HYDRODYNAMIC LIMIT OF A DISORDERED LATTICE GAS

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ABSTRACT. We consider a model of lattice gas dynamics in $\mathbb{Z}^d$ in the presence of disorder. If the particle interaction is only mutual exclusion and if the disorder field is given by i.i.d. bounded random variables, we prove the almost sure existence of the hydrodynamical limit in dimension $d \geq 3$. The limit equation is a non linear diffusion equation with diffusion matrix characterized by a variational principle.

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1. INTRODUCTION

Hopping motion of particles between spatially distinct locations is one of the fundamental transport mechanisms in solids and it has been extensively used in a variety of models, including electron conduction in disordered systems under a tight binding approximation. The interested reader is refereed to [5] for a detailed physical review.
From a mathematical point of view, hopping motion is often modeled as an interacting particle system in which each particle performs a random walk over the sites of an ordered lattice like $\mathbb{Z}^d$, with jump rates depending, in general, on the interaction with the nearby particles and, possibly, on some external field. Typically the interaction between the particles is assumed to be short range with an hard core exclusion rule (multiple occupancy of any site is forbidden) and only jumps between nearest neighbors sites are allowed. In the conduction models the hard core exclusion condition reflects the underlying Pauli exclusion principle for electrons. The main focus of the mathematical and physics literature on hopping motion models has been the understanding of transport properties and particularly of the collective diffusive behavior (see for instance [34]).
In this paper we consider an interacting particle system related to conduction of free electrons in doped crystals that can be described as follows. A particle sitting on a site x of the cubic lattice $\mathbb{Z}^d$ waits an exponential time and then attempts to jump to a neighbor site y. If the site y is occupied then the jump is canceled otherwise it is realized with a rate c^{α}_{xy} depending only on the values $(\alpha_x,\, \alpha_y)$ of some external quenched disorder field $\{\alpha_x\}_{x\in\mathbb{Z}^d}$ that, for simplicity, is assumed to be a collection of i.i.d. bounded random variables. Our assumptions on the transition rates are quite general. We require them to be translation covariant, strictly bounded and positive (to avoid trapping phenomena), and to satisfy the detailed balance condition w.r.t. to the (product) Gibbs measure $\mu^{\alpha}(\eta) \propto e^{-H^{\alpha}(\eta)}$, $H_{\alpha}(\eta) = -\sum_x \alpha_x \eta_x$, where η_x is the particle occupation number at site x. These requirements are general enough to include some popular models like the Random Trap and the Miller–Abrahams models, but not other models like the Random Barrier Model in which

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the jumps rates between x, y is assumed to depend only on the unoriented bond $[x, y]$ [20]. For a detailed derivation of the Hamiltonian H^α in the tight-binding approximation and a discussion of the regime of its validity we refer to [5].

Since in the linear–response regime the conductivity in a solid is linked to the diffusion matrix via the Einstein relation (see [34]), our main target has been the study of the bulk diffusion of the disordered lattice gas discussed above. Our main result states that, for $d \geq 3$, for almost any realization of the random field α, the diffusively rescaled system has hydrodynamical limit given by a non linear differential equation

$$\partial_t m = \nabla \cdot (D(m)\nabla m)$$

where $m(t, \theta)$ denotes the macroscopic density function at time t at the point θ of the d–dimensional torus in $\mathbb{R}^d$ with unit volume and the non random matrix $D(\cdot)$ is the diffusion matrix. Moreover, we give a variational characterization of the matrix $D(m)$ in terms of the distribution of the random field α similar to the usual Green–Kubo formula and we prove that $\inf_m D(m) > 0$ and that $D(\cdot)$ is continuous in the open interval $(0, 1)$.
We remark that the above result without the restriction on the dimension d, was already announced in [29] several years ago together with some sketchy ideas for its proof. However the details of the proof have never since been published and some of the technical estimates suggested in [29] turned out to be troublesome even in the absence of disorder (symmetric simple exclusion model) as explained in [15], chapter 6. Therefore we decided to tackle again the problem but we were forced to take a different route w.r.t. that indicated in [29].
We also observe that the problem of collective behavior in disordered lattice gas has been discussed mathematically in other papers, but, to the best of our knowledge, only for models with either homogeneous equilibrium measures (see for example [28], [16] for the one–dimensional Random Barrier model and its Brownian version) or with periodicity in the random field α allowing to solve directly the generalized Fick's law (see [32] and [38] for the one–dimensional Random Trap model having random field α of period 2) or finally for models satisfying the so called "gradient condition" (see below) [25]. From the physical point of view, diffusion of lattice gases in systems with site disorder has been studied mainly by means of simulations and more or less rough approximations like mean field . We refer the interested reader to [19], [20], [21], [22], [24] and to [17] for an iterative procedure to compute corrections to the mean–field approximation.

The main technical features of the model considered here are the absence of translation invariance (for a given disorder configuration) and the non validity of the so called gradient condition. This condition corresponds to the Fick's law of fluid mechanics according to which the current can be written as the gradient of some function. Since the continuity equation states that $\partial_t m = \nabla \cdot J$, J being the macroscopic current, the main problem is to derive J from the family of microscopic instantaneous currents $j^\alpha_{x,y}(\eta) := c^\alpha_{x,y}(\eta)\big(\eta_x - \eta_y\big)$, defined as the difference between the rate at which a particle jumps from x to y and the rate at which a particle jumps from y to x. The gradient condition (the Fick's law) is satisfied if, for each disorder configuration α, there exists a local function $h^\alpha(\eta)$ such that $j^\alpha_{x,x+e}(\eta) = \tau_{x+e}h^\alpha(\eta) - \tau_x h^\alpha(\eta)$ for any $x \in \mathbb{Z}^d$, where $\tau_x h^\alpha(\eta) := h^{\tau_x\alpha}(\tau_x\eta)$ and $\tau_x\eta$, $\tau_x\alpha$ denote the particle and disorder configurations η, α translated by the vector x.
If the system satisfies the gradient condition, the derivation of J is not too difficult (see [23] and reference therein). It is however simple to check (as in [34], p. 182) that our

system never satisfies the gradient condition except for constant disorder field α. We thus have to appeal to the methods developed by Varadhan [36], Quastel [30] and Varadhan-Yau [37] (see also [23] and references therein) for studying the hydrodynamic limit of non disordered non gradient systems. There the main idea is to prove a generalized Fick's law of the form

$$j_{0,e}^{\alpha} \approx \sum_{e' \in \mathcal{E}} D_{e,e'}(m_\ell)(\eta_{e'} - \eta_0) + \mathcal{L}^\alpha g \tag{1.1}$$

for a suitable non random matrix $D(m)$, where m_ℓ is the particle density in a cube centered in the origin of mesoscopic side ℓ, $g(\alpha, \eta)$ is a local function and $\mathcal{L}$ is the generator of the dynamics.

One (among many others) main difficulty in proving such an approximation for a disordered system is due to the fact that the disorder itself induces strong fluctuations in the gradient density field as it is easily seen by taking, for any fixed disorder configuration α, the average w.r.t. to the Gibbs measure μ^α of (1.1). By construction the current $j_{0,e}^{\alpha}$ and the fluctuation term $\mathcal{L}g$ have in fact zero average while the average of $\eta_{e'} - \eta_0$ (we neglect the factor $D(m_\ell)$ for simplicity) is in general $O(1)$ because of the disorder. However, and this is a key input, the average over the disorder of the Gibbs average of $\mu^\alpha\big(\eta_{e'} - \eta_0\big)$ vanishes and therefore one can hope to tame the disorder induced fluctuations in the gradient of the density field by first smearing them out using suitable spatial averages and then by appealing to the ergodic properties of the disorder field α, at least in high enough dimension. It turns out that the above sketchy plan works as soon as $d \geq 3$.

We conclude this short introduction with a plan of the paper. In section 2 we fix the notation, describe the model and state the main results. In section 3 and section 4 we discuss most of the "high level" technical tools (entropy estimates, perturbation theory, spectral gap bounds) and complete the proof of the main theorems following the standard route of non gradient systems, modulo some key technical results. In section 5 we discuss in detail the problem of the fluctuations of the gradient density field induced by the disorder. Section 6 is devoted to the proof of several technical bounds while in section 7 we discuss at length central limit variance, closed and exact forms in our context together with our own interpretation of the long jump method described in [29]. Finally some very technical estimates are collected in an appendix at the end.

We finish by saying that most of the material presented here is based on the unpublished thesis [15] written by one of us (A.F) where an expanded version of several of the arguments used in this paper can be found.

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2. NOTATION, THE MODEL AND MAIN RESULTS

In this section we fix the notation, we define the model and state our main result.

2.1. **Notation.**
Geometric setting. We consider the d dimensional lattice $\mathbb{Z}^d$ with sites $x = \{x_1, \dots, x_d\}$, canonical basis $\mathcal{E}$ and norm $|x| = \max\{|x_1|, \dots, |x_d|\}$. The bonds of $\mathbb{Z}^d$ are non oriented couple of adjacent sites and a generic bond will be denoted by b.
The cardinality of a finite subset $\Lambda \subset \mathbb{Z}^d$ is denoted by $|\Lambda|$ and $\mathbb{F}$ denotes the set of all nonempty finite subsets of $\mathbb{Z}^d$.
Given $\ell \in \mathbb{N}$ we denote by Λ_ℓ the cube centered at the origin of side $2\ell + 1$. If $\ell = 2j + 1$ we also set $Q_\ell = \Lambda_j$. The same cubes centered at x will be denoted by $\Lambda_{x,\ell}$ and $Q_{x,\ell}$ respectively. More generally, for any $V \subset \mathbb{Z}^d$ and $x \in \mathbb{Z}^d$, we will set $V_x := V + x$.
Next, given $e \in \mathcal{E}$ and $\ell = 2\ell' + 1$ with $\ell' \in \mathbb{N}$, we let

$$\Lambda_\ell^{1,e} := \Lambda_{-(\ell'+1)e,\ell'}\,, \quad \Lambda_\ell^{2,e} := \Lambda_{\ell' e,\ell'}\,, \quad \Lambda_\ell^{e} := \Lambda_\ell^{1,e} \cup \Lambda_\ell^{2,e}. \tag{2.1}$$

Finally, given $\epsilon \in (0,1)$ such that $\epsilon^{-1} \in \mathbb{N}$, we define the discrete torus of spacing ϵ by $\mathbb{T}^d_\epsilon := \mathbb{Z}^d/\epsilon^{-1}\mathbb{Z}^d$. The usual d–dimensional torus $\mathbb{R}^d/\mathbb{Z}^d$ (with unite volume) will instead be denoted by $\mathbb{T}^d$. $\mathcal{M}_1(\mathbb{T}^d)$ will denote the set of positive Borel measures on $\mathbb{T}^d$ with total mass bounded by 1, endowed of the weak topology, while $\mathcal{M}_2 \subset \mathcal{M}_1$ will denote the set of measures in $\mathcal{M}_1$ which are absolutely continuous w.r.t. the Lebesgue measure with density ρ satisfying $\|\rho\|_\infty \le 1$.

Spatial averages. We will make heavy use of spatial averages and it is better to fix from the beginning some handy notation. Given $\Lambda \in \mathbb{F}$ and $\ell \in \mathbb{N}$, the spatial average of $\{f_x\}_{x\in\mathbb{Z}^d}$ in $\Lambda \cap \ell\,\mathbb{Z}$ will be denoted by $\mathrm{Av}^{(\ell)}_{x\in\Lambda} f_x$. When $\ell = 1$ we will simply write $\mathrm{Av}_{x\in\Lambda} f_x$.
Next, given $e \in \mathcal{E}$ and two odd integers $\ell = 2\ell' + 1$, $s = 2s' + 1$ such that $\frac{s}{\ell} \in \mathbb{N}$, we let $Q_s^{(\ell)} := \ell\,\mathbb{Z}^d \cap Q_s$. Notice that, if we divide the cube $\Lambda_s^{1,e}$ in cubes of side ℓ, the centers of these cubes form the set $Q_{x,s}^{(\ell)}$ with $x = -(s' + 1)e$.
With these notation we define the (ℓ, s, e) spatial average around $y \in \mathbb{Z}^d$ by

$$\mathrm{Av}^{\ell,s}_{z,y} f_z := \frac{1}{(s/\ell)} \sum_{i=0}^{(s/\ell)-1} \mathrm{Av}_{x\in Q_s^{(\ell)}} f_{y+x+(\ell'+i\ell-s')e}. \tag{2.2}$$

The motivation of introducing such a spatial average will be discussed in subsection 4.2.

The disorder field. We assume the disorder to be described by a collection of real i.i.d random variables $\alpha := \{\alpha_x\}_{x\in\mathbb{Z}^d}$ such that $\sup_x |\alpha_x| \le B$ for some finite constant B. The corresponding product measure on $\Omega_{\mathrm{D}} := [-B, B]^{\mathbb{Z}^d}$ will be denoted by $\mathbb{P}$. Expectation w.r.t. $\mathbb{P}$ will be denoted by $\mathbb{E}$.
Notice that, for any given $\epsilon \in (0,1)$ such that ϵ^{-1} is an odd integer, the random field α induces in a natural way a random field on $\mathbb{T}^d_\epsilon$ via the identification of $\mathbb{T}^d_\epsilon$ with the cube $Q_{1/\epsilon}$. For notation convenience the induced random field will always be denoted by α.
Finally, given $\alpha \in \Omega_{\mathrm{D}}$ and $\Lambda \subset \mathbb{Z}^d$, we define $\alpha_\Lambda := \{\alpha_x\}_{x\in\Lambda}$.

The particle configuration space. Our particle configuration space is $\Omega = S^{\mathbb{Z}^d}$, $S = \{0,1\}$ endowed with the discrete topology, or $\Omega_\Lambda = S^\Lambda$ for some $\Lambda \in \mathbb{F}$. When $\Lambda = \mathbb{T}^d_\epsilon$ we will simply write Ω_ϵ. Given $\eta \in \Omega$ and $\Lambda \subset \mathbb{Z}^d$ we denote by η_Λ the natural projection over Ω_Λ. Given two sites $x, y \in \mathbb{Z}^d$ and a particle configuration η we denote by $\eta^{x,y}$ and η^x the configurations obtained from η by exchanging the values of η at x, y and by "flipping" the value of η at x respectively. More precisely,

$$\left(\eta^{x,y}\right)_z := \begin{cases} \eta_y & \text{if } z = x \\ \eta_x & \text{if } z = y \\ \eta_z & \text{otherwise} \end{cases}, \qquad (\eta^x)_z := \begin{cases} 1 - \eta_x & \text{if } z = x \\ \eta_z & \text{otherwise.} \end{cases}$$

Sometimes we will write $\eta^{x,y} := S_{x,y}\eta$ and call $S_{x,y}$ the *exchange operator* between x and y. Finally, given a probability measure μ on Ω_Λ, we will denote by $\mathrm{Var}_\mu(\xi)$ the variance of the random variable ξ w.r.t. μ, by $\mu(\xi;\xi')$ its covariance with the random variable ξ' and by $\mu(\xi,\xi')$ the scalar product between ξ and ξ' in the Hilbert space $L^2(\Omega_\Lambda, d\mu)$.

Local functions. If f is a measurable function on $\tilde{\Omega} := \Omega_{\mathrm{D}} \times \Omega$, the support of f, denoted by Δ_f, is the smallest subset of $\mathbb{Z}^d$ such that $f(\alpha,\eta)$ depends only on $\alpha_{\Delta_f}, \sigma_{\Delta_f}$ and f is called *local* if Δ_f is finite. By $\|f\|_\infty$ we mean the supremum norm of f. Given two sites $x, y \in \mathbb{Z}^d$ we define

$$\begin{aligned} \nabla_{x,y} f(\alpha,\eta) &:= f(\alpha,\eta^{x,y}) - f(\alpha,\eta), \\ \nabla_x f(\alpha,\eta) &:= f(\alpha,\eta^x) - f(\alpha,\eta). \end{aligned}$$

We write $\mathbb{G}$ for the set of measurable, local and bounded functions g on $\tilde{\Omega}$ and for any $g \in \mathbb{G}$ we introduce the formal series $\underline{g}$

$$\underline{g} := \sum_{x\in\mathbb{Z}^d} \tau_x g$$

where $\tau_x f(\alpha,\eta) := f(\tau_x\alpha, \tau_x\eta)$ and $\tau_x\alpha$ and $\tau_x\eta$ are the disorder and particle configurations translated by $x \in \mathbb{Z}^d$ respectively:

$$(\tau_x\alpha)_z := \alpha_{x+z}, \quad (\tau_x\eta)_z := \eta_{x+z}.$$

Although the above series is only formal, by the locality of g, the gradient $\nabla_{x,y}\, \underline{g}$ is meaningful for any $x, y \in \mathbb{Z}^d$.

Limits. Given n parameters $\ell_1, \dots \ell_n$ we use the compact notation $\lim_{\ell_n\to\ell_n',\dots,\ell_1\to\ell_1'}$ for the ordered limits $\lim_{\ell_n\to\ell_n'} \dots \lim_{\ell_1\to\ell_1'}$. The same convention is valid when "lim" is replaced by "$\limsup$" or "$\liminf$".

2.2. **The model.** In this subsection we describe the lattice gas model at the microscopic scale ϵ for a given disorder configuration α.

Gibbs measures. Given an external chemical potential $\lambda \in \mathbb{R}$, the Hamiltonian of the system in the set $\Lambda \subset \mathbb{Z}^d$ is defined as

$$H^\alpha_\Lambda(\eta) = -\sum_{x\in\Lambda} (\alpha_x + \lambda)\eta_x$$

and the corresponding grand canonical Gibbs measure on Ω_Λ, denoted by $\mu^{\alpha,\lambda}_\Lambda$, is simply the product measure

$$\mu^{\alpha,\lambda}_\Lambda(\eta) := \frac{1}{Z^{\alpha,\lambda}_\Lambda} \exp(-H^\alpha_\Lambda(\eta)) \tag{2.3}$$

where $Z^{\alpha,\lambda}_\Lambda$ is such that $\mu^{\alpha,\lambda}_\Lambda(\Omega_\Lambda)=1$.
For our purposes it is important to introduce also the canonical measures $\nu^\alpha_{\Lambda,m}$. Let $N_\Lambda(\eta)=\sum_{x\in\Lambda}\eta_x$ and let $m\in[0,\frac{1}{|\Lambda|},\dots,1]$. Then

$$\nu^\alpha_{\Lambda,m}(\cdot)=\mu^{\alpha,\lambda}_\Lambda(\cdot\mid N_\Lambda=m|\Lambda|) \tag{2.4}$$

The random variable N_Λ will usually be referred to as the number of particles and $m_\Lambda:=N_\Lambda/|\Lambda|$ as the particle density or simply the density. The set of all canonical measure $\nu^\alpha_{\Lambda,m}$ as m varies in $[0,\frac{1}{|\Lambda|},\dots,1]$ will be denoted by $\mathcal{M}^\alpha(\Lambda)$. Notice that $\nu^\alpha_{\Lambda,m}$ does not depend on the chemical potential λ. However, as it is well known [6], the canonical and grand canonical Gibbs measures are closely related if the chemical potential λ is canonically conjugate to the density m in the sense that the *average density* w.r.t. $\mu^{\alpha,\lambda}_\Lambda$ is equal to m. With this in mind, for any $m\in[0,1]$, we define the *empirical chemical potential* $\lambda_\Lambda(\alpha,m)$ as the unique value of λ such that $\mu^{\alpha,\lambda}_\Lambda(N_\Lambda)=m|\Lambda|$, the *annealed chemical potential* $\lambda_0(m)$ as the unique λ such that $\mathbb{E}\big[\mu^{\alpha,\lambda}(\eta_0)\big]=m$ and the corresponding static compressibility $\chi(m)$ as $\chi(m)=\mathbb{E}\big[\mu^{\alpha,\lambda_0(m)}(\eta_0;\eta_0)\big]$. Since $\frac{\partial}{\partial\lambda}\mu^{\alpha,\lambda}_\Lambda(f)=\mu^{\alpha,\lambda}_\Lambda(f;N_\Lambda)$ for any local function f, we get the following thermodynamic relations:

$$\frac{\partial}{\partial m}\lambda_\Lambda(\alpha,m)=\big[\mu^{\alpha,\lambda_\Lambda(\alpha,m)}_\Lambda(m_\Lambda;N_\Lambda)\big]^{-1}\quad\text{and}\quad\frac{\partial}{\partial m}\lambda_0(m)=\chi(m)^{-1}.$$

Notation warning. *From now on, in order to keep the notation to an acceptable level, we need to adopt the following shortcuts whenever no confusion arises.*

i) *Most of the times the label α will be omitted. That means that quantities like $\mu^\lambda_\Lambda(f)$ will actually be random variables w.r.t the disorder α. Moreover, the label λ of the chemical potential will be omitted when $\lambda=0$.*
ii) *If the region Λ on which the Gibbs measures or, later, the generator of the dynamics are defined coincides with $\mathbb{T}^d_\epsilon$, then the suffix Λ will be simply replaced by ϵ while if $\Lambda=\mathbb{Z}^d$ it will simply be dropped (i.e. $\mu_\epsilon:=\mu^\alpha_{\mathbb{T}^d_\epsilon}$).*
iii) *The symbol $\mu^{\lambda(m)}_\Lambda$ will always denote the grand canonical Gibbs measure on Ω_Λ with empirical chemical potential $\lambda_\Lambda(\alpha,m)$.*
iv) *The letter c will denote a generic positive constant depending only on d and B that may vary from estimate to estimate.*

The dynamics. The lattice gas dynamics we are interested in is the continuous time Markov chain on Ω_ϵ described by the Markov generator $\epsilon^{-2}\mathcal{L}_\epsilon$ where $\mathcal{L}_\epsilon:=\mathcal{L}_{\mathbb{T}^d_\epsilon}$ and for any $\Lambda\subset\mathbb{Z}^d$

$$\mathcal{L}_\Lambda f(\eta)=\sum_{b\subset\Lambda}\mathcal{L}_b f(\eta)$$

where, for any bond $b=\{x,y\}$,

$$\mathcal{L}_{x,y}f(\eta):=c^\alpha_{x,y}(\eta)\nabla_{x,y}f(\eta)$$

The non-negative real quantities $c^\alpha_{x,y}(\eta)$ are the *transition rates* for the process. They are defined as

$$c^\alpha_{x,x+e}(\eta)=f_e(\alpha_x,\eta_x,\alpha_{x+e},\eta_{x+e})\quad\forall x\in\mathbb{Z}^d,\ e\in\mathcal{E}$$

where f_e is a generic bounded function on $\big([-B,B]\times S\big)^2$ such that $f_e(a,s,a',s')=f_e(a',s',a,s)$ and $f_e\geq c>0$ for a suitable constant c. Thanks to this definition the transition rates are *translation covariant*, i.e.

$$c^{\alpha}_{x+z,y+z}(\eta) = c^{\tau_z\alpha}_{x,y}(\tau_z\eta) \qquad \forall z \in \mathbb{Z}^d.$$

The key hypothesis on the transition rates is the *detailed balance condition* w.r.t the Gibbs measures μ^{λ}_{Λ}, $\Lambda \subset \mathbb{Z}^d$ and $\lambda \in \mathbb{R}$, *i.e.*

$$f_e(a,s,a',s') = f_e(a,s',a',s)e^{-(s'-s)(a'-a)} \qquad \forall e \in \mathcal{E},\ a,a' \in [-B,B],\ s,s' \in S$$

which implies that the generator $\mathcal{L}_\Lambda$ becomes a selfadjoint operator on $L^2(\mu^{\lambda}_{\Lambda})$ for any λ. Actually, since the moves of the Markov chain generated by $\mathcal{L}_\Lambda$ do not change the number of particles, for any canonical Gibbs measure $\nu \in \mathcal{M}(\Lambda)$ the operator $\mathcal{L}_\Lambda$ is selfadjoint on $L^2(\nu)$ with a positive spectral gap

$$\operatorname{gap}(\mathcal{L}_\Lambda,\nu) := \inf\Big\{ \frac{\nu(f,-\mathcal{L}_\Lambda f)}{\operatorname{Var}_\nu(f)};\ \operatorname{Var}_\nu(f) \neq 0 \Big\} \tag{2.5}$$

and the corresponding Markov chain is irreducible on $\{\eta \in \Omega_\Lambda :\ N_\Lambda(\eta) = n\}$ for any $n \in [0,1,\dots,|\Lambda|]$.

Given $g \in \mathbb{G}$ we denote by $\mathcal{L}g$ the function $\sum_{b\subset\mathbb{Z}^d}\mathcal{L}_b\, g$. Given $\Delta \subset \Lambda$ and a probability measure μ on Ω_Λ, for any f with support inside Λ we will set

$$\mathcal{D}_\Delta(f;\mu) := \frac{1}{2}\sum_{b\subset\Delta}\mu\big(c_b(\nabla_b f)^2\big).$$

Notice that, if $\Delta = \Lambda$ and μ is either a grand canonical or a canonical measure on Λ, then the above expression is nothing but the Dirichlet form of the Markov chain generated by $\mathcal{L}_\Lambda$ w.r.t. μ.

Finally, given a probability measure μ on Ω_ϵ and $T > 0$, we denote by $\mathbb{P}^{\alpha,\mu}_T$ the distribution at time T of the Markov chain on $\mathbb{T}^d_\epsilon$ with generator $\epsilon^{-2}\mathcal{L}^\alpha_\epsilon$ and initial distribution μ, and by $\mathbb{P}^{\alpha,\mu}$ the induced probability measure on the Skorohod space $D([0,T],\Omega_\epsilon)$ (see [4]). The expectation w.r.t. $\mathbb{P}^{\alpha,\mu}$ will be denoted by $\mathbb{E}^{\alpha,\mu}$. Notice that, in turn, $\mathbb{P}^{\alpha,\mu}$ induces a probability measure $Q^{\alpha,\mu}$ on $D([0,T],\mathcal{M}_1)$ by the formula $\mathbb{P}^{\alpha,\mu}\circ\pi_\epsilon^{-1}$, where

$$\pi_\epsilon(\eta) := \operatorname{Av}_{x\in\mathbb{T}^d_\epsilon}\eta_x\,\delta_{\epsilon x} \in \mathcal{M}_1(\mathbb{T}^d)$$

denotes the empirical measure.

Warning In all the above measures, the crucial dependence on the parameter $\epsilon > 0$ does not appear in the various symbols in order to keep the notation to an acceptable level.

2.3. **Main results.** Our first result concerns the existence and regularity of the diffusion matrix $D(m)$ corresponding to the usual Green-Kubo matrix (see [34], proposition 2.2 page 180).

Theorem 2.1. *Let $d \geq 3$. Then for any density $m \in (0,1)$ there exists a unique symmetric $d\times d$ matrix $D(m)$, such that*

$$\big(a, D(m)a\big) = \frac{1}{2\chi(m)}\inf_{g\in\mathbb{G}}\sum_{e\in\mathcal{E}}\mathbb{E}\Big[\ \mu^{\alpha,\lambda_0(m)}\Big(c^\alpha_{0,e}\big(a_e(\eta_e-\eta_0)+\nabla_{0,e}\underline{g}\big)^2\Big)\Big] \quad \forall a \in \mathbb{R}^d. \tag{2.6}$$

Moreover $D(m)$ is continuous in the open interval $(0,1)$ and

$$0 < c^{-1}\mathbb{I} \leq D(m) \leq c\mathbb{I} \quad \forall m \in (0,1)$$

for some positive constant c.

Remark 2.2. *We actually expect the matrix D to be continuously extendable to the closed interval $[0,1]$. In particular we expect that $D(m)$ converges to the diffusion matrix of the random walk of a single particle in the random environment α as m goes to zero, as confirmed by simulations (see* [22]*).*

In order to state the next main result we need the following definition.

Definition 2.3. *Given a Lebesgue absolutely continuous measure $m(\theta)d\theta \in \mathcal{M}_2(\mathbb{T}^d)$, a sequence of probability measures μ^ϵ on Ω_ϵ is said to correspond to the macroscopic profile $m(\cdot)$ if, under μ^ϵ, the random variable π_ϵ in $\mathcal{M}_1(\mathbb{T}^d)$ converges in probability to $m(\theta)d\theta$ as $\epsilon \downarrow 0$, i.e. for any smooth function H on $\mathbb{T}^d$ and any $\delta > 0$*

$$\lim_{\epsilon\downarrow 0} \mu^\epsilon\big(\big| \operatorname{Av}_{x\in\mathbb{T}^d_\epsilon} H(\epsilon x)\eta_x - \int_{\mathbb{T}^d} H(\theta)m(\theta)d\theta \,\big| > \delta \big) = 0\,.$$

With the above definition the existence of the hydrodynamical limit for almost all disorder configurations reads as follows.

Theorem 2.4. *Let $d \geq 3$, let $T > 0$ and assume that $D(m)$ can be continuously extended to the closed interval $[0,1]$. Then almost all disorder configurations α satisfy the following property. Let $m_0(\theta) \in \mathcal{M}_2$ and suppose that the Cauchy problem*

$$\begin{cases} \partial_t m(t,\theta) = \nabla_\theta \Big(D\big(m(t,\theta)\big) \nabla_\theta\, m(t,\theta) \Big) \\ m(0,\theta) = m_0(\theta) \end{cases} \tag{2.7}$$

has a unique weak solution $m \in C([0,T],\mathcal{M}_2)$ satisfying the energy estimate

$$\int_0^T ds \int_{\mathbb{T}^d} d\theta \, |\nabla_\theta m(t,\theta)|^2 < \infty. \tag{2.8}$$

Let also $\{\mu^\epsilon\}_{\epsilon>0}$ be a sequence of probability measures on Ω_ϵ corresponding to the macroscopic density profile $m_0(\theta)$. Then the measure Q^{α,μ^ϵ} converges weakly to the probability measure on $D([0,T],\mathcal{M}_1)$ concentrated on the path $\{m(t,\theta)d\theta\}_{t\in[0,T]}$. In particular, for any $0 \leq t \leq T$, the sequence of time dependent probability measures $\{\mathbb{P}^{\alpha,\mu^\varepsilon}_t\}_{\epsilon>0}$ corresponds to the macroscopic density profile $m(t,\theta)$, i.e. for any smooth function H on $\mathbb{T}^d$ and any $\delta > 0$

$$\lim_{\epsilon\downarrow 0} \mathbb{P}^{\alpha,\mu_\epsilon}_t \big(\big| \operatorname{Av}_{x\in\mathbb{T}^d_\epsilon} H(\epsilon x)\eta_x - \int_{\mathbb{T}^d} H(\theta)m(t,\theta)d\theta \,\big| > \delta \big) = 0. \tag{2.9}$$

The thesis remains valid also if $D(m)$ has no continuous extension provided that one assumes instead that, for some fixed $\rho \in (0,1)$, there exists a sequence of probability measures μ^ϵ_* on Ω_ϵ such that

$$H[\mu^\epsilon|\mu^\epsilon_*] = o(\epsilon^{-d}) \qquad \textit{and} \qquad \inf_\epsilon \inf_{x\in\mathbb{T}^d_\epsilon} \min\big(\mu^\epsilon_*(\eta_x), 1-\mu^\epsilon_*(\eta_x)\big) \geq \rho, \tag{2.10}$$

where $H[\cdot|\cdot]$ denotes the relative entropy.

Remark 2.5. *Notice that condition (2.10) becomes rather natural if the initial profile $m_0(\cdot)$ satisfies $\rho \leq m_0(\theta) \leq 1-\rho$ for any $\theta \in \mathbb{T}^d$.*

3. Plan of the proof of the two main theorems

The proof of theorem 2.1 will be given in section 7.4 and it is based on more or less standard techniques. The proof of theorem 2.4 is more involved and it can be divided into several steps that we illustrate in what follows. In order to work in the simplest possible setting, in the sequel we assume that the diffusion matrix D can be continuously extended to the closed interval $[0,1]$. Only at the end (see subsection 4.8) we will explain how to treat the other case.

Let us begin with some remarks on the weak interpretation of (2.7) and (2.8). Let $A(m)$, $m \in [0,1]$, be a $d \times d$ matrix such that $A'(m) = D(m)$ so that

$$\big(D(m(t,\theta))\nabla_\theta m(t,\theta)\big)_e = \sum_{e' \in \mathcal{E}} \partial_{\theta_{e'}} A_{e,e'}(m(t,\theta))\,, \quad \forall e \in \mathcal{E}.$$

It is simple to prove (see appendix of [15]) that given $m \in D([0,T],\mathcal{M}_2)$ there is a measurable function $m(t,\theta)$ univocally defined up to sets of zero Lebesgue measure such that $m_t = m(t,\theta)d\theta$ for any $t \in [0,T]$ (see appendix of [15]). In what follows, we will often identify m with the funtion $m(t,\theta)$.
A path $m \in D([0,T],\mathcal{M}_2)$ is called a weak solution of (2.7) if $m(0,\cdot) = m_0(\cdot)$ and

$$\Phi(m,H) = 0 \quad \forall H \in C^{1,2}([0,T] \times \mathbb{T}^d)$$

where

$$\begin{aligned}\Phi(m,H) := &\int_{\mathbb{T}^d} m(T,\theta)H(T,\theta)\,d\theta - \int_{\mathbb{T}^d} m(0,\theta)H(0,\theta)\,d\theta - \int_0^T \int_{\mathbb{T}^d} m(s,\theta)\partial_s H(s,\theta)\,d\theta\,ds \\ &- \sum_{e,e'} \int_0^T \int_{\mathbb{T}^d} A_{e,e'}\big(m(s,\theta)\big)\,\partial^2_{\theta_e,\theta_{e'}} H(s,\theta)\,d\theta\,ds.\end{aligned} \tag{3.1}$$

Moreover, $m \in D([0,T],\mathcal{M}_2)$ satisfies the energy estimate (2.8) if

$$\sup_{e \in \mathcal{E}} \sup_{H \in C^1([0,T]\times\mathbb{T}^d)} \int_0^T \int_{\mathbb{T}^d} \big(2\,m\,\partial_{\theta_e} H - H^2\big)d\theta\,ds < \infty. \tag{3.2}$$

Warning. In what follows, we will introduce some other mesoscopic scales in addition to the microscopic scale ϵ. For example, we will introduce some positive scale parameters a, b and consider the mesoscopic scales $\left[\frac{a}{\epsilon}\right]$ and $\left[\frac{b}{\epsilon}\right]$, where $[\cdot]$ denotes the integer part. For simplicity of notation these new scales will be denoted only by $\frac{a}{\epsilon}$ and $\frac{b}{\epsilon}$. Moreover, we will introduce the scale n where n is a positive odd integer. The property of n to be odd will be always understood.

