrak{M} \cap H^{1,\frac{1}{2}-0}$ with $\dim(\mathfrak{M} / P_0\mathcal{H}) \leq 1$. Moreover, $\psi \in \mathfrak{M}$ belongs to $L^2$ if and only if $\langle \psi, V1 \rangle = 0$.*

We notice that Lemma 3.iii gives us a criterion to distinguish in $\mathfrak{M}$ a subspace (with codimension $\leq 1$) corresponding to the eigenfunctions of the operator $H = H_0 + V = -\Delta + V|_{L^2}$. We would like to associate the (possible) complementary subspace to a zero-energy resonance.

[8] In other words, while $-\Delta$ has only a *right* inverse $G_0$ on $\mathcal{S}'(\mathbb{R}^d)$, the restriction $-\Delta|_{\mathfrak{M}}$ has a *bilateral* inverse $G_0|_{\mathfrak{N}}$

However, before formulating a precise definition, we relate this approach to other possible definitions of zero energy-resonances, in particular to the ones used in Sec. 3 (in particular [Ik], [TDMB], [JK], [Ag]). A connection with the approach outlined in Note 7 is possible too.

**Proposition 4.3** *Assume $V \in (\mathrm{I})_3$. Then the following propositions are equivalent:*

(i) (***standard definition***) *there exists a* distributional *solution $\psi_1 \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$ of the stationary Schrödinger equation $(-\Delta + V)\,\psi_1 = 0$ such that $(1+|\cdot|^2)^{-\frac{\gamma}{2}}\psi_1 \in L^2(\mathbb{R}^3)$ for every $\gamma > 1/2$ but not for $\gamma = 0$.*

(ii) (***Agmon definition***) *there exists a $\psi_2 \in H^{2,-\gamma} \cap C_\infty$ (for every $\gamma > \frac{1}{2}$) such that $(1 + G_0 V)\,\psi_2 = 0$ but $\psi_2 \notin L^2$.*

(iii) (***Jensen-Kato definition***) *there exists a $\psi_3 \in H^{1,-\gamma}$ (for every $\gamma > \frac{1}{2}$) such that $(1 + G_0 V)\,\psi_3 = 0$ but $\psi_3 \notin L^2$.*

(iv) (***dual Jensen-Kato definition***) *there exists a $\phi \in H^{-1,\gamma}$ (for every $\gamma > \frac{1}{2}$) such that $(1 + V G_0)\,\phi = 0$ but $G_0\phi \equiv \psi_4 \notin L^2$.*

(v) (***Ikebe definition***) *there exists a $\psi_5 \in C_\infty(\mathbb{R}^3)$ such that $(1 + G_0 V)\,\psi_5 = 0$ but $\psi_5 \notin L^2$.*

*In addition, all the functions $\psi_i$ $(i = 1, \ldots, 5)$ are the same element of $\mathcal{S}'(\mathbb{R}^3)$ that will be denoted by $\psi_{\mathrm{res}}$. Moreover, if the potential satisfies (61) (and is a multiplication operator) then $\psi_{\mathrm{res}} \in H^2_{-a}(\mathbb{R}^3)$ and each one of the previous propositions implies that the meromorphic continuation of the resolvent, seen as a $\mathcal{B}(H_a, H^2_{-a})$-valued function, has a complex pole in $\kappa = 0$.*

**Definition 4.4 (Zero-energy resonances)** *We say that there is a zero-energy resonance for the pair $(H, H_0)$ if one of the conditions of Proposition 4.3 holds true.*

**Remark.** From the fact that $-\Delta G_0 = 1$ it follows straightforwardly that if $\psi \in \mathcal{S}'(\mathbb{R}^3)$ satisfy the equation

$$(1 + G_0 V)\,\psi = 0 \tag{64}$$

then it satisfies also the equation

$$-\Delta\psi + V\psi = 0 \tag{65}$$

provided that $V\psi$ is a well-defined distribution. [9] The converse statement is not true in general, since a *left* inverse of $-\Delta$ in $\mathcal{S}'$ does not exist.

**Proof of proposition.** We will show that (i) ⇒(ii) ⇒((iii) or (v))⇒(i). Moreover, (iii)⇔(iv) as a consequence of Lemma 3.ii .

