PROPAGATION PROPERTIES FOR SCHRÖDINGER OPERATORS AFFILIATED TO CERTAIN C*–ALGEBRAS

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Abstract. We consider anisotropic Schrödinger operators $H = -\Delta + V$ in $L^2(\mathbb{R}^n)$. To certain asymptotic regions F we assign asymptotic Hamiltonians H_F such that (a) $\sigma(H_F) \subset \sigma_{\text{ess}}(H)$, (b) states with energies not belonging to $\sigma(H_F)$ do not propagate into a neighbourhood of F under the evolution group defined by H. The proof relies on C*-algebra techniques. We can treat in particular potentials that tend asymptotically to different periodic functions in different cones, potentials with oscillation that decays at infinity, as well as some examples considered before by Davies and Simon in [3].

1. Introduction

This paper is concerned with propagation properties of scattering states of self-adjoint n-dimensional Schrödinger operators $H = -\Delta + V$ with potentials V having different asymptotics in different directions. We recall that the scattering states of H are defined by the property that, as the time t tends to $\pm\infty$, they propagate away from each bounded region of the configuration space $\mathbb{R}^n$ (at least in some time average [2]). In many situations, in particular if V is a bounded function, they can be identified with the states in the continuous spectral subspace of H. If the potential V tends to zero (or to some other constant) sufficiently rapidly at infinity, standard scattering theory provides a description of the behaviour of $e^{-itH}f$ for a scattering state f at large times t. In more complicated situations, in particular if the asymptotic behaviour of V is highly anisotropic, little is known about the propagation of the scattering states. One may expect that certain asymptotic regions of configuration space should be inaccessible to states of certain energies, as illustrated by the following two examples.

(1) In one dimension ($n = 1$), assume that $V(x) \to V_\pm$ as $x \to \pm\infty$, with $V_+ \neq V_-$. If for example $V_+ > V_-$, a state in the continuous spectral subspace of H with spectral support in the interval (V_-, V_+) will not propagate to the right. (2) In higher dimension ($n \geq 2$) consider a potential V approaching a periodic function

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V_0 as the argument x tends to infinity inside some cone $\mathcal{C} \subset \mathbb{R}^n$. In addition to the Hamiltonian $H = -\Delta + V$ one may introduce the periodic Schrödinger operator $H_0 = -\Delta + V_0$. Bearing in mind some hypothetical scattering theory (e^{-itH_0} should furnish a suitable comparison dynamics for the propagation inside $\mathcal{C}$), one could expect that scattering states of H with energy disjoint from the continuous spectrum of H_0 will not be able to propagate into $\mathcal{C}$ (such states may exist: the spectrum of H_0 has a band structure, whereas the continuous spectrum of H will depend also on the behaviour of V outside the cone $\mathcal{C}$ and thus could intersect some gap in the band spectrum of H_0).

Detailed results about the propagation and non-propagation of states for one-dimensional Schrödinger operators with different spatial asymptotics at $\pm\infty$ (in particular for the first example above) and for multi-dimensional operators periodic in all but one dimension have been obtained by Davies and Simon [3]. The investigation of these authors includes a careful spectral analysis of the Hamiltonians under study. We propose here a different method for obtaining non-propagation properties, based on a relatively recent approach to spectral theory in the framework of C*-algebras and without invoking any scattering theory. We shall in particular obtain the non-propagation result stated in the second example above. Below we give a brief non-technical description of this method.

Typically the potential V to be considered is an element of a C*-algebra $\mathcal{A}$ of bounded, continuous functions on $\mathbb{R}^n$. The functions in $\mathcal{A}$ are characterized by a specific asymptotic behaviour (for example asymptotic periodicity in certain cones). Then, by invoking the Neumann series for $(H-z)^{-1}$ (which is convergent for $\Im z$ large enough), one finds that the resolvent of the operator $H = -\Delta + V$ belongs to a C*-algebra $\mathfrak{C}_{\mathcal{A}}$ generated by products of elements of $\mathcal{A}$ (viewed as multiplication operators in $L^2(\mathbb{R}^n)$) and suitable functions of momentum. We shall say that H is affiliated to $\mathfrak{C}_{\mathcal{A}}$. A central concept is that of the spectrum of H relative to an ideal $\mathfrak{K}$ of $\mathfrak{C}_{\mathcal{A}}$ of the form $\mathfrak{K} = \mathfrak{C}_{\mathcal{K}}$, where $\mathcal{K}$ is an ideal of $\mathcal{A}$ and $\mathfrak{C}_{\mathcal{K}}$ is defined similarly to $\mathfrak{C}_{\mathcal{A}}$ (just replace $\mathcal{A}$ by $\mathcal{K}$ in the definition of $\mathfrak{C}_{\mathcal{A}}$). Let us denote this spectrum by $\sigma^{\mathcal{K}}(H)$ (a precise definition is given in Section 2). For $\mathcal{K} = \{0\}$, $\sigma^{\mathcal{K}}(H)$ is the usual spectrum of H; for $\mathcal{K} = C_0(\mathbb{R}^n)$ (the space of continuous functions converging to zero at infinity), $\mathfrak{K}$ will be the ideal of compact operators and $\sigma^{\mathcal{K}}(H)$ the essential spectrum $\sigma_{\text{ess}}(H)$ of H. For an ideal $\mathcal{K}$ of $\mathcal{A}$ such that $C_0(\mathbb{R}^n) \subset \mathcal{K}$, $\sigma^{\mathcal{K}}(H)$ is a subset of $\sigma_{\text{ess}}(H)$. A typical non-propagation result will assert that scattering states of H with spectral support disjoint from $\sigma^{\mathcal{K}}(H)$ will essentially never be localized in certain spatial domains W determined by $\mathcal{K}$.

In these considerations the ideal $\mathcal{K}$ will be given in terms of the asymptotic behaviour of the elements of $\mathcal{A}$ in some neighbourhood of infinity, and the theory will apply if the spatial domains W associated to $\mathcal{K}$ cover or intersect this neighbourhood of infinity. In the second example mentioned at the beginning $\mathcal{K}$ could be the set of functions in $\mathcal{A}$ that are asymptotically periodic in the cone $\mathcal{C}$ and tend to zero in directions not belonging to $\mathcal{C}$, W could be the intersection of $\mathcal{C}$ with the complement of a compact set and $\sigma^{\mathcal{K}}(H)$ would be the spectrum of H_0.

Our treatment consists in the introduction of a compactification $\mathcal{X}$ of $\mathbb{R}^n$ related to the algebra $\mathcal{A}$ (in fact $\mathcal{X}$ will be the character space of $\mathcal{A}$). The ideal $\mathcal{K}$ is determined by some closed subset F of the frontier $\mathcal{X} \setminus \mathbb{R}^n$ of $\mathbb{R}^n$ in $\mathcal{X}$ and W is a subset of $\mathbb{R}^n$ which is sufficiently close to F. In the cited example one may think, intuitively, of F as the part of the compactification attached at infinity to the cone $\mathcal{C}$.

In these considerations the behaviour of various objects under translations is important. The algebra $\mathcal{A}$ is assumed to be invariant under translations; then the natural action of the translation group in $\mathbb{R}^n$ has a continuous extension to the compactification $\mathcal{X}$ of $\mathbb{R}^n$ and F must be invariant under this extension in order to allow the application of a result from the theory of crossed products of C*-algebras to determine the spectrum $\sigma^{\mathcal{K}}(H)$.

In Section 2 we present some of the C*-algebraic concepts that are useful in the spectral theory of self-adjoint operators. In Section 3 we discuss some algebras $\mathcal{A}$ of continuous functions on $\mathbb{R}^n$, the associated compactifications of $\mathbb{R}^n$ and the continuous extension of the translation group to these compactifications. Section 4 contains a few remarks on crossed product algebras, and in Section 5 we give the proof of an abstract theorem on non-propagation. An application of this theorem to Schrödinger Hamiltonians that are asymptotically periodic in several cones is studied in detail in Section 6, and in Section 7 we mention other classes of Hamiltonians that can be treated in a similar way.

We refer to Chapter 8 of [1] for a presentation of the algebraic approach to spectral theory, and we use the terminology and results of [10] for the theory of C*-algebras. The algebras $\mathfrak{C}_{\mathcal{A}}$ and $\mathfrak{C}_{\mathcal{K}}$ mentioned above have the structure of a crossed product; Reference [4] contains a description of such algebras that is well adapted to applications in spectral theory.

Finally we point out that various generalizations of our results are possible with almost no extra effort (cf. also [1], [4], [8] and [9]). Local singularities of the potential can easily be taken into account. The kinetic energy $-\Delta$ can be replaced by $h(P)$, an arbitrary continuous function of momentum satisfying $|h(p)| \to \infty$ when $|p| \to \infty$. Instead of the configuration space $\mathbb{R}^n$, one can work with any abelian locally compact group X. The case $X = \mathbb{Z}^n$ leads to finite difference operators.