3.1. **Tightness.** The first step toward the proof of theorem 2.4 is to show that, for all disorder configurations α, if $\{\mu^\epsilon\}_{\epsilon>0}$ is a sequence of probability measures on Ω_ϵ then the sequence of measures on $D([0,T],\mathcal{M}_1)$, $\{Q^{\alpha,\mu^\epsilon}\}_{\epsilon>0}$, is relatively compact. For this purpose it is enough to use the Garsia-Rodemich-Rumsey inequality as done in [23], chapter 7, section 6.

3.2. **Regularity properties of the limit points.** In the second step one proves that, for almost all α, given a sequence $\{\mu^\epsilon\}_{\epsilon>0}$ of probability measures on Ω_ϵ, any limit point Q^α of the sequence $\{Q^{\alpha,\mu^\epsilon}\}_{\epsilon>0}$ is concentrated on paths enjoying a certain regularity property. For this purpose we first observe that, for any α, Q^α must satisfy $Q^\alpha\big(\,C([0,T],\mathcal{M}_2)\,\big) = 1$, since for any $\eta \in \Omega_\epsilon$, $H \in C(\mathbb{T}^d)$ and $b \subset \mathbb{T}^d_\epsilon$

$$\big|\,\pi_\epsilon(\eta)[H]\,\big| \le \mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}|H(\epsilon x)| \quad \text{and} \quad \big|\,\pi_\epsilon(\eta^b)[H] - \pi_\epsilon(\eta)[H]\,\big| \le 2\,\|H\|_\infty \epsilon^d.$$

Moreover, if the sequence of $\{\mu^\epsilon\}_{\epsilon>0}$ corresponds to the macroscopic profile $m_0(\theta)$, then necessarily

$$Q^\alpha\big(\,m \in C([0,T],\mathcal{M}_2)\ :\ m(0,\theta) = m_0(\theta)\,\big) = 1 \quad \forall \alpha. \tag{3.3}$$

The key result here, whose proof will be given later in section 4.7, is the following. Given a path $\eta(\cdot) \in D([0,T],\Omega_\epsilon)$, $x \in \mathbb{T}^d_\epsilon$ and $\ell \in \mathbb{N}$, let $m_{x,\ell}(t)$ be the particle density of $\eta(t)$ in the cube $Q_{x,\ell}$. Then we have

Lemma 3.1. (Energy estimate). *Let $d \geq 3$, let $T > 0$ and assume that $D(m)$ can be continuously extended to the closed interval $[0,1]$. Then almost any disorder configurations α have the following property. For any sequence $\{\mu^\epsilon\}_{\epsilon>0}$ of probability measures on Ω_ϵ and any $e \in \mathcal{E}$*

$$\sup_{b>0} \limsup_{a\downarrow 0, \epsilon\downarrow 0} \mathbb{E}^{\alpha,\mu^\epsilon}\Big(\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\int_0^T \Big[\frac{m_{x+\frac{b}{\epsilon}e,\frac{a}{\epsilon}}(s) - m_{x,\frac{a}{\epsilon}}(s)}{b}\Big]^2 ds\Big) < +\infty. \tag{3.4}$$

Moreover any limit point Q^α of the sequence $\{Q^{\alpha,\mu^\epsilon}\}_{\epsilon>0}$ satisfies

$$Q^\alpha\Big\{ m \in C([0,T],\mathcal{M}_2) \ : \mathrm{l.h.s.\ of\ } (3.2\ ) < \infty\Big\} = 1. \tag{3.5}$$

3.3. **Microscopic identification of the hydrodynamic equation.** In the third step of the proof one identifies at the microscopic level the hydrodynamic equation. It is convenient to introduce some more notation. Given $e,e' \in \mathcal{E}$, two positive numbers a,b and a smooth function H on $[0,T]\times\mathbb{T}^d$, we set

$$\begin{aligned}\bar H_{b,a,\epsilon} := \mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\Big[H(T,\epsilon x)\eta_x(T) - H(0,\epsilon x)\eta_x(0) - \int_0^T ds\,\eta_x(s)\partial_s H(s,\epsilon x)\Big] \\ + \sum_{e,e'\in\mathcal{E}}\int_0^T ds\,\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\nabla^\epsilon_e H(s,\epsilon x) D_{e,e'}\big(m_{x,\frac{a}{\epsilon}}(s)\big)\Big[\frac{m_{x+\frac{b}{\epsilon}e',\frac{a}{\epsilon}}(s) - m_{x-\frac{b}{\epsilon}e',\frac{a}{\epsilon}}(s)}{2b}\Big]\end{aligned} \tag{3.6}$$

where $\nabla^\epsilon_e H(s,\epsilon x) := \frac{1}{\epsilon}\big[H(s,\epsilon x+\epsilon e) - H(s,\epsilon x)\big]$.

The following theorem, whose proof will be discussed in a little while, corresponds to the microscopic identification of the hydrodynamical equation.

Theorem 3.2. *Let $d \geq 3$, let $T > 0$ and assume that $D(m)$ can be continuously extended to the closed interval $[0,1]$. Then almost all disorder configurations α have the following property. For any sequence $\{\mu^\epsilon\}_{\epsilon>0}$ of probability measures on Ω_ϵ, any $\delta > 0$ and any $H \in C^{1,2}([0,T]\times\mathbb{T}^d)$*

$$\limsup_{b\downarrow 0,\, a\downarrow 0,\, \epsilon\downarrow 0} \mathbb{P}^{\alpha,\mu^\varepsilon}\big(|\bar H_{b,a,\epsilon}| > \delta\big) = 0. \tag{3.7}$$

The proof of theorem 2.4, given Lemma 3.1 and theorem 3.2, now follows by more or less standard arguments and it can be found in section 1.5 of [15].

4. Proof of theorem 3.2 modulo some technical steps

In this section we prove theorem 3.2 modulo certain technical results that will be discussed in the remaining sections. Following [37] the first main step is to reduce the proof of the theorem to the eigenvalue estimates of certain symmetric operators, via the entropy inequality and the Feynman–Kac formula. To this aim we define $j_{x,x+e}$ as the instantaneous current through the oriented bond $\{x,x+e\}$, i.e. as the difference between the rate at which a particle jumps from x to $x+e$ and the rate at which a particle jumps from $x+e$ to x. It is simple to check that

$$j_{x,x+e}(\eta) = c_{x,x+e}(\eta)(\eta_x - \eta_{x+e})$$

and

$$\mathcal{L}_\epsilon \eta_x = \sum_{e\in\mathcal{E}} (-j_{x,x+e}(\eta) + j_{x-e,x}(\eta)).$$

In particular (see lemma 5.1, appendix 1 in [23], or [15]), for any smooth $H(t,x)$, integration by parts and stochastic calculus show that

$$\begin{aligned} &\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\big[H(T,\epsilon x)\eta_x(T) - H(0,\epsilon x)\eta_x(0)\big] = \\ &\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\int_0^T \partial_s H(s,\epsilon x)\eta_x(s)ds + \epsilon^{-1}\sum_{e\in\mathcal{E}}\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\int_0^T \nabla^\epsilon_e H(s,\epsilon x) j_{x,x+e} ds + M(T) \end{aligned} \tag{4.1}$$

where $M(\cdot)$ is a martingale w.r.t $\mathbb{P}^{\mu^\epsilon}$ satisfying

$$\mathbb{P}^{\mu^\epsilon}\big[\,|M(T)| > \delta\,\big] \le c(H)\,\delta^{-2}\epsilon^d \qquad \forall \delta > 0. \tag{4.2}$$

In order to benefit of the ergodicity of the system, it is convenient to replace the current $j_{x,x+e}$ in (4.1) by its local average around x. To this aim let us introduce a new scale parameter ℓ, that will be sent to ∞ after the limit $\epsilon \downarrow 0$. Then, because of the smoothness of the function H, for any $\ell \gg 1$ one can safely replace in the r.h.s. of (4.1) the current $j_{x,x+e}$ by a local average $\mathrm{Av}_{y:|y-x|\le \ell_1} j_{y,y+e}$, $\ell_1 := \ell - \sqrt{\ell}$, in the sense that, for any $\delta > 0$

$$\lim_{\epsilon\downarrow 0}\mathbb{P}^{\mu^\epsilon}\Big[\,\big|\epsilon^{-1}\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\int_0^T \nabla^\epsilon_e H(s,\epsilon x)\big[j_{x,x+e} - \mathrm{Av}_{y:|y-x|\le\ell_1} j_{y,y+e}\big]\,ds\,\big| > \delta\Big] = 0. \tag{4.3}$$

The key observation in the theory of non-gradient systems is that, thanks again to stochastic calculus,

$$\lim_{\epsilon\downarrow 0}\mathbb{P}^{\mu^\epsilon}\big[\,\big|\epsilon^{-1}\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\int_0^T \nabla^\epsilon_e H(s,\epsilon x)\tau_x\mathcal{L}g\,ds\,\big| > \delta\,\big] = 0 \qquad \forall\delta>0,\ \forall g\in\mathbb{G} \tag{4.4}$$

and similarly for $\mathrm{Av}_{y:|y-x|\le\ell_1}\tau_y\mathcal{L}g$ in place of $\tau_x\mathcal{L}g$.

In conclusion, thanks to (4.1), (4.2), (4.3) and (4.4), in order to prove (3.7) it is enough to show that for almost all disorder configuration α and for any $e\in\mathcal{E}$

$$\begin{aligned} \inf_{g\in\mathbb{G}}\ \limsup_{b\downarrow 0,\, a\downarrow 0,\, \ell\uparrow\infty,\, \epsilon\downarrow 0}\ \mathbb{E}^{\mu^\epsilon}\Big(\,\Big|\int_0^T \epsilon^{-1}\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\nabla^\epsilon_e H(s,\epsilon x)\Big[\mathrm{Av}_{y:|y-x|\le\ell_1}(j_{y,y+e}+\tau_y\mathcal{L}g) \\ + \sum_{e'\in\mathcal{E}} D_{e,e'}(m_{x,\frac{a}{\epsilon}})\big[\frac{m_{x+\frac{b}{\epsilon}e',\frac{a}{\epsilon}} - m_{x-\frac{b}{\epsilon}e',\frac{a}{\epsilon}}}{2b/\epsilon}\big]\Big]\,ds\,\Big|\Big) = 0. \end{aligned} \tag{4.5}$$

We next reduce (4.5) to certain equilibrium eigenvalue estimates by means of the entropy inequality and the Feynman-Kac formula (see proposition A.8). Let us recall the former: given two probability measures π,π' on the same probability space, for any $\beta>0$ and any bounded and measurable function f,

$$\pi(f) \le \beta^{-1}\big\{H(\pi\,|\,\pi') + \ln\big(\pi'(e^{\beta f})\big)\big\} \tag{4.6}$$

where $H(\pi\,|\,\pi')$ denotes the entropy of π w.r.t. π'. It is simple to verify that, for any initial distribution μ on Ω_ϵ, the relative entropy between the path measure $\mathbb{P}^\mu$ starting from μ and the equilibrium path measure $\mathbb{P}^{\mu_\epsilon}$ starting from the grand canonical measure μ_ϵ with zero chemical potential, satisfies

$$H\big(\mathbb{P}^\mu\,|\,\mathbb{P}^{\mu_\epsilon}\big) \le c\,\epsilon^{-d}.$$

Therefore, for any $\gamma > 0$ and any function h on $[0,T] \times \Omega_\epsilon$

$$\mathbb{E}^\mu\big(\big|\int_0^T h(s,\eta(s))ds\big|\big) \le \frac{c}{\gamma} + \frac{\epsilon^d}{\gamma}\ln \mathbb{E}^{\mu_\epsilon}\Big(\exp\big\{\gamma\epsilon^{-d}\big|\int_0^T h(s,\eta(s))ds\big|\big\}\Big). \tag{4.7}$$

The Feynman–Kac formula (see proposition A.8) now shows that,

$$\begin{aligned}&\frac{\epsilon^d}{\gamma}\ln \mathbb{E}^{\mu_\epsilon}\Big(\exp\big\{\gamma\epsilon^{-d}\big(\pm\int_0^T h(s,\eta(s))ds\big)\big\}\Big) \le \\ &\int_0^T \sup spec_{L^2(\mu_\epsilon)}\big\{\pm h(s,\cdot) + \gamma^{-1}\epsilon^{d-2}\mathcal{L}_\epsilon\big\}\, ds\ .\end{aligned} \tag{4.8}$$

We now apply the above reasoning to the function $h(s,\eta) =$ integrand of (4.5). Since for any $\epsilon > 0$ $\sup_{s\in[0,T]}\sup_{x\in\mathbb{T}^d}|\nabla^\epsilon_e H(s,\epsilon x)| \le c(H)$, after a suitable reparametrization of γ, in order to prove (4.5) it is enough to prove the following key eigenvalue estimate.

Proposition 4.1. *Let $d \ge 3$. Then, almost all disorder configurations α have the following property. For all $\gamma > 0$*

$$\inf_{g\in\mathbb{G}}\ \limsup_{b\downarrow 0,\, a\downarrow 0,\, \ell\uparrow\infty,\, \epsilon\downarrow 0}\ \sup_J \sup spec_{L^2(\mu_\epsilon)}\big\{\epsilon^{-1}\bar{J}^g_{b,a,\ell,\epsilon} + \gamma\epsilon^{d-2}\mathcal{L}\big\} \le 0 \tag{4.9}$$

where

$$\begin{aligned}\bar{J}^g_{b,a,\ell,\epsilon} &:= \mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}J(\epsilon x)\Big[\mathrm{Av}_{y:|y-x|\le\ell_1}(j_{y,y+e} + \tau_y\mathcal{L}g)\\ &+ \sum_{e'\in\mathcal{E}} D_{e,e'}(m_{x,\frac{a}{\epsilon}})\big[\frac{m_{x+\frac{b}{\epsilon}e',\frac{a}{\epsilon}} - m_{x-\frac{b}{\epsilon}e',\frac{a}{\epsilon}}}{2b/\epsilon}\big]\Big]\end{aligned} \tag{4.10}$$

and J varies in $\{J \in C(\mathbb{T}^d)\ :\ \|J\|_\infty \le 1\}$.

4.1. **Some technical tools to bound the spectrum.** Before we turn to the proof of proposition 4.1, let us introduce some tools to deal with the eigenvalue problem posed in (4.9). We begin by recalling a useful subadditivity property of the supremum of the spectrum of a selfadjoint operator and explain its role in the so–called *localization technique*.

Given a finite family $\{X_i\}_{i\in I}$ of self-adjoint operators on $L^2(\mu_\epsilon)$,

$$\sup spec_{L^2(\mu_\epsilon)}\{\sum_{i\in I} X_i\} \le \sum_{i\in I}\sup spec_{L^2(\mu_\epsilon)}\{X_i\}, \tag{4.11}$$

and similarly with $\sum_{i\in I}$ replaced by $\mathrm{Av}_{i\in I}$. The subadditivity property allows one to exploits the *localization* method which is best explained by means of an example, although the underlying idea has a much wider application. Let $\epsilon > 0$, $\ell < \frac{1}{\epsilon}$ and for any $x \in \mathbb{T}^d_\epsilon$ let f_x be a local function with support in $\Lambda_{x,\ell}$. Recall the definition of $\mathcal{M}(\Lambda_{x,\ell})$ as the set of canonical Gibbs measures on $\Lambda_{x,\ell}$. Then

$$\begin{aligned}&\sup spec_{L^2(\mu_\epsilon)}\big\{\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}f_x + \epsilon^{d-2}\mathcal{L}_\epsilon\big\}\\ &\le \mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\sup spec_{L^2(\mu_\epsilon)}\big\{f_x + c\,\epsilon^{-2}\mathrm{Av}_{b\in\Lambda_{x,\ell}}\mathcal{L}_b\big\}\\ &\le \mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\sup_{\nu\in\mathcal{M}(\Lambda_{x,\ell})}\sup spec_{L^2(\nu)}\big\{f_x + c\,\epsilon^{-2}\mathrm{Av}_{b\in\Lambda_{x,\ell}}\mathcal{L}_b\big\}\end{aligned} \tag{4.12}$$

where the former inequality follows from $\epsilon^d\mathcal{L}_\epsilon \le c\,\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\mathrm{Av}_{b\in\Lambda_{x,\ell}}\mathcal{L}_b$ together with the subadditivity property, while the latter can be easily checked.

Next we state a very general result on $\sup spec_{L^2(\nu)}\{\mathfrak{L}+\beta V\}$, where $\mathfrak{L}$ is an ergodic reversible Markov generator on a finite set E with invariant measure μ, and whose proof is based on perturbation theory for selfadjoint operators (see e.g. [23]).

Proposition 4.2. *Let* $\mathrm{gap}(\mathfrak{L},\mu)$ *be the spectral gap of* $\mathfrak{L}$ *in* $L^2(\mu)$ *and let, for* $\beta > 0$ *and* $V : E \mapsto \mathbb{R}$,

$$\lambda_\beta := \sup spec_{L^2(\mu)}\{\mathfrak{L}+\beta V\}.$$

Assume without loss of generality $\mu(V) = 0$. *If*

$$2\beta\ \mathrm{gap}(\mathfrak{L},\mu)^{-1}\|V\|_\infty < 1$$

then

$$0 \le \lambda_\beta \le \frac{\beta^2}{1-2\beta\ \mathrm{gap}(\mathfrak{L},\mu)^{-1}\|V\|_\infty}\,\mu\Big(V,(-\mathfrak{L})^{-1}\,V\Big).$$

The above proposition suggests that in order to prove proposition 4.1 we must be able to estimate:

(1) the spectral gap of the generator $\mathcal{L}_\Lambda$ in a generic box Λ;
(2) the H_{-1} norm appearing above.

We begin with the first one.

Proposition 4.3. [10] *Let* $\Lambda \subset \mathbb{Z}^d$ *be a parallelepiped with longest side* ℓ. *Then, for all disorder configurations* α *and all* $\nu \in \mathcal{M}(\Lambda)$,

$$\mathrm{gap}(\mathcal{L}_\Lambda;\nu) \ge c\,\ell^{-2} \tag{4.13}$$

In particular, for all disorder configurations and all $\nu \in \mathcal{M}(\Lambda)$, *the following Poincaré inequality holds*

$$\mathrm{Var}_\nu(f) \le c\,\ell^2\mathcal{D}_\Lambda(f;\nu) \tag{4.14}$$

Remark 4.4. *The key aspect of the above result is the uniformity in the disorder configuration. Its proof is based on some clever technique developed recently in* [13] *to deal with the Kac model for the Boltzmann equation and extended in* [12] *and* [11] *to other kind of diffusions. For other models of lattice gas dynamics like the dilute Ising lattice gas in the Griffiths regime the above uniformity will no longer be available and a more sophisticated analysis is required (see* [15] *for a discussion).*

Let us now tackle with the H_{-1} norm. Unfortunately that will requires the proof of some technical bounds that, on a first reading, can be just skipped.

Following the theory of non disordered non-gradient systems, we introduce the space $\mathcal{G} \subset \mathbb{G}$ defined as

$$\mathcal{G} := \{g \in \mathbb{G}\,:\,\exists\,\Lambda \in \mathbb{F}\ \text{ such that, }\ \forall\alpha \text{ and } \forall\nu \in \mathcal{M}^\alpha(\Lambda)\,,\ \nu(g) = 0\,\}. \tag{4.15}$$

Equivalently, $\mathcal{G}$ can be defined as the set of functions $g \in \mathbb{G}$ such that there exists $\Lambda \in \mathbb{F}$ and $h \in \mathbb{G}$ with $g = \mathcal{L}_\Lambda h$. Since the dynamics is reversible w.r.t. Gibbs measures, this second characterization assures an integration by parts property that will play an important role in the sequel. More precisely, if $g = \mathcal{L}_\Lambda h$, then, for any Λ' containing Λ and any $\nu \in \mathcal{M}(\Lambda')$, $\nu\big(g,f\big) = \nu(h,\mathcal{L}_\Lambda f)$. Moreover, if V and Δ are such that $\Lambda_x \subset V$ for any $x \in \Delta$, then for any $A > 0$ and $\nu \in \mathcal{M}(V)$,

$$\Big|\nu\big(\sum_{x\in\Delta}\tau_x g,f\big)\Big| \le c(g)|\Delta|^{\frac12}\mathcal{D}_V(f;\nu)^{\frac12} \le A\,c(g)|\Delta| + A^{-1}\,c(g)\mathcal{D}_V(f;\nu) \tag{4.16}$$

where, for some suitable constant $c(\Lambda)$,

$$c(g) := c(\Lambda) \sup_{\alpha} \sup_{\nu\in\mathcal{M}^{\alpha}(V)} \big(\nu(h^2)\big)^{\frac{1}{2}}. \tag{4.17}$$

A first simple consequence of integration by parts (see chapter 7 of [23] and section 1.16 of [15] for a proof) is the following bound.

Proposition 4.5. *Let $g \in \mathcal{G}$ have support included in Λ_s. Then for any disorder configuration α, any $\gamma > 0$ and any family of functions $F := \{f_x\}_{x\in\mathbb{T}^d_\epsilon}$ on Ω_ϵ,*

$$\begin{aligned}\sup spec_{L^2(\mu_\epsilon)}\{\epsilon^{-1}\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}(\tau_x g\, f_x) + \gamma\epsilon^{d-2}\mathcal{L}\} \le \\ \epsilon^{-1}c(g,\|F\|_\infty)\|\nabla F\|_\infty + \sup spec_{L^2(\mu_\epsilon)}\{c(g)\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon} f_x^2 + \frac{1}{2}\gamma\epsilon^{d-2}\mathcal{L}\},\end{aligned} \tag{4.18}$$

where $\|F\|_\infty := \sup_{x\in\mathbb{T}^d_\epsilon}\|f_x\|_\infty$ and $\|\nabla F\|_\infty := \sup_{x\in\mathbb{T}^d_\epsilon}\sup_{b\subset\Lambda_{x,s}}\|\nabla_b f_x\|_\infty$.

In the space $\mathcal{G}$ it is also possible to introduce a H_{-1} norm closely related to that given by perturbation theory (see proposition 4.2 above).

Given positive integers ℓ, s with $s^2 \le \ell$ and $f, g \in \mathcal{G}$ with $\Delta_f, \Delta_g \subset \Lambda_s$, for any canonical or grand canonical Gibbs measure μ on Λ_ℓ we define

$$V_\ell(f,g;\mu) := (2l)^{-d}\mu\Big(\sum_{|x|\le\ell_1}\tau_x f, \big(-\mathcal{L}_{\Lambda_\ell}\big)^{-1}\sum_{|x|\le\ell_1}\tau_x g\Big). \tag{4.19}$$

If Λ_ℓ is replaced by $\Lambda_{z,\ell}$ and the above sums are over $x \in \Lambda_{z,\ell_1}$ we will simply write $V_{z,\ell}(f,g;\mu)$ and if $f = g$ we write $V_\ell(g;\mu)$ or $V_{z,\ell}(g;\mu)$.

It is simple to check that $V_\ell(g;\mu)$ can be variationally characterized as follows:

$$\begin{aligned} V_\ell(g;\mu) &= (2l)^{-d}\sup_h\big\{2\mu\big(\sum_{|x|\le\ell_1}\tau_x g, h\big) - \mathcal{D}_{\Lambda_\ell}(h;\mu)\big\} \\ &= (2l)^{-d}\sup_h \frac{\mu(\sum_{|x|\le\ell_1}\tau_x g, h)^2}{\mathcal{D}_{\Lambda_\ell}(h;\mu)}\end{aligned} \tag{4.20}$$

where $\sup_h$ is taken among the non constant functions with support contained in Λ_ℓ.

The variational characterization allows one to derive some simple bounds on $V_\ell(g;\mu)$. Let Δ be a box such that $\Delta_g \subset \Delta \subset \Lambda_s$ and for any $x \in \mathbb{Z}^d$ let $\mathcal{F}_x$ be the σ–algebra generated by m_{Δ_x} and $\{\eta_y\}_{y\notin\Delta_x}$. Then, for any function h,

$$\begin{aligned}\mu(\tau_x g, h) &= \mu\big(\mu(\tau_x g; h\,|\,\mathcal{F}_x)\big) \le \mu\big(\mathrm{Var}_\mu(\tau_x g\,|\,\mathcal{F}_x)^{\frac{1}{2}}\mathrm{Var}_\mu(h\,|\,\mathcal{F}_x)^{\frac{1}{2}}\big) \\ &\le \mu\big(\mathrm{Var}_\mu(\tau_x g\,|\,\mathcal{F}_x)\big)^{\frac{1}{2}}\mu\big(\mathrm{Var}_\mu(h\,|\,\mathcal{F}_x)\big)^{\frac{1}{2}}\end{aligned}$$

which implies that

$$\mu\big(\sum_{|x|\le\ell_1}\tau_x g, h\big)^2 \le c\sum_{|x|\le\ell_1}\mu\big(\mathrm{Var}_\mu(\tau_x g\,|\,\mathcal{F}_x)\big)\sum_{|x|\le\ell_1}\mu\big(\mathrm{Var}_\mu(h\,|\,\mathcal{F}_x)\big). \tag{4.21}$$

If we appeal now to the Poincaré inequality

$$\mathrm{Var}_\mu(h\,|\,\mathcal{F}_x) \le cs^2\sum_{b\subset\Delta_x}\mu\big(c_b(\nabla_b h)^2\,|\,\mathcal{F}_x\big),$$

the last sum in (4.21) is bounded by $c\,s^{d+2}\mathcal{D}_{\Lambda_\ell}(h;\mu)$. Recalling (4.20), for any $\ell > s^2$ we finally get

$$V_\ell(g;\mu) \le c\,s^{d+2}\mathrm{Av}_{|x|\le\ell_1}\mu\big(\mathrm{Var}_\mu(\tau_x g\,|\,\mathcal{F}_x)\big). \tag{4.22}$$

In particular

$$V_\ell(g;\mu) \le c\, s^{d+2} \|g\|_\infty^2. \tag{4.23}$$

In order to benefit of the ergodicity of the random field, it is natural to define, for any $m \in (0,1)$ and any $g \in \mathcal{G}$,

$$V_m(g) := \lim_{\ell\uparrow\infty} (2\ell)^{-d}\, \mathbb{E}\Big[\mu^{\lambda_0(m)}\big(\sum_{|x|\le \ell_1} \tau_x g, (-\mathcal{L}_{\Lambda_\ell})^{-1} \sum_{|x|\le \ell_1} \tau_x g\big)\Big] \tag{4.24}$$

where, we recall, $\lambda_0(m)$ is the annealed chemical potential corresponding to the particle density m. If $m = 0, 1$ we simply set $V_m(g) = 0$ for any $g \in \mathcal{G}$. In section 7 we will prove, among other results, that the limit appearing in (4.24) exists finite and that it defines a semi–inner product on $\mathcal{G}$ (see theorem 7.2 there). With this definition we have the following result.

Lemma 4.6. *Let $g \in \mathcal{G}$. Then*

$$\limsup_{\ell\uparrow\infty,\, \epsilon\downarrow 0} \mathrm{Av}_{|x|\le\frac{1}{\epsilon}} \sup_{\nu\in\mathcal{M}(\Lambda_{x,\ell})} V_{x,\ell}\big(g;\nu\big) \le \sup_{m\in[0,1]} V_m(g). \tag{4.25}$$

Proof. As in [23], chapter 7, lemma 4.3, we introduce a scale parameter k, with $k \uparrow \infty$ after $\ell \uparrow \infty$, and partition Λ_ℓ in cubes of side $2k+1$. More precisely, we define $\Lambda_\ell^{(k)} := \Lambda_\ell \cap (2k+1)\mathbb{Z}^d$ and write $\Lambda_\ell = B_{k,l} \cup \big(\cup_{z\in\Lambda_\ell^{(k)}} \Lambda_{z,k}\big)$ where $B_{k,\ell} := \Lambda_\ell \setminus \cup_{z\in\Lambda_\ell^{(k)}} \Lambda_{z,k}$. Then, by proceeding as in [23] and by using the variational characterization (4.20) together with the integration by parts formula (4.16), for any $\nu \in \mathcal{M}(\Lambda_\ell)$ we get

$$V_\ell(g;\nu) \le (2\ell)^{-d} \sup_{\underline{h}} \big\{ \sum_{z\in\Lambda_\ell^{(k)}} F_z(h_z;\nu) \big\} + c(g)\sqrt{k\ell^{-1} + k^{-\frac{1}{2}}}\, \big\} \tag{4.26}$$

where $c(g)$ is as in (4.17), $F_z(h_z;\nu) := 2\sum_{y\in\Lambda_{z,k_1}} \nu(\tau_y g, h_z) - \mathcal{D}_{\Lambda_{z,k}}(h_z;\nu)$ and the supremum $\sup_{\underline{h}}$ is taken over all families $\underline{h} = \{h_z\}_{z\in\Lambda_\ell^{(k)}}$ such that h_z depends only on $\eta_{\Lambda_{z,k}}$ and $\mathcal{D}(h_z;\nu) \le c(g)k^d$.

Actually it is simple to check that in (4.26) we can restrict the supremum to families $\underline{h}$ that satisfy the extra condition $\|h\|_\infty \le c(g)c_k$ for some constant c_k depending on k.

Therefore, if m is the particle density associated to the canonical measure ν and thanks to the equivalence of ensembles (see lemmas A.4 and A.3), for any disorder configuration α we get

$$\begin{aligned}
&\big|\sum_{z\in\Lambda_\ell^{(k)}} F_z(h_z;\nu) - \sum_{z\in\Lambda_\ell^{(k)}} F_z(h_z;\mu_{\Lambda_\ell}^{\lambda(m)})\big| \le c(g)c_k,\\
&\big|\sum_{z\in\Lambda_\ell^{(k)}} F_z(h_z;\mu_{\Lambda_\ell}^{\lambda(m)}) - \sum_{z\in\Lambda_\ell^{(k)}} F_z(h_z;\mu^{\lambda_0(m)})\big| \le c(g)c_k\, \ell^d \,\big|m - \mu^{\lambda_0(m)}(m_{\Lambda_\ell})\big|.
\end{aligned}$$

Thanks to the previous observations we finally obtain

$$\mathrm{Av}_{|x|\le\frac{1}{\epsilon}} \sup_{\nu\in\mathcal{M}(\Lambda_{x,\ell})} V_{x,\ell}\big(g,\nu\big) \le c\sqrt{k\ell^{-1} + k^{-\frac{1}{2}}} + c_k\ell^{-d} + c_k\Theta_1 + c_{k,\ell}\Theta_2$$

where $c_{k,\ell}$ is a positive constant depending on k,ℓ such that $\lim_{k\uparrow\infty,\ell\uparrow\infty} c_{k,\ell}=1$ and

$$\Theta_1 := \mathrm{Av}_{|x|\le\frac{1}{\epsilon}} \sup_{m\in[0,1]} \big| m-\mu^{\lambda_0(m)}(m_{\Lambda_{x,\ell}})\big|,$$

$$\Theta_2 := \mathrm{Av}_{|x|\le\frac{1}{\epsilon}} \sup_{m\in[0,1]} \tau_x\Big(\sup_{\underline{h}}\big\{(2k)^{-d}\mathrm{Av}_{z\in\Lambda_\ell^{(k)}} F_z(h_z,\mu^{\lambda_0(m)})\big\}\Big),$$

and $\sup_{\underline{h}}$ is as before.

It is clear that, by considering a fixed density m in the definition of Θ_1 and Θ_2, for almost all disorder configurations α, Θ_1 is negligible as $\ell\uparrow\infty,\epsilon\downarrow 0$. Moreover, because of the ergodicity of the random field α and of the variational characterization (4.20), it is also clear that for almost all disorder configurations α

$$\limsup_{\ell\uparrow\infty,\epsilon\downarrow 0}\Theta_2 \le \mathbb{E}\big(V_k(g;\mu^{\lambda_0(m)})\big)$$

To handle the supremum over $m\in[0,1]$ requires some simple additional observations based on compactness of $[0,1]$ and lemma A.3 (see e.g section 1.13 in [15]). □

4.2. **Back to the proof of proposition 4.1.** Given the technical tools developed in the previous paragraph, let us now complete the proof of proposition 4.1 modulo some non trivial results to be proved later on.