**(i) ⇒(ii).** Suppose that $\psi_1 \equiv \psi \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$ satisfy the Schrödinger equation in distributional sense. From the fact that the kernel of $-\Delta$ is closed in $\mathcal{S}'(\mathbb{R}^3)$ it follows, by a decomposition argument, that the general solution of equation (65) must be in the form

$$\psi = G_0 V\psi + \varphi \tag{66}$$

where $\varphi$ is an harmonic distribution, i.e. $\Delta\varphi = 0$. We are going to show that from our assumptions it follows that $\varphi = 0$.

[9] This condition is satisfied for our class of potentials if $\psi \in L^2_{\mathrm{loc}}$

Indeed $\varphi \in L^2_{-\gamma}(\mathbb{R}^3)$ (in the following we understand $\gamma \in (\frac{1}{2}, +\infty)$). This follows from (66) by noticing that $\psi \in L^2_{-\gamma}(\mathbb{R}^3)$ by hypothesis [10] and that $G_0 V \psi$ also belongs to $L^2_{-\gamma}(\mathbb{R}^3)$, as can be proved by using the explicit form of $G_0$ as convolution operator and Jensen inequality.

In particular $\varphi$ is an harmonic distribution that belongs to $L^1_{\text{loc}}$, and then it is (identifiable with) a smooth function in $C^\infty(\mathbb{R}^3)$ (see [LL], Th. 9.3).

Moreover, $\varphi$ must be *bounded.* In order to see this, suppose that $\varphi$ is unbounded. Then it is possible to find a sequence $\{\mathrm{x}_n\}_{n\in\mathbb{N}}$ such that $|\varphi(\mathrm{x}_n)| \geq n$. The continuity of $\varphi$ implies that this sequence cannot have accumulation points. Then it is possible to extract a subsequence $\left\{\mathrm{x}_{n_j} \equiv \mathrm{w}_j\right\}_{j\in\mathbb{N}}$ such that:

(i) $|\varphi(\mathrm{w}_j)| \geq n_j \geq j$ and $|\mathrm{w}_j| \geq 2^j$

(ii) if $\delta_j = \frac{1}{2}|\mathrm{w}_j|$ then $B_{\delta_j}(\mathrm{w}_j) \cap B_{\delta_l}(\mathrm{w}_l) = \emptyset$ if $j \neq l$.

By using the disjointedness of the balls $B_j \equiv B_{\delta_j}(\mathrm{w}_j)$ we get that

$$\begin{aligned}
\|\varphi\|_{L^2_{-\gamma}} &= \int_{\mathbb{R}^3} |\varphi(\mathrm{y})|^2 \,(1+|\mathrm{y}|^2)^{-\gamma}\, d\mathrm{y} \geq \\
&\geq \sum_{j\in\mathbb{N}} \int_{B_j} |\varphi(\mathrm{y})|^2 \,(1+|\mathrm{y}|^2)^{-\gamma}\, d\mathrm{y} \geq \\
&\geq \sum_{j\in\mathbb{N}} \inf_{\mathrm{y}\in B_j} \left\{(1+|\mathrm{y}|^2)^{-\gamma}\right\} \int_{B_j} |\varphi(\mathrm{y})|^2 \, d\mathrm{y}
\end{aligned}$$

¿From the fact that $\Delta\varphi = 0$ it follows that $|\varphi|^2$ is a subharmonic function, i.e. $\Delta|\varphi|^2 \geq 0$. Then we can apply the mean value inequality getting

$$\begin{aligned}
\|\varphi\|_{L^2_{-\gamma}} &\geq \sum_{j\in\mathbb{N}} \inf_{\mathrm{y}\in B_j} \left\{(1+|\mathrm{y}|^2)^{-\gamma}\right\} \frac{4}{3}\pi\delta_j^3 \,|\varphi(\mathrm{w}_j)|^2 \\
&\geq \sum_{j\in\mathbb{N}} C \frac{\delta_j^3 \;\; j^2}{(1+(|\mathrm{w}_j|+\delta_j)^2)^\gamma}
\end{aligned}$$

By using the claimed properties of the sequence $\{\mathrm{w}_j\}_{j\in\mathbb{N}}$ one gets that the previous series diverges, against the fact that $\varphi \in L^2_{-\gamma}(\mathbb{R}^3)$. Then $\varphi$ is bounded.