2. C*-Algebras and Generalized Essential Spectra

If H is a self-adjoint operator in a Hilbert space $\mathcal{H}$, the spectral theorem allows one to associate an operator $\eta(H)$ to a large class of functions $\eta : \mathbb{R} \to \mathbb{C}$. We shall be here concerned with the set $C_0(\mathbb{R})$ consisting of all continuous functions $\eta : \mathbb{R} \to \mathbb{C}$ that vanish at infinity (i.e. satifying $\lim_{x\to\pm\infty}\eta(x) = 0$). Some parts of the spectrum of H can easily be characterized in terms of these functions: (i) a number $\lambda \in \mathbb{R}$ belongs to the spectrum $\sigma(H)$ of H if $\eta(H) \neq 0$ whenever $\eta \in C_0(\mathbb{R})$ and $\eta(\lambda) \neq 0$, (ii) λ belongs to the essential spectrum $\sigma_{\mathrm{ess}}(H)$ of H if $\eta(H)$ is a non-compact operator whenever $\eta \in C_0(\mathbb{R})$ and $\eta(\lambda) \neq 0$.

If $\mathfrak{C}$ is a C*-algebra of bounded operators in $\mathcal{H}$ such that $\eta(H) \in \mathfrak{C}$ for each $\eta \in C_0(\mathbb{R})$, then H is said to be *affiliated* to $\mathfrak{C}$. A sufficient condition for H to be affiliated to $\mathfrak{C}$ is the requirement that $(H - z)^{-1} \in \mathfrak{C}$ for some complex number $z \notin \sigma(H)$.

The preceding situation can be viewed as a special case of the following abstract definition:

Definition 1. *(a)* An observable affiliated to a C*-algebra $\mathfrak{C}$ *is a* $*$*-homomorphism* *from the C*-algebra $C_0(\mathbb{R})$ to $\mathfrak{C}$ (i.e. a linear mapping $\Phi : C_0(\mathbb{R}) \to \mathfrak{C}$ satisfying $\Phi(\xi\eta) = \Phi(\xi)\Phi(\eta)$ and $\Phi(\eta)^* = \Phi(\overline{\eta})$ if $\xi, \eta \in C_0(\mathbb{R})$).*

(b) The spectrum $\sigma(\Phi)$ of the observable Φ is defined as the set of real numbers λ such that $\Phi(\eta) \neq 0$ whenever $\eta(\lambda) \neq 0$. $\sigma(\Phi)$ is a closed subset of $\mathbb{R}$.

Now let $\mathfrak{K}$ be a (closed, self-adjoint, bilateral) ideal in $\mathfrak{C}$. We denote by $\hat{\mathfrak{C}} \equiv \mathfrak{C}/\mathfrak{K}$ the associated quotient C*-algebra and by Π the canonical $*$-homomorphism of $\mathfrak{C}$ onto $\hat{\mathfrak{C}}$. If Φ is an observable affiliated to $\mathfrak{C}$, then clearly $\Pi \circ \Phi$ determines an observable affiliated to $\hat{\mathfrak{C}}$.

Definition 2. *The spectrum $\sigma(\Pi\circ\Phi)$ of the observable $\Pi\circ\Phi$ (relative to $\hat{\mathfrak{C}}$) is called* the $\mathfrak{K}$-essential spectrum *of Φ and will be denoted by $\sigma_{\mathfrak{K}}(\Phi)$: $\sigma_{\mathfrak{K}}(\Phi) \equiv \sigma(\Pi \circ \Phi)$. Equivalently, a real number λ belongs to $\sigma_{\mathfrak{K}}(\Phi)$ if and only if $\Phi(\eta) \notin \mathfrak{K}$ whenever $\eta \in C_0(\mathbb{R})$ is such that $\eta(\lambda) \neq 0$.*

To motivate the present terminology, let us consider the situation introduced at the beginning, where $\mathfrak{C}$ is a C*-subalgebra of $B(\mathcal{H})$ and Φ^H is the observable determined by a self-adjoint operator H affiliated to $\mathfrak{C}$ (so $\Phi^H(\eta) = \eta(H)$). Assume that $\mathfrak{C}$ contains the ideal $K(\mathcal{H})$ of all compact operators in $\mathcal{H}$. Then $\sigma_{K(\mathcal{H})}(\Phi^H)$ is just the essential spectrum $\sigma_{\mathrm{ess}}(H)$ of the self-adjoint operator H.

Now let us observe that, if $\mathfrak{K}_1$ and $\mathfrak{K}_2$ are two ideals in $\mathfrak{C}$ satisfying $\mathfrak{K}_1 \subset \mathfrak{K}_2$, then $\sigma_{\mathfrak{K}_2}(\Phi) \subset \sigma_{\mathfrak{K}_1}(\Phi) \subset \sigma(\Phi)$. In particular, if H is a self-adjoint operator affiliated to a C*-subalgebra $\mathfrak{C}$ of $B(\mathcal{H})$ and if $\mathfrak{K}$ is an ideal in $\mathfrak{C}$ with $K(\mathcal{H}) \subset \mathfrak{K}$, then $\sigma_{\mathfrak{K}}(\Phi^H) \subset \sigma_{\mathrm{ess}}(H)$.

It seems that the $\mathfrak{K}$-essential spectrum for $\mathfrak{K}$ different from $\{0\}$ and $K(\mathcal{H})$ has never been used before.

One of the interesting aspects of the preceding framework in the study of self-adjoint operators in a Hilbert space $\mathcal{H}$ is as follows. Let $\mathfrak{C}$ be a C*-subalgebra of $B(\mathcal{H})$ and consider a class Θ of self-adjoint operators H affiliated to $\mathfrak{C}$ such that, for some ideal $\mathfrak{K}$ of $\mathfrak{C}$, the $*$-homomorphisms $\Pi \circ \Phi^H$ do not depend on H. So all members H of Θ have the same $\mathfrak{K}$-essential spectrum $\sigma_{\mathfrak{K}}$. In some situations it is rather easy to determine $\sigma_{\mathfrak{K}}$: although the quotient C*-algebra $\hat{\mathfrak{C}}$ will not be identifiable with a subalgebra of $B(\mathcal{H})$, it may be possible to specify a faithful representation of $\hat{\mathfrak{C}}$ in a Hilbert space $\hat{\mathcal{H}}$ (an injective $*$-homomorphism $\pi : \hat{\mathfrak{C}} \to B(\hat{\mathcal{H}})$) and a simple self-adjoint operator $\hat{H}$ in $\hat{\mathcal{H}}$ affiliated to $\pi(\hat{\mathfrak{C}})$ such that $\pi\left[\Pi(\eta(H))\right] = \eta(\hat{H})$ for all $\eta \in C_0(\mathbb{R})$ and all $H \in \Theta$. Then $\sigma_{\mathfrak{K}}$ is just the spectrum of the (presumably simple) operator $\hat{H}$. Examples will be considered in Sections 6 and 7.

The following result will be used in Section 5:

Lemma 1. *Let $\mathfrak{K}$ be an ideal in a C*-algebra $\mathfrak{C}$ and Φ an observable affiliated to $\mathfrak{C}$. If $\eta \in C_0(\mathbb{R})$ is such that $\eta(\mu) = 0$ for all $\mu \in \sigma_{\mathfrak{K}}(\Phi)$, then $\Phi(\eta) \in \mathfrak{K}$.*

Proof. (i) Let $\lambda \in \mathbb{R} \setminus \sigma_{\mathfrak{K}}(\Phi)$. There are a number $\varepsilon > 0$ and a function $\theta \in C_0(\mathbb{R})$ such that $|\theta(\mu)| > \varepsilon$ for all $\mu \in (\lambda-\varepsilon, \lambda+\varepsilon)$ and $\Phi(\theta) \in \mathfrak{K}$. Now let $\xi \in C_0(\mathbb{R})$ be such that $\mathrm{supp}\xi \subset (\lambda-\varepsilon, \lambda+\varepsilon)$. Since $\xi/\theta \in C_0(\mathbb{R})$, we have $\Phi(\xi) = \Phi(\theta)\Phi(\xi/\theta) \in \mathfrak{K}$. In conclusion: each λ in $\mathbb{R} \setminus \sigma_{\mathfrak{K}}(\Phi)$ has an open neighbourhood $\mathcal{V}_\lambda$ with the property that $\Phi(\xi) \in \mathfrak{K}$ for each $\xi \in C_0(\mathbb{R})$ having support in $\mathcal{V}_\lambda$.

(ii) Since $\mathfrak{K}$ is norm-closed, it is enough to establish the conclusion of the lemma under the additional assumption that η has compact support in $\mathbb{R} \setminus \sigma_{\mathfrak{K}}(\Phi)$. Choose a finite collection of numbers $\lambda_1, \ldots, \lambda_M \in \mathrm{supp}\eta$ such that $\mathrm{supp}\eta \subset \cup_k \mathcal{V}_{\lambda_k}$ and a corresponding partition of unity on $\mathrm{supp}\eta$, i.e. a collection of functions ξ_k in $C_0(\mathbb{R})$ such that $\mathrm{supp}\xi_k \subset \mathcal{V}_{\lambda_k}$ and $\sum_{k=1}^M \xi_k(\lambda) = 1$ for all $\lambda \in \mathrm{supp}\eta$. Since $\Phi(\xi_k) \in \mathfrak{K}$ by (i), we get $\Phi(\eta) = \sum_{k=1}^M \Phi(\eta)\Phi(\xi_k) \in \mathfrak{K}$. $\square$

3. Some Abelian C*-Algebras

If Y is a locally compact, Hausdorff space, we denote by $C_b(Y)$ the abelian C*-algebra of all bounded, continuous complex functions defined on Y. If G is a closed subset of Y, we set $C^G(Y) = \{\varphi \in C_b(Y) \mid \varphi(y) = 0,\ \forall y \in G\}$. Certain C*-subalgebras of $C_b(Y)$ will be important further on, in particular the algebras $C_b^u(Y)$ and $C_0(Y)$ consisting respectively of all bounded, uniformly continuous functions and of all continuous functions vanishing at infinity. In fact $C_0(Y)$ is an ideal of $C_b(Y)$.