The basic idea would be to benefit of the ergodicity of the model by means of the *localization technique* discussed in subsection 4.1. Unfortunately, the function $\bar J^g_{b,a,\ell,\epsilon}$ appearing in (4.9) cannot be written as $\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon} f_x$ (or as a more complex spatial average) for suitable functions f_x having support independent of ϵ. We will need some subtle techniques developed for non-gradient systems in order to approximate $\bar J^g_{b,a,\ell,\epsilon}$ with such a spatial average. There is however one piece of $\bar J^g_{b,a,\ell,\epsilon}$, namely the density "gradient" $\big(2b/\epsilon\big)^{-1}\big[m_{x+\frac{b}{\epsilon}e',\frac{a}{\epsilon}} - m_{x-\frac{b}{\epsilon}e',\frac{a}{\epsilon}}\big]$ which can be conveniently written as a suitable spatial average. To this aim recall the definition (2.2) of the spatial average $\mathrm{Av}^{\ell,s}_{z,y}$ and define for any particle configuration η, $m_\ell^{1,e}$, $m_\ell^{2,e}$ and m_ℓ^e to be the particle density associated to η in the sets $\Lambda_\ell^{1,e},\Lambda_\ell^{1,e}$ and Λ_ℓ^e defined in (2.1) respectively. It is then simple to check the following identity (which motivates the introduction of $\mathrm{Av}^{\ell,s}_{z,y}$):

$$\mathrm{Av}^{\ell,s}_{z,y}\tau_z\frac{m_\ell^{2,e}-m_\ell^{1,e}}{\ell} = \tau_y\frac{m_s^{2,e}-m_s^{1,e}}{s}. \tag{4.27}$$

Let now $n,\frac{a}{\epsilon},\frac{b}{\epsilon}$ be odd integers such that $\frac{a}{n\epsilon}\in\mathbb{N}$ and $\frac{b}{a}\in\mathbb{N}$. Then, it is simple to check that

$$\mathrm{Av}_{u=0}^{\frac{2b}{a}-1}\tau_{x_u}\frac{m^{2,e}_{\frac{a}{\epsilon}}-m^{1,e}_{\frac{a}{\epsilon}}}{a/\epsilon} = \frac{m_{x+\frac{b}{\epsilon}e,\frac{a}{\epsilon}}-m_{x-\frac{b}{\epsilon}e,\frac{a}{\epsilon}}}{2b/\epsilon} \tag{4.28}$$

where

$$x_u := x+\Big(u\frac{a}{\varepsilon}-\frac{b}{\epsilon}+\frac{1}{2}\big(\frac{a}{\epsilon}-1\big)+1\Big)e.$$

Therefore, if we define

$$\mathrm{Av}^*_{z,x}f_z := \mathrm{Av}_{u=0}^{\frac{2b}{a}-1}\mathrm{Av}^{n,\frac{a}{\epsilon}}_{z,x_u}f_z \tag{4.29}$$

(when necessary we will also add the versor $e \in \mathcal{E}$ into the notation by writing $\mathrm{Av}^{*,e}_{z,x}$), thanks to (4.27) and (4.28) we obtain:

$$\mathrm{Av}^*_{z,x} \frac{m_n^{2,e} - m_n^{1,e}}{n} = \frac{m_{x+\frac{b}{\epsilon}e,\frac{a}{\epsilon}} - m_{x-\frac{b}{\epsilon}e,\frac{a}{\epsilon}}}{2b/\epsilon} . \tag{4.30}$$

If the above conditions on $n, \frac{a}{\epsilon}, \frac{b}{\epsilon}$ are not satisfied, we extend the definition of $\mathrm{Av}^*_{z,x}$ by replacing in (4.29) $\frac{a}{\epsilon}$, $\frac{b}{\epsilon}$, $\frac{2b}{a}$ with r_1, r_2 and $\frac{2r_2}{r_1}$ respectively, where r_1 is the smallest odd number in $n\mathbb{Z}$ such that $\frac{a}{\epsilon} \le r_1$ and r_2 is the smallest odd number in $r_1\mathbb{Z}$ such that $\frac{b}{\epsilon} \le r_2$.

Warning. In the sequel, for the sake of simplicity we will always assume $n, \frac{a}{\epsilon}, \frac{b}{\epsilon}$ to be odd integers such that $\frac{a}{n\epsilon} \in \mathbb{N}$ and $\frac{b}{a} \in \mathbb{N}$. The way to treat the general case is shortly discussed right after section 4.5.

It is convenient to introduce also $\mathrm{Av}^\star_{z,x}$ defined as the dual average of $\mathrm{Av}^*_{z,x}$, *i.e.*

$$\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\Big(f_x \left(\mathrm{Av}^*_{z,x} g_z \right) \Big) = \mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\Big(g_x \left(\mathrm{Av}^\star_{z,x} f_z \right) \Big). \tag{4.31}$$

The explicit formula of $\mathrm{Av}^\star_{z,x} f_z$ can be easily computed and it is similar to the formula of $\mathrm{Av}^*_{z,x} f_z$.

We introduce at this point some special functions related to the gradient of the density field. Given two integers $0 \le n \le s$, $e \in \mathcal{E}$ and a grand canonical measure μ on an arbitrary set Λ containing Λ^e_s, we write

$$m_n^{2,e} - m_n^{1,e} = \psi^e_{n,s} + \phi^e_{n,s} , \quad \text{with} \quad \phi^e_{n,s} := \mu\big[m_n^{2,e} - m_n^{1,e} \,|\, \mathcal{F}^e_s \big] , \tag{4.32}$$

where $\mathcal{F}^e_s$ is the σ–algebra generated by m^e_s. Notice that, in absence of disorder, the function $\phi^e_{n,s}$ would be identically equal to zero and that $\psi^e_{n,s} \in \mathcal{G}$ for all $n < s$, since $\nu(\psi^e_{n,s}) = 0$ for all $\nu \in \mathcal{M}(\Lambda)$ and all Λ containing Λ^e_s. Thanks to (4.22) with $\Delta := \Lambda^e_n$ and $s := n$ and thanks to the equivalence of ensembles (see lemma A.5), given $\ell \ge n^2$ it is easy to check that

$$V_\ell\big(\frac{\psi^e_{n,n}}{n}; \nu \big) \le c \quad \forall \nu \in \mathcal{M}(\Lambda_\ell), \qquad V_\ell\big(\frac{\psi^e_{n,n}}{n}; \mu^{\lambda_0(m)} \big) \le c\, m(1-m). \tag{4.33}$$

Using decomposition (4.32) we can now write $\bar{J}^g_{b,a,\ell,\epsilon}$ as

$$\bar{J}^g_{b,a,\ell,\epsilon} = \sum_{j=0}^{5} \mathrm{Av}_{x\in\mathbb{T}^d_\epsilon} J(\epsilon x)\, \psi^{(j)}_x$$

where (we omit in the notation the suffix b, a, ℓ, ϵ, g)

$$\psi_x^{(0)} := \mathrm{Av}_{y:|y-x|\le \ell_1}\Big[j_{y,y+e} + \tau_y \mathcal{L} g + \sum_{e'\in\mathcal{E}} D_{e,e'}(m_{x,\ell})\tau_y \frac{\psi^{e'}_{n,n}}{n}\Big],$$

$$\psi_x^{(1)} := \sum_{e'\in\mathcal{E}} D_{e,e'}(m_{x,\ell})\Big[\tau_x \frac{\psi^{e'}_{n,n}}{n} - \mathrm{Av}_{y:|y-x|\le\ell_1}\tau_y \frac{\psi^{e'}_{n,n}}{n}\Big]$$

$$\psi_x^{(2)} := \sum_{e'\in\mathcal{E}} \Big[D_{e,e'}(m_{x,\frac{a}{\epsilon}}) - D_{e,e'}(m_{x,\ell})\Big]\tau_x \frac{\psi^{e'}_{n,n}}{n}$$

$$\psi_x^{(3)} := \sum_{e'\in\mathcal{E}} D_{e,e'}(m_{x,\frac{a}{\epsilon}})\Big[\mathrm{Av}^{*,e'}_{z,x}\tau_z \frac{\psi^{e'}_{n,n}}{n} - \tau_x \frac{\psi^{e'}_{n,n}}{n}\Big]$$

$$\psi_x^{(4)} := \sum_{e'\in\mathcal{E}} D_{e,e'}(m_{x,\frac{a}{\epsilon}})\Big[\frac{m_{x+\frac{b}{\epsilon}e',\frac{a}{\epsilon}} - m_{x-\frac{b}{\epsilon}e',\frac{a}{\epsilon}}}{2b/\epsilon} - \mathrm{Av}^{*,e'}_{z,x}\tau_z \frac{m_n^{2,e'} - m_n^{1,e'}}{n}\Big]$$

$$\psi_x^{(5)} := \sum_{e'\in\mathcal{E}} D_{e,e'}(m_{x,\frac{a}{\epsilon}})\mathrm{Av}^{*,e'}_{z,x}\tau_z \frac{\phi^{e'}_{n,n}}{n}$$

and we define

$$\Omega_j := \sup spec_{L^2(\mu_\epsilon)}\big\{\epsilon^{-1}\mathrm{Av}_x J(\epsilon x)\psi_x^{(j)} + \gamma\epsilon^{d-2}\mathcal{L}_\epsilon\big\} \quad j = 0,\dots,5\,.$$

Then, thanks to the subadditivity of "$\sup spec$", proposition 4.1 follows from the next result.

Proposition 4.7. *Let $d \ge 3$ and $\gamma > 0$. Then, for almost any disorder configuration α,*

$$\inf_{g\in\mathbb{G}} \limsup_{n\uparrow\infty,\ell\uparrow\infty,\epsilon\downarrow 0} \sup_J \Omega_0 \le 0 \tag{4.34}$$

and, for any $j = 1,\dots,5$,

$$\limsup_{n\uparrow\infty,b\downarrow 0,a\downarrow 0,\ell\uparrow\infty,\epsilon\downarrow 0} \sup_J \Omega_j \le 0 \tag{4.35}$$

where J varies in $\{J \in C(\mathbb{T}^d) \,:\, \|J\|_\infty \le 1\}$.

The proof of proposition 4.7 is best divided into several pieces according to the value of the index j.

4.3. **The term Ω_0.** Let us first prove (4.34). By localizing on cubes of side $2\ell+1$ (see (4.12)) and using the regularity of $J(\cdot)$, it is enough to prove that for almost any disorder configuration α,

$$\inf_{g\in\mathbb{G}} \limsup_{n\uparrow\infty,\ell\uparrow\infty,\epsilon\downarrow 0} \mathrm{Av}_{x\in\mathbb{T}^d_\epsilon} \sup_{|\beta|\le 1} \sup_m \sup_{L^2(\nu_{\Lambda_{x,\ell},m})} spec \,\big\{\epsilon^{-1}\beta\,\mathrm{Av}_{y:|y-x|\le\ell_1}\tau_y\psi_m^{(n,g)} + c\ell^{-d}\epsilon^{-2}\mathcal{L}_{\Lambda_{x,\ell}}\big\} \le 0 \tag{4.36}$$

where

$$\psi_m^{(n,g)} := j_{0,e} + \mathcal{L} g + \sum_{e'\in\mathcal{E}} D_{e,e'}(m)\frac{\psi^{e'}_{n,n}}{n}.$$

Since $\epsilon \downarrow 0$ before $\ell \uparrow \infty$ and since for any ℓ large enough, any $|y-x| \le \ell_1$ and any $\nu \in \mathcal{M}(\Lambda_{x,\ell})$, $\nu\big(\tau_y\,\psi_m^{(n,g)}\big) = 0$, we can appeal to perturbation theory (see proposition 4.2) and conclude that it is enough to show that

$$\inf_{g\in\mathbb{G}} \lim_{n\uparrow\infty,\ell\uparrow\infty,\epsilon\downarrow 0} \mathrm{Av}_{x\in\mathbb{T}^d_\epsilon} \sup_{m\in[0,1]} V_{\Lambda_{x,\ell}}\big(\psi^{(n,g)}_m, \nu_{\Lambda_{x,\ell},m}\big) = 0 \tag{4.37}$$

where $V_{x,\ell}$ has been defined right after (4.19). A minor modification of the proof of lemma 4.6 shows that (4.37) follows from

$$\inf_{g\in\mathbb{G}} \limsup_{n\uparrow\infty} \sup_{m\in[0,1]} V_m\big(\psi^{(n,g)}_m\big) = 0 \qquad \forall d \geq 3 \tag{4.38}$$

(see (4.24) for the definition of V_m) which, in turn, follows from theorem 7.22.

4.4. **The three terms $\mathbf{\Omega_1}$, $\mathbf{\Omega_2}$, $\mathbf{\Omega_3}$.** Let us prove (4.35) for $j = 1, 2, 3$. In what follows, by means of proposition 4.5, we will reduce the eigenvalues estimate Ω_1, Ω_2 and Ω_3 to the Two Blocks estimate (see subsection A.7). To this aim, by integrating by parts, we can write

$$\epsilon^{-1}\mathrm{Av}_x J(\epsilon x)\psi^{(j)}_x = \sum_{e'\in\mathcal{E}} \epsilon^{-1}\mathrm{Av}_x \tau_x \frac{\psi^{e'}_{n,n}}{n} \cdot B^{(j)}_x \quad \forall j = 1, 2, 3$$

where

$$\begin{aligned}
B^{(1)}_x &:= J(\epsilon x) D_{e,e'}(m_{x,\ell}) - \mathrm{Av}_{y:|y-x|\leq \ell_1} J(\epsilon y) D_{e,e'}(m_{y,\ell})\\
B^{(2)}_x &:= J(\epsilon x)\big[D_{e,e'}(m_{x,\frac{a}{\epsilon}}) - D_{e,e'}(m_{x,\ell})\big]\\
B^{(3)}_x &:= \mathrm{Av}^{\star,e'}_{z,x} J(\epsilon z) D_{e,e'}\big(m_{z,\frac{a}{\epsilon}}\big) - J(\epsilon x) D_{e,e'}\big(m_{x,\frac{a}{\epsilon}}\big).
\end{aligned}$$

Notice that, for any $b \subset \Lambda_{x,n}$,

$$\nabla_b B^{(1)}_x = \nabla_b B^{(2)}_x = 0, \quad |\nabla_b B^{(3)}_x| \leq c\, n\frac{\epsilon}{a}\mathrm{Osc}\big(D, c\frac{\epsilon^d}{a^d}\big).$$

Therefore, using proposition 4.5, it is enough to prove that for almost any disorder configuration α, given $\gamma > 0$,

$$\limsup_{b\downarrow 0, a\downarrow 0, \ell\uparrow\infty,\epsilon\downarrow 0} \sup_J \sup spec_{L^2(\mu_\epsilon)}\big\{\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\big(B^{(j)}_x\big)^2 + \frac{1}{2}\gamma\epsilon^{d-2}\mathcal{L}_\epsilon\big\} = 0 \qquad \forall j = 1, 2, 3. \tag{4.39}$$

Since D can be approximated by Lipschitz functions and J is smooth, (4.39) can be derived from the Two Blocks estimate (see subsection A.7). For simplicity of notation, let us consider the case $j = 2$ (the case $j = 1$ is simpler, while $j = 3$ is a slight variation) and D Lipschitz continuous. Since $\big(B^{(2)}_x\big)^2 \leq c\,\big|\, m_{x,\ell} - m_{x,\frac{a}{\epsilon}}\,\big|$, by introducing a scale parameter k such that $k \uparrow \infty$ after $a \downarrow 0, \ell \uparrow \infty$ and $\epsilon \downarrow 0$, we can estimate

$$\big(B^{(2)}_x\big)^2 \leq c\,\mathrm{Av}_{|y|\leq\ell}\mathrm{Av}_{|z|\leq\frac{a}{\epsilon}}|m_{x+y,k} - m_{x+z,k}| + c\,\frac{k}{\ell} + c\frac{k}{a/\epsilon}.$$

At this point, by the subadditivity (4.11) of "$\sup spec$", the thesis follows from the Two Blocks estimate.

4.5. **The term $\mathbf{\Omega_4}$.** The proof of (4.35) for $j = 4$ is based on the Two Blocks estimate. Notice that, thanks to (4.30), the function $\psi^{(4)}_x$ entering in the definition of Ω_4 is either identically equal to zero if $n, \frac{a}{\epsilon}, \frac{b}{\epsilon}$ are odd integers such that $\frac{a}{n\epsilon} \in \mathbb{N}$ and $\frac{b}{a} \in \mathbb{N}$, or it can be written as

$$\psi^{(4)}_x = \sum_{e'\in\mathcal{E}} D_{e,e'}(m_{Q_{x,\frac{a}{\epsilon}}})\Big[\frac{m_{x+\frac{b}{\epsilon}e',\frac{a}{\epsilon}} - m_{x-\frac{b}{\epsilon}e',\frac{a}{\epsilon}}}{2b/\epsilon} - \frac{m_{x+r_2e',r_1} - m_{x-r_2e',r_1}}{2r_2}\Big] \tag{4.40}$$

where r_1, r_2 have been defined in subsection 4.2. By the Two Blocks estimate it is simple to check that for any $\gamma > 0$ and for almost any disorder configuration α

$$\lim_{a\downarrow 0,\epsilon\downarrow 0} \sup spec_{L^2(\mu_\epsilon)}\big\{\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\big|m_{x,\frac{a}{\epsilon}} - m_{x,r_1}\big| + \gamma\epsilon^{d-2}\mathcal{L}_\epsilon\big\} = 0 \tag{4.41}$$

$$\lim_{a\downarrow 0,\epsilon\downarrow 0} \sup_{|w|\le 2\frac{a}{\epsilon}} \sup spec_{L^2(\mu_\epsilon)}\big\{\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\big|m_{x,\frac{a}{\epsilon}} - m_{x+w,\frac{a}{\epsilon}}\big| + \gamma\epsilon^{d-2}\mathcal{L}_\epsilon\big\} = 0 \tag{4.42}$$

(hint: introduce the scale parameter k with $a \downarrow 0, k \uparrow \infty, \epsilon \downarrow 0$ and write $m_{x,s} = \mathrm{Av}_{y\in\Lambda_{x,s}} m_{y,k} + O(k/s)$ for $s = \frac{a}{\epsilon}, r_1$).
In (4.40) we can substitute r_1 by $\frac{a}{\epsilon}$ (thanks to (4.41)) and after that in the numerators we can substitute r_2 by $\frac{b}{\epsilon}$ (thanks to (4.42)). In order to conclude is enough to observe that $\epsilon^{-1}\big|\frac{1}{b/\epsilon} - \frac{1}{r_2}\big| \le c\,\frac{a}{b^2}$ which goes to 0.

4.6. **The term Ω_5.** The proof of (4.35) for $j = 5$ is based on the key results of section 5 and it is one place where the restriction on the dimension $d \ge 3$ is crucial for us. We refer the reader to the beginning of section 5 for an heuristic justification of the above condition. Here it is enough to say that the main contribution to the term Ω_5 comes from the fluctuations in the density field induced by the fluctuations of the *disorder field*.

By the subadditivity of "$\sup spec$" we only need to prove that for almost all α, given $e, e' \in \mathcal{E}$ and $\gamma > 0$,

$$\limsup_{n\uparrow\infty,\, b\downarrow 0,\, a\downarrow 0,\, \epsilon\downarrow 0} \sup_J \sup spec_{L^2(\mu_\epsilon)}\big\{\epsilon^{-1}\mathrm{Av}_x J(\epsilon x) D_{e,e'}(m_{x,\frac{a}{\epsilon}})\mathrm{Av}^{*,e'}_{z,x}\tau_z\frac{\phi^{e'}_{n,n}}{n} + \gamma\epsilon^{d-2}\mathcal{L}_\epsilon\big\} \le 0. \tag{4.43}$$

Recall the definition of $\mathrm{Av}^{*,e'}_{z,x}$ and x_u given in (4.29). Then, thanks again to the subadditivity of "$\sup spec$", the "$\sup spec$" in the l.h.s. of (4.43) is bounded from above by

$$\mathrm{Av}_{u=0}^{\frac{2b}{a}-1} \sup spec_{L^2(\mu_\epsilon)}\big\{\epsilon^{-1}\mathrm{Av}_x J(\epsilon x) D_{e,e'}(m_{x,\frac{a}{\epsilon}})\mathrm{Av}^{n,\frac{a}{\epsilon}}_{z,x_u}\tau_z\frac{\phi^{e'}_{n,n}}{n} + \gamma\epsilon^{d-2}\mathcal{L}_\epsilon\big\}. \tag{4.44}$$

Observe that $\mathrm{Av}^{n,\frac{a}{\epsilon}}_{z,x_u}\tau_z\frac{\phi^{e'}_{n,n}}{n}$ has support inside $\Lambda^{e'}_{x_u,\frac{a}{\epsilon}}$. We would like at this point to localize on boxes of side length of order $O\big(\frac{a}{\epsilon}\big)$ in such a way that $D_{e,e'}(m_{x,\frac{a}{\epsilon}})$ becomes a constant. To this aim, given $u \in \{0,\dots,\frac{2b}{a}-1\}$ and $x \in \mathbb{T}^d_\epsilon$, we set

$$\Delta_{x,u} := \begin{cases} Q_{x,10\frac{a}{\epsilon}} & \text{if } Q_{x,\frac{a}{\epsilon}} \cap \Lambda_{x_u,2\frac{a}{\epsilon}} \ne \emptyset \\ Q_{x,\frac{a}{\epsilon}} & \text{otherwise} \end{cases}.$$

and we observe that either $\Delta_{x,u}$ is disjoint from or completely contains $\Lambda_{x_u,2\frac{a}{\epsilon}}$. Therefore, if in (4.44) we could replace the term $D_{e,e'}(m_{x,\frac{a}{\epsilon}})$ by the new term $D_{e,e'}(m_{\Delta_{x,u}})$, then it would be simple to check (by localizing on boxes $\Lambda_{x_u,2\frac{a}{\epsilon}}$) that all what is needed is that for $d \ge 3$, for all $T \in \mathbb{N}$ and for almost all α,

$$\limsup_{n\uparrow\infty,a\downarrow 0,\epsilon\downarrow 0} \mathrm{Av}_{x\in\mathbb{T}^d_\epsilon} \sup_{|\beta|\le T} \sup_{\nu\in\mathcal{M}(\Lambda_{x,2\frac{a}{\epsilon}})} \sup spec_{L^2(\nu)}\big\{\epsilon^{-1}\beta\mathrm{Av}^{n,\frac{a}{\epsilon}}_{z,x}\tau_z\frac{\phi^{e'}_{n,n}}{n} + \epsilon^{-2}\mathrm{Av}_{b\in\Lambda_{x,2a/\epsilon}}\mathcal{L}_b\big\} \le 0 \tag{4.45}$$

Section 5 is devoted to the proof of (4.45) (see theorem 5.3 there).
Therefore, it remains to prove that for $d \geq 3$, for almost all α and for any $\gamma > 0$

$$\limsup_{n\uparrow\infty, b\downarrow 0, a\downarrow 0, \epsilon\downarrow 0} \sup_J \mathrm{Av}_{u=0}^{\frac{2b}{a}-1} \sup spec_{L^2(\mu_\epsilon)} \big\{ \epsilon^{-1} \mathrm{Av}_{x\in\mathbb{T}^d_\epsilon} J(\epsilon x) \times \big[D_{e,e'}(m_{x,\frac{a}{\epsilon}}) - D_{e,e'}(m_{\Delta_{x,u}}) \big] \mathrm{Av}_{z,x_u}^{n,\frac{a}{\epsilon}} \tau_z \frac{\phi^{e'}_{n,n}}{n} + \gamma \epsilon^{d-2} \mathcal{L}_\epsilon \big\} \leq 0. \tag{4.46}$$

Notice that the only values of u which contribute to the $\mathrm{Av}_{u=0}^{\frac{2b}{a}-1}$ above, in what follows called "bad values", are those for which $Q_{x,\frac{a}{\epsilon}} \neq \Delta_{x,u}$ for some $x \in \mathbb{T}^d_\epsilon$. It is easy to check that the cardinality of the bad values of u is of order $O(1)$ for any fixed $x \in \mathbb{T}^d_\epsilon$. Thus we only need to bound the "$\sup spec$" appearing in (4.46) by $o(\frac{b}{a})$, uniformly in u in the bad set. Thanks to (4.30) and (4.32) we can write

$$\mathrm{Av}_{z,x_u}^{n,\frac{a}{\epsilon}} \tau_z \frac{\phi^{e'}_{n,n}}{n} = \frac{m_{x_u+\frac{b}{\epsilon},\frac{a}{\epsilon}} - m_{x_u-\frac{b}{\epsilon},\frac{a}{\epsilon}}}{2b/\epsilon} - \mathrm{Av}_{z,x_u}^{n,\frac{a}{\epsilon}} \tau_z \frac{\psi^{e'}_{n,n}}{n} \tag{4.47}$$

Then, the contribution in (4.46) coming from the first addendum in the r.h.s. of (4.47) is not larger than $O(\frac{1}{b})$ and therefore negligible.
Let us consider the contribution of the second addendum. An integration by parts shows that

$$\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon} J(\epsilon x) \big(D_{e,e'}(m_{x,\frac{a}{\epsilon}}) - D_{e,e'}(m_{\Delta_{x,u}}) \big) \mathrm{Av}_{z,x_u}^{n,\frac{a}{\epsilon}} \tau_z \frac{\psi^{e'}_{n,n}}{n} = \mathrm{Av}_{x\in\mathbb{T}^d_\epsilon} \tau_x \frac{\psi^{e'}_{n,n}}{n} B_{x,u}$$

where the functions $B_{x,u}$ satisfy $\|B_{x,u}\| \leq c$ together with

$$|\nabla_b B_{x,u}| \leq c \frac{n\epsilon}{a} Osc(D_{e,e'}, c\frac{\epsilon^d}{a^d}) \quad \forall b \in \Lambda^{e'}_{x,n}.$$

Moreover, $B_{x,u}$ is a particular spatial average (dual to $\mathrm{Av}_{z,x_u}^{n,\frac{a}{\epsilon}}$) of $J(\epsilon z)\big(D_{e,e'}(m_{z,\frac{a}{\epsilon}}) - D_{e,e'}(m_{\Delta_{z,u}})\big)$. Therefore, by proposition 4.5 and the Two Blocks estimate (see subsection A.7), the contribution of the second addendum is also negligible (see also the discussion at the end of subsection 4.4).

4.7. **Proof of the energy estimate.** In this subsection we prove lemma 3.1. It is simple to check that

$$\text{spatial–time average in } (3.4) = \sup_{H\in\mathcal{H}_b} H^\star_{b,a,\epsilon} \tag{4.48}$$

where $\mathcal{H}_b := \{H \text{ smooth on } [0,T]\times\mathbb{T}^d,\ \|H\|_\infty \leq \frac{1}{b}\}$ and

$$H^\star_{b,a,\epsilon} := \mathrm{Av}_{x\in\mathbb{T}^d_\epsilon} \int_0^T \Big(2H(s,\epsilon x) \Big[\frac{m_{x+\frac{b}{\epsilon}e,\frac{a}{\epsilon}}(s) - m_{x,\frac{a}{\epsilon}}(s)}{b} \Big] - H(s,\epsilon x)^2 \Big) ds.$$

In what follows let H belong to $\mathcal{H}_b$. By the entropy inequality and the Feynman-Kac formula (see (4.6) and (4.7)), for any $\gamma > 0$,

$$\mathbb{E}^{\mu^\epsilon}\big(H^\star_{2b,a,\epsilon}\big) \leq \frac{\kappa}{\gamma} - \gamma \mathrm{Av}_{x\in\mathbb{T}^d_\epsilon} \int_0^T ds H(s,\epsilon x)^2 + \int_0^T ds \sup_{L^2(\mu_\epsilon)} spec \big\{ \epsilon^{-1} \gamma \mathrm{Av}_{x\in\mathbb{T}^d_\epsilon} 2H(s,\epsilon x) \Big[\frac{m_{x+\frac{2b}{\epsilon}e,\frac{a}{\epsilon}}(s) - m_{x,\frac{a}{\epsilon}}(s)}{2b/\epsilon} \Big] + \epsilon^{d-2}\mathcal{L}_\epsilon \big\}. \tag{4.49}$$

It is convenient to introduce a free scale parameter n, with $n \uparrow \infty$ after $a \downarrow 0$ and $\epsilon \downarrow 0$, and write the gradient of masses appearing in (4.49) as $\mathrm{Av}^*_{z,x}\tau_z\big(\frac{\psi^e_{n,n}}{n}+\frac{\phi^e_{n,n}}{n}\big)$ (see (4.30) and (4.32)).
By the definition of $\mathrm{Av}^*_{z,x}$, the subadditivity of $\sup spec$ and theorem 5.3,

$$\limsup_{n\uparrow\infty,a\downarrow 0,\epsilon\downarrow 0}\int_0^T ds \sup_{L^2(\mu_\epsilon)} spec\big\{\epsilon^{-1}\gamma\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}2H(s,\epsilon x)\mathrm{Av}^*_{z,x}\tau_z\frac{\phi^e_{n,n}}{n}+\epsilon^{d-2}\mathcal{L}_\epsilon\big\}\le 0.$$

Let us consider, for fixed b,n,a,

$$\sup spec_{L^2(\mu_\epsilon)}\big\{\epsilon^{-1}\gamma\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}2H(s,\epsilon x)\mathrm{Av}^*_{z,x}\tau_z\frac{\psi^e_{n,n}}{n}+\epsilon^{d-2}\mathcal{L}_\epsilon\big\}. \tag{4.50}$$

Thanks to the definition of the dual average $\mathrm{Av}^\star_{z,x}$ we can write

$$\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}2H(s,\epsilon x)\mathrm{Av}^*_{z,x}\tau_z\frac{\psi^e_{n,n}}{n}=\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}a_x\tau_x\frac{\psi^e_{n,n}}{n}$$

where $a_x := \mathrm{Av}^\star_{z,x}2H(s,\epsilon z)$. Since $\mathrm{Av}^\star_{z,x}$ is translationally invariant w.r.t. x and H is regular, we can proceed as at the very beginning of this section and safely replace $\tau_x\frac{\psi^e_{n,n}}{n}$ by a local average $\mathrm{Av}_{|y-x|\le\ell_1}\tau_y\frac{\psi^e_{n,n}}{n}$, $\ell\gg n$, to get

$$(4.50)\le\sup spec_{L^2(\mu_\epsilon)}\big\{\epsilon^{-1}\gamma\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}a_x\mathrm{Av}_{|y-x|\le\ell_1}\tau_y\frac{\psi^e_{n,n}}{n}+\epsilon^{d-2}\mathcal{L}_\epsilon\big\}+c(H)\gamma\epsilon\ell^2. \tag{4.51}$$

By the usual trick of localizing on boxes $\Lambda_{x,\ell}$ and proposition 4.2, if ϵ is small enough then the first term in the r.h.s. of (4.51) is bounded from above by

$$c\,\gamma^2\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}a_x^2\sup_{\nu\in\mathcal{M}(\Lambda_{x,l})}V_{x,\ell}\big(\frac{\psi^{e'}_{n,n}}{n};\nu\big)$$

which in turn, thanks to (4.33), is bounded from above by

$$c\,\gamma^2\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}a_x^2\le c^*\,\gamma^2\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}H(s,\epsilon x)^2$$

for some suitable positive constant c^*. Let us now choose γ so small that $c^*\gamma^2-\gamma<0$. Then, by the previous estimates, if ϵ is small enough,

$$\limsup_{n\uparrow\infty,a\downarrow 0,\epsilon\downarrow 0}\ \text{r.h.s. of (4.49)}\le\frac{\kappa}{\gamma}+(c^*\gamma^2-\gamma)\int_0^T\int_{\mathbb{T}^d}H(s,\theta)^2d\theta\,ds\le\frac{\kappa}{\gamma}. \tag{4.52}$$

In order to conclude the proof it is enough to observe that there exists a finite set $\mathcal{H}^*_b\subset\mathcal{H}_b$ depending on b such that

$$\sup_{H\in\mathcal{H}_b}H^\star_{b,a,\epsilon}\le 1+\sup_{H\in\mathcal{H}^*_b}H^\star_{b,a,\epsilon}$$

so that

$$\limsup_{n\uparrow\infty,a\downarrow 0,\epsilon\downarrow 0}\mathbb{E}^{\mu^\epsilon}\big(\sup_{H\in\mathcal{H}_b}H^\star_{b,a,\epsilon}\big)\le 1+\limsup_{n\uparrow\infty,a\downarrow 0,\epsilon\downarrow 0}\mathbb{E}^{\mu^\epsilon}\big(\sup_{H\in\mathcal{H}^*_b}H^\star_{b,a,\epsilon}\big)\le 1+\frac{\kappa}{\gamma} \tag{4.53}$$

thus allowing to conclude the proof of (3.4).
Let us now sketch the proof of (3.5). Since $C^1([0,T]\times\mathbb{T}^d)$ has a countable base, by Beppo–Levi theorem it is enough to prove that there exists a constant c_0 such that, given $H_1,\dots,H_n$ in $C^1([0,T]\times\mathbb{T}^d)$, then

$$\int dQ(m)\Big[\sup_{i=1,\dots,n}\int_0^T\int_{\mathbb{T}^d}\big(2\,m(s,\theta)\frac{\partial}{\partial\theta_e}H_i(s,\theta)-H_i(s,\theta)^2\big)d\theta\,ds\Big]\le c_0. \tag{4.54}$$

By the Lebesgue density theorem and the dominated convergence theorem, the l.h.s. of (4.54) is equal to $\lim_{a\downarrow 0}\mathbb{E}_Q\left(\Theta^{(a)}\right)$ where, for any $\nu\in D([0,T],\mathcal{M}_1)$,

$$\Theta^{(a)}(\nu):=\sup_{i=1,\dots,n}\int_0^T\int_{\mathbb{T}^d}\big(2\,\nu^{(a)}(s,\theta)\frac{\partial}{\partial\theta_e}H_i(s,\theta)-H_i^2(s,\theta)\big)d\theta\,ds, \tag{4.55}$$

with

$$\nu^{(a)}(s,\theta):=\frac{1}{(2a)^d}\nu_s\big(\{\theta'\in\mathbb{T}^d:\sup_{i=1,\dots,d}|\theta'_i-\theta_i|\le a\}\big).$$

It is simple to prove (see [15], section 1.18) that

$$\lim_{a\downarrow 0}\int dQ(m)\big(\Theta^{(a)}(m)\big)\le\limsup_{a\downarrow 0,\epsilon\downarrow 0}\int dQ^{\alpha,\,\mu^\epsilon}(\nu)\big(\Theta^{(a)}(\nu)\big)=$$

$$\limsup_{b\downarrow 0,a\downarrow 0,\epsilon\downarrow 0}\mathbb{E}^{\alpha,\mu^\epsilon}\Big(\sup_{i=1,\dots,n}\int_0^T\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\big(2m_{x,\frac{a}{\epsilon}}(s,\epsilon x)\big[\frac{H_i(s,\epsilon x+be)-H_i(s,\epsilon x)}{b}\big]-H_i^2(s,\epsilon x)\big)\Big).$$

By integrating by parts and observing that

$$\sup_{H\in\mathcal{H}_b}H^\star_{b,a,\epsilon}=\sup_{H\in C^1([0,T]\times\mathbb{T}^d)}H^\star_{a,b,\epsilon},$$

the thesis follows from (4.53).