By Liouville's theorem, a bounded harmonic function on $\mathbb{R}^d$ is constant and a constant function that belongs to $L^2_{-\gamma}(\mathbb{R}^3)$ (for every $\gamma > \frac{1}{2}$) must be identically zero. Then $\varphi = 0$.

This proves that $\psi$ satisfies equation (64) in distributional sense. It remains to be shown that $\psi \in H^{2,-\gamma} \cap C_\infty$.

Since $\psi = G_0 V \psi$, an iterative or "bootstrap" argument based upon the smoothing properties of the operator $G_0$ (see, for example, [LL] Sec. 10.2) shows that $\psi \in C^{0,\alpha}(\mathbb{R}^3)$, the space of uniformly Hölder-continuous functions of order $\alpha$, for every $\alpha < 1/2$.

Moreover, $\psi \in L^\infty(\mathbb{R}^3)$. Recall that the hypothesis $V \in (\mathrm{I})_n$ implies that there exist $R > 0$ and $\varepsilon > 0$ such that $V$ is continuous in $\mathbb{R}^3 \setminus B_R$ and

$$|V(\mathrm{y})| \leq \frac{C}{|\mathrm{y}|^{n+\varepsilon}} \qquad \text{for } |\mathrm{y}| > R. \tag{67}$$

[10] Recall that $f \in L^2_s$ if and only if $(1+|\cdot|^2)^{\frac{s}{2}} f \in L^2$ .

Choose $\gamma_0 = \frac{1}{2} + \frac{\varepsilon}{2}$ (with $\varepsilon$ from (67)). Fix a compact set $K$ such that $K \supseteq B_{2R}$. The continuity of $\psi$ implies that it is bounded over the compact set $K$. To show the boundness on $\mathbb{R}^3 \setminus K$, we use the explicit form of the distributional kernel of $G_0$ getting

$$\begin{aligned} |\psi(\mathrm{x})| &\leq \int_{\mathbb{R}^3 \setminus B_1(\mathrm{x})} \frac{1}{|\mathrm{x}-\mathrm{y}|} |V(\mathrm{y})\psi(\mathrm{y})| \; dy + \int_{B_1(\mathrm{x})} \frac{1}{|\mathrm{x}-\mathrm{y}|} |V(\mathrm{y})\psi(\mathrm{y})| \; dy \\ &\equiv \psi_1(\mathrm{x}) + \psi_2(\mathrm{x}) \end{aligned}$$

The first term can be easily bounded, by noticing that

$$|\psi_1(\mathrm{x})| \leq \int_{\mathbb{R}^3} |V(\mathrm{y})| \, |\psi(\mathrm{y})| \; dy < +\infty$$

where the last inequality follows from the fact that $V\psi \in L^1_{\mathrm{loc}}$ and that for $V \in (\mathrm{I})_2$ one has

$$\int_{\mathbb{R}^3 \setminus B_{2R}(0)} |V(\mathrm{y})| \, |\psi(\mathrm{y})| \; dy \leq \left( \int_{\mathbb{R}^3 \setminus B_{2R}(0)} |V(\mathrm{y})|^2 \, (1+|\mathrm{y}|)^{2\gamma_0} dy \right)^{1/2} \|\psi\|_{L^2_{-\gamma_0}} < +\infty. \tag{68}$$

As for the second term, by a simple change of variables we get

$$\begin{aligned} |\psi_2(\mathrm{x})| &\leq \int_{B_1(\mathrm{x})} \frac{1}{|\mathrm{x}-\mathrm{y}|} |V(\mathrm{y})\psi(\mathrm{y})| \; dy \\ &\leq \int_{B_1(0)} \frac{1}{|\mathrm{w}|} |V(\mathrm{w}+\mathrm{x})\psi(\mathrm{w}+\mathrm{x})| \; dw \\ &\leq \operatorname*{Sup}_{\mathrm{y} \in B_1(\mathrm{x})} \{|V(\mathrm{y})| \, (1+|\mathrm{y}|)^{\gamma}\} \int_{B_1(0)} \frac{1}{|\mathrm{w}|} (1+|\mathrm{w}+\mathrm{x}|)^{-\gamma} \, |\psi(\mathrm{w}+\mathrm{x})| \; dw \\ &\leq \operatorname*{Sup}_{\mathrm{y} \in B_1(\mathrm{x})} \{|V(\mathrm{y})| \, (1+|\mathrm{y}|)^{\gamma}\} \left( \int_{B_1(0)} \frac{1}{|\mathrm{w}|^2} \; dw \right)^{1/2} \|\psi\|_{L^2_{-\gamma}} \end{aligned} \tag{69}$$