Throughout this paper we set $X = \mathbb{R}^n$. Let Y be as above and assume that X acts on Y as a group of homeomorphisms: so if α_x denotes the homeomorphism in Y associated to the element $x \in X$, we have $\alpha_x \circ \alpha_{x'} = \alpha_{x+x'}$. The mapping $X \times Y \ni (x, y) \mapsto \alpha_x(y) \in Y$ is assumed continuous. Then α induces a representation of the group X by $*$-automorphisms of $C_b(Y)$ as well as of various C*-subalgebras of $C_b(Y)$: for $\varphi \in C_b(Y)$ and $x \in X$, define $a_x(\varphi) \in C_b(Y)$ by $[a_x(\varphi)](y) = \varphi(\alpha_x(y))$ $(y \in Y)$. On $C_b^u(X)$ the representation is continuous. We observe that a C*-subalgebra of the form $C^G(Y)$ is invariant under this automorphism group (i.e. $a_x\left[C^G(Y)\right] \subset C^G(Y)$) if and only if the closed set G is invariant under each α_x.

Let $\mathcal{A}$ be a unital C*-subalgebra of $C_b(X)$ containing $C_0(X)$. We denote its character space $\Omega(\mathcal{A})$ by $\mathcal{X}$ and we recall that $\mathcal{X}$ is a compactification of X, i.e. $\mathcal{X}$ is a compact topological space and there is a homeomorphism i from X to a dense subset of $\mathcal{X}$ (see e.g. §8.1 of [6]). For $x \in X$, the character $i(x)$ is given by the formula $[i(x)](\varphi) = \varphi(x)$, for $\varphi \in \mathcal{A}$. We write $\mathcal{Z} = \mathcal{X} \setminus i(X)$ and call it the *frontier* of X in $\mathcal{X}$. By the Gelfand Theorem, $\mathcal{A}$ is isomorphic to the C*-algebra $C(\mathcal{X})$ of continuous functions on $\Omega(\mathcal{A})$. We shall use the notation $\mathcal{G} : C[\Omega(\mathcal{A})] \to \mathcal{A}$ for the inverse of the Gelfand isomorphism. The C*-subalgebra $C^{\mathcal{Z}}(\mathcal{X})$ (consisting of continuous functions on $\mathcal{X}$ that vanish on the frontier $\mathcal{Z}$ of $\mathcal{X}$) can be naturally identified with $C_0(X)$, more precisely $C_0(X) = \mathcal{G}C^{\mathcal{Z}}[\Omega(\mathcal{A})]$. There is a one-to-one correspondence between (self-adjoint, closed) ideals $\mathcal{K}$ of $\mathcal{A}$ and closed subsets G of $\mathcal{X}$, given by $\mathcal{K} = \mathcal{G}C^G(\mathcal{X})$ (Theorem 3.4.1. of [7]). In particular each closed subset F of the frontier $\mathcal{Z}$ determines an ideal $\mathcal{K}^F$ in $\mathcal{A}$, viz. $\mathcal{K}^F = \mathcal{G}C^F(\mathcal{X})$. It is clear that such an ideal contains $C_0(X)$.

Suppose now that the C*-algebra $\mathcal{A}$ considered above is contained in $C_b^u(X)$ and invariant under translations, i.e. such that $a_x\mathcal{A} \subset \mathcal{A}$ for all $x \in X$, with $[a_x(\varphi)](y) = \varphi(x + y)$. Since $\mathcal{A} \subset C_b^u(X)$, the mapping $x \mapsto a_x(\varphi)$ is norm continuous for each $\varphi \in \mathcal{A}$. Furthermore the action of X on itself (given as $\alpha_x(y) = x + y$) induces a continuous representation ρ of X by homeomorphisms of the character space $\mathcal{X} = \Omega(\mathcal{A})$: for $\tau \in \mathcal{X}$ the character $\rho_x\tau$ is defined as $[\rho_x\tau](\varphi) = \tau[a_x(\varphi)]$. For $y \in X$, set $\tau_y = i(y)$; then $\rho_x\tau_y = \tau_{x+y}$ $(x \in X)$.

We end this section with a result which will be useful in the examples presented further on. Let $\tau \in \mathcal{X}$ be a character of $\mathcal{A}$. A neighbourhood base of τ in $\mathcal{X}$ is given by the collection $\{\mathcal{V}_{\mathcal{F},\varepsilon}(\tau)\}$, where ε varies over $(0, \infty)$ and $\mathcal{F}$ over all finite families

$\{\varphi_1, \dots, \varphi_m\}$ of elements of $\mathcal{A}$ and where $\mathcal{V}_{\mathcal{F},\varepsilon}(\tau) = \{\tau' \in \mathcal{X} \mid |\tau'(\varphi_i) - \tau(\varphi_i)| < \varepsilon$ for each $\varphi_i \in \mathcal{F}\}$.

Lemma 2. *Let $\mathcal{A}$ be a unital C*-subalgebra of $C_b(X)$. Let F be a closed subset of $\Omega(\mathcal{A})$ and $\mathcal{W}$ a neighbourhood of F. Then there exist $\varepsilon > 0$ and a finite family $\mathcal{F} = \{\varphi_1, \dots, \varphi_m\}$ of elements of $\mathcal{A}$ such that $F \subset \cup_{\tau \in F} \mathcal{V}_{\mathcal{F},\varepsilon}(\tau) \subset \mathcal{W}$.*

Proof. Let $\tau \in F$. Then $\mathcal{W}$ is a neighbourhood of τ, hence there are a finite family $\mathcal{F}(\tau)$ of elements of $\mathcal{A}$ and a number $\varepsilon(\tau) > 0$ such that $\mathcal{V}_{\mathcal{F}(\tau),\varepsilon(\tau)}(\tau) \subset \mathcal{W}$. Since F is compact, there are a finite number of points $\tau_1, \dots, \tau_M$ in F such that $F \subset \cup_{j=1}^M \mathcal{V}_{\mathcal{F}(\tau_j),\varepsilon(\tau_j)/2}(\tau_j)$. Let $\mathcal{F} = \cup_{j=1}^M \mathcal{F}(\tau_j)$ and $\varepsilon = \frac{1}{2}\min\{\varepsilon(\tau_1), \dots, \varepsilon(\tau_M)\}$. The result of the lemma is true if we can show that, for each $\tau \in F$, there is $j \in \{1, \dots, M\}$ such that $\mathcal{V}_{\mathcal{F},\varepsilon}(\tau) \subset \mathcal{V}_{\mathcal{F}(\tau_j),\varepsilon(\tau_j)}(\tau_j)$. Since clearly $\mathcal{V}_{\mathcal{F},\varepsilon}(\tau) \subset \mathcal{V}_{\mathcal{F}(\tau_j),\varepsilon}(\tau) \subset \mathcal{V}_{\mathcal{F}(\tau_j),\varepsilon(\tau_j)/2}(\tau)$ for each j, it is enough to show that for some $j \in \{1, \dots, M\}$ one has $\mathcal{V}_{\mathcal{F}(\tau_j),\varepsilon(\tau_j)/2}(\tau) \subset \mathcal{V}_{\mathcal{F}(\tau_j),\varepsilon(\tau_j)}(\tau_j)$.

To prove this last inclusion, observe that τ belongs to $\mathcal{V}_{\mathcal{F}(\tau_j),\varepsilon(\tau_j)/2}(\tau_j)$ for at least one value of j. Choose one of these values of j and let $\tau' \in \mathcal{V}_{\mathcal{F}(\tau_j),\varepsilon(\tau_j)/2}(\tau)$. By the triangle inequality one has for each $\varphi \in \mathcal{A}$:

$$|\tau_j(\varphi) - \tau'(\varphi)| \leq |\tau_j(\varphi) - \tau(\varphi)| + |\tau(\varphi) - \tau'(\varphi)|.$$

For every $\varphi \in \mathcal{F}(\tau_j)$ each term on the r.h.s. is less than $\varepsilon(\tau_j)/2$. Hence τ' belongs to $\mathcal{V}_{\mathcal{F}(\tau_j),\varepsilon(\tau_j)}(\tau_j)$. □

4. Some Crossed Product C*-Algebras

We consider some C*-subalgebras of the space $B(\mathcal{H})$ of all bounded, linear operators in the Hilbert space $\mathcal{H} = L^2(X)$. If $\varphi : X \to \mathbb{C}$ is a bounded, measurable function, we denote by $\varphi(Q)$ the operator of multiplication by φ in $\mathcal{H}$ and by $\varphi(P)$ the operator $\mathfrak{F}^*\varphi(Q)\mathfrak{F}$ (the operator of multiplication by φ in the momentum space), where $\mathfrak{F}$ is the Fourier transformation. A C*-subalgebra $\mathcal{A}$ of $C_b^u(X)$ will be identified with the subalgebra of $B(\mathcal{H})$ consisting of all multiplication operators $\varphi(Q)$ with $\varphi \in \mathcal{A}$.