4.8. **Hydrodynamic limit without regularity of the diffusion matrix.** In this last paragraph we shortly discuss the hydrodynamic limit when the regularity condition on the diffusion matrix is replaced by the two conditions at the end of theorem 2.4, in the sequel referred to as assumptions $A(\rho)$. The main idea here is to prove that one can safely introduce a density cutoff near the edges of the interval $(0,1)$, and for this purpose the main technical tool is the following result.

Lemma 4.8. *Assume that the sequence of initial probability measures μ^ϵ satisfy $A(\rho)$. Then there exists a constant $0<\bar\rho\le\rho$ such that, for any $T>0$ and any disorder configuration α,*

$$\lim_{\ell\uparrow\infty,\epsilon\downarrow 0}\mathbb{E}^{\mu^\epsilon}\Big(\int_0^T ds\,\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\left(\mathbb{1}_{\{m_{x,\ell}(s)<\bar\rho\}}+\mathbb{1}_{\{m_{x,\ell}(s)>1-\bar\rho\}}\right)\Big)=0. \tag{4.56}$$

Proof. For simplicity, we consider in (4.56) only the contribution coming from $\mathbb{1}_{\{m_{x,\ell}(s)<\bar\rho\}}$, the other one being similar. Given two probability measures μ_1, μ_2 on Ω_ϵ, we will write $\mu_1\le\mu_2$ if $\mu_1(f)\le\mu_2(f)$ for any function f which is increasing w.r.t. the partial order in Ω_ϵ given by $\eta\le\eta'\Leftrightarrow\eta(x)\le\eta'(x)\ \forall x\in\mathbb{T}^d_\epsilon$. It is then simple to check that our model is *attractive* [26] in the sense that $\mu_1\le\mu_2$ implies that $\mathbb{P}^{\mu_1}_t\le\mathbb{P}^{\mu_2}_t$ for any $t>0$ and for any disorder configuration α. Therefore, condition $A(\rho)$ implies that there exists $\lambda<0$ such that $\mu^\lambda_\epsilon\le\mu^\epsilon$ for any ϵ. Let now $\bar\rho:=\frac12\min\big(\frac{e^{\lambda-B}}{1+e^{\lambda-B}},\rho\big)$. Then, given $\beta>0$ and thanks to attractivity, the entropy inequality (4.6) and the identity $H[\mu^\epsilon|\mu^\epsilon_*]=H[\mathbb{P}^{\mu^\epsilon}|\mathbb{P}^{\mu^\epsilon_*}]$,

$$\begin{gathered}\mathbb{E}^{\mu^\epsilon}\big(\int_0^T ds\,\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\mathbb{1}_{m_{x,\ell}(s)<\bar\rho}\big)\le\\ \frac{1}{\beta}H\left[\mu^\epsilon|\mu^\epsilon_*\right]+\frac{1}{\beta}\ln\Big(\mu^\lambda_\epsilon\big(\exp\{\int_0^T ds\,\beta\,\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\mathbb{1}_{m_{x,\ell}(s)<\bar\rho}\}\big)\Big).\end{gathered} \tag{4.57}$$

Thanks to the Jensen's inequality and the reversibility of $\mathcal{L}_\epsilon$ w.r.t. μ^λ_ϵ the second addendum in the r.h.s. of (4.57) can be bounded by

$$\frac{1}{\beta}\ln\big(\mu^\lambda_\epsilon\big(\exp\{T\,\beta\,\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\mathbb{1}_{m_{x,\ell}<\bar\rho}\}\big)\big). \tag{4.58}$$

Let us call ν^λ the product measure on Ω_ϵ such that $\nu^\lambda(\eta_x) = \frac{e^{\lambda-B}}{1+e^{\lambda-B}}$. Then, it is simple to check that $\nu^\lambda \le \mu_\epsilon^\lambda$, which implies that

$$(4.58) \le \frac{1}{\beta}\ln\big(\,\nu^\lambda\big(\exp\{T\,\beta\,\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\mathbb{I}_{m_{x,\ell}<\bar\rho}\}\big)\,\big).$$

At this point, let us recall a general result based on the Herbst's argument and the logarithmic Sobolev inequality (see [1] for a complete discussion): for any $\gamma > 0$ and any function f on Ω_ϵ

$$\nu^\lambda(e^{\gamma f}) \le e^{c_f\,\gamma^2+\gamma\nu^\lambda(f)}$$

where $c_f := c\sum_{x\in\mathbb{T}^d_\epsilon}\|\nabla_x f\|^2_\infty$ and $c = c(B,\lambda)$ is a suitable constant independent of ϵ (c is related to the logarithmic Sobolev constant of the Bernoulli measure ν^λ).
Thus

$$\frac{1}{\beta}\ln\big(\,\nu^\lambda\big(\exp\{T\,\beta\,\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\mathbb{I}_{m_{x,\ell}<\bar\rho}\}\big)\,\big) \le c\,T^2\,\beta\,\epsilon^d\ell^d + T\,\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\nu^\lambda\big(\mathbb{I}_{m_{x,\ell}<\bar\rho}\big). \tag{4.59}$$

Since $\bar\rho < \nu^\lambda(\eta_0)$, by choosing $\beta^2 := H[\mu^\epsilon|\mu^\epsilon_*]/(T^2\,\epsilon^d\ell^d)$ the r.h.s. of (4.59) is negligible as $\ell\uparrow\infty, \epsilon\downarrow 0$. Since $H[\mu^\epsilon|\mu^\epsilon_*] = o(\epsilon^{-d})$, the thesis follows by collecting all the above estimates. □

Using the above result we are in position to discuss our density cutoof. Let us recall first that, given a generic continuous extensions $\bar D$ of D outside the interval $[\rho, 1-\rho]$, any weak solution $m(t,\theta)$ of the Cauchy problem (2.7), where D has been replaced by $\bar D$ and $\rho \le m_0(\theta) \le 1-\rho$ for any $\theta\in\mathbb{T}^d$, satisfies $\rho \le m(t,\theta) \le 1-\rho$ for any $0\le t\le T$ and any $\theta\in\mathbb{T}^d$. Let $\bar D$ be defined as

$$\bar D(m) := \begin{cases} D(\bar\rho) & \text{if } 0\le m\le\bar\rho \\ D(m) & \text{if } \bar\rho\le m\le 1-\bar\rho \\ D(1-\bar\rho) & \text{if } 1-\bar\rho\le m\le 1. \end{cases}$$

Let us explain next how one should modify the proof of theorem 3.2 in order to get the same result but with D replaced by $\bar D$ in the definition of $\bar H_{b,a,\epsilon}$ (in what follows this replacement will be understood without further notice). To this aim it is convenient to introduce the following shorter notation

$$\chi_{x,\ell} := \mathbb{I}_{m_{x,\ell}<\bar\rho} + \mathbb{I}_{m_{x,\ell}>1-\bar\rho}.$$

Then, thanks to lemma 4.8, equation (4.5) can be substituted by

$$\begin{aligned}\inf_{g\in\mathbb{G}}\inf_{r\ge 0}\limsup_{b\downarrow 0,a\downarrow 0,l\uparrow\infty,\epsilon\downarrow 0}\ \mathbb{E}^{\mu^\epsilon}\Big(\Big|\int_0^T \epsilon^{-1}\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\nabla^\epsilon_e H(s,\epsilon x)\Big[\mathrm{Av}_{y:|y-x|\le\ell_1}(j_{y,y+e}+\tau_y\mathcal{L}g)\\ +\sum_{e'\in\mathcal{E}}\bar D_{e,e'}(m_{x,\frac{a}{\epsilon}})\Big[\frac{m_{x+\frac{b}{\epsilon}e',\frac{a}{\epsilon}} - m_{x-\frac{b}{\epsilon}e',\frac{a}{\epsilon}}}{2b/\epsilon}\Big]\Big] - r\int_0^T ds\,\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\chi_{x,\ell}\,ds\Big|\Big) = 0\end{aligned}$$

and the main issue is to prove proposition 4.1 with $\bar J^g_{b,a,\ell,\epsilon}$ replaced by

$$\bar J^{g,r}_{b,a,\ell,\epsilon} := \Big\{\text{r.h.s. of } (4.10) \text{ with } D\to\bar D\Big\} - \epsilon\,r\int_0^T ds\,\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\chi_{x,\ell}\,.$$

In turn the proof of the modified version of proposition 4.1 is splitted into several steps, one for each term $\Omega^{(r)}_j$, $j = 0,1,\dots 5$, where now

$$\Omega^{(r)}_0 := \sup spec_{L^2(\mu_\epsilon)}\big\{\epsilon^{-1}\mathrm{Av}_x J(\epsilon x)\psi^{(0)}_x - r\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\chi_{x,\ell} + \gamma\epsilon^{d-2}\mathcal{L}\big\}.$$

and all the other Ω_j are unchanged. It thus remains to explain how the discussion in subsection 4.3 has to be modified in order to apply to $\Omega_0^{(r)}$. Because of the new definition of Ω_0, (4.36) has to be replaced by

$$\inf_{g\in\mathbb{G}}\ \inf_{r\geq 0}\ \limsup_{n\uparrow\infty,\ell\uparrow\infty,\epsilon\downarrow 0}\ \mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\ \sup_{|\beta|\leq T}\ \sup_{m\in[0,1]}$$
$$\Big[\sup_{L^2(\nu_{\Lambda_{x,\ell},m})} spec\left\{\epsilon^{-1}\beta\,\mathrm{Av}_{y:|y-x|\leq\ell_1}\tau_y\psi_m^{(n,g)}+\epsilon^{d-2}\mathcal{L}_\epsilon\right\}-r\,\mathbb{I}_{m<\bar\rho}-r\,\mathbb{I}_{m>1-\bar\rho}\Big]\leq 0 \tag{4.60}$$

where $D\to\bar D$ in the definition of $\psi_m^{(n,g)}$.
We observe that, provided $\epsilon\ell^{d+2}\ll 1$, the $\sup spec$ inside the square bracket in (4.60) is bounded by $c_g T^2$, for a suitable constant c_g depending on g. That follows immediately from perturbation theory (see proposition 4.2) and the estimate (4.33). Therefore, by choosing r large enough, we only need to prove (4.60) with $m\in[\bar\rho,1-\bar\rho]$ where $D(m)$ and $\bar D(m)$ coincide. Similarly one shows that the two "$\sup_{m\in[0,1]}$" appearing in (4.37) and (4.38) can be safley replaced by "$\sup_{m\in[\bar\rho,1-\bar\rho]}$".

5. Disorder induced fluctuations in the averaged gradient density field

In this section we analyze a key term that, as we seen in section 4, arises naturally when one tries to approximate spatial averages of the current with spatial averages of *gradients of the density profile*. Since the currents $j_{x,x+e}$ have, by construction, zero canonical expectation with respect to *any canonical measure* on any set $\Lambda\ni x,x+e$, in order to approximate $\mathrm{Av}_x\,j_{x,x+e}$ with suitable averages of gradients of the density field, one is forced to subtract from these gradients appropriate canonical expectations. Therefore, a key point in order to establish the hydrodynamical limit, is to prove that these "counter terms" vanish as $\epsilon\downarrow 0$. These kind of terms arise also in the hydrodynamical limit of non–disordered lattice gases (see [37], section 7) with short range interaction. In our context however their nature is quite different and, as we will show next, they are basically produced by fluctuations in the disorder field.

In order to be more precise recall first, for any given $e\in\mathcal{E}$, the notation $\Lambda_n^{1,e},\Lambda_n^{2,e}$ and $\Lambda_n^e:=\Lambda_n^{1,e}\cup\Lambda_n^{2,e}$ described in section 2.1, together with the associated densities $m_n^{1,e}:=m_{\Lambda_n^{1,e}}$, $m_n^{2,e}:=m_{\Lambda_n^{2,e}}$, $m_n^e:=m_{\Lambda_n^e}$.

Using the above notation and given two integers $n\leq s$ and a vector $e\in\mathcal{E}$, the basic object of our investigation is defined as (see (4.32)):

$$\phi_{n,s}:=\mu[m_n^{2,e}-m_n^{1,e}\,|\,m_s^e] \tag{5.1}$$

Notice that if the disorder configuration α was identical in the two cubes $\Lambda_n^{1,e}$ and $\Lambda_n^{2,e}$ then $\phi_{n,n}$ would be identically equal to zero. Moreover $\mathbb{E}\big(\phi_{n,s}\big)=0$ and $\mathbb{E}\big([\phi_{n,s}]^2\big)=O(n^{-d})$ uniformly in $s\geq n$.

Remark 5.1. *The fact that $\phi_{n,s}$ is small (on some average sense) with n and not with s is one of the main differences with non disordered lattice gases where, instead, the analogous term goes very fast to zero as $s\uparrow\infty$ (see* [37]*, section* 10*).*

The main result of this section is the proof that the contribution to the hydrodynamical limit of suitable spatial averages of $\frac{\phi_{n,n}}{n}$ is negligible as $\epsilon\downarrow 0$ at least in dimension $d\geq 3$.

In order to be more precise let us introduce the following equivalence relation.

Definition 5.2. *Given two families of functions $f_{x,n,a,\epsilon}(\alpha,\eta)$ and $g_{x,n,a,\epsilon}(\alpha,\eta)$ with $x \in \mathbb{T}^d_\epsilon, n \in \mathbb{N}, a > 0, \epsilon > 0$ we will write $f_x \approx g_x$ if, for any given $T > 0$ and for almost all disorder configurations α,*

$$\limsup_{n\uparrow\infty, a\downarrow 0,\epsilon\downarrow 0} \mathrm{Av}_{x\in\mathbb{T}^d_\epsilon} \sup_{|\beta|\le T} \sup_{\nu} \sup spec_{L^2(\nu)}\big\{\epsilon^{-1}\beta(f_x - g_x) + \epsilon^{-2}\mathrm{Av}_{b\in\Lambda_{x,2a/\epsilon}}\mathcal{L}_b\big\} \le 0$$

where $\sup_\nu$ is the supremum over ν in the set $\mathcal{M}(\Lambda_{x,2\frac{a}{\epsilon}})$ of all the canonical measures on $\Lambda_{x,2\frac{a}{\epsilon}}$.

We are now in a position to state our main result. Assume that a given direction e has been fixed once and for all and, given two integers $\ell \le s$ with $\frac{s}{\ell} \in \mathbb{N}$ and $x \in \mathbb{T}^d_\epsilon$, recall the definition of the spatial average $\mathrm{Av}^{\ell,s}_{z,x}$ given in (2.2).

Theorem 5.3. *For any $d \ge 3$*

$$\mathrm{Av}^{n,\frac{a}{\epsilon}}_{z,x}\,\tau_z\frac{\phi_{n,n}}{n} \approx 0\,.$$

Before discussing the plan of the proof of the theorem we would like to justify the restriction $d \ge 3$. If we pretend that the particle density is constant everywhere, say equal to m, then

$$\sup spec_{L^2(\nu)}\big\{\epsilon^{-1}\beta\mathrm{Av}^{n,\frac{a}{\epsilon}}_{z,x}\,\tau_z\frac{\phi_{n,n}}{n} + \epsilon^{-2}\mathrm{Av}_{b\in\Lambda_{x,2a/\epsilon}}\mathcal{L}_b\big\} \le$$
$$\epsilon^{-1}\beta\mathrm{Av}^{n,\frac{a}{\epsilon}}_{z,x}\,\tau_z\frac{\phi_{n,n}(m)}{n}$$

Since the typical fluctuations (in α) of the quantity

$$\epsilon^{-1}\mathrm{Av}^{n,\frac{a}{\epsilon}}_{z,x}\,\tau_z\frac{\phi_{n,n}(m)}{n}$$

are of the order of $\epsilon^{\frac{d-2}{2}}C(a,n)$, necessarily we must assume $d \ge 3$ since $\epsilon \downarrow 0$ before $a \downarrow 0$ and $n \uparrow \infty$.

5.1. **Plan of the proof of theorem 5.3.** The main difficulty in proving theorem 5.3 lyes in the fact that first $\epsilon \downarrow 0$ and only afterward $n \uparrow \infty$. In particular there is no hope to beat the diverging factor ϵ^{-1} appearing in definition 5.2 with the typical smallness $O(n^{-\frac{d+2}{2}})$ of $\frac{\phi_{n,n}}{n}$. The main idea is therefore first to try to prove that

$$\mathrm{Av}^{n,\frac{a}{\epsilon}}_{z,x}\,\tau_z\frac{\phi_{n,n}}{n} \approx \mathrm{Av}^{s,\frac{a}{\epsilon}}_{z,x}\,\tau_z\frac{\phi_{s,s}}{s} \tag{5.2}$$

where the new mesoscopic scale $s = s(\epsilon)$ diverges sufficiently fast as $\epsilon \downarrow 0$. By standard large deviations estimates (see lemma 6.7) it's simple to verify that, given $0 < \delta \ll 1$ and $0 < \gamma < 1$, for almost any disorder configuration α and $s = O(\epsilon^{-\gamma})$

$$\sup_{x\in\mathbb{T}^d_\epsilon} |\tau_x\,\phi_{s,s}| \le C s^{-\frac{d}{2}+\delta} \tag{5.3}$$

for any ϵ small enough. In particular, by a trivial L^∞ estimate,

$$\mathrm{Av}^{s,\frac{a}{\epsilon}}_{z,x}\,\tau_z\frac{\phi_{s,s}}{s} \approx 0 \quad \text{if} \quad \gamma > \frac{2}{d+2}. \tag{5.4}$$

The above simple reasoning suggests to define a first mesoscopic critical scale $s_\infty := \epsilon^{-\frac{2}{d+2}}$ above which things become trivial. It is important to outline that we will *not* be able to prove (5.2) with $s \gg s_\infty$ but only with $s = \bar{s}$ where $\bar{s} := \epsilon^\delta s_\infty$ and $0 < \delta \ll 1$ can be taken

arbitrarily small.
Once we have reached scale $\bar{s}$ we cannot simply use L^∞ bounds but we need to appeal to an improved version of the well known Two Blocks Estimate (see proposition 5.9 below) in order to conclude that $\mathrm{Av}_{z,x}^{\bar{s},\frac{a}{\epsilon}} \tau_z \frac{\phi_{\bar{s},\bar{s}}}{\bar{s}} \approx 0$.
We now explain the main steps in the proof of (5.2) with $s = \bar{s}$. As discussed in subsection 4.1, a main tool for estimating eigenvalues is given by localization together with perturbation theory. However, because of proposition 4.2, it turns out that this technique can be applied to prove (5.2) only if

$$\epsilon s^{d+2} \| \mathrm{Av}_{z,x}^{n,\frac{a}{\epsilon}} \tau_z \frac{\phi_{n,n}}{n} - \mathrm{Av}_{z,x}^{s,\frac{a}{\epsilon}} \tau_z \frac{\phi_{s,s}}{s} \|_\infty \le \mathrm{const},$$

that is if $\epsilon s^{d+2} \le \mathrm{const}$. In particular we see immediately that this approach cannot be used directly to prove (5.2) for $s = \bar{s}$, but only up to a new critical mesoscopic scale $s_0 := \epsilon^{-\frac{1}{d+2}}$.
Assuming that we have been able to replace $\mathrm{Av}_{z,x}^{n,\frac{a}{\epsilon}} \tau_z \frac{\phi_{n,n}}{n}$ with $\mathrm{Av}_{z,x}^{s_0,\frac{a}{\epsilon}} \tau_z \frac{\phi_{s_0,s_0}}{s_0}$, we face the problem to increase the mesoscopic scale from s_0 to $\bar{s}$.
The main observation now is that the L^∞ norm of the new quantity $\mathrm{Av}_{z,x}^{s_0,\frac{a}{\epsilon}} \tau_z \frac{\phi_{s_0,s_0}}{s_0}$ is at least smaller than $s_0^{-\frac{d+2}{2}}$ (see (5.3)) almost surely (here and in what follows we deliberately neglect the correction s^δ appearing in (5.3)). This means that the limit scale beyond which perturbation theory cannot be applied, previously equal to s_0, is now pushed up to a new scale s_1 given by

$$\epsilon s_1^{d+2} s_0^{-\frac{d+2}{2}} \le \mathrm{const} \;\Rightarrow\; s_1 = \epsilon^{-\frac{3}{2(d+2)}}$$

The above remark clearly suggests an inductive scheme on a sequence of length scales $\{s_k\}_{k\ge 0}$ given by

$$s_0 := \epsilon^{-\frac{1}{d+2}}; \qquad s_{k+1} := \epsilon^{-\frac{1}{d+2}} \sqrt{s_k}$$

in which one proves recursively, by means of localization on scale s_{k+1} combined together with perturbation theory, that

$$\mathrm{Av}_{z,x}^{s_k,\frac{a}{\epsilon}} \tau_z \frac{\phi_{s_k,s_k}}{s_k} - \mathrm{Av}_{z,x}^{s_{k+1},\frac{a}{\epsilon}} \tau_z \frac{\phi_{s_{k+1},s_{k+1}}}{s_{k+1}} \approx 0.$$

Notice that $\lim_{k\to\infty} s_k = s_\infty$ where $s_\infty = \epsilon^{-\frac{2}{d+2}}$ represents the limiting scale introduced at the beginning of this section.
A large but finite number of steps of the inductive scheme proves that

$$\mathrm{Av}_{z,x}^{n,\frac{a}{\epsilon}} \tau_z \frac{\phi_{n,n}}{n} - \mathrm{Av}_{z,x}^{\bar{s},\frac{a}{\epsilon}} \tau_z \frac{\phi_{\bar{s},\bar{s}}}{\bar{s}} \approx 0$$

where, as before, $\bar{s} = \epsilon^\delta s_\infty$. We remark that for this part of the proof we only need $d \ge 2$, while we will assume $d \ge 3$ when proving the improved version of the Two Blocks estimate (see proposition 5.9).

5.2. **Preliminary tools.** In this section we collect some general techniques that are common to all the steps of the proof of theorem 5.3. We recall that $\Lambda_{z,\ell}^e$ denotes the translated by z of the box Λ_ℓ^e.

Lemma 5.4. *Let $\ell_0 < \ell_1 < \ell_2$ be odd integers such that $\frac{\ell_2}{\ell_0} \in \mathbb{N}$. Let ν be an arbitrary canonical measure on the cube Λ_{ℓ_2} and let f be a function with support in $\Lambda^e_{\ell_1}$. Then*

$$\sup spec_{L^2(\nu)}\{\mathrm{Av}^{\ell_0,\ell_2}_{z,0}\tau_z f + \mathrm{Av}_{b\in\Lambda_{\ell_2}}\mathcal{L}_b\} \leq \mathrm{Av}^{\ell_0,\ell_2}_{z,0}\sup_{\nu'}\sup spec_{L^2(\nu')}\{\tau_z f + c\mathrm{Av}_{b\in\Lambda^e_{z,\ell_1}}\mathcal{L}_b\}$$

where ν' varies in $\mathcal{M}_\alpha(\Lambda^e_{z,\ell_1})$ and c is a suitable constant.

Proof. It is sufficient to observe that

$$\mathrm{Av}_{b\in\Lambda_{\ell_2}}\mathcal{L}_b \leq c\,\mathrm{Av}^{\ell_0,\ell_2}_{z,0}\big(\mathrm{Av}_{b\in\Lambda^e_{z,\ell_1}}\mathcal{L}_b\big)$$

and localize in the box Λ^e_{z,ℓ_1}. □

At this point, it is convenient to observe the factorization property of the average $\mathrm{Av}^{\ell,s}_{z,x}$ defined in (2.2): given odd integers ℓ, ℓ', L such that $\frac{\ell'}{\ell}, \frac{L}{\ell'} \in \mathbb{N}$, then

$$\mathrm{Av}^{\ell,L}_{z,x} f_z = \mathrm{Av}^{\ell',L}_{z,x}\big(\mathrm{Av}^{\ell,\ell'}_{w,z} f_w\big). \tag{5.5}$$

Proposition 5.5. *Let $d \geq 2$, $0 < \gamma \leq \gamma' < 1$ and $\gamma' < \frac{1}{d+2} + \frac{\gamma}{2}$. If either $\ell = n$ and $s = O\big(\epsilon^{-\frac{1}{d+2}}\big)$ or $\ell = O(\epsilon^{-\gamma})$ and $s = O(\epsilon^{-\gamma'})$, then*

$$\mathrm{Av}^{\ell,\frac{a}{\epsilon}}_{z,x}\tau_z\frac{\phi_{\ell,s}}{\ell} \approx \mathrm{Av}^{s,\frac{a}{\epsilon}}_{z,x}\tau_z\frac{\phi_{s,2s}}{s}.$$

Proof. By the factorization property (5.5), we have

$$\mathrm{Av}^{\ell,\frac{a}{\epsilon}}_{z,x}\tau_z\frac{\phi_{\ell,s}}{\ell} - \mathrm{Av}^{s,\frac{a}{\epsilon}}_{z,x}\tau_z\frac{\phi_{s,2s}}{s} = \mathrm{Av}^{s,\frac{a}{\epsilon}}_{z,x}\Big[\mathrm{Av}^{\ell,s}_{w,z}\tau_w\frac{\phi_{\ell,s}}{\ell} - \tau_z\frac{\phi_{s,2s}}{s}.\Big]$$

Therefore, by lemma 5.4, it is enough to prove that for any $T > 0$ and for almost all disorder configuration α

$$\limsup_{n\uparrow\infty,a\downarrow0,\epsilon\downarrow0}\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\sup_{|\beta|\leq T}\sup_{\nu\in\mathcal{M}_\alpha(\Lambda^e_{x,2s})} f_{x,\nu} \leq 0 \tag{5.6}$$

where

$$f_{x,\nu} := \sup spec_{L^2(\nu)}\Big\{\epsilon^{-1}\beta\Big[\mathrm{Av}^{\ell,s}_{z,x}\tau_z\frac{\phi_{\ell,s}}{\ell} - \tau_x\frac{\phi_{s,2s}}{s}\Big] + c\epsilon^{-2}\mathrm{Av}_{b\in\Lambda^e_{x,2s}}\mathcal{L}_b\Big\}$$

for a suitable constant c. Notice that $\tau_x\frac{\phi_{s,2s}}{s} = \nu\big(\mathrm{Av}^{\ell,s}_{z,x}\tau_z\frac{\phi_{\ell,s}}{\ell}\big)$ ν a.s..

Because of lemma 6.7, given $0 < \delta \ll 1$, for almost all α and ϵ small enough

$$\sup_{x\in\mathbb{T}^d_\epsilon}\Big\|\tau_x\frac{\phi_{\ell,s}}{\ell}\Big\|_\infty \leq \begin{cases}\ell^{-1} & \text{if } \ell = n\\ \ell^{-(d+2)/2+\delta} & \text{if } \ell = O(\epsilon^{-\gamma})\end{cases}$$

Thanks to the above bound and to the choice $\gamma' < \frac{1}{d+2} + \frac{\gamma}{2}$, for almost all α and ϵ small enough, we can apply proposition 4.2 together with lemma A.5 to get

$$\sup_{\nu\in\mathcal{M}_\alpha(\Lambda^e_{x,2s})} f_{x,\nu} \leq cT^2\ell^{-2}s^{d+2}\sup_m F(x,m) \tag{5.7}$$

where m varies among all possible particle densities in $\Lambda^e_{x,2s}$ and

$$F(x,m) := \mathrm{Var}_{\mu^{\lambda_x(m)}}\big(\mathrm{Av}^{\ell,s}_{z,x}\,\tau_z\phi_{\ell,s}\big)$$

and $\lambda_x(m) := \lambda_{\Lambda^e_{x,2s}}(m)$.

We claim that for almost all α and ϵ small enough

$$\sup_{x\in\mathbb{T}^d_\epsilon}\sup_m F(x,m)\le c\,s^{-2d+2\delta} \tag{5.8}$$

thus proving the proposition since $d\ge 2$. The proof of (5.8) follows exactly the same lines of the proof of proposition 6.5 with the main difference that it is necessery to use lemma 6.4 in order to control the empirical chemical potetials (see also section 4.7 in [15])

□

Proposition 5.6. *Let $d\ge 2$, $\frac{1}{d+2}\le\gamma\le\gamma'<1$ and $\gamma'<\frac{1}{d+2}+\frac{\gamma}{2}$. Set $s=O(\epsilon^{-\gamma})$ and $s'=O(\epsilon^{-\gamma'})$. Then*

$$\mathrm{Av}^{s,\frac{a}{\epsilon}}_{z,x}\tau_z\frac{\phi_{s,s}}{s}\approx\mathrm{Av}^{s,\frac{a}{\epsilon}}_{z,x}\tau_z\frac{\phi_{s,s'}}{s}.$$

Proof. By lemma 5.4 it is enough to prove that for any $T>0$ and for almost any disorder configuration α

$$\limsup_{a\downarrow 0,\epsilon\downarrow 0}\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\sup_{|\beta|\le T}\sup_{\nu\in\mathcal{M}_\alpha(\Lambda^e_{x,s'})}f_x\le 0$$

where

$$f_x:=\sup spec_{L^2(\nu)}\Big\{\epsilon^{-1}\beta\tau_x\Big[\frac{\phi_{s,s}}{s}-\frac{\phi_{s,s'}}{s}\Big]+c\epsilon^{-2}\mathrm{Av}_{b\in\Lambda^e_{x,s'}}\mathcal{L}_b\Big\}$$

for a suitable constant c. Notice that $\nu(\phi_{s,s})=\phi_{s,s'}$ ν a.s..

Because of lemma 6.7, given $0<\delta\ll 1$, for almost all α and ϵ enough small

$$\sup_{x\in\mathbb{T}^d_\epsilon}\Big\|\tau_x\frac{\phi_{s,s}}{s}\Big\|_\infty\le s^{-(d+2)/2+\delta}.$$

Thanks to the above bound and to the choice $\gamma'<\frac{1}{d+2}+\frac{\gamma}{2}$, for almost any α and ϵ small enough, we can apply proposition 4.2 together with lemma A.5 to get

$$\sup_{\nu\in\mathcal{M}_\alpha(\Lambda^e_{x,s'})}f_x\le T^2s^{-2}(s')^{d+2}\sup_m F(x,m), \tag{5.9}$$

where m varies among all possible particle densities in $\Lambda^e_{x,s'}$,

$$F(x,m):=\mathrm{Var}_{\mu^{\lambda_x(m)}}\big(\tau_x\phi_{s,s}\big)$$

and now $\lambda_x(m)=\lambda_{\Lambda^e_{x,s'}}(m)$.
We claim that for almost all α and ϵ small enough

$$\sup_{x\in\mathbb{T}^d_\epsilon}\sup_m F(x,m)\le c\,s^{-2d+2\delta} \tag{5.10}$$

thus proving the proposition because of the constraint on γ,γ',d. The proof of (5.8), requiring $d\ge 2$, follows exactly the same lines of the proof of proposition 6.5 with the main difference that it is necessery to use lemma 6.4 in order to control the empirical chemical potetials (see also section 4.6 in [15]) □

5.3. **From scale n to scale s_0.** Here we show how to replace the starting scale n with our first mesoscopic scale increasing with ϵ, $s_0 = O\big(\epsilon^{-\frac{1}{d+2}}\big)$.