where in the last inequality we applied the Schwartz theorem.
For every $\mathrm{y} \in B_1(\mathrm{x})$ we have $\frac{|\mathrm{x}|}{2} < |\mathrm{y}| < \frac{3}{2}|\mathrm{x}|$ (without loss of generality we can assume $R > 1$, so that the previous condition holds true). Recalling that $|\mathrm{x}| > 2R$ and using the decreasing condition (67) we obtain

$$\operatorname*{Sup}_{\mathrm{y} \in B_1(\mathrm{x})} \{|V(\mathrm{y})| \, (1+|\mathrm{y}|)^{\gamma}\} \leq C' \frac{(1+|\mathrm{x}|)^{\gamma}}{|\mathrm{x}|^{2+\varepsilon}} < C_R \tag{70}$$

where the last bound follows by choosing $\gamma \in (\frac{1}{2}, 2)$. From estimates (68), (69) and (70) it follows our claim $\psi \in L^{\infty}(\mathbb{R}^3)$.

It has been proved by Ikebe (see [Ik], Lemma 3.1) that a bounded and continuous solution of the equation (64) with $V \in (\mathrm{I})_2$ must vanish at infinity. Then $\psi \in C_{\infty}(\mathbb{R}^3)$.

Finally, we notice that $\Delta\psi = V\psi \in L^2_{\mathrm{loc}}$ and hence, by Sobolev inequalities, we get $\psi \in H^2_{\mathrm{loc}}$. Since, by hypothesis, $\psi$ belongs also to $L^2_{-\gamma}$ a theorem by Agmon (see [Ag] Lemma 5.1) implies that $\psi \in H^{2,-\gamma}$.

**(ii) $\Rightarrow$(iii) or (v).** Trivial.

**(iii) $\Rightarrow$(i).** From the fact that $G_0$ is the right inverse of $-\Delta$ it follows that $\psi_3$ is a distributional solution of the stationary Schrödinger equation (65). Since $\psi_3$ belongs to $H^{2,-\gamma} \subset L^2_{-\gamma}$ the claim follows.

**(v) $\Rightarrow$(i).** As before, we know that $\psi_5 \equiv \psi$ is a distributional solution of (65). The identity $\psi = G_0 V \psi$ and a result by Ikebe ([Ik], Lemma 3.2; see also [TDMB], Lemma 3.3) implies that $|\psi(\mathrm{x})| \leq C\ |\mathrm{x}|^{-1}$ as $|\mathrm{x}| \to \infty$, provided that $V \in (\mathrm{I})_3$. Taking into account the continuity of $\psi$, it follows that $\psi \in L^2_{-\gamma}(\mathbb{R}^3)$ for every $\gamma > \frac{1}{2}$. ■

**Remark.** We used the hypothesis $V \in (\mathrm{I})_3$ only to prove that (v)$\Rightarrow$(i). All the remaining results hold true under the weaker hypothesis that $V \in (\mathrm{I})_2$.

**Remark.** From the proof of the previous proposition it is possible to extract the following result concerning the asymptotic behavior and the local regularity of distributional solutions of the stationary Schrödinger equation: if $V \in (\mathrm{I})_2$ and $\psi \in L^2_{-\gamma}(\mathbb{R}^3)$ (with $\gamma > \frac{1}{2}$) solves equation (65) then $\psi_2 \in H^{2,-\gamma} \cap C_\infty$.

The "equivalence theorem" (Prop. 4.3) shows that – as expected from the physical point of view – the concept of (zero-energy) resonance is largely independent from the technical tools needed to define it. In this spirit, as pointed out by Enss [En], it would be interesting to give a characterization of resonances which involves only the time evolution and the position operator – analogous to the RAGE characterization of bound states and scattering states.

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