If $\mathcal{A}$ is a C*-subalgebra of $C_u^b(X)$, we write $\mathfrak{C}_\mathcal{A}$ for the norm closure in $B(\mathcal{H})$ of the set of finite sums of the form $\varphi_1(Q)\psi_1(P) + \cdots + \varphi_N(Q)\psi_N(P)$ with $\varphi_k \in \mathcal{A}$ and $\psi_k \in C_0(X)$. We mention the fact that, if $\mathcal{A} = C_0(X)$, then $\mathfrak{C}_\mathcal{A}$ is the ideal of all compact operators in $L^2(X)$.

If $\mathcal{A}$ is invariant under translations, then $\mathfrak{C}_\mathcal{A}$ is a C*-algebra isomorphic to the crossed product algebra $\mathcal{A} \rtimes X$ defined in terms of the action a_x of X on $\mathcal{A}$. In the proof of Lemma 6 we shall use the following result from the theory of crossed products: If $\mathcal{K}$ is an ideal in $\mathcal{A}$ that is invariant under translations, then the quotient C*-algebra $\mathfrak{C}_\mathcal{A}/\mathfrak{C}_\mathcal{K}$ is isomorphic to $[\mathcal{A}/\mathcal{K}] \rtimes X$. The point is that the general theory allows us to define the crossed product $[\mathcal{A}/\mathcal{K}] \rtimes X$ only by using the continuous action of X by $*$-automorphisms of $\mathcal{A}/\mathcal{K}$ (the quotient action); the fact that $\mathcal{A}/\mathcal{K}$ is not a C*-subalgebra of $B(\mathcal{H})$ does not matter.

Remark. If $V \in \mathcal{A}$, where $\mathcal{A}$ is a unital C*-subalgebra of $C_b^u(X)$, then the self-adjoint operator $H = -\Delta + V$ is affiliated to $\mathfrak{C}_\mathcal{A}$. This is easily seen from the fact that the Neumann series $[H - z]^{-1} = \sum_{k=0}^\infty (P^2 - z)^{-1}\left(-V(Q)[P^2 - z]^{-1}\right)^k$ converges in the norm of $B(\mathcal{H})$ if $\Im z$ is sufficiently large.

5. A Non-propagation Theorem

For our principal theorem and its corollary we consider the following

Framework. *$\mathcal{A}$ is a unital C^*-subalgebra of $C_b^u(X)$, invariant under translations and such that $C_0(X) \subset \mathcal{A}$, and $\mathfrak{C}_{\mathcal{A}}$ is the associated C^*-subalgebra of $B(\mathcal{H})$ introduced in §4 (with $\mathcal{H} = L^2(X)$). $\mathcal{X} = \Omega(\mathcal{A})$ is the character space of $\mathcal{A}$, F a translation invariant, closed subset of $\mathcal{Z} = \mathcal{X} \setminus i(X)$ and $C^F(\mathcal{X})$ the ideal in $C(\mathcal{X})$ determined by F. We set $\mathcal{K}^F \equiv \mathcal{G}C^F(\mathcal{X})$, which is a translation invariant ideal in $\mathcal{A}$. Then $\mathfrak{K}^F \equiv \mathfrak{C}_{\mathcal{K}^F}$ is an ideal in $\mathfrak{C}_{\mathcal{A}}$ that contains all the compact operators in $\mathcal{H}$.*

We shall work with families $\boldsymbol{W}$ of subsets of X such that their images through i in $\mathcal{X}$ are close to F. $\boldsymbol{W}$ will have the structure of a filter base, i.e. a non-void collection of non-void subsets of X such that for any $W_1, W_2 \in \boldsymbol{W}$ there is a $W \in \boldsymbol{W}$ with $W \subset W_1 \cap W_2$. If $\boldsymbol{W}$ is a filter base in X, then the family $\{i(W) \mid W \in \boldsymbol{W}\}$ is a filter base in $\mathcal{X}$ and we say that $\boldsymbol{W}$ is *adjacent to F* if all cluster points in $\mathcal{X}$ of this family $\{i(W) \mid W \in \boldsymbol{W}\}$ belong to F, i.e. if $\cap_{W \in \boldsymbol{W}} \overline{i(W)} \subset F$, where the closures are taken in $\mathcal{X}$. We observe that the set of these cluster points is non-empty since $\mathcal{X}$ is a compact space. In the majority of situations considered further on it will suffice to take for $\boldsymbol{W}$ the family $\{W = i^{-1}[\mathcal{W} \cap i(X) \mid \mathcal{W} \in \mathfrak{W}]\}$, where $\mathfrak{W}$ is a neighbourhood base of F in $\mathcal{X}$ (since $i(X)$ is dense in $\mathcal{X}$, each of these sets W is non-void).

If $\boldsymbol{W}$ is a filter base adjacent to F and $\varphi \in C(\mathcal{X})$ is such that $\varphi|_F = 0$, then given any $\varepsilon > 0$, there is some $W \in \boldsymbol{W}$ such that $\mid \varphi(\tau) \mid < \varepsilon$ for all $\tau \in i(W)$. In the sequel we shall denote by χ_W the characteristic function of W.

Theorem. *Let $\mathcal{A}$ and F be as in the Framework and let $\boldsymbol{W}$ a filter base in X that is adjacent to F. Let H be a self-adjoint operator in $\mathcal{H}$ affiliated to $\mathfrak{C}_{\mathcal{A}}$. Let $\varepsilon > 0$ and $\eta \in C_0(\mathbb{R})$ with supp$\eta \cap \sigma_{\mathfrak{K}^F}(\Phi^H) = \emptyset$. Then there is a $W \in \boldsymbol{W}$ such that*

$$\| \chi_W(Q)\eta(H) \| \le \varepsilon. \tag{1}$$

Proof. (i) We use the notation τ for characters in $\Omega(\mathcal{A})$ and observe that

$$\mathcal{K}^F = \mathcal{G}\{\varphi \in C(\mathcal{X}) \mid \varphi|_F = 0\} = \{\varphi \in \mathcal{A} \mid \tau(\varphi) = 0 \ \ \forall \tau \in F\}.$$

So if φ belongs to $\mathcal{K}^F$, then for each $\delta > 0$ there exists $W \in \boldsymbol{W}$ such that $|\tau(\varphi)| \le \delta$ $\forall \tau \in i(W)$. Thus, if $\varphi \in \mathcal{K}^F$, we have $\mid \varphi(x) \mid \le \delta$ for all $x \in W$.

(ii) By the hypothesis on the support of η we have $\eta(H) \in \mathfrak{K}^F$ (see Lemma 1). So there are a finite number of functions $\varphi_1, \dots, \varphi_N \in \mathcal{K}^F$ and $\psi_1, \dots, \psi_N \in C_0(X)$ such that

$$\| \eta(H) - \sum_{k=1}^{N} \varphi_k(Q)\psi_k(P) \| \le \varepsilon/2.$$

We also have

$$\begin{aligned} \| \chi_W(Q)\eta(H) \| &\le \sum_{k=1}^{N} \| \varphi_k \|_{L^\infty(W)} \| \psi_k \|_{L^\infty(X)} + \\ &\quad + \| \eta(H) - \sum_{k=1}^{N} \varphi_k(Q)\psi_k(P) \| . \end{aligned} \tag{2}$$

The first term in the r.h.s. of (2) can be made less than $\varepsilon/2$ by using the result of (i) with $\delta = \left[N \cdot \sup_{k=1,\dots,N} \| \psi_k \|_{L^\infty(X)}\right]^{-1} \cdot \varepsilon/2$, so the proof is finished □

Corollary. *Let $\mathcal{A}$, F, $\boldsymbol{W}$ and H be as in the Theorem. Then for each $\varepsilon > 0$ and each $\eta \in C_0(\mathbb{R})$ with $\operatorname{supp}\eta \subset \mathbb{R} \setminus \sigma_{\mathfrak{R}^F}(\Phi^H)$, there exists $W \in \boldsymbol{W}$ such that*

$$\| \chi_W(Q) e^{-itH} \eta(H) f \| \leq \varepsilon \| f \| \tag{3}$$

for all $t \in \mathbb{R}$ and all $f \in L^2(X)$.

(3) is a straightforward consequence of (1). Note the obvious fact that one may replace in the corollary $\{e^{-itH}\}$ by any bounded family of bounded operators commuting with H.

Remark. The corollary gives the precise meaning, in our framework, of the notion of non-propagation described in the Introduction. To be more specific, let us denote by $\operatorname{supp}_H(f)$ *the spectral support with respect to* H of the vector $f \in \mathcal{H}$, defined as follows in terms of the spectral measure E_H of H:

$$\lambda \notin \operatorname{supp}_H(f) \Leftrightarrow \exists \varepsilon > 0 \text{ such that } E_H(\lambda - \varepsilon, \lambda + \varepsilon) f = 0.$$

$\operatorname{supp}_H(f)$ is the smallest closed set $J \subset \mathbb{R}$ such that $E_H(J) f = f$. Then it follows easily that, under the hypotheses of the corollary, for each $\varepsilon > 0$ and each closed subset L of $\mathbb{R} \setminus \sigma_{\mathfrak{R}^F}(\Phi^H)$ there exists an element W of $\boldsymbol{W}$ such that

$$\| \chi_W(Q) e^{-itH} f \| \leq \varepsilon \| f \|$$

for all $t \in \mathbb{R}$ and all $f \in L^2(X)$ with $\operatorname{supp}_H(f) \subset L$.