Proposition 5.7. *Let $d \geq 3$. Then*

$$\mathrm{Av}_{z,x}^{n,\frac{a}{\epsilon}}\,\tau_z\Big[\frac{\phi_{n,n}}{n} - \frac{\phi_{n,s_0}}{n}\Big] \approx 0. \tag{5.11}$$

Proof. Without loss of generality, we assume that $\frac{s_0}{n} \in \mathbb{N}$ and similarly for $\frac{a/\epsilon}{n}$.
By the definition of $\mathrm{Av}_{z,x}^{n,\frac{a}{\epsilon}}$ and setting $B = Q_{a/\epsilon} \cap n\,\mathbb{Z}^d$, in order to prove (5.11) it is enough to show that

$$\limsup_{n\uparrow\infty, a\downarrow 0, \epsilon\downarrow 0} \mathrm{Av}_{x\in\mathbb{T}^d_\epsilon} \sup_{|\beta|\leq T} \sup_{\nu} \sup spec_{L^2(\nu)}\big\{\epsilon^{-1}\beta \mathrm{Av}_{z\in B+x}\tau_z f_{n,s_0} + c\,\epsilon^{-2}\mathrm{Av}_{b\in\Lambda_{x,3a/2\epsilon}}\mathcal{L}_b\big\} \leq 0 \tag{5.12}$$

where $f_{n,s_0} := \frac{\phi_{n,n}}{n} - \frac{\phi_{n,s_0}}{n}$ and ν varies in $\mathcal{M}(\Lambda_{x,3a/2\epsilon})$. The proof is nothing more than a careful writing of the spatial average $\mathrm{Av}_{z,x}^{n,\frac{a}{\epsilon}}$ together with the subadditivity property of $\sup spec$.
Setting $B' = Q_{s_0} \cap n\,\mathbb{Z}^d$, $Y = Q_{a/\epsilon} \cap s_0\mathbb{Z}^d$ we can write $B = \cup_{y\in Y}(B' + y)$ so that

$$\mathrm{Av}_{z\in B+x}\tau_z f_{n,s_0} = \mathrm{Av}_{y\in Y+x}\mathrm{Av}_{z\in B'+y}\tau_z f_{n,s_0}$$

and

$$\mathrm{Av}_{b\in\Lambda_{x,3a/2\epsilon}}\mathcal{L}_b \leq c\,\mathrm{Av}_{y\in Y+x}\mathrm{Av}_{b\in\Lambda_{y,2s_0}}\mathcal{L}_b\,.$$

By the subadditivity property of $\sup spec$, (5.12) is bounded from above by

$$\limsup_{n\uparrow\infty, a\downarrow 0, \epsilon\downarrow 0} \mathrm{Av}_{x\in\mathbb{T}^d_\epsilon} \sup_{|\beta|\leq T} \sup_\nu \sup spec_{L^2(\nu)}\{\epsilon^{-1}\beta \mathrm{Av}^{(n)}_{y\in Q_{s_0}}\tau_{x+y} f_{n,s_0} + c\,\epsilon^{-2}\mathrm{Av}_{b\in\Lambda_{x,2s_0}}\mathcal{L}_b\}$$

where ν varies among $\mathcal{M}(\Lambda_{x,2s_0})$ and $\mathrm{Av}^{(n)}_{y\in\Lambda} := \mathrm{Av}_{y\in\Lambda\cap n\mathbb{Z}^d}$.
At this point we can apply perturbation theory (see proposition 4.2): since $\lim_{n\uparrow\infty}\sup_{\epsilon>0}\epsilon s_0^{d+2}\|f_{n,s_0}\|_\infty = 0$, it is enough to prove that for almost all disorder α

$$\limsup_{n\uparrow\infty, \epsilon\downarrow 0}\frac{1}{n^2}\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\sup_{\nu\in\mathcal{M}(\Lambda_{x,2s_0})}\Psi^{(\nu)}_{s_0}\big(\tau_x\left[\phi_{n,n} - \phi_{n,s_0}\right]\big) = 0 \tag{5.13}$$

where

$$\Psi^{(\nu)}_{s_0}(f) = s_0^d\,\nu\big(\mathrm{Av}^{(n)}_{y\in Q_{s_0}}\tau_y f, (-\mathcal{L}_{\Lambda_{2s_0}})^{-1}\mathrm{Av}^{(n)}_{y\in Q_{s_0}}\tau_y f\big).$$

In order to prove (5.13) it is clearly sufficient to prove it with $\phi_{n,n}$ replaced by ϕ_{n,n^4}, provided one is able to show that for almost any disorder α

$$\limsup_{n\uparrow\infty, \epsilon\downarrow 0}\frac{1}{n^2}\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\sup_{\nu\in\mathcal{M}(\Lambda_{x,2s_0})}\Psi^{(\nu)}_{s_0}\big(\tau_x\left[\phi_{n,n} - \phi_{n,n^4}\right]\big) = 0. \tag{5.14}$$

We will concentrate only on the first step and refer the reader to section 4.5 in [15] for the details of the proof of (5.14).
Given $\nu \in \mathcal{M}(\Lambda_{2s_0})$ we first estimate $\Psi^{(\nu)}_{s_0}\big(\phi_{n,n^4} - \phi_{n,s_0}\big)$ as follows (a similar bound will then be applied to any translation by x).
Assume, without loss of generality, that $s_0 = N^4$ for some $N \in \mathbb{N}$ and set $\ell_k := k^4$ for any $k \in \mathbb{N}$. Then, given $0 < \rho \ll 1$, by Schwarz inequality,

$$\Psi^{(\nu)}_{s_0}(\phi_{n,n^4} - \phi_{n,s_0}) \leq c_\rho \sum_{k=n}^{N-1} k^{1+\rho}\,\Psi^{(\nu)}_{s_0}(\phi_{n,\ell_k} - \phi_{n,\ell_{k+1}}).$$

In order to estimate $\Psi^{(\nu)}_{s_0}(\phi_{n,\ell_k}-\phi_{n,\ell_{k+1}})$ we divide Q_{s_0} in cubes $\{Q_{i,k}\}_{i\in I_k}$ with side ℓ_k where, without loss of generality, we assume that $s_0/\ell_k\in\mathbb{N}$ and similarly for ℓ_k/n. Let $\bar Q_{i,k}$ be the cube of side $10\ell_k$ concentric to $Q_{i,k}$. Then by lemma 6.1 with

$$I:=I_{k+1},\quad \Lambda:=\Lambda_{2s_0},\quad \Lambda_i:=\bar Q_{i,k+1},\quad f_i:=\mathrm{Av}^{(n)}_{x\in Q_{i,k+1}}\tau_x[\phi_{n,\ell_k}-\phi_{n,\ell_{k+1}}]$$

we obtain (thanks also to lemma A.5)

$$\Psi^{(\nu)}_{s_0}(\phi_{n,\ell_k}-\phi_{n,\ell_{k+1}})\le c\,\ell_{k+1}^{d+2}\mathrm{Av}_{i\in I_{k+1}}\mathrm{Var}_{\mu^\lambda}\Big(\mathrm{Av}^{(n)}_{x\in Q_{i,k+1}}\tau_x[\phi_{n,\ell_k}-\phi_{n,\ell_{k+1}}]\Big) \tag{5.15}$$

where μ^λ is the grand canonical measure corresponding to ν.
Let now J be the set of possible densities on Λ_{2s_0}. Then, thanks to (5.15), it is enough to prove that, for ρ small enough and for almost any disorder α,

$$\lim_{n\uparrow\infty,\epsilon\downarrow0}\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\frac{1}{n^2}\sum_{k=n+1}^{N}k^{1+\rho}\ell_k^{d+2}\mathrm{Av}_{i\in I_k}\sup_{m\in J}\mathrm{Var}_{\mu^{\lambda(m)}_{\Lambda_{x,2s_0}}}\Big(\mathrm{Av}^{(n)}_{y\in x+Q_{i,k}}\tau_y\phi_{n,\ell_k}\Big)=0 \tag{5.16}$$

and similarly with ϕ_{n,ℓ_k} replaced by $\phi_{n,\ell_{k-1}}$.
Given $\gamma>0$ we set $J_k=\{\ell_k^{-\gamma},2\ell_k^{-\gamma},\dots,1-\ell_k^{-\gamma}\}$. Then, using (A.1), the variance in (5.16) can be bounded from above by

$$\mathrm{Var}_{\mu^{\lambda(\bar m)}_{x+\bar Q_{i,k}}}\Big(\mathrm{Av}^{(n)}_{y\in x+Q_{i,k}}\tau_y\phi_{n,\ell_k}\Big)+c\ell_k^{d-\gamma}$$

provided that $\bar m\in J_k$ satisfies $|\bar m-m|\le\ell_k^{-\gamma}$.
Therefore, by choosing γ large enough, we can replace in (5.16) $\mu^{\lambda(m)}_{\Lambda_{x,2s_0}}$ by $\mu^{\lambda(m)}_{x+\bar Q_{i,k}}$ and J by J_k. We can at this apply proposition 6.6 to get that

$$\sup_{m\in J_k}\mathrm{Var}_{\mu^{\lambda(m)}_{\bar Q_{i,k}+x}}\Big(\mathrm{Av}^{(n)}_{y\in x+Q_{i,k}}\tau_y\phi_{n,\ell_k}\Big)\le c\mathbb{I}_{\mathcal{A}^c_{x,i,k}}(\alpha){\ell_k}^{-2d+2\delta}+\mathbb{I}_{\mathcal{A}_{x,i,k}}(\alpha)\,, \tag{5.17}$$

where $\mathcal{A}_{x,i,k}$ is a set of disorder configurations in $x+\bar Q_{i,k}$ with $\mathbb{P}(\mathcal{A}_{x,i,k})\le e^{-c\,\ell_k^\delta}$, $\delta>0$.
Therefore

$$\begin{aligned}\text{l.h.s. of (5.16)}\ &\le\lim_{n\uparrow\infty,\epsilon\downarrow0}n^{-2}\sum_{k=n+1}^{N}k^{1+\rho}\ell_k^{d+2}\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\mathrm{Av}_{i\in I_k}\mathbb{I}_{\mathcal{A}_{x,i,k}}\\&+\lim_{n\uparrow\infty,\epsilon\downarrow0}n^{-2}\sum_{k=n+1}^{N}k^{1+\rho}\ell_k^{2-d+2\delta}.\end{aligned} \tag{5.18}$$

The second addendum in the r.h.s. of (5.18) is zero because of the definition of ℓ_k and the condition $d\ge3$.
Let us consider the first addendum in the r.h.s. of (5.18). By Chebyschev inequality, for any $q>0$ and any x,k

$$\begin{aligned}\mathbb{P}\big(\mathrm{Av}_{i\in I_k}\mathbb{I}_{\mathcal{A}_{x,i,k}}\ge\ell_k^{-q}\big)&\le\mathbb{P}\big(\exists i\in I_k\,:\,\mathbb{I}_{\mathcal{A}_{x,i,k}}\ge\ell_k^{-q}\big)\\&\le s_0^d\,\ell_k^{q-d}\,e^{-c\,\ell_k^\delta}.\end{aligned} \tag{5.19}$$

Moreover, by setting $\bar{\mathbb{I}}_{\mathcal{A}_{x,i,k}}=\mathbb{I}_{\mathcal{A}_{x,i,k}}-\mathbb{P}(\mathcal{A}_{x,i,k})$, we have for any $r\in\mathbb{N}$ and any x,k

$$\mathbb{P}\big(\mathrm{Av}_{i\in I_k}\mathbb{I}_{\mathcal{A}_{x,i,k}}\ge\ell_k^{-q}\big)\le c_r\ell_k^{2rq}\mathbb{E}\big[\big(\mathrm{Av}_{i\in I_k}\bar{\mathbb{I}}_{\mathcal{A}_{x,i,k}}\big)^{2r}\big]\le c_r'\ell_k^{2rq+dr}s_0^{-dr} \tag{5.20}$$

By taking the geometric average of the two estimates (5.19) and (5.20) we finally obtain

$$\mathbb{P}\big(\mathrm{Av}_{i\in I_k}\mathbb{I}_{\mathcal{A}_{x,i,k}}\ge l_k^{-q}\big)\le c(q,r)s_0^{-d(r-1)/2}.$$

It is enough at this point to choose q and r large enough, define

$$\Theta_\epsilon := \{\ \exists x \in \mathbb{T}^d_\epsilon : \mathrm{Av}_{i\in I_k} \mathbb{I}_{\mathcal{A}_{x,i,k}} \geq \ell_k^{-q} \quad \text{for some } k \leq N\ \},$$

and apply Borel-Cantelli lemma to get that also the first addendum in the r.h.s. of (5.18) is negligible. □

5.4. **From scale s_k to scale s_{k+1}.** Here we define precisely the sequence of length scales s_k and discuss the details of the inductive step $s_k \to s_{k+1}$ described section 5.1.
Let $\{a_k\}_{k\geq 0}$ be defined inductively by

$$a_0 = 1 \quad \text{and} \quad a_{k+1} = 1 + \big(\frac{1}{2} - \frac{1}{2^{k+1}}\big)a_k$$

It is easy to verify that the sequence $\{a_k\}_{k\geq 0}$ is increasing with $\lim_{k\to\infty} a_k = 2$. Let also $s_k := \epsilon^{-\frac{a_k}{d+2}}$.

Proposition 5.8. *Let $d \geq 2$. Then*

$$\mathrm{Av}^{s_k, \frac{a}{\epsilon}}_{z,x} \tau_z \frac{\phi_{s_k,s_k}}{s_k} \approx \mathrm{Av}^{s_{k+1}, \frac{a}{\epsilon}}_{z,x} \tau_z \frac{\phi_{s_{k+1},s_{k+1}}}{s_{k+1}} \qquad \forall k \geq 0. \tag{5.21}$$

Proof. In order to prove (5.21) observe that, by construction, the two exponents $\frac{a_k}{d+2}$ and $\frac{a_{k+1}}{d+2}$ satisfy the conditions of propositions 5.5 and 5.6 with $\gamma := \frac{a_k}{d+2}$ and $\gamma' := \frac{a_{k+1}}{d+2}$. Therefore we have the following chain of equivalences:

$$\mathrm{Av}^{s_k, \frac{a}{\epsilon}}_{z,x} \tau_z \frac{\phi_{s_k,s_k}}{s_k} \approx \mathrm{Av}^{s_k, \frac{a}{\epsilon}}_{z,x} \tau_z \frac{\phi_{s_k,s_{k+1}}}{s_k} \approx \mathrm{Av}^{s_{k+1}, \frac{a}{\epsilon}}_{z,x} \tau_z \frac{\phi_{s_{k+1},2s_{k+1}}}{s_{k+1}}$$

Finally, using again proposition 5.6 with $s = s_{k+1}$ and $s' = 2s$, we obtain (5.21). □

5.5. **Analysis of $\frac{\phi_{\bar{s},\bar{s}}}{\bar{s}}$ via an improved Two Blocks Estimate.** Here we describe the final step in the proof of theorem 5.3, namely we show that

$$\mathrm{Av}^{\bar{s}, \frac{a}{\epsilon}}_{z,x} \tau_z \frac{\phi_{\bar{s},\bar{s}}}{\bar{s}} \approx 0$$

where $\bar{s} = \epsilon^\delta s_\infty$ and $s_\infty = \epsilon^{-\frac{2}{d+2}}$ (see section 5.1). The basic tool is represented by the following improved version of the Two Blocks Estimate (see e.g. [23]), whose proof mainly relies on the same techniques used for proving proposition A.9 (see section 4.10 in [15]).

Proposition 5.9. *(Improved Two Blocks Estimate)*
Let $d \geq 3$, $0 < \gamma < \gamma' < 1$ and set $s = \epsilon^{-\gamma}$, $\ell = \epsilon^{-\gamma'}$. Then, for any r such that $0 < r < \min\big(\frac{2(1-\gamma')}{d+4}, \frac{\gamma}{2}\big)$ and for almost any disorder configuration α

$$\limsup_{a\downarrow 0, \epsilon\downarrow 0} \mathrm{Av}_{x\in\mathbb{T}^d_\epsilon} \sup_\nu \sup spec_{L^2(\nu)} \big\{ \epsilon^{-r} \mathrm{Av}^{\ell, \frac{a}{\epsilon}}_{w,x} Av^{s,\ell}_{z,w} \, |m^e_{z,s} - m^e_{w,\ell}| + \epsilon^{-2} \mathrm{Av}_{b\in\Lambda_{x,2\frac{a}{\epsilon}}} \mathcal{L}_b \big\} \leq 0$$

where ν varies among $\mathcal{M}(\Lambda_{x,2\frac{a}{\epsilon}})$.

Corollary 5.10. *Let $d \geq 3$ and $0 < \delta \ll 1$. Then*

$$\mathrm{Av}^{\bar{s}, \frac{a}{\epsilon}}_{z,x} \tau_z \frac{\phi_{\bar{s},\bar{s}}}{\bar{s}} \approx 0. \tag{5.22}$$

Proof. For simplicity of notation we omit the bar in $\bar{s}$ and we set $\Delta m := m_s^{2,e} - m_s^{1,e}$ and $N := N_{\Lambda_s^e}$.
Let $\hat{\phi}_{s,s}(\eta) = \mu_{\Lambda_s^e}^{\lambda(m_s^e(\eta))}(\Delta m)$. Then, by the equivalence of ensembles (see proposition A.4), it is enough to prove (5.22) with $\phi_{s,s}$ replaced by $\hat{\phi}_{s,s}$. Let m be a particle density on Λ_s^e that, without loss of generality, we can suppose in $(0, \frac{1}{2})$ and set $\lambda := \lambda_{\Lambda_s^e}(m)$ and $\lambda_0 := \lambda_0(m)$. Then, by Taylor expansion,

$$\mu^{\lambda}(\Delta m) = \mu^{\lambda_0}(\Delta m) + \mu^{\lambda_0}(\Delta m; N)(\lambda - \lambda_0) + \mu^{\lambda'}(\Delta m; N; N)(\lambda - \lambda_0)^2 \tag{5.23}$$

where λ' is between λ and λ_0.
Let us observe that $|\mu^{\lambda'}(\Delta m; N; N)| \leq c$, while by lemma A.3

$$|\lambda - \lambda_0| \leq c|1 - \frac{\mu^{\lambda_0}(m_s^e)}{m}|.$$

Moreover, $\mathbb{E}[\mu^{\lambda_0}(m_s^e)] = m$ and $\mathbb{E}[\mu^{\lambda_0}(\Delta m_s; N)] = 0$. Therefore, thanks to the large deviations estimate of lemma A.1 applied to the function $f(\alpha) := \frac{\mu^{\lambda_0}(\eta_0)}{m} - 1$, for any $\beta \in (0,1)$ and ϵ small enough

$$\mathbb{P}(|\lambda - \lambda_0| \geq s^{-\frac{d}{2}+\frac{\beta}{2}}) \leq \mathbb{P}(|\mathrm{Av}_{x \in \Lambda_s^e} \tau_x f| \geq \frac{1}{c} s^{-\frac{d}{2}+\frac{\beta}{2}}) \leq e^{-cs^{\beta}}.$$

A similar reasoning applies to the term $\mu^{\lambda_0}(\Delta m; N)$ if we consider instead the function $f(\alpha) := \mu^{\lambda_0}(\eta_0; \eta_0) - \mathbb{E}(\mu^{\lambda_0}(\eta_0; \eta_0))$. The above bounds together with the fact that the number of possible choices of m is polynomially bounded in s and together with Borel Cantelli lemma, implies in particular that for almost all disorder configuration α and for ϵ small enough

$$\sup_{x \in \mathbb{T}_\epsilon^d} \|\tau_x(\,\hat{\phi}_{s,s} - \mu^{\lambda_0(m_s^e)}(\Delta m)\,)\|_\infty \leq s^{-d+\beta}.$$

Thanks to the above estimate it is enough to prove (5.22) with $\phi_{s,s}$ replaced by $\mu^{\lambda_0(m_s^e)}(\Delta m)$, that is

$$\mathrm{Av}_{z,x}^{s,\frac{a}{\epsilon}} \mu^{\lambda_0(m_{z,s}^e)}\Big(\tau_z \frac{m_s^{2,e} - m_s^{1,e}}{s}\Big) \approx 0. \tag{5.24}$$

We assert that we only need to show that

$$\text{l.h.s. of (5.24)} \approx \mathrm{Av}_{z,x}^{\ell,\frac{a}{\epsilon}} \mu^{\lambda_0(m_{z,\ell}^e)}\Big(\tau_z \frac{m_\ell^{2,e} - m_\ell^{1,e}}{\ell}\Big) \tag{5.25}$$

where $\ell = \epsilon^{1-\rho}$ is a new mesoscopic scale with $0 < \rho < 1$ so small that $s < \ell$ and $\epsilon^{-1}\ell^{-\frac{d+2}{2}} \downarrow 0$ as $\epsilon \downarrow 0$. In fact, thanks to lemma A.1 applied with $f(\alpha) := \mu^{\lambda_0(m_{z,\ell}^e)}(\eta_0 - \eta_{\ell e})$, given $0 < \beta \ll 1$ for almost any disorder configuration α and for ε small enough the r.h.s. of (5.25) is bounded by $\ell^{-\frac{d+2}{2}+\beta}$. Because of our choice of ℓ, the r.h.s. of (5.25) is equivalent to 0.
Let us prove (5.25). To this aim, we observe that thanks to (5.5) and (4.27)

$$\text{l.h.s. of (5.24)} = \mathrm{Av}_{w,x}^{\ell,\frac{a}{\epsilon}} \mathrm{Av}_{z,w}^{s,\ell}\, \mu^{\lambda_0(m_{z,s}^e)}\Big(\tau_z \frac{\Delta m}{s}\Big),$$

$$\text{r.h.s. of (5.25)} = \mathrm{Av}_{w,x}^{\ell,\frac{a}{\epsilon}} \mathrm{Av}_{z,w}^{s,\ell}\, \mu^{\lambda_0(m_{w,\ell}^e)}\Big(\tau_z \frac{\Delta m}{s}\Big).$$

Therefore, we only need to prove that

$$\mathrm{Av}_{w,x}^{\ell,\frac{a}{\epsilon}} \mathrm{Av}_{z,w}^{s,\ell} \Big(\mu^{\lambda_0(m_{z,s}^e)}\big(\tau_z \frac{\Delta m}{s}\big) - \mu^{\lambda_0(m_{w,\ell}^e)}\big(\tau_z \frac{\Delta m}{s}\big)\Big) \approx 0.$$

Let us assume for the moment that, given $0 < \beta \ll 1$, for almost all disorder configuration α and ϵ small enough

$$\sup_{x\in\mathbb{T}^d_\epsilon} |\mu^{\lambda_0(m)}\big(\tau_x \Delta m\big) - \mu^{\lambda_0(m')}\big(\tau_x\Delta m\big)| \le c s^{-\frac{d}{2}+\beta}|m-m'| + cs^{-\frac{d}{2}-\beta} \quad \forall m,m' \in [0,1]. \tag{5.26}$$

Then it is simple to deduce (5.25) from (5.26) and proposition 5.9 with $\gamma = \frac{2}{d+2} - \delta$, $\gamma' = 1-\rho$ and $r = -\delta\beta + \frac{d+2}{2}\delta + \frac{2}{d+2}\beta$ by choosing suitable $0 < \beta \ll \delta \ll \rho \ll 1$.
It remains to prove (5.26). For simplicity of notation, let us consider only the case $x = 0$ (the general case is a simple variation). By continuity, we may assume $0 < m < m' < 1$ and by Taylor expansion,

$$\begin{aligned}|\mu^{\lambda_0(m')}\big(\Delta m\big) - \mu^{\lambda_0(m)}\big(\Delta m\big)| &= |\mu^{\lambda_0(\bar m)}(\Delta m; N)\lambda_0'(\bar m)(m'-m)| \\ &\le c\,|\frac{\mu^{\lambda_0(\bar m)}(\Delta m;N)}{\bar m}|(m'-m)\end{aligned}$$

where $m < \bar m < m'$. If we could restrict the possible values of $\bar m$ to $\{s^{-d}, 2s^{-d}, \dots, 1-s^{-d}\}$, then, by means of large deviations estimate as in the first part of the proof, we would obtain $\frac{1}{\bar m}|\mu^{\lambda_0(\bar m)}(\Delta m;N)| \le c\,s^{-\frac{d}{2}+\beta}$ for almost any disorder α and for ϵ small enough, thus implying (5.26). The complete proof requires some addional straightforward computations (see also section 4.10 in [15]). □

6. Some technical results needed in section 5

In this section we collect some technical results, mostly based on estimates of large deviations in the disorder field α, that are used in the proof of theorem 5.3. Our bounds mainly concern canonical or grand canonical variances of suitable spatial averages of local functions. Such variances arise naturally from eigenvalue estimates via perturbation theory. We have seen in fact that, when perturbation theory applies (see proposition 4.2), the maximal eigenvalue is bounded by an expression containing an H_{-1} norm that, in general, can be bounded from above by:

$$\nu\big(f, -\mathcal{L}_\Lambda^{-1} f\big) \le c\,\ell^2 \mathrm{Var}_\nu(f) \le c\,\ell^2\mathrm{Var}_\mu(f) \tag{6.1}$$

where ν is a canonical measure on the cube Λ of side ℓ with disorder α, μ is the corresponding grand canonical measure (with suitable empirical chemical potential) and f is a (mean zero w.r.t. ν) function. Above we used the spectral gap bound $\mathrm{gap}(\mathcal{L}_\Lambda) \ge c\ell^{-2}$ together with lemma A.5.

When the function f is the spatial average of local functions $\{f_i\}_{i\in I}$ each with support much smaller than Λ it is possible to do better than (6.1). We have in fact:

Lemma 6.1. *Let Λ be a box in $\mathbb{Z}^d$ and $\{\Lambda_i\}_{i\in I}$ be a family of cubes $\Lambda_i \subset \Lambda$ with side R satisfying*

$$\big|\{i \in I \,:\, x\in\Lambda_i\}\big| \le 10^{10d} \quad \forall x\in\Lambda.$$

Let $f = \mathrm{Av}_{i\in I} f_i$ where, for any $i \in I$ and for all α, f_i has support in Λ_i and has zero mean w.r.t. any canonical measure on Λ_i. Then, for any canonical measure ν on Λ with disorder configuration α,

$$\nu\big(f, -\mathcal{L}_\Lambda^{-1} f\big) \le c\,R^2|I|^{-1}\mathrm{Av}_{i\in I}\nu\big(\,\mathrm{Var}_\nu(f_i\,|\,\mathcal{F}_i)\,\big).$$

Proof. Let $\mathcal{F}_i := \sigma(m_{\Lambda_i}, \eta_x$ with $x \notin \Lambda_i)$ and observe that

$$\nu(f_i, g) = \nu\big(\nu(f_i; g\,|\,\mathcal{F}_i)\big) \quad \forall\, g$$

Thus, by Schwarz and Poincaré inequalities and the diffusive scaling of the spectral gap

$$\begin{aligned}|\nu(f,g)| &\le c\,R\,\mathrm{Av}_{i\in I}\,\nu\Big\{\,\big[\mathrm{Var}_\nu(f_i\,|\,\mathcal{F}_i)\mathcal{D}_{\Lambda_i}\big(g;\nu(\cdot|\mathcal{F}_i)\big)\big]^{1/2}\,\Big\}\\ &\le c\,R\,|I|^{-1/2}\Big(\mathrm{Av}_{i\in I}\,\nu\big(\mathrm{Var}_\nu(f_i\,|\,\mathcal{F}_i)\big)\Big)^{1/2}\mathcal{D}_\Lambda(g;\nu)^{1/2}.\end{aligned}$$

It is enough now to take $g=-\mathcal{L}_\Lambda^{-1}f$. □

6.1. **Variance bounds.** One of the key issues is to provide sharp enough upper bounds (see proposition 6.5 below) on the variance

$$\mathrm{Var}_{\mu^{\lambda_0(m)}}\big(\mathrm{Av}_{x\in\Lambda_k}\tau_x\phi_{n,s}\big) \tag{6.2}$$

where n,s,k are positive integers satisfying $n\le s\le k$ and $m\in(0,\frac12)$ and $\phi_{n,s}$ has been defined in (5.1). Actually the method developed below is very general and it can be used to estimate also other similar variances, like for example (6.2) with $\lambda_0(m)$ replaced by the empirical chemical potential $\lambda_{\Lambda_k}(\alpha,m)$.

It is convenient to define first some additional convenient notation besides those already defined at the beginning of section 5:

$$\begin{aligned}\hat\phi_{n,s}(\eta) &:= \mu^{\lambda(m_s^e)}_{\Lambda_s^e}(m_n^{2,e}-m_n^{1,e})\\ \xi_0(m) &:= \mu^{\lambda_0(m)}(\,m_n^{2,e}-m_n^{1,e}\,;N_{\Lambda_n^e})\\ \xi(m) &:= \mu^{\lambda(m)}_{\Lambda_s^e}(\,m_n^{2,e}-m_n^{1,e}\,;N_{\Lambda_s^e})\\ \sigma_0^2(m) &:= \mu^{\lambda_0(m)}(\,m_s^e;N_{\Lambda_s^e})\\ \sigma^2(m) &:= \mu^{\lambda(m)}_{\Lambda_s^e}(\,m_s^e;N_{\Lambda_s^e}),\end{aligned} \tag{6.3}$$

where $N_{\Lambda_n^e}$, $N_{\Lambda_s^e}$ denote the particle number respectively in the box Λ_n^e and Λ_s^e. Let us recall the definition of static compressibility $\chi(m)=\mathbb{E}\big(\mu^{\lambda_0(m)}(\,\eta_0;\eta_0\,)\big)$.
Moreover, given $0<\delta\ll 1$ and a site x, we define the events:

$$\begin{aligned}\mathcal{M}_x(m) &:= \{|m_{x,s}^e(\eta)-m|\ge\sqrt{m}\,s^{-\frac d2+\frac\delta2}\,\}\\ \mathcal{A}_x^{(1)}(m) &:= \Big\{\,\frac1m|m-\mu^{\lambda_0(m)}(m_{x,s}^e)|\ge s^{-\frac d2+\frac\delta2}\,\Big\}\\ \mathcal{A}_x^{(2)}(m) &:= \Big\{\,\Big|\tau_x\frac{\sigma_0^2(m)}{\chi(m)}-1\Big|\ge s^{-\frac d2+\frac\delta2}\,\Big\}\end{aligned} \tag{6.4}$$

Remark 6.2. *Notice that the first event is an event for the particles configuration η while all the others are events for the disorder field.*

Lemma 6.3. *There exists $s_0(\delta)$ such that the following holds for any $s\ge s_0(\delta)$. Assume $n\le s$, $4s^{-d+\delta}\le m\le 1/2$, $\eta\notin\mathcal{M}_x(m)$ and $\alpha\notin\mathcal{A}_x^{(1)}(m)\cup\mathcal{A}_x^{(2)}(m)$. Then, for any site y,*

$$\Big|\nabla_y[\tau_x\hat\phi_{n,s}](\eta)-\frac{(1-2\eta_y)}{2s^d}\tau_x\frac{\xi_0(m)}{\chi(m)}\Big|\le c\,s^{-d}\Big\{\frac{s^{-d}}{m}+\frac{1}{\sqrt m}s^{-\frac d2+\frac\delta2}\Big\}. \tag{6.5}$$

Proof. By Lagrange theorem we can write

$$\nabla_y[\tau_x\hat\phi_{n,s}](\eta)=\int_{m_{x,s}^e(\eta)}^{m_{x,s}^e(\eta^y)}\tau_x\frac{\xi(m')}{\sigma^2(m')}\,dm'. \tag{6.6}$$

Assume m' in the interval with end-points $m_{x,s}^e(\eta)$ and $m_{x,s}^e(\eta^y)$. Then, by lemma A.2,

$$\xi_0(m')\le c\,m',\quad \xi(m')\le c\,m',\quad \sigma_0^2(m')\ge c\,m',\quad \sigma^2(m')\ge c\,m',\quad \chi(m')\ge c\,m.$$

Moreover, since $\eta \notin \mathcal{M}_x(m)$, $m' \geq c\, m$ if s is large enough depending on δ. Therefore, by lemma $A.3$

$$\Big|\tau_x \frac{\xi(m')}{\sigma^2(m')} - \tau_x \frac{\xi_0(m')}{\sigma_0^2(m')}\Big| \leq \frac{c}{m}|m' - \mu^{\lambda_0(m')}(m^e_{x,s})|, \tag{6.7}$$

$$\Big|\tau_x \frac{\xi_0(m')}{\sigma_0^2(m')} - \tau_x \frac{\xi_0(m)}{\sigma_0^2(m)}\Big| \leq \frac{c}{m}|m' - m| \leq \frac{c}{m}|m^e_{x,s}(\eta) - m| + \frac{c}{m}s^{-d}, \tag{6.8}$$

$$\Big|\tau_x \frac{\xi_0(m)}{\sigma_0^2(m)} - \tau_x \frac{\xi_0(m)}{\chi(m)}\Big| \leq c\Big|\tau_x \frac{\sigma_0^2(m)}{\chi(m)} - 1\Big|. \tag{6.9}$$

By lemma A.3 and the assumption $\alpha \notin \mathcal{A}^{(1)}_x(m)$, the r.h.s. of (6.7) can be bounded from above by

$$\frac{c}{m}|m - \mu^{\lambda_0(m)}(m^e_{x,s})| + \frac{c}{m}s^{-d} \leq c\big[\, s^{-\frac{d}{2}+\frac{\delta}{2}} + \frac{1}{m}s^{-d}\big]. \tag{6.10}$$

Similarly, the contribution of the r.h.s. of (6.8) together with (6.9) can be bounded from above by

$$c\Big[\frac{s^{-d}}{m} + \frac{1}{\sqrt{m}}s^{-\frac{d}{2}+\frac{\delta}{2}}\Big] \tag{6.11}$$

The thesis follows immediately from (6.6) together with (6.10), (6.11). ☐

Lemma 6.4. *There exists $s_0(\delta)$ such that the following holds for any $s \geq s_0(\delta)$. Let $n \leq s$, $m \in (0, \frac{1}{2})$, and let $\lambda(\alpha)$ be a bounded measurable function such that for any disorder configuration α*

$$|\lambda(\alpha) - \lambda_0(m)| \leq s^{-\frac{d}{2}+\frac{\delta}{4}}. \tag{6.12}$$

Then, for any $s \geq s_0(\delta)$ and any finite set $\Delta \subset \mathbb{Z}^d$,

$$\mathbb{P}\Big(\mu^{\lambda(\alpha)}\big(\cup_{x\in\Delta}\mathcal{M}_x(m)\big) \geq |\Delta| e^{-s^{\delta/2}}\Big) \leq c\, e^{-s^{\delta/2}}. \tag{6.13}$$

Proof. By the Chebyshev inequality and the translation invariance of $\mathbb{P}$, the l.h.s. of (6.13) can be bounded from above by $\exp(s^{\delta/2})\mathbb{E}\big[\mu^{\lambda(\alpha)}(\mathcal{M}_0(m))\big]$.
Let us bound the term

$$e^{s^{\frac{\delta}{2}}}\mathbb{E}\big[\, \mu^{\lambda(\alpha)}\big(m^e_s - m \geq \sqrt{m}\, s^{-\frac{d}{2}+\frac{\delta}{2}}\big)\big]. \tag{6.14}$$

Thanks again to Chebyshev inequality, for any $0 < t < 1$ (6.14) can be bounded from above by

$$e^{s^{\frac{\delta}{2}} - 2t\sqrt{m}s^{\frac{d}{2}+\frac{\delta}{2}}}\mathbb{E}\big[\prod_{x\in\Lambda^e_s} \mu^{\lambda(\alpha)}\big(e^{t(\eta_x - m)}\big)\big]. \tag{6.15}$$

Using the basic assumption (6.12) and Lagrange theorem, it is not difficult to see that

$$\mu^{\lambda(\alpha)}\big(e^{t(\eta_x-m)}\big) \leq (1 + c\,tms^{-\frac{d}{2}+\frac{\delta}{4}})\mu^{\lambda_0(m)}\big(e^{t(\eta_x-m)}\big)$$

so that (6.15) is bounded from above by

$$e^{s^{\frac{\delta}{2}} - 2t\sqrt{m}s^{\frac{d}{2}+\frac{\delta}{2}} + c\,tms^{\frac{d}{2}+\frac{\delta}{4}}}\mathbb{E}\big[\mu^{\lambda_0}\big(e^{t(\eta_0-m)}\big)\big]^{2s^d}.$$

Since $e^x \leq 1 + x + 2\,x^2$ if $|x| \leq 1$, the above expression is bounded from above by

$$\exp\big(s^{\frac{\delta}{2}} - 2t\sqrt{m}\, s^{\frac{d}{2}+\frac{\delta}{2}} + c\,t\,ms^{\frac{d}{2}+\frac{\delta}{4}} + c\,t^2ms^d\big).$$

The thesis follows by choosing t such that $t^2 m = s^{-d+\delta/2}$. $\square$

We are finally in a position to state our main bound on the variance appearing in (6.2).

Proposition 6.5. *For $d \geq 2$ there exists $s_0(\delta)$ such that the following holds for any $s \geq s_0(\delta)$. Let $m \in (0, \frac{1}{2})$ and let $n \leq s \leq k \leq 1000s$. Then there exists a measurable set $\mathcal{A}$ with $\mathbb{P}(\mathcal{A}) \leq k^{2d} e^{-cs^{\delta/2}}$ such that*

$$\mathrm{Var}_{\mu^{\lambda_0(m)}}(\mathrm{Av}_{x\in\Lambda_k}\tau_x\phi_{n,s}) \leq c\mathbb{I}_{\mathcal{A}^c}(\alpha)s^{-2d+2\delta} + \mathbb{I}_{\mathcal{A}}(\alpha). \tag{6.16}$$

Proof. Let us consider first the case of "low density" $m \leq 4s^{-d+\delta}$.
Since $|\tau_x\phi_{n,s}| \leq c\, m^e_{x,s}$, $|\mathrm{Av}_{x\in\Lambda_k}\tau_x\phi_{n,s}| \leq c\, m_{\Lambda_{2k}}$ and therefore the l.h.s. of (6.16) can be bounded from above by

$$\mu^{\lambda_0(m)}(m^2_{\Lambda_{2k}}) \leq c(k^{-d}m + m^2) \leq cs^{-2d+2\delta}.$$

Let us now consider the "high density" case $m \geq 4s^{-d+\delta}$.
By the equivalence of ensembles (see proposition A.4), in the l.h.s. of (6.16) $\phi_{n,s}$ can be substituted by $\hat{\phi}_{n,s}$ with an error of order s^{-2d}. Therefore, by the Poincaré inequality

$$\mathrm{Var}_{\mu^{\lambda_0(m)}}(f) \leq c\, m\, \mu^{\lambda_0(m)}(\sum_y |\nabla_y f|^2), \tag{6.17}$$

it is enough to estimate

$$c\, m\, \mu^{\lambda_0(m)}\Big[\frac{1}{k^{2d}}\sum_{y\in\Lambda_{2k}}\big(\sum_{x\in\Lambda_k\cap\Lambda_{y,s}}\nabla_y[\tau_x\hat{\phi}_{n,s}]\big)^2\Big]. \tag{6.18}$$

To this aim we set (recall (6.4))

$$\begin{aligned}
&\mathcal{M} := \cup_{x\in\Lambda_k}\mathcal{M}_x(m) \qquad \mathcal{A}_0 := \big\{\mu^{\lambda_0(m)}\big(\mathcal{M}\big) \geq k^d\exp(-s^{\delta/2})\,\big\},\\
&\mathcal{A}_1 := \cup_{x\in\Lambda_k}\mathcal{A}^{(1)}_x(m) \qquad \mathcal{A}_2 := \cup_{x\in\Lambda_k}\mathcal{A}^{(2)}_x(m)\\
&\mathcal{A}_3 := \cup_{y\in\Lambda_{2k}}\Big\{\,\Big|\mathrm{Av}_{x\in\Lambda_k\cap\Lambda_{y,s}}\Big[\tau_x\,\frac{\xi_0(m)}{\chi(m)}\Big]\,\Big| \geq |\Lambda_k\cap\Lambda_{y,s}|^{-\frac{1}{2}+\frac{\delta}{2d}}\,\Big\},\\
&\mathcal{A} := \mathcal{A}_0\cup\mathcal{A}_1\cup\mathcal{A}_2\cup\mathcal{A}_3.
\end{aligned}$$

We first estimate

$$\mathbb{I}_{\mathcal{A}^c}(\alpha)m\,\mu^{\lambda_0(m)}\Big[\mathbb{I}_{\mathcal{M}^c}\frac{1}{k^{2d}}\sum_{y\in\Lambda_{2k}}\big(\sum_{x\in\Lambda_k\cap\Lambda_{y,s}}\nabla_y[\tau_x\hat{\phi}_{n,s}]\big)^2\Big]. \tag{6.19}$$

By lemma 6.3, for s large enough (6.19) can be bounded from above by

$$\frac{c}{k^{2d}}\mathbb{I}_{\mathcal{A}^c}(\alpha)\mathrm{Av}_{y\in\Lambda_k}\Big[\frac{1}{s^d}\sum_{x\in\Lambda_k\cap\Lambda_{y,s}}\tau_x\frac{\xi_0(m)}{\chi(m)}\Big]^2 + c\,\frac{s^{-d+\delta}}{k^d} \tag{6.20}$$

By straightforward computations and the definition of $\mathcal{A}_3$ the first addendum in (6.20) can be bounded by $c\,k^{-d}s^{-d+\delta}$. Moreover, because of the definition of $\mathcal{A}_0$, expression (6.19) with $\mathbb{I}_{\mathcal{M}^c}$ replaced by $\mathbb{I}_{\mathcal{M}}$ can be bounded by $c\,s^{2d}e^{-s^{\delta/2}}$.
In conclusion

$$\mathbb{I}_{\mathcal{A}^c}(\alpha)m\,\mu^{\lambda_0(m)}\Big[\frac{1}{k^{2d}}\sum_{y\in\Lambda_{2k}}\big(\sum_{x\in\Lambda_k\cap\Lambda_{y,s}}\nabla_y[\tau_x\hat{\phi}_{n,s}]\big)^2\Big] \leq c\,\big[\,k^{-d}s^{-d+\delta} + s^{2d}e^{-s^{\delta/2}}\,\big].$$

It remains to prove that $\mathbb{P}(\mathcal{A}) \leq k^{2d}e^{-c\,s^{\delta/2}}$. To this aim we set

$$f_1(\alpha) := 1 - \mu^{\lambda_0(m)}(\eta_0)\,,$$
$$f_2(\alpha) := 1 - \mu^{\lambda_0(m)}(\eta_0;\eta_0)/\sigma_0^2(m)\,,$$
$$f_3(\alpha) := \big(\mu^{\lambda_0(m)}(\eta_{ne};\eta_{ne}) - \mu^{\lambda_0(m)}(\eta_0;\eta_0)\big)/\chi(m).$$

By lemma 6.4 $\mathbb{P}(\mathcal{A}_0) \le c\,e^{-s^{\delta/2}}$ while $\mathbb{P}(\mathcal{A}_x^{(1)}(m))$ and $\mathbb{P}(\mathcal{A}_x^{(2)}(m))$ can be bounded from above by $e^{-c\,s^{\delta}}$ by means of lemma $A.1$ with $f = f_1$ and $f = f_2$ respectively. Therefore

$$\mathbb{P}(\mathcal{A}_1) + \mathbb{P}(\mathcal{A}_2) \le k^d e^{-c\,s^{\delta}} \tag{6.21}$$

In order to bound $\mathbb{P}(\mathcal{A}_3)$ we observe that

$$\mathrm{Av}_{x\in\Lambda_k\cap\Lambda_{y,s}}\tau_x\frac{\xi_0(m)}{\chi(m)} = \mathrm{Av}_{z\in\Lambda_n^{1,e}}\mathrm{Av}_{x\in\Lambda_k\cap\Lambda_{y,s}}\tau_{x+z}f_3\,.$$

Thus

$$\mathcal{A}_3 \subset \cup_{y\in\Lambda_{2k}}\cup_{z\in\Lambda_n^{1,e}}\mathcal{A}_3(y,z)$$

where

$$\mathcal{A}_3(y,z) = \{|\mathrm{Av}_{x\in\Lambda_k\cap\Lambda_{y,s}}\tau_{x+z}f_3| \ge |\Lambda_k\cap\Lambda_{y,s}|^{-\frac{1}{2}+\frac{\delta}{2d}}\}\,.$$

Using once more lemma A.1 we get

$$\mathbb{P}(\mathcal{A}_3(y,z)) \le \exp(-cs^{\frac{d-1}{d}\delta})$$

and the proof is complete. □

We conclude this part with a slight modification of proposition 6.5.

Proposition 6.6. *Let $n \le s$ be positive integers and let $0 < \delta \ll 1$. Let also $\gamma > 0$ and set $J_s = \{1/s^\gamma, 2/s^\gamma, \dots 1 - 1/s^\gamma\}$. Then there exists a set $\mathcal{A}$ of disorder configurations α in Λ_{2s} satisfying*

$$\mathbb{P}(\mathcal{A}) \le s^\gamma\, e^{-c\,s^\delta}$$

and such that, for s large enough depending on δ,

$$\sup_{m\in J_s}\mathrm{Var}_{\mu^{\lambda(m)}_{\Lambda_{2s}}}\Big(\mathrm{Av}^{(n)}_{x\in\Lambda_s}\tau_x\phi_{n,s}\Big) \le c\mathbb{I}_{\mathcal{A}^c}(\alpha)s^{-2d+2\delta} + \mathbb{I}_{\mathcal{A}}(\alpha) \tag{6.22}$$

where $\mathrm{Av}^{(n)}_{x\in\Lambda_s} := \mathrm{Av}_{x\in\Lambda_s\cap n\mathbb{Z}^d}$.

Proof. The proposition can be proved as proposition 6.5 with some slight modifications that we comment. For any $m \in J_s$ it is convenient to define $\mathcal{M}(m)$, $\mathcal{A}_1(m)$, and $\mathcal{A}_2(m)$ as done respectively for $\mathcal{M}$, $\mathcal{A}_1$, and $\mathcal{A}_2$ in the proof of proposition 6.5 and to set

$$\mathcal{A}_0(m) := \{\mu^{\lambda(m)}_{\Lambda_{2s}}(\mathcal{M}) \ge s^d\exp(-s^{\delta/2})\},$$
$$\mathcal{A}_3(m) := \Big\{\Big|\mathrm{Av}^{(n)}_{x\in\Lambda_s}\tau_x\frac{\xi_0(m)}{\chi(m)}\Big| \ge s^{-\frac{d}{2}+\frac{\delta}{2}}\Big\}.$$

Then one sets again $\mathcal{A}(m) := \mathcal{A}_0(m)\cup\mathcal{A}_1(m)\cup\mathcal{A}_2(m)\cup\mathcal{A}_3(m)$, $\mathcal{A} := \cup_{m\in J_s}\mathcal{A}(m)$. By the same arguments as in the proof of proposition 6.5 one obtains (6.22).
Let us prove the estimate $\mathbb{P}(\mathcal{A})$ or, equivalently, that for any $m \in J_s$ $\mathbb{P}(\mathcal{A}(m)) \le e^{-cs^\delta}$. For this purpose, given $m \in J_s$, it is convenient to define

$$\mathcal{B}(m) := \{\,|\lambda_{\Lambda_{2s}}(m) - \lambda_0(m)| \ge s^{-\frac{d}{2}+\frac{\delta}{4}}\,\}$$

and write

$$\mathbb{P}(\mathcal{A}(m)) \le \mathbb{P}(\mathcal{B}(m)) + \mathbb{P}(\mathcal{B}^c(m)\cap\mathcal{A}_0(m)) + \mathbb{P}(\mathcal{A}_1(m)) + \mathbb{P}(\mathcal{A}_2(m)) + \mathbb{P}(\mathcal{A}_3(m)). \tag{6.23}$$

Let us suppose $0 < m \leq \frac{1}{2}$. Then lemma A.3 implies that

$$\left|\lambda_{\Lambda_{2s}}(m) - \lambda_0(m)\right| \leq c\left|1 - m^{-1}\mu^{\lambda_0(m)}(m_{\Lambda_{2s}})\right|.$$

Thanks to the above estimate and to lemma A.1 applied with $f := 1 - m^{-1}\mu^{\lambda_0(m)}(\eta_0)$, the first term in the r.h.s. of (6.23) is smaller than $e^{-c\,s^{\delta/2}}$. The second term is smaller than $e^{-cs^{\delta/2}}$ by lemma 6.4. Moreover, $\mathbb{P}(\mathcal{A}_1(m))$ and $\mathbb{P}(\mathcal{A}_2(m))$ can be bounded by $s^d e^{-c\,s^\delta}$ as in the proof of proposition 6.5.
Finally, let us consider $\mathbb{P}(\mathcal{A}_3(m))$. For simplicity of notation we restrict to the case $d = 1$ and we write

$$n \sum_{x\in\Lambda_s\cap n\mathbb{Z}} \tau_x \frac{\xi_0(m)}{\chi(m)} = \Big(\sum_{x\in\Lambda_s\cap 2n\mathbb{Z}} \sum_{z\in\Lambda^e_{1,n}} \tau_{x+z} f\Big) + \Big(\sum_{x\in\Lambda_s\cap(2n+n)\mathbb{Z}} \sum_{z\in\Lambda^e_{1,n}} \tau_{x+z} f\Big) \tag{6.24}$$

where $f := \chi(m)^{-1}\big(\mu^{\lambda_0(m)}(\eta_0;\eta_0) - \mu^{\lambda_0(m)}(\eta_{ne};\eta_{ne})\big)$. We remark that in both the addenda in the r.h.s. of (6.24) the appearing functions have disjoint support and form a set of cardinality $O(k^d)$, moreover $\mathbb{E}(f) = 0$. Therefore, by the same arguments used in the proof of lemma A.1, we obtain that $\mathbb{P}(\mathcal{A}_3(m)) \leq e^{-cs^\delta}$. □

6.2. **An L^∞ bound.** We conclude this section with a simple L^∞ bound on $|\tau_x\phi_{s,s'}|$ when s scales as an inverse power of ϵ.

Lemma 6.7. *Let $0 < \gamma < 1$ and $0 < \delta \ll 1$ and set $s = O(\epsilon^{-\gamma})$. Then, for almost all configuration disorder α and ϵ small enough,*

$$\sup_{x\in\mathbb{T}^d_\epsilon} |\tau_x\phi_{s,s'}| \leq c\,s^{-\frac{d}{2}+\delta} \quad \forall s' \in [s, \epsilon^{-1}]. \tag{6.25}$$

Proof. By the equivalence of ensembles it is enough to prove (6.25) with $\phi_{s,s'}$ replaced by $\hat\phi_{s,s'}$. Using lemma $A.3$ we get

$$|\hat\phi_{s,s'} - \mu^{\lambda_0(m^e_{s'})}\big(m^{1,e}_s - m^{2,e}_s\big)| \leq c\,|m^e_{s'}(\eta) - \mu^{\lambda_0}(m^e_{s'})| \tag{6.26}$$

and similarly upon translation by x.
Let us define

$$\mathcal{D}_x(m) := \{\, |m - \mu^{\lambda_0(m)}(m^e_{x,s'})| \geq (s')^{-\frac{d}{2}+\delta}\,\}$$
$$\mathcal{D}'_x(m) := \{\, |\mu^{\lambda_0(m)}\big(\tau_x(m^{1,e}_s - m^{2,e}_s)\big)| \geq s^{-\frac{d}{2}+\delta}\,\}$$
$$\mathcal{D} := \cup_m \cup_{x\in\mathbb{T}^d_\epsilon} \big(\mathcal{D}_x(m) \cup \mathcal{D}'_x(m)\big)$$

where, in the last formula, m varies among all possible values of $m^e_{s'}$.
$\mathbb{P}(\mathcal{D}_x(m))$ and $\mathbb{P}(\mathcal{D}'_x(m))$ can now be estimated from above by $e^{-cs^{2\delta}}$ thanks to lemma $A.1$ applied to $f(\alpha) = \mu^{\lambda_0(m)}(\eta_0) - m$ and $f(\alpha) = \mu^{\lambda_0(m)}(\eta_0;\eta_0) - \mathbb{E}\big[\mu^{\lambda_0(m)}(\eta_0;\eta_0)\big]$ respectively. Therefore, $\mathbb{P}(\mathcal{D}) \leq \epsilon^{-2d}e^{-cs^{2\delta}}$ and a simple use of Borel-Cantelli lemma proves the thesis. □

7. Central Limit Theorem Variance

In this section we investigate the structure of the space $\mathcal{G}$ that we recall was defined as (see (4.15))

$$\mathcal{G} := \{g \in \mathbb{G} \,:\, \exists\,\Lambda \in \mathbb{F} \text{ such that, } \forall\alpha \text{ and } \forall\nu \in \mathcal{M}^\alpha(\Lambda)\,,\ \nu(g) = 0\,\}$$

endowed with the non negative semi-inner product

$$V_m(f,g) := \lim_{\ell\uparrow\infty} V_m^{(\ell)}(f,g) \tag{7.1}$$

where

$$V_m^{(\ell)}(f,g) := (2l)^{-d}\,\mathbb{E}\Big[\mu^{\lambda_0(m)}\big(\sum_{|x|\le\ell_1}\tau_x f, (-\mathcal{L}_{\Lambda_\ell})^{-1}\sum_{|x|\le\ell_1}\tau_x g\big)\Big]\,,\quad m\in(0,1)$$

with $\ell_1 := \ell-\sqrt{\ell}$. For $m=0,1$ we simply define $V^{(m)}(f,g)=V_\ell^{(m)}(f,g)=0$
In all what follows we fix a density $m\in(0,1)$ that, most of the times, will not appear inside the notation and we denote by $\mathbb{P}^*$ the annealed probability measure on $\tilde{\Omega}$ characterized by

$$\mathbb{P}^*(d\alpha,d\eta)=\mathbb{P}(d\alpha)\mu^{\alpha,\lambda_0(m)}(d\eta).$$

We remark that $\mathbb{P}^*$ is translation invariant and we write $\mathbb{E}^*$ for the corresponding expectation.

7.1. **The pre-Hilbert space $\mathcal{G}$.** In what follows we prove that the semi–inner product V is well defined and that the subspace generated by the currents $j_{0,e}$, $e\in\mathcal{E}$, and by the fluctuations $\mathcal{L}g$, $g\in\mathbb{G}$, is dense in $\mathcal{G}$. To this aim we need to generalize the standard theory ([23] and references therein), based on closed and exact forms, to the disordered case. The main new feature in the disordered case is a richer structure of the space of closed forms which requires a proper analysis.
We begin with a *table of calculus* that can be easily checked as in the non disordered case. For any $f\in\mathcal{G}$, $u\in\mathbb{G}$ and $e\in\mathcal{E}$ let

$$t_e(f) := \sum_{x\in\mathbb{Z}^d}(x,e)\mathbb{E}^*(\eta_x,f),\qquad (f,u)_0 := \sum_{x\in\mathbb{Z}^d}\mathbb{E}^*(\tau_x f,u).$$

Lemma 7.1. *For any $f\in\mathcal{G}$, $u\in\mathbb{G}$ and $e,e'\in\mathcal{E}$*

$$\begin{aligned}
&V(f,\mathcal{L}u) = -(f,u)_0, && V(\mathcal{L}u,\mathcal{L}u)=\sum_{e\in\mathcal{E}}\frac{1}{2}\mathbb{E}^*\left(c_{0,e}(\nabla_{0,e}\underline{u})^2\right),\\
&V(j_{0,e},g) = -t_e(f), && V(j_{0,e},j_{0,e'})=\frac{1}{2}\mathbb{E}^*\left(\,c_{0,e}(\nabla_{0,e}\eta_0)^2\right)\delta_{e,e'},\\
&V(j_{0,e},\mathcal{L}u) = -\frac{1}{2}\mathbb{E}^*\left(c_{0,e}\nabla_{0,e}\underline{u}\cdot\nabla_{0,e}\eta_0\,\right).
\end{aligned}$$

The main result of this paragraph is the following.

Theorem 7.2.

i) For any $f,g\in\mathcal{G}$ the limit $V(f,g) := \lim_{\ell\uparrow\infty}V_\ell(f,g)$ exists, it is finite and it defines a non negative semi–inner product on $\mathcal{G}$. In particular $V(f) := \lim_{\ell\uparrow\infty}V_\ell(f,f)$ is well defined.

ii) For any $f\in\mathcal{G}$

$$\begin{aligned}
V(f) &= \sup_{a\in\mathbb{R}^d}\sup_{u\in\mathbb{G}}\Big\{2V(f,\sum_{e\in\mathcal{E}}a_e j_e+\mathcal{L}u)-V(\sum_{e\in\mathcal{E}}a_e j_e+\mathcal{L}u)\Big\}\\
&= \sup_{a\in\mathbb{R}^d}\sup_{u\in\mathbb{G}}\Big\{\sum_{e\in\mathcal{E}}2a_e t_e(f)+2(f,u)_0-\sum_{e\in\mathcal{E}}\frac{1}{2}\mathbb{E}^*\left(c_{0,e}(a_e\nabla_e\eta_0-\nabla_e\underline{u})^2\right)\Big\}\,.
\end{aligned} \tag{7.2}$$

iii) The subspace

$$\Big\{ \sum_{e\in\mathcal{E}} a_e j_{0,e} + \mathcal{L}u \ : \ a \in \mathbb{R}^d,\ u \in \mathbb{G} \Big\} \tag{7.3}$$

is dense in $\mathcal{G}$ endowed of the semi–inner product V.

Notice that lemma 7.1 proves that the two expressions appearing in the r.h.s. of the first equality sign in (7.2) are equal.
Before proving the theorem we need to introduce the notion of closed and exact forms together with their generalization to the disordered case and prove few preliminary results. We refer the reader to [15] for a complete treatment.

Definition 7.3. *A form on Ω is a family $\xi = \{\xi_b\}_{b\subset\mathbb{Z}^d}$ of functions $\xi_b : \Omega \to \mathbb{R}$. It is called closed if, given $\eta \in \Omega$ and bonds $b_1, \dots, b_n$ with $\eta = S_{b_n} \circ \cdots \circ S_{b_2} \circ S_{b_1}(\eta)$, then*

$$\sum_{i=1}^{n} \xi_{b_i}(\eta_{i-1}) = 0 \quad \textit{where} \quad \eta_0 := \eta,\ \eta_i := S_{b_i} \circ \cdots \circ S_{b_2} \circ S_{b_1}(\eta)\ \ \forall i = 1, \dots, n.$$

The expression $\sum_{i=1}^{n} \xi_{b_i}(\eta_{i-1})$ can be thought of as the integral of the form ξ on the closed path $\eta_0 = \eta, \eta_1, \dots, \eta_n = \eta$. It can be proved, see [15], that a form on Ω is closed if and only if it satisfies the following properties **P.1**, **P.2** and **P.3**.

P.1. Let $a, v, w \in \mathbb{Z}^d$ with $|v| = |w| = 1$ and $v \pm w \neq 0$. We set $c = a + v + w$, $x = a + v$, $x' = a + w$, $b_1 = \{a, x\}$, $b_2 = \{x, c\}$, $b_1' = \{a, x'\}$, $b_2' = \{x', c\}$. Then

$$\xi_{b_1} \circ S_{b_2} \circ S_{b_1} + \xi_{b_2} \circ S_{b_1} + \xi_{b_1} = \xi_{b_1'} \circ S_{b_2'} \circ S_{b_1'} + \xi_{b_2'} \circ S_{b_1'} + \xi_{b_1'}.$$

P.2. For any couple of bonds $b_1, b_2 \subset \mathbb{Z}^d$ such that $b_1 \cap b_2 = \emptyset$,

$$\xi_{b_2} \circ S_{b_1} + \xi_{b_1} = \xi_{b_1} \circ S_{b_2} + \xi_{b_2}.$$

P.3 . For any bond $b \subset \mathbb{Z}^d$,

$$\xi_b \circ S_b + \xi_b = 0.$$

The above characterization allows us to generalize the definition of closed forms to the disorder case.

Definition 7.4. *A form in $L^2(\mathbb{P}^*)$ is a family of functions $\xi = \{\xi_b\}_{b\subset\mathbb{Z}^d}$ with $\xi_b \in L^2(\mathbb{P}^*)$.*
A form ξ is called closed if it satisfies properties **P.1***,* **P.2** *and* **P.3** *where equalities are in $L^2(\mathbb{P}^*)$. A form $\xi = \{\xi_b\}_{b\subset\mathbb{Z}^d}$ if called exact is $\xi_b = \nabla_b \underline{u}$ for some $u \in \mathbb{G}$. A form ξ is called translation covariant if $\tau_x \xi_b = \xi_{b+x}$ for any $x \in \mathbb{Z}^d$, $b \subset \mathbb{Z}^d$.*

It is easy to check that exact forms are automatically closed and translation covariant. Given a closed form ξ in $L^2(\mathbb{P}^*)$ the form on Ω $\{\xi_b(\alpha, \cdot)\}_{b\subset\mathbb{Z}^d}$ is a closed form on Ω for almost any disorder configuration α.
In what follows by a form we will always mean a form in $L^2(\mathbb{P}^*)$.

Definition 7.5. *A family of functions $\xi = \{\xi_e\}_{e\in\mathcal{E}}$, $\xi_e \in L^2(\mathbb{P}^*)$, is called the germ of the form $\xi' = \{\xi_b'\}_{b\subset\mathbb{Z}^d}$ if $\xi'_{x,x+e} = \tau_x \xi_e$ for any $x \in \mathbb{Z}^d$ and $e \in \mathcal{E}$.*

It follows that ξ' is automatically translation covariant as soon as it is generated by a germ ξ.
Within the subset of closed and translation covariant forms we consider the special family $\{\mathfrak{U}^e\}_{e\in\mathcal{E}}$ defined by

$$\mathfrak{U}^e_{x,x+e'}(\eta) := \delta_{e,e'}(\eta_{x+e} - \eta_x)\,, \quad \forall x \in \mathbb{Z}^d,\ e, e' \in \mathcal{E}.$$

It is simple to check that the form $\mathfrak{U}^e$ is not exact. Finally, we define Ξ_C as the set of germs of closed forms and

$$\Xi_0 := \{\xi = \{\xi_e\}_{e\in\mathcal{E}} \,:\, \exists a \in \mathbb{R}^d, u \in \mathbb{G} \quad \text{with} \quad \xi_e = a_e\mathfrak{U}^e + \nabla_e \underline{u} \quad \forall e \in \mathcal{E}\,\}.$$

We remark that $\Xi_0 \subset \Xi_C$ and that Ξ_C is a closed subspace in $L^2(\otimes^d\mathbb{P}^*)$. A deeper result is given by the following density theorem.

Theorem 7.6. $\Xi_C = \overline{\Xi}_0$ *in* $L^2(\otimes^d\mathbb{P}^*)$.

Proof. The proof follows closely the proof of theorem 4.14 in appendix 3 of [23] with the exception of the last step. As in [23] it can be proved that for any $\xi \in \Xi_C$ there exists a germ $\omega \in \Xi_C$ with the following properties:

i) $\xi - \omega \in \overline{\Xi}_0$;
ii) ω can written as $\omega = \omega_- + \omega_+$ with $\omega_\pm = \{\omega_{\pm,e}\}_{e\in\mathcal{E}}$, $\omega_{\pm,e}(\alpha,\eta) = \omega_{\pm,e}(\alpha,\eta_0,\eta_e)$ such that $\forall e \in \mathcal{E}$

$$\begin{aligned}
&\omega_{-,e}(\alpha,\eta_0,\eta_{2e}) - \omega_{-,e}(\alpha,\eta_0,\eta_e) = \omega_{-,e}(\alpha,\eta_e,\eta_{2e}),\\
&\omega_{+,e}(\alpha,\eta_{-e},\eta_e) - \omega_{+,e}(\alpha,\eta_0,\eta_e) = \omega_{+,e}(\alpha,\eta_{-e},\eta_0)\,.
\end{aligned} \tag{7.4}$$

It remains to prove that $\omega \in \overline{\Xi}_0$. Because of (7.4), $\forall e \in \mathcal{E}$ there exists $a_{\pm,e} \in L^2(\mathbb{P})$ such that $\omega_{\pm,e} = a_{\pm,e}(\alpha)(\eta_e - \eta_0)$. Lemma 7.7 then completes the proof of the theorem.

Lemma 7.7. *Let* $\omega \in \Xi_C$ *such that for any* $e \in \mathcal{E}$ *there exists* $a_e \in L^2(\mathbb{P})$ *with* $\omega_e = a_e(\alpha)(\eta_e - \eta_0)$. *Then* $\omega \in \overline{\Xi}_0$.

Proof. By subtracting $\sum_{e\in\mathcal{E}} \mathbb{E}(a_e)\,\mathfrak{U}^e$ from the germ ω, we can assume that $\mathbb{E}(a_e) = 0$ for any $e \in \mathcal{E}$. In what follows we denote the form generated by the germ ω by the same symbol ω.

Given $x \in \mathbb{Z}^d$ let $\eta^{(x)} \in \Omega$ the configuration with just one particle at x and let $\{b_1,\dots,b_r\}$ be a sequence of bonds such that $\eta^{(x)} = S_{b_r} \circ \cdots \circ S_{b_2} \circ S_{b_1}(\eta^{(0)})$. Define

$$g_x(\alpha) = \sum_{i=1}^r \omega_{b_i}(\alpha,\eta_{i-1}) \quad \text{where} \quad \eta_i := S_{b_i} \circ \cdots \circ S_{b_2} \circ S_{b_1}(\eta^{(0)}) \quad \forall i = 1,\dots,r. \tag{7.5}$$

Notice that, since $\{\omega_b(\alpha,\cdot)\}_{b\subset\mathbb{Z}^d}$ is a closed form on Ω for almost any α, the definition of g_x does not depend on the particular choice of the bonds $b_1,\dots,b_r$ and the family $\{g_x\}_{x\in\mathbb{Z}^d}$ satisfies

$$g_{x+e} - g_x = \tau_x a_e \quad \forall x \in \mathbb{Z}^d,\ e \in \mathcal{E}\,.$$

Therefore, by setting $h_n := -\sum_{x\in\Lambda_n} g_x(\alpha)\eta_x$, we get

$$\nabla_b h_n = \omega_b \qquad \forall\, n \in \mathbb{N},\ b \in \Lambda_n \tag{7.6}$$

In order to conclude the proof it is enough to show that

$$\lim_{n\uparrow\infty} \omega_e^{(n)} = \omega_e \quad \forall e \in \mathcal{E} \quad \text{where} \quad \omega_b^{(n)} := \frac{1}{(2n)^d}\nabla_b\,\underline{h}_n \in \Xi_0.$$

By translation covariance and (7.6) $\nabla_{0,e}\tau_x h_n = \omega_e$ if $-x, -x+e \in \Lambda_n$. Thus, for any $e \in \mathcal{E}$, we can write

$$\omega_e^{(n)} = \frac{(2n+1)^{d-1}}{(2n)^{d-1}}\omega_e + \frac{1}{(2n)^d}\sum_{\substack{x\in\Lambda_n\\ x_e=n}} \tau_{-x}\nabla_{x,x+e}h_n + \frac{1}{(2n)^d}\sum_{\substack{x\in\Lambda_n-e\\ x_e=-n-1}} \tau_{-x}\nabla_{x,x+e}h_n\,. \tag{7.7}$$

and we are left with the proof that the second and third term in the r.h.s.of (7.7) tend to 0 in $L^2(\mathbb{P}^*)$. Let us consider the second term (the third one being similar). By Schwarz inequality and the identity

$$\nabla_{x,x+e}h_n = -g_x(\alpha)(\eta_{x+e}-\eta_x) \quad \forall x \in \Lambda_n \text{ with } x_e = n,$$

it is enough to show that

$$\lim_{n\uparrow\infty} \frac{1}{n^{d+1}} \sum_{\substack{x\in\Lambda_n\\ x_e=n}} \mathbb{E}(g_x^2) = 0. \tag{7.8}$$

To this aim, given $x=(x_1,\dots,x_d)$, we choose the bonds $b_1,\dots,b_r$ in the definition (7.5) in such a way that $\eta_i = \eta^{(y_i)}$ where y_0 is the origin of $\mathbb{Z}^d$, $y_r := x$ and in general $y_0, y_1, \dots, y_r$ are the points encountered by moving in $\mathbb{Z}^d$ first from $(0,\dots,0)$ to $(x_1,0,\dots,0)$ in the first direction, then from $(x_1,0,\dots,0)$ to $(x_1,x_2,0,\dots,0)$ in the second direction and so on until arriving to x.
Given this choice, it is simple to verify that for any $x\in\Lambda_n$ and $e\in\mathcal{E}$ there exists $z_e\in\Lambda_n$ and an integer $k_e\in[0,n]$ such that

$$g_x^2 \le c\sum_{e\in\mathcal{E}}\Big(\sum_{s=0}^{k_e}\tau_{z_e+se}\, a_e\Big)^2.$$

Therefore, in order to prove (7.8), we need to show that

$$\lim_{n\uparrow\infty} \sup_{k=0,1,\dots,n} \frac{1}{n^2}\mathbb{E}\Big(\big(\sum_{s=0}^{k}\tau_{se}\, a_e\big)^2\Big) \quad \forall e\in\mathcal{E}.$$

To this aim, for simplicity of notation, we fix $e\in\mathcal{E}$ and we write a_s in place of $\tau_{se}a_e$. Moreover, for any $r\in\mathbb{N}$ we set $a_s^{(r)} := \mathbb{E}[a_s \mid \alpha_{\Lambda_{se,r}}]$. Since $a_s^{(r)} = \tau_{se}a_0^{(r)}$ and $\mathbb{E}(a_s^{(r)})=0$, we have for any $0\le k\le n$

$$\begin{aligned}\frac{1}{n^2}\mathbb{E}\Big(\big(\sum_{s=0}^{k}a_s\big)^2\Big) &\le 2\frac{1}{n^2}\mathbb{E}\Big(\big(\sum_{s=0}^{k}[a_s-a_s^{(r)}]\big)^2\Big) + 2\frac{1}{n^2}\mathbb{E}\Big(\big(\sum_{s=0}^{k}a_s^{(r)}\big)^2\Big)\\ &\le 2\mathbb{E}\Big(\big(a_0-a_0^{(r)}\big)^2\Big) + \frac{c(r)}{n}.\end{aligned}$$

and the thesis follows. □

The connection between the forms and the space $\mathcal{G}$ endowed with the semi–inner product $V(f,g)$ is clarified by next proposition, which can be proved, following [23] and [37], as explained in section 5.5 of [15].