In the situation just described, let us take for H a self-adjoint Schrödinger operator affiliated to $\mathfrak{C}_{\mathcal{A}}$. Then, if f is a unit vector with $\operatorname{supp}_H(f) \subset L$, one has $\| \chi_W(Q) e^{-itH} f \| \leq \varepsilon$ for all $t \in \mathbb{R}$. In physical terms: the probability of finding f localized in W is less than ε^2 at all times. If K is a compact subset of X and if the preceding vector f belongs to the absolutely continuous subspace of H, then there is $t_0 \in \mathbb{R}$ such that $\| \chi_K(Q) e^{-itH} f \| \leq \varepsilon$ for all $t > t_0$ [2]. It follows that $\| \chi_{K^c \cap W^c}(Q) e^{-itH} f \|^2 \geq 1 - 2\varepsilon^2$ for all $t > t_0$, which (for small ε) essentially means that f describes a state that will propagate into the complement $(K \cup W)^c$ of the set $K \cup W$. If f belongs to the singularly continuous subspace of H, a similar conclusion is true, except that some averaging over time may be necessary [2]: there are $0 < t_0 < t_1$ such that $(t - t_0)^{-1} \int_{t_0}^{t} \| \chi_{K^c \cap W^c}(Q) e^{-i\tau H} f \|^2 \, d\tau \geq 1 - 3\varepsilon^2$ for all $t > t_1$.

6. Example: Non-propagation in Multicrystalline Systems

As an application we present in some detail the situation where the potential V of a Schrödinger Hamiltonian becomes asymptotically periodic, with different periodic limit functions in different cones (the more general case in which the limit functions are only almost periodic can be treated analogously). More precisely V will belong to the C*-algebra $\mathcal{A}$ introduced below, so that $H = -\Delta + V$ will be affiliated to $\mathfrak{C}_{\mathcal{A}}$.

Let S be the unit sphere in $X = \mathbb{R}^n$. For $j = 1, \dots, N$ let Γ_j a periodic lattice in X and Σ_j a non-empty open subset of S, with $\overline{\Sigma_j} \cap \overline{\Sigma_k} = \emptyset$ if $j \neq k$. We denote by $C_j(X)$ the C*-algebra $C_j(X) = \{\varphi \in C_b^u(X) \mid \varphi(x+\gamma) = \varphi(x) \ \ \forall x \in X, \ \forall \gamma \in \Gamma_j\}$ and we define $\mathcal{A}$ as the set of bounded, uniformly continuous complex functions φ on X such that for any $j \in \{1, \dots, N\}$ there exists $\varphi_j \in C_j(X)$ such that $\lim_{r\to\infty} |\varphi(r\omega) - \varphi_j(r\omega)| = 0$ for all $\omega \in \Sigma_j$, uniformly in ω on each compact subset of Σ_j. The (uniquely determined) collection $\{\varphi_j \mid j = 1, \dots, N\}$ corresponding to $\varphi \in \mathcal{A}$ will be called *the asymptotic functions of* φ. If Σ is some subset of Σ_j and $R > 0$, let $W_j^R(\Sigma)$ be the subset $\{r\omega \mid r > R, \omega \in \Sigma\}$ of X.

The application of the results of Section 5 leads to the following non-propagation property into the cone subtended by Σ_j:

Proposition 1. *Let $V \in \mathcal{A}$ be real and denote its asymptotic functions by $\{V_j\}$. Set $H = -\Delta + V$. Fix a number $j \in \{1, \dots, N\}$ and choose $\eta \in C_0(\mathbb{R})$ with suppη disjoint from the spectrum of the periodic Schrödinger operator $-\Delta + V_j$. Then, given a compact subset K of Σ_j and $\varepsilon > 0$, there is $R \in (0, \infty)$ such that for each $f \in L^2(X)$ we have*

$$\sup_{t\in\mathbb{R}} \| \chi_{W_j^R(K)}(Q) e^{-itH} \eta(H) f \| \leq \varepsilon \| f \| . \tag{4}$$

The proof will be given in a series of lemmas. The validity of (4) is obtained by combining the last two lemmas (Lemma 5 and Lemma 6) with the Corollary given in Section 5 and with the Remark at the end of Section 4. The estimate (4) gives a precise meaning to the statement made in the second example of the Introduction that states with spectral support away from certain subsets of $\mathbb{R}$ will not propagate into the asymptotic part of the cone $\mathcal{C}_j$ subtended by Σ_j.

We shall use the following notations: $W_j^R \equiv W_j^R(\Sigma_j) = \{r\omega \mid r > R > 0, \omega \in \Sigma_j\} \subset X$, and $\mathbb{T}_j = X/\Gamma_j$ (the class of $z \in X$ in $\mathbb{T}_j$, denoted by ζ, is given as $\zeta = \{x \in X \mid x = z + \gamma, \gamma \in \Gamma_j\}$). We observe that $C_j(X)$ is isomorphic to $C(\mathbb{T}_j)$ and that the correspondence $\varphi \mapsto \varphi_j$ defines a $*$-homomorphism Ψ_j from $\mathcal{A}$ to $C_j(X)$.

Lemma 3. *(a) $\mathcal{A}$ is a unital C*-algebra containing $C_0(X)$ and invariant under translations.*

(b) The $*$-homomorphism $\Psi_j : \mathcal{A} \to C_j(X)$ is surjective.

Proof. (a) It is clear that $\mathcal{A}$ contains $C_0(X)$ and the constants. To see that $\mathcal{A}$ is closed, let $\{\varphi^{(k)}\}$ be a Cauchy sequence in $\mathcal{A}$, denote by $\varphi \in C_b^u(X)$ its limit and by $\varphi_j^{(k)}$ $(j = 1, \dots, N)$ the asymptotic functions of $\varphi^{(k)}$. Let us show that, for fixed j, $\{\varphi_j^{(k)} \mid k \in \mathbb{N}\}$ is Cauchy in the norm of $C_b^u(X)$. We have for any $\gamma \in \Gamma_j$:

$$|\varphi_j^{(k)}(x) - \varphi_j^{(l)}(x)| = |\varphi_j^{(k)}(x+\gamma) - \varphi_j^{(l)}(x+\gamma)| \leq |\varphi_j^{(k)}(x+\gamma) - \varphi^{(k)}(x+\gamma)| + \\ + |\varphi^{(k)}(x+\gamma) - \varphi^{(l)}(x+\gamma)| + |\varphi^{(l)}(x+\gamma) - \varphi_j^{(l)}(x+\gamma)|.$$

Fix $\varepsilon > 0$. The second term on the r.h.s. is less than $\varepsilon/3$ for all x and all γ if $k, l > L$ for some $L \in \mathbb{N}$. For fixed k and l the first and the third term are less than $\varepsilon/3$ in the sup norm (with respect to x) since for each fixed $x \in X$ and any $R > 0$

one may find $\gamma \in \Gamma_j$ such that $x+\gamma \in W_j^R(K)$ if K is a compact subset of Σ_j with non-empty interior.

Now define $\varphi_j = \lim_{k\to\infty} \varphi_j^{(k)}$ and observe that $\varphi_j \in C_j(X)$. By another $\varepsilon/3$ type argument one then finds that these functions φ_j are asymptotic functions of φ.

Let us show that $\mathcal{A}$ is invariant under translations. If $x \in X$ and $\varphi \in \mathcal{A}$ with asymptotic functions $\{\varphi_j\}$, then the collection $\{a_x(\varphi_j)\}$ are asymptotic functions of $a_x(\varphi)$, hence $a_x(\varphi) \in \mathcal{A}$:

$$|a_x(\varphi)(r\omega) - a_x(\varphi_j)(r\omega)| = \left|\varphi\left[r\left(\omega + \frac{x}{r}\right)\right] - \varphi_j\left[r\left(\omega+\frac{x}{r}\right)\right]\right| \to 0 \quad \text{as} \quad r \to \infty,$$

uniformly in ω belonging to compact subsets of Σ_j.

(b) Let $\psi \in C_j(X)$. Let $\varphi = \theta\psi$, where θ is a function in $C_b^u(X)$ that is homogeneous of degree zero outside the unit ball of X and satisfies $\theta(x) = 1$ on W_j^1 and $\theta(x) = 0$ on W_k^1 if $k \neq j$. Then $\varphi \in \mathcal{A}$, with asymptotic functions $\varphi_j = \psi$, $\varphi_k = 0$ for $k \neq j$. $\square$

We next show that there is a canonical identification of $\mathbb{T}_j$ with a closed subset $\mathcal{T}_j$ of $\mathcal{Z} = \mathcal{X} \setminus i(X)$, where $\mathcal{X} = \Omega(\mathcal{A})$ as before. This is a direct consequence of the fact that there is a surjective $*$-homomorphism $\Phi_j : C(\mathcal{Z}) \to C(\mathbb{T}_j)$, deduced from $\Psi_j : \mathcal{A} \to C_j(X)$, which is a surjective $*$-homomorphism with kernel including $C_0(X)$, and from the natural isomorphisms $\mathcal{A}/C_0(X) \cong C(\mathcal{Z})$ and $C_j(X) \cong C(\mathbb{T}_j)$. Below we shall make the construction as explicit as possible.

For this we introduce a mapping $i_j : \mathbb{T}_j \to \mathcal{X}$ that associates to $\zeta \in \mathbb{T}_j$ the character $i_j(\zeta) \equiv \tau_\zeta^{(j)} \in \mathcal{X}$ given as $\tau_\zeta^{(j)}(\varphi) = \varphi_j(z)$, where φ_j is the j-th asymptotic function of φ and z is any representative of the class ζ. In other terms $\tau_\zeta^{(j)} = \tau_z \circ \Psi_j$, where τ_z is interpreted as a character of $C(\mathbb{T}_j)$. We set $\mathcal{T}_j = i_j(\mathbb{T}_j)$.

Lemma 4. *(a) $\mathcal{T}_j$ is contained in $\mathcal{X} \setminus i(X)$.*

(b) The correspondence $\zeta \mapsto \tau_\zeta^{(j)}$ is injective and continuous.

(c) The set $\mathcal{T}_j$ is closed.

(d) The set $\mathcal{T}_j$ is invariant under all translations.

Proof. (a) If $\zeta \in \mathbb{T}_j$, then $\tau_\zeta^{(j)}$ does not belong to $i(X)$: if $x \in X$, choose a function $\varphi \in C_0(X)$ such that $\varphi(x) \neq 0$: then $\varphi_j = 0$, so that $\tau_\zeta^{(j)}(\varphi) = 0 \neq \varphi(x) = [i(x)](\varphi)$. Thus $\tau_\zeta^{(j)} \neq i(x)$ for each $x \in X$.

(b) Assume that $z_1 - z_2 \notin \Gamma_j$. Choose $\varphi_j \in C_j(X)$ such that $\varphi_j(z_1) \neq \varphi_j(z_2)$ and let $\varphi \in \mathcal{A}$ be such that $\varphi_j = \Psi_j(\varphi)$ (Lemma 3.(b)). Then $\tau_{\zeta_1}^{(j)}(\varphi) = \varphi_j(z_1) \neq \varphi_j(z_2) = \tau_{\zeta_2}^{(j)}(\varphi)$, so $\tau_{\zeta_1}^{(j)} \neq \tau_{\zeta_2}^{(j)}$. Thus the mapping i_j is injective. Its continuity is easy to establish.

(c) $\mathcal{T}_j$ is the continuous image of the compact space $\mathbb{T}_j$, hence it is a compact subset of $\mathcal{Z}$.

(d) Let $z \in X$ be a representative of $\zeta \in \mathbb{T}_j$. For $x \in X$, denote the class of $z+x$ in $\mathbb{T}_j$ by $\zeta + \xi$. Then

$$\left[\rho_x(\tau_\zeta^{(j)})\right](\varphi) = \tau_\zeta^{(j)}[a_x(\varphi)] = [a_x(\varphi)]_j(z) = [a_x(\varphi_j)](z) = \varphi_j(z+x) = \tau_{\zeta+\xi}^{(j)}(\varphi).$$

Hence $\rho_x(\tau_\zeta^{(j)}) = \tau_{\zeta+\xi}^{(j)}$, the character associated to the class of $z+x$ in $\mathbb{T}_j$. We conclude that $\mathcal{T}_j$ is a single orbit under the representation ρ of X in $\mathcal{X}$. $\square$

We need to find subsets of $i^{-1}[\mathcal{W}\cap i(X)]$ easy to express in terms of the geometry of X, for arbitrary neighbourhoods $\mathcal{W}$ of $\mathcal{T}_j$.

Lemma 5. *Let $\mathcal{W}$ be a neighbourhood of $\mathcal{T}_j$ in $\mathcal{X}$ and set $W = i^{-1}[\mathcal{W}\cap i(X)]$. Given any compact subset K of Σ_j there is $R \in (0,\infty)$ such that $W_j^R(K) \subset W$.*

Proof. By Lemma 2 applied to $F = \mathcal{T}_j$, there are $\varepsilon > 0$ and a finite family $\mathcal{F} = \{\varphi_1, \dots, \varphi_m\}$ of elements of $\mathcal{A}$ such that $\cup_{\tau\in\mathcal{T}_j} \mathcal{V}_{\mathcal{F},\varepsilon}(\tau) \subset \mathcal{W}$, in other terms such that $\cap_i\{\tau' \in \mathcal{X} \mid |\tau'(\varphi_i) - \tau_\zeta^{(j)}(\varphi_i)| < \varepsilon\} \subset \mathcal{W}$ for each $\zeta \in \mathbb{T}_j$. Upon restricting to characters τ' belonging to $i(X)$ and denoting the j-th asymptotic function of φ_i by $\varphi_{i,j}$, we get immediately that $\cap_i\{x \in X \mid |\varphi_i(x) - \varphi_{i,j}(x)| < \varepsilon\} \subset \mathcal{W}$. Now for each i there is $R_i \in (0,\infty)$ such that $|\varphi_i(x) - \varphi_{i,j}(x)| < \varepsilon$ for all $x \in W_j^{R_i}(K)$. Then clearly the assertion of the Lemma holds for $R = \max\{R_1, \dots, R_m\}$. $\square$

We finally specify the $\mathfrak{K}^{\mathcal{T}_j}$-essential spectrum of H.

Lemma 6. *The set $\sigma_{\mathfrak{K}^{\mathcal{T}_j}}(\Phi^H)$ coincides with the (band) spectrum of the periodic Schrödinger operator $H_j = -\Delta + V_j$.*

Proof. Let us denote by $\Pi_j : \mathfrak{C}_\mathcal{A} \to \mathfrak{C}_\mathcal{A}/\mathfrak{K}^{\mathcal{T}_j}$ the canonical $*$-homomorphism. By definition, $\sigma_{\mathfrak{K}^{\mathcal{T}_j}}(\Phi^H)$ is the spectrum of the observable $\Pi_j \circ \Phi^H$ affiliated to $\mathfrak{C}_\mathcal{A}/\mathfrak{K}^{\mathcal{T}_j}$. It is enough to show that $\mathfrak{C}_\mathcal{A}/\mathfrak{K}^{\mathcal{T}_j}$ is isomorphic to $\mathfrak{C}_{C_j(X)}$ and that the image of $\Pi_j \circ \Phi^H$ under this isomorphism is Φ^{H_j}; this will conclude the proof, since isomorphisms of C*-algebras leave the spectra of observables invariant.

For any $M \in \mathbb{N}$, $\varphi_1, \dots, \varphi_M \in \mathcal{A}$ and $\psi_1, \dots, \psi_M \in C_0(X)$ we set

$$\Theta_j\left[\sum_{i=1}^M \varphi_i(Q)\psi_i(P)\right] = \sum_{i=1}^M \varphi_{i,j}(Q)\psi_i(P),$$

where $\varphi_{i,j} \in C_j(X)$ is the j-th asymptotic function of φ_i. By the discussion in Section 4, Θ_j extends to a surjective $*$-homomorphism $\mathfrak{C}_\mathcal{A} \to \mathfrak{C}_{C_j(X)}$ with kernel $\mathfrak{K}^{\mathcal{T}_j} = [\mathcal{G}C^{\mathcal{T}_j}(\mathcal{X})] \rtimes X$. A simple argument in terms of the Neumann series shows that $\Theta_j[(H-z)^{-1}] = (H_j - z)^{-1}$, so that $\Theta_j(\Phi^H) = \Phi^{H_j}$. $\square$

Remark. Since $K(\mathcal{H}) = \mathfrak{C}_{C_0(X)} = \mathfrak{K}^\mathcal{Z} \subset \mathfrak{K}^{\mathcal{T}_j}$, we have $\cup_{j=1}^N \sigma(H_j) \subset \sigma_{\text{ess}}(H)$. The behaviour of the bounded, uniformly continuous function V outside the cones $\{\mathcal{C}_j\}_{j=1,\dots,N}$ is submitted to no constraint and the asymptotic functions V_j are not related. So it is possible to have a large set $\sigma_{\text{ess}}(H) \setminus \cup_{j=1}^N \sigma(H_j)$ on which the result of the Proposition (non-propagation into the asymptotic part of $\cup_{j=1}^N \mathcal{C}_j$) is relevant and non-trivial. Of course the simplest situation is that where $N = 1$, and the general case ($N > 1$) can be reduced to it since Proposition 1 involves only one value of j.

7. Other Examples, Comments

Some other examples will be discussed briefly in the present section. Most of the proofs consist in suitable adaptations of arguments already used above and we shall just sketch them.

Example 1. Potentials that are asymptotically periodic in a half-space.

This is the example that is the most close to that treated in [3]. It is also related to Section 6.

Let us write $X = \mathbb{R} \times X'$. For a periodic lattice Γ_+ of X, we denote by $C_+(X)$ the C*-algebra of all complex continuous functions on X that are Γ_+-periodic. We shall consider the unital C*-algebra $\mathcal{A}_+$ of all bounded, uniformly continuous functions $\varphi : X \to \mathbb{C}$ such that there exist a (necessarily unique) element $\varphi_+ \in C_+(X)$ such that $\mid \varphi(x_1, x') - \varphi_+(x_1, x') \mid \to 0$ when $x_1 \to +\infty$, uniformly in $x' \in X'$. As before we call φ_+ *the asymptotic function* of φ. It is easy to see that $\mathcal{A}_+$ is invariant under translations.