Proposition 7.8. *Given $f\in\mathcal{G}$ and $e\in\mathcal{E}$ there exists a function $\phi_e\in\mathbb{G}$ such that*

$$\sup_{\xi\in\Xi_0}\Theta_f(\xi) \le \liminf_{\ell\uparrow\infty}V_\ell(f) \le \limsup_{\ell\uparrow\infty}V_\ell(f) \le \sup_{\xi\in\Xi_C}\Theta_f(\xi) \tag{7.9}$$

where

$$\Theta_f(\xi) := \sum_{e\in\mathcal{E}} 2\,\mathbb{E}^*\left(c_{0,e}\phi_e\xi_e\right) - \sum_{e\in\mathcal{E}}\frac{1}{2}\mathbb{E}^*\left(c_e\xi_e^2\right).$$

Moreover, given $a\in\mathbb{R}^d$ and $u\in\mathbb{G}$,

$$\Theta_f\big(\sum_{e\in\mathcal{E}}(-a_e\mathfrak{U}^e+\nabla_e\underline{u})\big) = \sum_{e\in\mathcal{E}}2a_e t_e(f) + 2(f,u)_0 - \sum_{e\in\mathcal{E}}\frac{1}{2}\mathbb{E}^*\left(c_{0,e}(a_e\nabla_e\eta_0-\nabla_e\underline{u})^2\right). \tag{7.10}$$

We are finally in a position to prove theorem 7.2. We first observe that theorem 7.6 proves that the inequalities in (7.9) are actually equalities so that $V(f) = \lim_{\ell\uparrow\infty} V_\ell(f)$ exists and it is given by (7.2). Moreover, because of (4.23), $V(f) < \infty$ so that, by polarization, $V(f,g)$ exists finite for any $f, g \in \mathcal{G}$ and it defines a semi–inner product. The density of the subspace (7.3) follows at once from the first equality in (7.2). $\square$

7.2. **The method of long jumps revisited.** In this paragraph we consider, for any $e \in \mathcal{E}$, a particular sequence $\{W_n^e/n\}_{n\in\mathbb{N}}$ in the space $\mathcal{G}$ which is asymptotically equivalent to the sequence

$$2m(1-m)\lambda_0'(m)\frac{\psi_{n,n}^e}{n}, \quad n \in \mathbb{N}$$

where $\psi_{n,n}^e$ has been defined in (4.32) as

$$\psi_{n,n}^e = m_n^{2,e} - m_n^{1,e} - \mu\big[m_n^{2,e} - m_n^{1,e} \,|\, m_n^e\big]$$

The functions W_n^e have been introduced in [29] in order to depress the extra fluctuations produced by the disorder and are defined as

$$W_n^e := \mathrm{Av}_{x\in\Lambda_n^{1,e}}\mathrm{Av}_{y\in\Lambda_n^{2,e}}\, w_{x,y} \quad \text{where} \quad w_{x,y} := \big(1 + e^{-(\alpha_x-\alpha_y)(\eta_x-\eta_y)}\big)(\eta_y - \eta_x).$$

We remark that, for any bond $b = \{x,y\}$, the quantities $c_{x,y} := 1 + e^{-(\alpha_x-\alpha_y)(\eta_x-\eta_y)}$ are a possible choice of transition rates compatible with our general assumptions (see section 2.2). Therefore, for generic $x, y \in \mathbb{Z}^d$ $c_{x,y}$ can be thought of as the rate of the (long) jump from x to y and viceversa. In a sense the rates $c_{x,y}$, $x, y \in \mathbb{Z}^d$, define a new process with arbitrarily long jumps but still reversibile w.r.t. the Gibbs measure of the system.

Remark 7.9. *The role of the function W_n^e here is very different from that indicated in* [29]*. In our approach and for reasons that will appear clearly in the next subsection, we are interested in computing the asymptotic of the semi–inner product $V(j_{0,e'}, \frac{\psi_{n,n}^e}{n})$ as $n \uparrow \infty$. Our strategy to compute $V(j_{0,e'}, \frac{\psi_{n,n}^e}{n})$ is to replace (in $\mathcal{G}$) $\frac{\psi_{n,n}^e}{n}$ with $\frac{W_n^e}{n}$ and then to exploit some nice integration by parts properties pointed out in* [29] *(see below).*
In [29] *instead, the main idea is first to approximate, as $\epsilon \downarrow 0$, the microscopic current $j_{0,e}$ with a fluctuation term $\mathcal{L}g$ plus a linear combination of the $\frac{W_k^{e'}}{k}$, $e' \in \mathcal{E}$, on a scale k that must diverge as $\epsilon \downarrow 0$ like $\epsilon^{-\frac{2}{d+2}}$. The second step indicated in* [29] *is to replace $\frac{W_k^e}{k}$ with*

$$2m(1-m)\lambda_0'(m)\frac{(m_k^{2,e} - m_k^{1,e})}{k}$$

Such a step is very similar to the main result of this subsection described at the beginning but, at the same time, very different. The first main difference is that our mesoscopic scale n is not linked with ϵ. The second difference is that our functions $\psi_{n,n}^e$ represent (discrete) gradient of the density minus their canonical average. Such a counter term, discussed at length in section 5, is absent in the approach of [29].

Our main result is given by the following theorem

Theorem 7.10. *For any $e \in \mathcal{E}$*

$$\lim_{n\uparrow\infty} V\Big(\frac{W_n^e}{n} - 2m(1-m)\lambda_0'(m)\frac{\psi_{n,n}^e}{n}\Big) = 0. \tag{7.11}$$

We will use the above result only to compute the limit of $V(j_{0,e'}, \frac{\psi^e_{n,n}}{n})$. Indeed, as pointed in [29], the function $w_{x,y}$ satisfies the following integration by parts property: for any $\Lambda \in \mathbb{F}$ with $\Lambda \ni x, y$ and any $\nu \in \mathcal{M}(\Lambda)$

$$\nu(w_{x,y}g) = \nu((\eta_x - \eta_y)\nabla_{x,y}g).$$

By the above property and lemma 7.1 it is simple to check that, for any $e, e' \in \mathcal{E}$, $V\Big(j_{0,e'}, \frac{W^e_n}{n}\Big) = -2m(1-m)\delta_{e,e'}$. Therefore, by theorem 7.10, we get

$$\lim_{n\uparrow\infty} V\Big(j_{0,e'}, \frac{\psi^e_{n,n}}{n}\Big) = -\chi(m)\delta_{e,e'}, \quad \forall e, e' \in \mathcal{E}. \tag{7.12}$$

Proof. In order to prove theorem 7.10 it is convenient to introduce some notation. First, we fix the vector $e \in \mathcal{E}$ which will be often omit in the notation. Moreover we introduce the following equivalence relation.

Definition 7.11. *Given two sequences of functions $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ such that f_n and g_n have support in Λ^e_n, we write $f_n \approx g_n$ if*

$$\lim_{n\uparrow\infty} V\Big(\frac{f_n - \mu[f_n \,|\, m^e_n]}{n} - \frac{g_n - \mu[g_n \,|\, m^e_n]}{n}\Big) = 0.$$

• **Step 1**: $f_n := W_n - \mu[W_n|\, m^{1,e}_n, m^{2,e}_n] \approx 0$.

For any $x \in \mathbb{Z}^d$, let $\nu_{x,n}$ be the random canonical measure $\mu[\cdot \,|\, \mathcal{F}_{x,n}]$ where $\mathcal{F}_{x,n}$ is the σ–algebra generated by $\tau_x m^{1,e}_n$, $\tau_x m^{2,e}_n$ and η_y with $y \notin \Lambda^e_{x,n}$. Let us observe that

i) $\mu^{\lambda_0(m)}(\tau_x f_n, g) = \mu^{\lambda_0(m)}\big(\,\nu_{x,n}(\tau_x W_n; g)\,\big)$ for any function g;

ii) W_n can be written as sum of functions f of the following form

$$f = \mathrm{Av}_{z\in\Lambda^{1,e}_n}\tau_z h\, \mathrm{Av}_{z'\in\Lambda^{2,e}_n}\tau_{z'} h'$$

where h and h' depend only on α_0 and η_0.

Because of $i)$ and $ii)$ and thanks to the the variational characterization (4.20) of $V_\ell(\cdot, \mu^{\lambda_0(m)})$, it is enough to prove that, for a function f as in $ii)$,

$$\lim_{n\uparrow\infty}\lim_{k\uparrow\infty} \frac{1}{n^2 k^d}\,\mathbb{E}\big[\sup_{g\in\mathbb{G}}\{\phi(g)/\mathcal{D}_{\Lambda_k}(g;\mu^{\lambda_0(m)})\}\big] = 0 \tag{7.13}$$

where

$$\phi(g) := \big[\sum_{|x|\le k_1} \mu^{\lambda_0(m)}\big(\nu_{x,n}(\tau_x f; g)\big)\big]^2, \quad k_1 := k - \sqrt{k}.$$

By proposition A.6, for any $\delta > 0$ there exists $\ell_0 \in \mathbb{N}$ such that, if $n \ge \ell \ge \ell_0$, then

$$\nu_{x,n}(\tau_x f; g)^2 \le \frac{c(\ell)}{n^d}\mathcal{D}(g;\nu_{x,n}) + \frac{\delta}{n^d}\mathrm{Var}_{\nu_{x,n}}(g) + \frac{c}{n^d}\mathrm{Var}_{\nu_{x,n}}(g)\vartheta_{\Lambda^e_{x,n},\ell}(\alpha) \tag{7.14}$$

where, for any given $\gamma > 0$ and $\ell \ge \ell_1(\gamma) \ge \ell_0$,

$$\mathbb{P}(\vartheta_{\Lambda^e_{x,n},\ell}(\alpha) \ge \gamma) \le e^{-c(\gamma,\ell)n^d} \tag{7.15}$$

for a suitable constant $c(\gamma,\ell)$. Using the spectral gap estimate (4.13), the r.h.s. of (7.14) can be bounded by

$$\mathcal{D}(g;\nu_{x,n})\big(c(\ell) + c\,\delta\, n^2 + c\, n^2\vartheta_{\Lambda^e_{x,n},\ell}(\alpha)\big)n^{-d}$$

and therefore, by Schwarz inequality,

$$\phi(g) \le \mathcal{D}_{\Lambda_k}(g;\mu^{\lambda_0(m)})\big(c(\ell)k^d + c\,\delta k^d n^2 + c\,n^2 \sum_{|x|\le k_1} \vartheta_{\Lambda^e_{x,n},\ell}\big).$$

By taking the limits $\delta \downarrow 0$, $\ell \uparrow \infty$, $n \uparrow \infty$, $k \uparrow \infty$ (from right to left), in order to prove (7.13) the thesis follows since $\lim_{n\uparrow\infty} \mathbb{E}\big(\vartheta_{\Lambda^e_n,\ell}(\alpha)\big) = 0$ because of (7.15).

• **Step 2**: $\mu[W_n|\, m_n^{1,e}, m_n^{2,e}] \approx 2m(1-m)\lambda_0'(m)\psi^e_{n,n}$.
The proof is based on the following lemma, which follows easily from the variational characterization of $V_\ell(\cdot,\mu^{\lambda_0(m)})$ given in (4.20).

Lemma 7.12. *Let, for any $n \in \mathbb{N}$, f_n, $h_n \in \mathbb{G}$ be such that*
i) $\Delta_{f_n} \subset \Lambda^e_n$*;*
ii) $\sup_n \|h_n\|_\infty < \infty$ *and* $\lim_{n\uparrow\infty} n^d \mathbb{E}\big[\mu^{\lambda_0(m)}\big(h_n^2\big)\big] = 0$*;*
iii) $|f_n| \le |h_n|$.
Then $f_n \approx 0$.

Thanks to the estimates given in the Appendix it can be proved (see [15]) that condition *ii)* of the lemma is satisfied by any of the following sequences:

$$\{n^{-d}\}_{n\in\mathbb{N}},\quad \big\{\,(m-m_{\Delta_n})^2\,\big\}_{n\in\mathbb{N}},\quad \big\{\mathbb{I}_{\{|m-m_{\Delta_n}|\ge c\}}\big\}_{n\in\mathbb{N}},\quad \big\{\big(m-\mu^{\lambda_0(m)}(m_n^{i,e})\big)^2\big\}_{n\in\mathbb{N}}$$

where $i = 1,2$, $c > 0$ and Δ_n is either one of the sets Λ^e_n, $\Lambda_n^{1,e}$, $\Lambda_n^{2,e}$.
As in [29] we define the (random w.r.t. α) function $F_n(m_1,m_2)$ as

$$F_n(m_1,m_2) = \mu^{\lambda(m_1)}_{\Lambda_n^{1,e}} \otimes \mu^{\lambda(m_2)}_{\Lambda_n^{2,e}}\big(W_n\big)$$

It is not difficult to show that $F_n(m_1,m_2)$ has the explicit expression

$$F_n(m_1,m_2) = m_1 - m_2 + e^{\lambda_{1,n}(m_1)-\lambda_{2,n}(m_2)}(1-m_1)m_2 - e^{\lambda_{2,n}(m_2)-\lambda_{1,n}(m_1)}m_1(1-m_2).$$

The main reason to introduce $F_n(m_1,m_2)$ is that

$$\mu[W_n|\, m_n^{1,e}, m_n^{2,e}] \approx F_n(m_n^{1,e}, m_n^{2,e}). \tag{7.16}$$

This equivalence follows at once from the equivalence of the ensembles together with lemma 7.12 applied to $f_n = \mu[W_n|\, m_n^{1,e}, m_n^{2,e}] - F_n(m_n^{1,e}, m_n^{2,e})$ and $h_n = c\,n^{-d}$ for a large enough constant c.
Next, again by lemma 7.12 applied with $h_n = \mathbb{I}_{\{|m-m_n^{1,e}|\ge c_m\}} + \mathbb{I}_{\{|m-m_n^{2,e}|\ge c_m\}}$, $c_m = (m\wedge(1-m))/2$, we get that

$$F_n(m_n^{1,e}, m_n^{2,e}) \approx F_n(m_n^{1,e}, m_n^{2,e})\,\mathbb{I}_m \tag{7.17}$$

where $\mathbb{I}_m := \mathbb{I}_{\{|m-m_n^{1,e}|\le c_m\}}\mathbb{I}_{\{|m-m_n^{2,e}|\le c_m\}}$.
Next, by Taylor expansion around the arithmetic mean of $m_n^{1,e}$ and $m_n^{2,e}$, we write

$$F_n(m_n^{1,e}, m_n^{2,e}) =$$
$$F_n(m_n^e,m_n^e) + \frac{\partial F_n}{\partial m_1}(m_n^e,m_n^e)(m_n^{1,e}-m_n^e) + \frac{\partial F_n}{\partial m_1}(m_n^e,m_n^e)(m_n^{2,e}-m_n^e) + R_n(m_n^{1,e},m_n^{2,e})$$

Then, the zero order contribution $F_n(m^e_n,m^e_n)\mathbb{I}_m$ is negligible, $F_n(m^e_n,m^e_n)\mathbb{I}_m \approx 0$, since $F_n(m^e_n,m^e_n) \approx 0$ because of definition 7.11 and $F_n(m^e_n,m^e_n)(1-\mathbb{I}_m) \approx 0$ again by lemma 7.12.

The second order error term, $R_n(m_n^{1,e}, m_n^{2,e})\mathbb{I}_m$, is negligible because of lemma 7.12 applied with $h_n = c\big[(m_n^{1,e} - m_n^e)^2 + (m_n^{2,e} - m_n^e)^2\big]$. Notice that it is here that the characteristic function $\mathbb{I}_m$ plays an important role since the second derivatives of $F_n(m_1, m_2)$ diverge as m_i tends to 0 or to 1.
Let us now examine the relevant first order terms. We claim that for $i = 1, 2$

$$\frac{\partial F_n}{\partial m_i}(m_n^e, m_n^e)(m_n^{i,e} - m_n^e)\mathbb{I}_m + (-1)^i 2m_n^e(1 - m_n^e)\lambda'_{i,n}(m_n^e)(m_n^{i,e} - m_n^e)\mathbb{I}_m \approx 0 \quad (7.18)$$

and

$$2m_n^e(1 - m_n^e)(\lambda'_{i,n}(m_n^e) - \lambda'_0(m))(m_n^{i,e} - m_n^e)\mathbb{I}_m \approx 0. \quad (7.19)$$

where $\lambda_{i,n} := \lambda_{\Lambda_n^{i,e}}$.
Before proving (7.18) and (7.19) let us summarize what we have obtained so far. Thanks to (7.16), (7.17), the above discussion of the Taylor expansion and (7.18) together with (7.19)

$$\mu[W_n |\, m_n^{1,e}, m_n^{2,e}] \approx 2m_n^e(1 - m_n^e)\lambda'_0(m)(m_n^{2,e} - m_n^{1,e})\mathbb{I}_m$$

Using once more lemma 7.12 it is now rather simple to remove the factor $\mathbb{I}_m$ and to replace m_n^e with m, thus concluding the proof.
We are left with the proof of (7.18) and (7.19).
Let us prove (7.18) for $i = 1$. By computing $\frac{\partial F_n}{\partial m_1}$ it is simple to check that the l.h.s. of (7.18) is equal to

$$\begin{aligned} &(e^{\lambda_{1,n}(m_n^e) - \lambda_{2,n}(m_n^e)} - 1)m_n^e(\lambda'_{1,n}(m_n^e)(1 - m_n^e) - 1)(m_n^{1,e} - m_n^e)\mathbb{I}_m + \\ &(e^{\lambda_2(m_n^e) - \lambda_{1,n}(m_n^e)} - 1)(1 - m_n^e)(\lambda'_{1,n}(m_n^e)m_n^e - 1)(m_n^{1,e} - m_n^e)\mathbb{I}_m. \end{aligned} \quad (7.20)$$

It is enough to show that both addenda in (7.20) are equivalent to 0 and for simplicity we deal with only with the first one. Since $\sup_n \|\lambda'_{1,n}(m_n^e)\mathbb{I}_m\|_\infty \le k_m$ for a suitable constant k_m depending on m, using the estimate $|e^z - 1| \le e^{|z|}|z|$ valid for any $z \in \mathbb{R}$ and thanks to lemma A.3 we obtain

$$\begin{aligned} |\text{ first term in (7.20) }| &\le k_m\, |\lambda_{1,n}(m_n^e) - \lambda_{2,n}(m_n^e)|\, |m_n^{(i)} - m_n^e|\mathbb{I}_m \\ &\le k'_m\Big(\sum_{i=1,2}\big(m_n^e - \mu^{\lambda_0(m_n^e)}(m_n^{i,e})\big)^2 + (m_n^{1,e} - m_n^e)^2\Big). \end{aligned} \quad (7.21)$$

The claim follows by applying lemma 7.12 with h_n equal to the r.h.s. of (7.21).
Let us prove (7.19). By Schwarz inequality, it is enough to apply lemma 7.12 with $h_n := (\lambda'_{i,n}(m_n^e) - \lambda'_0(m))^2\mathbb{I}_m + (m_n^{i,e} - m_n^e)^2$. In order to verify condition $ii)$ of lemma 7.12 for h_n, thanks to the boundedness of $(\lambda'_{i,n}(m_n^e) - \lambda'_0(m))^2\mathbb{I}_m$ uniformly in n, we only need to prove that

$$\lim_{n\uparrow\infty} n^d\mathbb{E}\big[\mu^{\lambda_0(m)}\big((\lambda'_{i,n}(m_n^e) - \lambda'_0(m))^4\mathbb{I}_m\big)\big] = 0$$

or equivalently

$$\lim_{n\uparrow\infty} n^d\mathbb{E}\Big[\mu^{\lambda_0(m)}\Big(\big\{\mathrm{Av}_{x\in\Lambda_n^{i,e}}\big[\mu^{\lambda_{i,n}(m_n^e)}(\eta_x;\eta_x) - \mathbb{E}\mu^{\lambda_0(m)}(\eta_0;\eta_0)\big]\big\}^4\Big)\Big] = 0 \quad (7.22)$$

Let $g_x(\lambda) := \mu^\lambda(\eta_x;\eta_x)$ and observe that l.h.s. of (7.22) is bounded from above by

$$c\lim_{n\uparrow\infty} n^d\mathbb{E}\Big[\mu^{\lambda_0(m)}\big(A_n^{(1)} + A_n^{(2)} + A_n^{(3)}\big)\Big] \quad (7.23)$$

where

$$A_n^{(1)} = \big\{ \mathrm{Av}_{x\in\Lambda_n^{i,e}} [\, g_x(\lambda_{i,n}(m_n^e)) - g_x(\lambda_{i,n}(m))\,]\big\}^4$$
$$A_n^{(2)} = \big\{ \mathrm{Av}_{x\in\Lambda_n^{i,e}} [\, g_x(\lambda_{i,n}(m)) - g_x(\lambda_0(m))\,]\,\big\}^4$$
$$A_n^{(3)} = \big\{ \mathrm{Av}_{x\in\Lambda_n^{i,e}} [\, g_x(\lambda_0(m)) - \mathbb{E}\mu^{\lambda_0(m)}(\eta_0;\eta_0)\,]\big\}^4.$$

By lemma $A.3$, $A_n^{(1)} \le c\,(m_n^e - m)^4$ and $A_n^{(2)} \le c(m - \mu^{\lambda_0(m)}(m_n^{i,e}))^4$. At this point (7.23) follows by simple considerations for sum of centered independent random variables.

□

7.3. **The subspace orthogonal to the fluctuations.** Here we introduce a convenient Hilbert space $\mathcal{H}$ containing $\mathcal{G}$ and we describe the orthogonal subspace in $\mathcal{H}$ of the space of fluctuations $\{\mathcal{L}g \,:\, g\in\mathbb{G}\}$.

Definition 7.13. *Let $\mathcal{N} := \{g\in\mathcal{G} \,:\, V(g)=0\}$ and let $\mathcal{H}$ be the completion of the pre-Hilbert space $\mathcal{G}/\mathcal{N}$. With an abuse of notation, we write V for the scalar product in $\mathcal{H}$ induced by the semi-inner product V in $\mathcal{G}$.*

The sets

$$\mathcal{L}\mathbb{G} := \{\mathcal{L}g \,:\, g\in\mathbb{G}\}, \qquad \mathcal{L}\mathcal{G} := \{\mathcal{L}g \,:\, g\in\mathcal{G}\}$$

can be considered as subsets of $\mathcal{H}$ in a natural way. Our main result proves that for any $e\in\mathcal{E}$ the sequence $\{\psi_{n,n}^e/n\}_{n\in\mathbb{N}}$ converges in $\mathcal{H}$ to some limit point ψ_e and that the set $\{\psi_e\}_{e\in\mathcal{E}}$ forms a basis of $\mathcal{L}\mathbb{G}^\perp$. The Cauchy property of the sequence $\{\psi_{n,n}^e/n\}_{n\in\mathbb{N}}$ follows by a telescopic estimate based on the variance bounds discussed in subsection 6.1. To this aim the following lemma is crucial.

Lemma 7.14. *Given $k\in\mathbb{N}$ let $f\in\mathcal{G}$ be such that $\Delta_f\subset\Lambda_k$. Then*

$$V(f) \le c\,k^{d+2}\mathbb{E}\big(\mathrm{Var}_{\mu^{\lambda_0(m)}}(\mathrm{Av}_{x\in\Lambda_k}\tau_x f)\big).$$

Proof. We first estimate $V_\ell(f)$ for $\ell\gg 1$ by means of lemma 6.1. To this aim we partition the cube Λ_{ℓ_1} into non overlapping cubes $\{\Lambda_{x_i,k}\}_{i\in I}$ of side $2k+1$ and write

$$\mathrm{Av}_{x\in\Lambda_{\ell_1}}\tau_x f = \mathrm{Av}_{i\in I}\big(\mathrm{Av}_{x\in\Lambda_{x_i,k}}\tau_x f\big)$$

Therefore, by applying lemma 6.1 with $\Lambda=\Lambda_\ell$ and $\Lambda_i=\Lambda_{x_i,2k}$, we obtain

$$V_\ell(f) \le c\,k^{d+2}\mathrm{Av}_{i\in I}\mathrm{Var}_{\mu^{\lambda_0(m)}}\big(\mathrm{Av}_{x\in\Lambda_{x_i,k}}\tau_x f\big).$$

It is enough now to take the expectation w.r.t. α and then the limit $\ell\uparrow\infty$. □

Lemma 7.14 and proposition 6.5 allow us to prove the key technical estimate of this subsection:

Lemma 7.15. *Let $d\ge 2$, $n\le s\le k\le 100s$ be positive integers and $0<\delta\ll 1$. Then*

$$V(\phi_{n,s}^e - \phi_{n,k}^e) \le c\,s^{2-d+\delta} \quad \forall e\in\mathcal{E} \tag{7.24}$$

for any s large enough ($s\ge s_0(\delta)$).

Proof. Since $\phi_{n,s}^e - \phi_{n,k}^e\in\mathcal{G}$ has support in Λ_k, by lemma 7.14 we obtain

$$V(\phi_{n,s}^e - \phi_{n,k}^e) \le c\,k^{d+2}\sum_{r=s,k}\mathbb{E}\big(\mathrm{Var}_{\mu^{\lambda_0(m)}}(\mathrm{Av}_{x\in\Lambda_k}\tau_x\phi_{n,r}^e)\big).$$

The thesis now follows from proposition 6.5. □

We also need a density result.

Lemma 7.16. *$\mathcal{L}\mathcal{G}$ and $\mathcal{L}\mathbb{G}$ have the same closure in $\mathcal{H}$.*

Proof. We fix $g \in \mathbb{G}$ and we prove that $\mathcal{L}g = \lim_{s\uparrow\infty} \mathcal{L}(g - g_s)$ where $g_s = \mu[g \,|\, m_s]$, i.e. that $\lim_{s\uparrow\infty} V(\mathcal{L}g_s) = 0$. To this aim we define $X_s := \{x : s-1 \le |x| \le s+1\}$. Then lemma 7.1 implies that

$$V(\mathcal{L}g_s) = \sum_{e\in\mathcal{E}} \frac{1}{2}\mathbb{E}^* \big(c_{0,e}(\sum_{x\in X_s} \nabla_{0,e}\tau_x g_s)^2\big). \tag{7.25}$$

Let $\hat g_s(\alpha,\eta) := \mu_{\Lambda_s}^{\lambda(m_s(\eta))}(g)$. By the equivalence of ensembles (see lemma A.4), in (7.25) g_s can be substituted by $\hat g_s$ with an error bounded by $c\,s^{-2}$. By lemma$A.3$, $|\nabla_{0,e}\tau_x\hat g_s| \le c\,s^{-d}$ which, thanks to (7.25) with g_s replaced with $\hat g_s$, implies that $V(\mathcal{L}g_s) \le c\,s^{-2}$. □

We are ready for the first result about the structure of the space $\mathcal{L}\mathbb{G}^\perp$.

Proposition 7.17. *Let $d \ge 3$ and $e \in \mathcal{E}$. Then the sequence*

$$\psi^e_{1,s} = \eta_e - \eta_0 - \mu[\eta_e - \eta_0 \,|\, m^e_s]$$

converges to some element $\psi_e \in \mathcal{L}\mathbb{G}^\perp$ as $s \uparrow \infty$.
Moreover,

$$\lim_{s\uparrow\infty} \frac{\psi^e_{n,s}}{n} = \psi_e \quad \forall n \in \mathbb{N}. \tag{7.26}$$

Proof. We fix $0 < \delta \ll 1$. By lemma 7.15, if $i \in \mathbb{N}$ is large enough and $i^3 \le s \le (i+1)^3$,

$$V(\psi^e_{1,i^3} - \psi^e_{1,s}) \le c\,i^{\,3(2-d+\delta)}.$$

Since $d \ge 3$, it is enough to prove that the sequence $\{\psi^e_{1,i^3}\}_{i\in\mathbb{N}}$ is Cauchy. This follows by applying again lemma 7.15 to get

$$\sum_{i=1}^\infty V^{\frac12}(\psi^e_{1,i^3} - \psi^e_{1,(i+1)^3}) \le \sum_{i=1}^\infty c\,i^{\,\frac32(2-d+\delta)} < \infty.$$

Next we prove that ψ_e, the limit point of $\{\psi^e_{1,s}\}_{s\in\mathbb{N}}$, belongs to $\mathcal{L}\mathbb{G}^\perp$. To this aim, by lemmas 7.1 and 7.16, we need to show that

$$\lim_{s\uparrow\infty} \sum_{x\in\mathbb{Z}^d} \mathbb{E}[\,\mu^{\lambda_0(m)}(\psi^e_{1,s}, \tau_x g)\,] = 0 \quad \forall g \in \mathcal{G},$$

or similarly (by translation invariance of the random field α)

$$\lim_{s\uparrow\infty} \sum_{x\in\mathbb{Z}^d} \mathbb{E}[\,\mu^{\lambda_0(m)}(\phi^e_{1,s}, \tau_x g)\,] = 0 \quad \forall g \in \mathcal{G},$$

where we recall $\phi^e_{1,s} = \mu[\eta_e - \eta_0 \,|\, m^e_s]$. To this aim we set

$$\Delta_s := \{x \in \mathbb{Z}^d \,:\, (x + \Delta_g) \cap \Lambda^e_s \ne \emptyset \text{ and } (x+\Delta_g) \cap (\Lambda^e_s)^c \ne \emptyset\,\}.$$

Since $g \in \mathcal{G}$,

$$\sum_{x\in\mathbb{Z}^d} \mathbb{E}[\,\mu^{\lambda_0(m)}(\phi^e_{1,s}, \tau_x g)\,] = \mathbb{E}[\,\mu^{\lambda_0(m)}(\phi^e_{1,s}, \sum_{x\in\Delta_s} \tau_x g)\,]. \tag{7.27}$$

We estimate the r.h.s. of (7.27) by Schwarz inequality. Let us observe that

$$\mathbb{E}[\mathrm{Var}_{\mu^{\lambda_0(m)}}\big(\sum_{x\in\Delta_s} \tau_x g\big)\,] \le c_g\, s^{-d+1}. \tag{7.28}$$

for some finite constant c_g. Therefore, in order to conclude the proof, it is enough to show

$$\mathbb{E}[\operatorname{Var}_{\mu^{\lambda_0(m)}}(\phi^e_{1,s})] \le c\, s^{-d}. \tag{7.29}$$

By the equivalence of ensemble (see lemma A.5) and Poincaré inequality for Glauber dynamics, we obtain

$$\operatorname{Var}_{\mu^{\lambda_0(m)}}(\phi^e_{1,s}) \le c\, s^{-2d} + c\operatorname{Var}_{\mu^{\lambda_0(m)}}(\hat\phi^e_{1,s}) \le c\, s^{-2d} + c\, s^d \mu^{\lambda_0(m)}\big((\nabla_0\hat\phi^e_{1,s})^2\big) \tag{7.30}$$

where $\hat\phi^e_{1,s}$ has been defined in (6.3).
By lemma A.3 the last term in (7.30) is bounded by $c\, s^{-d}$ thus proving (7.29).
Finally we prove (7.26). To this aim, by writing

$$\psi^e_{n,s} = \frac{1}{n}\sum_{v=0}^{n-1} \operatorname{Av}_{x\in\Lambda^{1,e}_n}\big(\eta_{x+(v+1)e} - \eta_{x+ve} - \mu[\eta_{x+(v+1)e} - \eta_{x+ve}\,|\, m^e_s]\big),$$

and by the observation that $\tau_x f = f$ for any $f\in\mathcal{H}$ and $x\in\mathbb{Z}^d$, it is enough to prove that for any given $x\in\mathbb{Z}^d$

$$V(\mu[\eta_e - \eta_0\,|\, m^e_{x,s}] - \mu[\eta_e - \eta_0\,|\, m^e_s]) \tag{7.31}$$

goes to 0 as $s\uparrow\infty$. As in the proof of lemma 7.15 (7.31) is bounded from above by $c(\delta)s^{2-d+\delta}$ for any $0<\delta\ll 1$. □

We are now able to exhibit a basis of $\mathcal{L}\mathbb{G}^\perp$ related to the functions $\frac{\psi^e_{n,n}}{n}$ with $n\in\mathbb{N}$ and $e\in\mathcal{E}$.