Since we imposed no conditions on the behaviour of φ outside a remote half-space, we cannot determine the character space $\mathcal{X}$ of $\mathcal{A}_+$ precisely. But it is straightforward to show that the torus $\mathbb{T}_+ = X/\Gamma_+$ is a closed invariant subset of its frontier and that any neighbourhood of $\mathbb{T}_+$ in $\mathcal{X}$ contains $\{(x_1, x') \in X \mid x_1 > R\}$ for some $R > 0$ large enough (use Lemma 2).

It is also easy to show that the quotient C*-algebra $\mathfrak{C}_{\mathcal{A}_+}/\mathfrak{K}^{\mathbb{T}_+}$ is isomorphic to $\mathfrak{C}_{C_+(X)}$ in such a way that $\varphi(Q)\psi(P)$ corresponds to $\varphi_+(Q)\psi(P)$; here $\varphi \in \mathcal{A}_+$, $\varphi_+ \in C_+(X)$ is its asymptotic function and $\psi \in C_0(X)$.

By applying the results of Section 5 and the discussion above one gets

Proposition 2. *Let $V \in \mathcal{A}_+$ be real and let $V_+ \in C_+(X)$ be its asymptotic function. Let $H = -\Delta + V$ and $H_+ = -\Delta + V_+$ be the associated self-adjoint operators in $\mathcal{H} = L^2(X)$. Let $\eta \in C_0(\mathbb{R})$ with supp$\eta \cap \sigma(H_+) = \emptyset$. Then for each $\varepsilon > 0$ there exist $R > 0$ such that*

$$\| \chi(Q_1 \geq R) e^{-itH} \eta(H) f \| \leq \varepsilon \| f \|$$

for all $f \in \mathcal{H}$ and all $t \in \mathbb{R}$.

Of course, one can also introduce the C*-algebra $\mathcal{A}_-$, consisting in all bounded, uniformly continuous functions that become Γ_--periodic (for some other periodic lattice Γ_-) at $x_1 = -\infty$, uniformly in the orthogonal variable x'. The elements of $\mathcal{A} = \mathcal{A}_- \cap \mathcal{A}_+$ are bounded, uniformly continuous functions which have (different) periodic limits at $x_1 = \pm\infty$; their behaviour in a vertical strip is submitted to no constraint.

Example 2. Potentials with asymptotic vanishing oscillation.

We say that a function $\varphi \in C_b(X)$ has *asymptotic vanishing oscillation*, and we write $\varphi \in VO(X)$, if the function $x \mapsto \sup_{|y|\leq 1} \mid \varphi(x+y) - \varphi(x) \mid$ is of class $C_0(X)$. We remark that $VO(X)$ contains the C*-algebra $C^{\rm rad}(X)$ of all complex continuous functions on X that have radial limits at infinity, uniformly in all directions. Sums of the form $\varphi_0 + \varphi_1$ with $\varphi_0 \in C_0(X)$ and φ_1 continuous and homogeneous of degree zero outside a ball are the most general elements of $C^{\rm rad}(X)$. Note that $VO(X)$ is considerably larger than $C^{\rm rad}(X)$. A C^1-function φ with $\partial_j \varphi \in C_0(X)$ for all j is in $VO(X)$. This includes $\varphi(x) = \phi\left[(1 + |x|)^p\right]$ for $p < 1$ and ϕ, ϕ' continuous and bounded. We point out that Proposition 3 will be particularly simple to interpret for potentials in $C^{\rm rad}(X)$.

Let $\mathcal{X}$ be the character space of $VO(X)$. By identifying X with its homeomorphic image in $\mathcal{X}$, we can express this character space as the disjoint union $\mathcal{X} = X \sqcup \mathcal{Z}$.

The nice feature is that $VO(X)$ is the largest unital, translation invariant C*-subalgebra of $C_b^u(X)$ such that all the elements of $\mathcal{Z}$ are fixed points under the extension of the action of the group X. This was used in [9] to show that, for $V \in VO(X)$, the essential spectrum of the Schrödinger operator $H = -\Delta + V$ is $[\min V(X)_{\rm asy}, \infty)$, where *the asymptotic range* of V is defined as $V(X)_{\rm asy} = \cap_K \overline{V(X \setminus K)}$ with K varying over all compact subsets of X. This result is specific to the class $VO(X)$.

The frontier $\mathcal{Z}$ is not easy to understand, so we shall consider only closed sets $F \subset \mathcal{Z}$ which are suitably related to a given potential V. Let $\hat{V}$ be the continuous extension of V to $\mathcal{X}$ and G a closed subset of $\mathbb{R}$ such that its interior G^o meets $V(X)_{\rm asy}$. We set $F = \hat{V}^{-1}(G) \cap \mathcal{Z}$; it is a closed, non-void subset of $\mathcal{Z}$, and it is automatically invariant under translations (this is the point which makes our analysis possible without extra information on $\mathcal{Z}$). To apply the Theorem, one has to find in X a filter base adjacent to F and to calculate the $\mathfrak{K}^F$-essential spectrum of H. For the latter problem we proceed as in [9], where more details can be found.

The set $\sigma_{\mathfrak{K}^F}(H)$ is the spectrum in $\mathfrak{C}_{VO(X)}/\mathfrak{K}^F$ of the image of the observable Φ^H through the canonical $*$-homomorphism $\mathfrak{C}_{VO(X)} \to \mathfrak{C}_{VO(X)}/\mathfrak{K}^F$. The quotient C*-algebra $\mathfrak{C}_{VO(X)}/\mathfrak{K}^F$ is isomorphic to $C(F) \rtimes X$ (crossed product constructed in terms of the trivial action of X on $C(F)$). The latter can be embedded in the direct sum $\oplus_{\tau\in F}\mathbb{C} \rtimes X \cong \oplus_{\tau\in F} C_0^P(X)$, where $C_0^P(X)$ is the C*-subalgebra of $B(\mathcal{H})$ of all the operators of the form $\psi(P)$, with $\psi \in C_0(X)$. This leads to a $*$-homomorphism $\Pi_F : \mathfrak{C}_{VO(X)} \to \oplus_{\tau\in F} C_0^P(X)$ with kernel $\mathfrak{K}^F$. This $*$-homomorphism maps $\varphi(Q)\psi(P)$ to $(\hat{\varphi}(\tau)\psi(P))_{\tau\in F}$, where $\hat{\varphi}$ is the continuous extension of φ to $\mathcal{X}$. With the Neumann series for the resolvent, it follows easily that the observable Φ^H is mapped to $\left(\Phi^{H_\tau}\right)_{\tau\in F}$, where $H_\tau = -\Delta + \hat{V}(\tau)$ (self-adjoint operator in $\mathcal{H}$). It follows that $\sigma_{\mathfrak{K}^F}(H) = \cup_{\tau\in F}\sigma(H_\tau) = \Big[\min \hat{V}(F), \infty\Big)$. By taking into account the definition of the closed set F and the fact that $\hat{V}(\mathcal{Z}) = V(X)_{\rm asy}$ one easily shows that $\hat{V}(F) = G \cap V(X)_{\rm asy}$, thus $\sigma_{\mathfrak{K}^F}(H) = [\min\{G \cap V(X)_{\rm asy}\}, \infty)$.

We next indicate a suitable filter base in X adjacent to F. For any compact subset K of X we set $W_K = V^{-1}(G) \cap K^c$. The assumption $G^o \cap V(X)_{\rm asy} \neq \emptyset$ implies that $W_K \neq \emptyset$. Since $W_K \cap W_{K'} = W_{K\cup K'}$, $\boldsymbol{W} = \{W_K\}_K$ is a filter base in X. Then (all closures are taken in $\mathcal{X}$):

$$\cap_K \overline{W_K} = \cap_K \overline{[V^{-1}(G) \cap K^c]} \subset \cap_K \left[\overline{V^{-1}(G)} \cap \overline{K^c}\right] \subset$$
$$\subset \hat{V}^{-1}(G) \cap \left[\cap_K \overline{K^c}\right] = \hat{V}^{-1}(G) \cap \mathcal{Z} = F.$$

Thus $\boldsymbol{W}$ is adjacent to F. By applying the Corollary in Section 5 we obtain

Proposition 3. *Let $V \in VO(X)$ and consider the self-adjoint operator $H = -\Delta + V$ in $\mathcal{H} = L^2(X)$, which defines an observable affiliated to $\mathfrak{C}_{VO(X)}$. Let $G \subset \mathbb{R}$ be a closed set such that $G^o \cap V(X)_{\rm asy} \neq \emptyset$, and let $\varepsilon > 0$. Then for each $\eta \in C_0(\mathbb{R})$ with supp$\eta \subset (-\infty, \min\{G \cap V(X)_{\rm asy}\})$ there is a compact subset K of X such that*

$$\| \chi_{V^{-1}(G)\setminus K}(Q) e^{-itH} \eta(H) f \| \leq \varepsilon \| f \|$$

for all $f \in \mathcal{H}$ and all $t \in \mathbb{R}$.

To illustrate this result, let us take $G = [\lambda, \infty)$ with $\min[V(X)_{\mathrm{asy}}] < \lambda < \max[V(X)_{\mathrm{asy}}]$. Then, roughly, scattering states at energies situated below λ will not propagate into the asymptotic part of the set $\{x \in X \mid V(x) \geq \lambda\}$. For a one-dimensional system with a slowly oscillating potential, this corresponds to tunneling through an infinite sequence of more and more widely separated barriers of increasing length. The effective parts of these barriers, for states with energy less than λ, occupy the intervals $\{x \in \mathbb{R}\backslash K \mid V(x) \geq \lambda\}$ for some compact set $K \subset \mathbb{R}$ (as an example one may consider a potential that is asymptotically of the form $\cos(|x|^\beta)$ with $0 < \beta < 1$; the essential spectrum of the associated Sturm-Liouville operator will often be continuous, cf. Theorem 4 in [5], in particular vectors with a spectral support in the interval $(-1, \lambda)$ will propagate away from each compact set K and thus undergo tunneling of the indicated type). For multi-dimensional systems there are various possibilities: for a spherically symmetric slowly oscillating potential there will be an infinite sequence of spherically symmetric barriers arranged (as a function of the radial variable r) in analogy with the one-dimensional case; for a potential having radial limits ($V \in C^{\mathrm{rad}}(X)$) there will be no propagation into the asymptotic part of the cone subtended by $\{\omega \in S \mid \lim_{r\to\infty} V(r\omega) \geq \lambda\}$; for certain slowly oscillating non-spherically symmetric potentials there may be an infinite collection of inaccessible regions of increasing size towards infinity.

The above type of behaviour is specific to the class $VO(X)$ and is related to the fact that the action of translations on the frontier $\mathcal{Z}$ is trivial. If V would tend at infinity to a periodic function for instance, the connection between the localization in energy and the domains of non-propagation has a different nature, as seen in Section 6.

Example 3. Potentials with cartesian anisotropy.

We shall work here with $X = \mathbb{R}^2 \equiv \mathbb{R}_1 \times \mathbb{R}_2$. The generalization to arbitrary dimension is straightforward.

For $j = 1, 2$ let us denote by $\mathcal{A}_j$ the C*-algebra of all continuous functions $\varphi : \mathbb{R}_j \to \mathbb{C}$ such that the limits $c_j^\pm = \lim_{x_j\to\pm\infty} \varphi(x_j)$ exist. Then $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$ is an unital C*-subalgebra of $C_b^u(X)$ that is invariant under translations by elements of X.

Let $\widetilde{\mathbb{R}_j} = \mathbb{R}_j \cup \{-\infty_j, +\infty_j\}$ the two-point compactification of $\mathbb{R}_j$. Then $\widetilde{\mathbb{R}_j}$ is the spectrum of $\mathcal{A}_j$ and $\mathcal{X} = \widetilde{\mathbb{R}_1} \times \widetilde{\mathbb{R}_2}$ is the spectrum of $\mathcal{A}$. The closed invariant subsets of the frontier $\mathcal{Z} = \mathcal{X} \setminus X$ are as follows: the four corners $\{(\pm\infty_1, \pm\infty_2)\}$, the four edges $\widetilde{\mathbb{R}_1} \times \{\pm\infty_2\}$ and $\{\pm\infty_1\} \times \widetilde{\mathbb{R}_2}$ and all their unions. We shall illustrate our Theorem for $F = \{(+\infty_1, +\infty_2)\}$ and for $F = \widetilde{\mathbb{R}_1} \times \{+\infty_2\}$.

So let us consider the self-adjoint operator $H = -\Delta + V$ in $\mathcal{H} = L^2(X)$, for $V \in \mathcal{A}$ a real function. It is easy to show that V has the following property (which in fact characterizes the elements of $\mathcal{A}$): for $j, k \in \{1, 2\}$ and $k \neq j$, the limits $V_j^\pm(x_j) = \lim_{x_k\to\pm\infty_k} V(x)$ exist uniformly in $x_j \in \mathbb{R}_j$ and define elements of $\mathcal{A}_j$. The values taken by the continuous extension of V to $\mathcal{X}$ on the four edges coincide respectively with $V_j^\pm$, $j = 1, 2$. Its values at the four corners will be denoted by c^{++}, c^{+-}, c^{-+}, c^{--}. Note that for example $c^{++} = \lim_{x_1\to+\infty_1} V_1^+(x_1) = \lim_{x_2\to+\infty_2} V_2^+(x_2)$.

Let us consider the operator $H_1^+ = -\Delta + V_1^+ = \left(-\Delta_1 + V_1^+\right) \otimes 1_2 + 1_1 \otimes (-\Delta_2)$ in the representation $L^2(\mathbb{R}_1) \otimes L^2(\mathbb{R}_2)$. Its spectrum equals $[a_1^+, \infty)$, where a_1^+ is the infimum of the spectrum of the operator $H^{1,+} = -\Delta_1 + V_1^+$ acting in $\mathcal{H}_1 = L^2(\mathbb{R}_1)$. Three other operators of this kind are available and, with obvious notations, we have $\sigma_{\text{ess}}(H) = \big[\min\{a_1^+, a_1^-, a_2^+, a_2^-\}, \infty\big)$. This follows quite easily from our arguments and was proved in a greater generality in §3 of [9]. Remark also that the spectrum of $H^{\pm\pm} = -\Delta + c^{\pm\pm}$ is $[c^{\pm\pm}, \infty)$ and that inequalities such as $a_1^+ \leq \min\{c^{-+}, c^{++}\}$ are true.

A neighbourhood base of the point $\{(+\infty_1, +\infty_2)\}$ is composed of all the rectangles $\{(y_1, +\infty_1] \times (y_2, +\infty_2] \mid y_1 \in \mathbb{R}_1, y_2 \in \mathbb{R}_2\}$ and a neighbourhood base of $\widetilde{\mathbb{R}_1} \times \{+\infty_2\}$ consists of the rectangles $\{\widetilde{\mathbb{R}_1} \times (y_2, +\infty_2] \mid y_2 \in \mathbb{R}_2\}$. We get:

Proposition 4. *Let $\varepsilon > 0$.*

(a) For any $\eta \in C_0(\mathbb{R})$ with supp$\eta \subset (-\infty, c^{++})$ there exist $y_1 \in \mathbb{R}_1, y_2 \in \mathbb{R}_2$ such that for all $f \in \mathcal{H}$

$$\| \chi(Q_1 > y_1)\chi(Q_2 > y_2)e^{-itH}\eta(H)f \| \leq \varepsilon \| f \| .$$

(b) For any $\eta \in C_0(\mathbb{R})$ with supp$\eta \subset (-\infty, a_1^+)$ there exists $y_2 \in \mathbb{R}_2$ such that for all $f \in \mathcal{H}$

$$\| \chi(Q_2 > y_2)e^{-itH}\eta(H)f \| \leq \varepsilon \| f \| .$$

So, at energies below a_1^+ there is no propagation towards $x_2 = +\infty_2$. At energies comprised between a_1^+ and c^{++} this becomes possible, but then the observable Q_2 cannot diverge through positive values. Results of this type are by no means trivial in the sense that, in certain situations, propagation away from any compact subset of X does occur at energies as above (under suitable assumptions on V there will be intervals of purely absolutely continuous spectrum of H in the considered energy range, and associated states must propagate to infinity by the results of [2]).

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References

1. W. O. Amrein, A. Boutet de Monvel and V. Georgescu, *C_0-Groups, Commutator Methods and Spectral Theory of N-Body Hamiltonians*, Birkhäuser-Verlag, Basel-Boston-Berlin, 1996.
2. W. O. Amrein and V. Georgescu, *On the Characterization of Bound States and Scattering States in Quantum Mechanics*, Helv. Phys. Acta **46** (1973), 635-658.
3. E. B. Davies and B. Simon, *Scattering Theory for Systems with Different Spatial Asymptotics on the Left and Right*, Commun. Math. Phys. **63** (1978), 277-301.
4. V. Georgescu and A. Iftimovici, *C*-Algebras of Energy Observables: I. General Theory and Bumps Algebras*, preprint 00-520 at *http : //www.ma.utexas.edu/mp-arc/*.
5. A. Ya. Gordon, *Pure Point Spectrum under 1-Parameter Perturbations and Instability of Anderson Localizaton*, Commun. Math. Phys. **164** (1994), 489-505.
6. A. Guichardet, *Special Topics in Topological Algebras*, Gordon and Breach, New York, 1968.
7. R. V. Kadison and J. R. Ringrose, *Fundamentals of the Theory of Operator Algebras*, Academic Press, New York, 1983.
8. M. Măntoiu, *On a Class of Anisotropic Schrödinger Operators*, preprint 01-201 at *http://www.ma.utexas.edu/mp−arc/*.
9. M. Măntoiu, *C*-Algebras, Dynamical Systems at Infinity and the Essential Spectrum of Generalized Schrödinger Operators*, preprint 01-298 at *http://www.ma.utexas.edu/mp−arc/* and to appear in J. Reine Angew. Math..
10. G. J. Murphy, *C*-Algebras and Operator Theory*, Academic Press, Boston, 1990.