Theorem 7.18. *Let $d\ge 3$. Then*

$$\lim_{n\uparrow\infty}\frac{\psi^e_{n,n}}{n} = \psi_e \quad \forall e\in\mathcal{E} \tag{7.32}$$

where ψ_e is as in proposition 7.17. Moreover,

$$V(j_{0,e'},\psi_e) = -\chi(m)\delta_{e',e} \quad \forall e,e'\in\mathcal{E} \tag{7.33}$$

and $\{\psi_e\}_{e\in\mathcal{E}}$ forms a basis of $\mathcal{L}\mathbb{G}^\perp$.

Proof. For any $n\in\mathbb{N}$ let $k_n\in\mathbb{N}$ be such that $(k_n-1)^3 < n \le k_n^3$. Then, by lemma 7.15, $V(\psi^e_{n,n}/n - \psi^e_{n,k_n^3}/n)\downarrow 0$ as $n\uparrow\infty$. Therefore, thanks to (7.26),

$$\lim_{n\uparrow\infty} V^{\frac12}(\psi_e - \frac{\psi^e_{n,n}}{n}) = \lim_{n\uparrow\infty} V^{\frac12}(\psi_e - \frac{\psi^e_{n,k_n}}{n}) \le \lim_{n\uparrow\infty}\frac{1}{n}\sum_{i=1}^{\infty} V^{\frac12}(\psi^e_{n,i^3} - \psi^e_{n,(i+1)^3}) \tag{7.34}$$

and the last series is converging by lemma 7.15. Thus (7.32) follows.
At this point, (7.33) follows from theorem 7.10. Let us prove that $\{\psi_e\}_{e\in\mathcal{E}}$ forms a basis of $\mathcal{L}\mathbb{G}^\perp$. Let P be the orthogonal projection of $\mathcal{H}$ on $\mathcal{L}\mathbb{G}^\perp$. Then, $\mathcal{L}\mathbb{G}^\perp$ has dimension non larger than d since, by theorem 7.2, it is generated by $\{Pj_{0,e}\}_{e\in\mathcal{E}}$. By (7.33) $\{\psi_e\}_{\in\mathcal{E}}$ is a set of d independent vectors belonging to $\mathcal{L}\mathbb{G}^\perp$ and therefore a basis of $\mathcal{L}\mathbb{G}^\perp$. □

Remark 7.19. *Let us make an observation which will reveal useful in the proof of the continuity of the diffusion matrix $D(m)$ (see next subsection).*
Since the constant c appering in (7.24) does not depend on the density m and thanks to the estimate (7.34), the statement (7.32) in the above theorem can be strengthed as

$$\lim_{n\uparrow\infty}\sup_{m\in(0,1)} V_m\big(\frac{\psi^e_{n,n}}{n} - \psi_e\big) = 0 \quad \forall e\in\mathcal{E}.$$

7.4. **Decomposition of currents.** In this subsection we prove the characterization and the regularity of the diffusion matrix $D(m)$ stated in theorem 2.1 and we prove also theorem 7.22, which is crucial for the estimate of Ω_0 (see subsection 4.3). In what follows, we assume $d \geq 3$.

Denoting by P the orthogonal projection of $\mathcal{H}$ on $\mathcal{L}\mathbb{G}^{\perp}$, thanks to theorem 7.18, for a suitable $d \times d$ matrix $D(m)$ we can write

$$j_{0,e} = -\sum_{e' \in \mathcal{E}} D_{e,e'}(m)\psi_{e'} + (1-P)(j_{0,e}) \qquad \forall e \in \mathcal{E}. \tag{7.35}$$

By taking the scalar product of both sides of (7.35) with $j_{0,e'}$, thanks to lemma 7.1 and (7.33), we obtain

$$D_{e,e'}(m) = \frac{1}{\chi(m)} V_m(Pj_{0,e}, Pj_{0,e'})$$

thus proving that $D(m)$ is a non-negative symmetric matrix. In particular, $D(m)$ can be characterized as the unique symmetric $d \times d$ matrix such that

$$(a, D(m)a) = \frac{1}{\chi(m)} V_m\big(P(\sum_{e\in\mathcal{E}} a_e j_{0,e})\big) \quad \forall a \in \mathbb{R}^d. \tag{7.36}$$

Since the r.h.s. of (7.36) can be written as

$$\inf_{g \in \mathbb{G}} \frac{1}{\chi(m)} V_m\big(\sum_{e\in\mathcal{E}} a_e j_{0,e} - \mathcal{L}g\big),$$

by lemma 7.1 the matrix $D(m)$ corresponds to the one described in proposition 2.1.

In the following lemmas we describe some properties of the diffusion matrix $D(m)$.

Lemma 7.20. *There exists $c > 0$ such that $c\mathbb{I} \leq D(m) \leq c^{-1}\mathbb{I}$ for any $m \in (0,1)$.*

Proof. Given $a \in \mathbb{R}^d$ we set $w := \sum_{e\in\mathcal{E}} a_e \psi_e$ and $v := \sum_{e\in\mathcal{E}} a_e P j_{0,e}$. Then (7.36) and lemma 7.1 imply the upper bound

$$(a, D(m)a) = \frac{1}{\chi(m)} V_m(v,v) \leq \frac{1}{\chi(m)} V_m(\sum_{e\in\mathcal{E}} a_e j_{0,e}) \leq c\,\|a\|^2.$$

In order to prove the lower bound we observe that, by theorem 7.18, $V_m(v,w) = -\chi(m)\|a\|^2$ while, thanks to (4.33), $V_m(w) \leq c\,m(1-m)\|a\|^2$. Therefore, by Schwarz inequality,

$$(a, D(m)a) \geq \frac{1}{\chi(m)} \frac{V_m(v,w)^2}{V_m(w)} \geq c\,\|a\|^2$$

thus proving the lemma. □

Lemma 7.21. *$D(m)$ is a continuous function on $(0,1)$.*

Proof. Let $0 < \beta$ and $0 < \delta < \frac{1}{2}$. We observe that the limit point ψ_e of the sequence $\frac{\psi^e_{n,n}}{n}$ depends on the closure of $\mathcal{G}/\mathcal{N}$ and therefore on m. Therefore, it is convenient to denote it by $\psi_e^{(m)}$. Moreover, thanks to remark 7.19 and lemma 7.20, there exists $n_0 \in \mathbb{N}$ such that

$$\|D\|_\infty \sup_{m\in(0,1)} V_m\big(\psi_e^{(m)} - \frac{\psi^e_{n,n}}{n}\big)^{\frac{1}{2}} \leq \beta \quad \forall e \in \mathcal{E} \quad \forall n \geq n_0 \tag{7.37}$$

where, $\|D\|_\infty := \sup_{e,e'\in\mathcal{E}} \|D_{e,e'}\|_\infty$.
Together with (7.35), this implies that, for any given $m \in (0,1)$, we can find $g_m \in \mathcal{G}$ such that

$$V_m\big(j_{0,e} + \sum_{e'\in\mathcal{E}} D_{e,e'}(m)\frac{\psi^{e'}_{n_0,n_0}}{n_0} + \mathcal{L}g_m\big)^{\frac{1}{2}} \le 2\beta. \tag{7.38}$$

Since $\lambda_0(m)$ is a smooth function of $m \in (0,1)$ and thanks to Lemma 7.1, (7.38) remains valid if V_m is replaced by $V_{m'}$, where m' is arbitrary inside an open interval I_m containing m. In what follows we restrict to the density interval $[\delta, 1-\delta]$. Thanks to compactness and interpolation and thanks to (4.33), there exists a continuos matrix $D^{(\beta)}(\cdot)$ and a family of functions $g^{(\beta)}_m$, $m \in [\delta, 1-\delta]$, such that $\|D^{(\beta)}_{e,e'}\|_\infty \le \|D_{e,e'}\|_\infty$ and

$$V_m\big(j_{0,e} + \sum_{e'\in\mathcal{E}} D^{(\beta)}_{e,e'}(m)\frac{\psi^{e'}_{n_0,n_0}}{n_0} + \mathcal{L}g^{(\beta)}_m\big)^{\frac{1}{2}} \le 3\beta \quad \forall m \in [\delta, 1-\delta]$$

and therefore

$$V_m\big(j_{0,e} + \sum_{e'\in\mathcal{E}} D^{(\beta)}_{e,e'}(m)\psi^{(m)}_{e'} + \mathcal{L}g^{(\beta)}_m\big)^{\frac{1}{2}} \le 4\beta \quad \forall m \in [\delta, 1-\delta] \tag{7.39}$$

From the above formula and (7.35), we have

$$Pj_{0,e} = -\sum_{e'\in\mathcal{E}} D_{e,e'}(m)\psi^{(m)}_{e'} = -\sum_{e'\in\mathcal{E}} D^{(\beta)}_{e,e'}(m)\psi^{(m)}_{e'} + \xi^{(m)}_e \quad \forall m \in [\delta, 1-\delta]$$

where $V_m(\xi^{(m)}_e)^{\frac{1}{2}} \le 4\beta$. By taking the scalar product with $j_{0,e'}$ we obtain (thanks to theorem 7.18)

$$|\chi(m)\big(D_{e,e'}(m) - D^{(\beta)}_{e,e'}(m)\big)| \le 4V_m(j_{0,e})^{\frac{1}{2}}\beta \quad \forall m \in [\delta, 1-\delta],$$

that is $|D_{e,e'}(m) - D^{(\beta)}_{e,e'}(m)| \le c(\delta)\beta$, thus proving that $D_{e,e'}(\cdot)$ is continuous on $[\delta, 1-\delta]$. □

We are now able to prove our main result.

Theorem 7.22. *Let $d \ge 3$. Then given $\delta > 0$*

$$\inf_{g\in\mathbb{G}} \limsup_{n\uparrow\infty} \sup_{m\in[\delta,1-\delta]} V_m\Big(j_{0,e} + \mathcal{L}g + \sum_{e'\in\mathcal{E}} D_{e,e'}(m)\frac{\psi^{e'}_{n,n}}{n}\Big) = 0. \tag{7.40}$$

Moreover, if D has continuous extension to $\{0,1\}$, (7.40) is valid with $\delta = 0$.

Proof. (7.40) is a simple consequence of the estimates exhibited in the proof of lemma 7.21. Let us observe that, given $\beta > 0$, by defining $g^{(\beta)}_m$ as in the above proof, then

$$\limsup_{n\uparrow\infty} \sup_{m\in[\delta,1-\delta]} V_m\Big(j_{0,e} + \mathcal{L}g^{(\beta)}_m + \sum_{e'\in\mathcal{E}} D_{e,e'}(m)\frac{\psi^{e'}_{n,n}}{n}\Big) \le c\,\beta. \tag{7.41}$$

In order to define a function g independent of m, it is enough to proceed as in the proof of corollary 5.9, chapter 7, [23]. If D has continuous extension to $\{0,1\}$ then it is simple to extend (7.41) to all $[0,1]$. □

Appendix A

In this final appendix we have collected several technical results used in the previous sections.

A.1. **Large deviations estimates.** .

Lemma A.1. *Let* $f = f(\alpha)$ *be a mean-zero local function and* $\Lambda \in \mathbb{F}$ *be such that* $(\Delta_f + x) \cap (\Delta_f + y) = \emptyset$ *for any* $x, y \in \Lambda$. *Then*

$$\mathbb{P}[\,|\mathrm{Av}_{x\in\Lambda}\tau_x f\,| \geq \delta\,] \leq 2e^{-\frac{\delta^2|\Lambda|}{4\|f\|_\infty^2}} \qquad \forall \delta > 0.$$

Proof. Given $t > 0$, since $\mathbb{E}(f) = 0$ and $e^x - x \leq e^{x^2}$ for any $x \geq 0$,

$$e^{tf} = \sum_{n=0}^{\infty} \frac{(tf)^n}{n!} \leq e^{t\|f\|_\infty} - \|f\|_\infty t \leq e^{\|f\|_\infty^2}.$$

Therefore, thanks to the conditions on f and Λ,

$$\mathbb{P}[\,\mathrm{Av}_{x\in\Lambda}\tau_x f \geq \delta\,] \leq e^{-t\delta}\mathbb{E}(e^{t\,\mathrm{Av}_{x\in\Lambda}\tau_x f}) = e^{-t\delta}[\mathbb{E}(e^{tf|\Lambda|^{-1}})]^{|\Lambda|} \leq e^{-t\delta + t^2\|f\|_\infty^2|\Lambda|^{-1}}.$$

The thesis follows by taking $t := \delta|\Lambda|/(2\|f\|_\infty^2)$ and by considering the above estimates with f replaced by $-f$. $\square$

A.2. **Equilibrium bounds.**

Lemma A.2. *Given* $\Lambda \in \mathbb{F}$ *and* $\lambda \in \mathbb{R}$ *we define* $m := \mu^\lambda(m_\Lambda)$ *and* $a_m := \min(m, 1-m)$. *Then, for any* $\Delta \subset \Lambda$ *and any function* f *such that* $\Delta_f \subset \Lambda$,

a) $c|\Delta|m \leq \mu(N_\Delta) \leq c^{-1}|\Delta|m$,

b) $c\,|\Delta|(1-m) \leq \mu(|\Delta| - N_\Delta) \leq c^{-1}|\Delta|(1-m)$,

c) $c|\Delta|a_m \leq \mu(N_\Delta; N_\Delta) \leq c^{-1}|\Delta|a_m$,

d) $|\mu(f; N_\Lambda)| \leq c\|f\|_\infty \min\Big(|\Delta_f|a_m, \sqrt{|\Delta_f|a_m}\,\Big)$.

Proof. In what follows we assume $m \leq \frac{1}{2}$.
$a)$ and $b)$ can be easily derived from the boundedness of the random field α. Let us prove $c)$. The upper bound follows by observing that $\mu(N_\Delta; N_\Delta) \leq \mu(\Delta)$ and by applying $a)$. In order to prove the lower bound, let us introduce the set $W := \{x \in \Lambda : \mu(\eta_x) \leq \frac{1}{2}\}$. Since $|W| \geq |\Lambda|/2$ and thanks to $a)$,

$$\mu(N_\Lambda; N_\Lambda) \geq \mu(N_W; N_W) \geq \frac{1}{2}\mu(N_W) \geq c\,m|\Lambda|$$

thus proving the lower bound in $c)$ with Δ replaced by Λ. In order to consider the general case, we define $m' = \mu(m_\Delta)$. Then by the previous arguments, $\mu(N_\Delta; N_\Delta) \geq c\,m'|\Delta|$ which, by $a)$, is bounded from below by $c\,m|\Delta|$.
Let us prove $d)$. By Schwarz inequality and $c)$

$$|\mu(f; N_\Lambda)| \leq \mu(f; f)^{\frac{1}{2}}\mu(N_{\Delta_f}; N_{\Delta_f})^{\frac{1}{2}} \leq c\,\mu(f; f)^{\frac{1}{2}}\sqrt{m\,|\Delta_f|}$$

Since $\mu(f; f) \leq \|f\|_\infty^2$, it remains to prove that $\mu(f; f) \leq c\,m\|f\|_\infty^2|\Delta_f|$. To this aim let η^* be the configuration with no particle. Then, thanks to $a)$,

$$\mu(f; f) \leq \mu\big((f - f(\eta^*))^2\big) \leq \|f\|_\infty^2\mu(N_{\Delta_f}) \leq c\,\|f\|_\infty^2|\Delta_f|.$$

$\square$

Lemma A.3. *For any $\lambda, \lambda' \in \mathbb{R}$, $\Lambda \in \mathbb{F}$ and any function f with $\Delta_f \subset \Lambda$,*

$$|\mu^{\lambda'}(f) - \mu^{\lambda}(f)| \le c\|f\|_\infty |\Delta_f|\, |\mu^{\lambda'}(m_\Lambda) - \mu^{\lambda}(m_\Lambda)|, \tag{A.1}$$

$$|\mu^{\lambda'}(\eta_x;\eta_x) - \mu^{\lambda}(\eta_x;\eta_x)| \le c\, |\mu^{\lambda'}(m_\Lambda) - \mu^{\lambda}(m_\Lambda)| \quad \forall x \in \Lambda. \tag{A.2}$$

For any $m, m' \in (0,1)$ and any local function f,

$$|\mu^{\lambda_0(m')}(\eta_0) - \mu^{\lambda_0(m)}(\eta_0)| \le c\, |m' - m|, \tag{A.3}$$

$$|\mu^{\lambda_0(m')}(\eta_0;\eta_0) - \mu^{\lambda_0(m)}(\eta_0;\eta_0)| \le c\, |m' - m|, \tag{A.4}$$

$$|\mu^{\lambda_0(m')}(f) - \mu^{\lambda_0(m)}(f)| \le c(|\Delta_f|)\, \|f\|_\infty |m' - m| \tag{A.5}$$

for a suitable constant $c(|\Delta_f|)$ depending on $|\Delta_f|$.
Moreover, for any $\Lambda \in \mathbb{F}$ and any $m \in (0,1)$,

$$|\lambda_\Lambda(m) - \lambda_0(m)| \le \frac{c}{m(1-m)} |m - \mu^{\lambda_0(m)}(m_\Lambda)|. \tag{A.6}$$

Proof. It is simple to derive (A.2), (A.4) and (A.5) from (A.1) and (A.3).
Let us prove (A.1). By setting $\lambda(s) := \lambda_\Lambda(s)$, $m := \mu^{\lambda}(m_\Lambda)$ and $m' := \mu^{\lambda'}(m_\Lambda)$, we have

$$\mu^{\lambda'}(f) - \mu^{\lambda}(f) = \int_m^{m'} \frac{\partial}{\partial s} \mu^{\lambda(s)}(f)\, ds = \int_m^{m'} \mu^{\lambda(s)}(f; N_{\Delta_f}) \lambda'(s) ds.$$

By lemma A.2, $|\mu^{\lambda(s)}(f; N_{\Delta_f})\lambda'(s)| \le c\|f\|_\infty |\Delta_f|$, thus concluding the proof of (A.1).
In order to prove (A.3), we observe that

$$\mu^{\lambda_0(m')}(\eta_y) - \mu^{\lambda_0(m)}(\eta_y) = \int_m^{m'} \frac{d}{ds} \mu^{\lambda_0(s)}(\eta_0) ds = \int_m^{m'} \frac{\mu^{\lambda_0(s)}(\eta_0;\eta_0)}{\mathbb{E}\mu^{\lambda_0(s)}(\eta_0;\eta_0)} ds.$$

Thanks to the boundedness of the random field α, the last integrand is bounded, thus proving (A.3).
Let us prove (A.6). By Lagrange theorem

$$m = \mu_\Lambda^{\lambda(m)}(m_\Lambda) = \mu^{\lambda_0(m)}(m_\Lambda) + \mu^{\lambda}(m_\Lambda; N_\Lambda)\big(\lambda_\Lambda(m) - \lambda_0(m)\big)$$

for a suitable λ between $\lambda_\Lambda(m)$ and $\lambda_0(m)$. In order to conclude the proof, it is enough to apply lemma A.2. □

A.3. **Equivalence of ensembles.** In this paragraph we compare multi-canonical and multi-grand canonical expectations. The following results can be proved by the same methods developed in [6] with strong simplifications since here the grand canonical measures are product (see [15] for a complete treatment).

In what follows we fix $\Delta \in \mathbb{F}$ and we partition it as $\Delta = \cup_{i=1}^k \Delta_i$. Moreover, chosen a set $\mathbf{N} = \{N_i\}_{i=1}^k$ of possible particle numbers in each atom Δ_i, we define the multi-grand canonical measure $\bar{\mu}$ and the multi-canonical measure $\bar{\nu}$ as

$$\bar{\mu} := \otimes_{i=1}^k \mu_{\Delta_i}^{\lambda(m_i)} \quad \text{where} \quad m_i := \frac{N_i}{|\Delta_i|},$$
$$\bar{\nu} := \mu(\cdot \mid N_{\Delta_i} = N_i \ \ \forall i = 1, \dots, k).$$

Then we have the following main results (for the latter see also proposition 3.3 in [7]).

Proposition A.4. *(Equivalence of ensembles)*
Let $\gamma, \delta \in (0,1)$ and f be a local function such that $|\Delta_i| \geq \delta|\Delta|$, for any $i = 1, \ldots, k$, $\Delta_f \subset \Delta$ and $|\Delta_f| \leq |\Delta|^{1-\gamma}$.
Then there exist constants c_1, c_2, depending respectively on γ, δ, k and δ, k, such that

$$|\Delta| \geq c_1 \quad \Rightarrow \quad |\bar{\nu}(f) - \bar{\mu}(f)| \leq c_2 \, \|f\|_\infty \frac{|\Delta_f|}{|\Delta|}.$$

Lemma A.5. *Let $\delta \in (0,1)$ and f be a local function such that $\Delta_f \subset \Delta$ and $|\Delta_i \setminus \Delta_f| \geq \delta|\Delta_i|$ for any $i = 1, \cdots, k$.*
Then there exist constants c_1, c_2, depending respectively on k and k, δ, such that

$$|\Delta_i| \geq c_1 \quad \forall i = 1, \ldots, k \quad \Rightarrow \quad \bar{\nu}(|f|) \leq c_2 \, \bar{\mu}(|f|) \quad \textit{and} \quad \mathrm{Var}_{\bar{\nu}}(f) \leq c_2 \mathrm{Var}_{\bar{\mu}}(f),$$

A.4. **Some special equilibrium covariances.** In this paragraph we estimate the canonical covariance between a generic function and a function which can be written as the spatial average of local functions. We observe that the bound we provide differs from the standard Lu-Yau's Two Blocks Estimate (see [27]) by an additional term depending on the random field α and satisfying a large deviations estimate.

In what follows we fix functions $h, h' \in \mathbb{G}$, depending only on α_0 and η_0, such that $\|h\|_\infty, \|h'\|_\infty \leq 1$. Moreover, for any positive integer L, we denote by R_L the set of boxes with sides of length in $[L, 100L]$.

Proposition A.6. *Given $0 < \delta < \frac{1}{2}$ there exists $\ell_0 \in \mathbb{N}$ having the following property.*
Let $\ell, L \in \mathbb{N}$ be such that $\ell_0 \leq \ell \leq L$ and let $V, W \in R_L$ with $V \cap W = \emptyset$. Then, for any $\nu \in \mathcal{M}(V)$ and any function $g \in \mathbb{G}$,

$$\nu(\mathrm{Av}_{v \in V} \tau_v h; g)^2 \leq \frac{c(\ell)}{|V|} \mathcal{D}(g;\nu) + \frac{c\delta}{|V|} \mathrm{Var}_\nu(g) + \frac{c}{|V|} \mathrm{Var}_\nu(g) 1\!\!1_{\{m \in I_\delta\}} \vartheta_{V,\ell}(\alpha) \tag{A.7}$$

where $m := \nu(m_V)$ and $I_\delta := [\delta, 1-\delta]$. Moreover, for any $\gamma > 0$ there exists $\ell_1 = \ell_1(\gamma) \geq \ell_0$ such that

$$\ell_1 \leq \ell \leq L \quad \Rightarrow \quad \mathbb{P}(\, \vartheta_{V,\ell}(\alpha) \geq \gamma \,) \leq e^{-c(\gamma,\ell)L^d}. \tag{A.8}$$

Finally, for any $\nu \in \mathcal{M}(V \cup W)$ and any function $g \in \mathbb{G}$,

$$\begin{aligned} &\nu(\mathrm{Av}_{v \in V} \tau_v h \cdot \mathrm{Av}_{w \in W} \tau_w h'; g)^2 \leq \\ &\qquad \frac{c(\ell)}{|\Lambda|} \mathcal{D}(g;\nu) + \frac{c\delta}{|\Lambda|} \mathrm{Var}_\nu(g) + \frac{c}{|\Lambda|} \mathrm{Var}_\nu(g) \big(\vartheta_{V,\ell}(\alpha) + \vartheta_{W,\ell}(\alpha) \big) \end{aligned} \tag{A.9}$$

Proof. We first prove (A.7) by referring, for many steps, to the proof of proposition A.1 in [7]. Let us fist introduce some useful notation.
We fix a partition $V = \cup_{i \in I} Q_i$, with $Q_i \in R_\ell$, and define $N_i := N_{Q_i}$, $m_i := N_{Q_i}/|Q_i|$, $h_i := \sum_{x \in Q_i} \tau_x h$, $\mathcal{F} := \sigma(m_i \,|\, i \in I)$ and for $s \in [0,1]$

$$A_i(m) := \frac{\mu_V^{\lambda(m)}(h_i; N_i)}{\mu_V^{\lambda(m)}(N_i; N_i)} - \frac{\mathbb{E}\mu^{\lambda_0(m)}(h_0; \eta_0)}{\mathbb{E}\mu^{\lambda_0(m)}(\eta_0; \eta_0)} \qquad B_i(s) := \frac{\mu_{Q_i}^{\lambda(s)}(h_i; N_i)}{\mu_{Q_i}^{\lambda(s)}(N_i; N_i)} - \frac{\mu_V^{\lambda(m)}(h_i; N_i)}{\mu_V^{\lambda(m)}(N_i; N_i)}.$$

As in [7], if $m \notin I_\delta$ then it is enough to apply Schwarz inequality and lemma A.5 to obtain the thesis, otherwise it is convenient to bound the l.h.s. of (A.7) as

$$\nu(\mathrm{Av}_{v \in V} \tau_v h; g)^2 \leq 2\nu\big(\, \nu(\mathrm{Av}_{v \in V} \tau_v h; g \,|\, \mathcal{F}) \,\big)^2 + 2\nu\big(\, \nu(\mathrm{Av}_{v \in V} \tau_v h \,|\, \mathcal{F}); g \,\big)^2 \tag{A.10}$$

As in [7] we can bound the first addendum in the r.h.s. of (A.10) by $c(\ell)\mathcal{D}(g;\nu)$ and the second one by

$$c\mathrm{Var}_\nu(g)\Big(\frac{1}{\ell^d L^\delta}+\frac{1}{L^d}\sum_{i\in I}\mathrm{Var}_{\mu_V^{\lambda(m)}}(\xi_i^\gamma)\Big) \tag{A.11}$$

where, for an arbitrarily fixed γ, $\xi_i^\gamma(\eta):=\mu_{Q_i}^{\lambda(m_i(\eta))}(h_i-\gamma N_i)$. Let us explain how to proceed. Thanks to Poincaré inequality for Glauber dynamics we obtain

$$\mathrm{Var}_{\mu_V^{\lambda(m)}}(\xi_i^\gamma)\le c\sum_{x\in Q_i}\mu_V^{\lambda(m)}((\nabla_x\xi_i^\gamma)^2). \tag{A.12}$$

By choosing $\gamma=\frac{\mathbb{E}\,\mu^{\lambda_0(m)}(h_0;\eta_0)}{\mathbb{E}\,\mu^{\lambda_0(m)}(\eta_0;\eta_0)}$ it is simple to check that

$$\nabla_x\xi_i^\gamma=A_i+(-1)^{\eta_x}\int_{m_i(\eta)}^{m_i(\eta^x)}B_i(s)ds.$$

By writing

$$B_i(s)=\frac{|Q_i|}{\mu_V^{\lambda(m)}(N_i;N_i)}\Big(\int_{\mu_V^{\lambda(m)}(m_i)}^{s}\frac{\mu_{Q_i}^{\lambda(s')}(h_i;N_i;N_i)}{\mu_{Q_i}^{\lambda(s')}(N_i;N_i)}ds'+\\ \frac{\mu_{Q_i}^{\lambda(s)}(h_i;N_i)}{\mu_{Q_i}^{\lambda(s)}(N_i;N_i)}\int_{\mu_V^{\lambda(m)}(m_i)}^{s}\frac{\mu_{Q_i}^{\lambda(s')}(N_i;N_i;N_i)}{\mu_{Q_i}^{\lambda(s')}(N_i;N_i)}ds'\Big),$$

by lemma A.2 and the condition $m\in I_\delta$ we obtain that $|B_i(s)|\le\frac{c}{\delta}|s-\mu_V^{\lambda(m)}(m_i)|$ and therefore

$$|\nabla_x\xi_i^\gamma|\le A_i+\frac{c}{\delta}\big|\,m_i(\eta)-\mu_V^{\lambda(m)}(m_i)\big|+\frac{c}{\delta}\ell^{-d}. \tag{A.13}$$

By (A.11), (A.12) and (A.13) it is simple to conclude the proof if ℓ is large enough and

$$\vartheta_{V,\ell}(\alpha):=\sup_{m\in M_V}\mathrm{Av}_{i\in I}A_i(m)^2\quad\text{where}\quad M_V=\big\{\frac{1}{|V|},\frac{2}{|V|},\dots,1-\frac{1}{|V|}\big\}.$$

By standard arguments (as for lemma 3.9 in [7]) (A.9) can be derived from (A.7).
Let us prove (A.8). By lemmas A.2 and A.3

$$|A_i|\le\frac{c}{m(1-m)}\Big(|\mu^{\lambda_0(m)}(h_i;m_i)-\mathbb{E}\mu^{\lambda_0(m)}(h_0;\eta_0)|+\\ |(\mu^{\lambda_0(m)}(N_i;m_i)-\mathbb{E}\mu^{\lambda_0(m)}(\eta_0;\eta_0)|+|m-\mu^{\lambda_0(m)}(m_V)|\Big).$$

Therefore it is enough to prove that given a function $f=f(\alpha_0)$ with $\|f\|_\infty\le1$ then for any $\gamma>0$ there exists $\ell_1=\ell_1(\gamma)$ such that

$$\mathbb{P}\big(\mathrm{Av}_{i\in I}(\mathrm{Av}_{x\in Q_i}\tau_xf-\mathbb{E}(f))^2\ge\gamma\big)\le 2e^{-\frac{c\gamma^2L^d}{\ell^d}}\qquad\forall\ell\ge\ell_1.$$

To this aim we define $f_i:=(\mathrm{Av}_{x\in Q_i}\tau_xf-\mathbb{E}(f))^2$ and $\bar f_i:=f_i-\mathbb{E}(f_i)$. Then by lemma $A.1$, for any $0<\delta<1$,

$$\mathbb{E}(f_i)\le\mathbb{P}\big(|\mathrm{Av}_{x\in Q_i}\tau_xf-\mathbb{E}(f)|\ge\delta\big)+\delta^2\le 2e^{-c\,\delta^2\ell^d}+\delta^2.$$

Therefore, by choosing δ small enough and ℓ large enough, $\mathbb{E}(f_i)\le\frac{\gamma}{2}$ for any $i\in I$ and (by applying again lemma A.1)

$$\mathbb{P}(\,\mathrm{Av}_{i\in I} f_i \geq \gamma\,) \leq \mathbb{P}(\,\mathrm{Av}_{i\in I}\bar{f}_i \geq \frac{\gamma}{2}\,) \leq 2e^{-c\,\gamma^2|I|} \tag{A.14}$$

thus concluding the proof. □

A.5. **Moving Particle Lemma.** Given $x, y \in \mathbb{Z}^d$ we define

$$z_i := (y_1, y_2, \dots, y_i, x_{i+1}, \dots, x_d) \qquad \forall i = 0, \dots, d$$

and write $\gamma_{x,y}$ for the path connecting $z_0 = x$ to z_1 by moving along the first direction, then connecting z_1 to z_2 by moving along the second direction and so on until arriving to $z_d = y$. We denote by $|\gamma_{x,y}|$ the length of the path $\gamma_{x,y}$.

Lemma A.7. *(Moving Particles Lemma)*
Given a box Λ and $\nu \in \mathcal{M}(\Lambda)$ then

$$\nu\big(\,(\nabla_{x,y} f)^2\,\big) \leq c\,|\gamma_{x,y}| \sum_{b\in\gamma_{x,y}} \nu\big(\,(\nabla_b f)^2\,\big) \qquad \forall x, y \in \Lambda,\ f \in \mathbb{G}.$$

The above lemma is well known for non disordered systems (see for example [35]). We learned from J.Quastel the generalization to the disordered case.

A.6. **An application of Feynman-Kac formula.** The following proposition can be derived from the Feynman-Kac formula as explained in [23]. We report only the statement.

Let X be a finite set on which it is defined a probability measure ν and a Markov generator $\mathfrak{L}$ reversible w.r.t. ν. We denote by $\mathbb{E}_\nu$ the expectation w.r.t. the Markov process having infinitesimal generator $\mathfrak{L}$ and initial distribution ν and by x_t the configuration at time t.

Proposition A.8. *Let $V : \mathbb{R}_+ \times X \to \mathbb{R}$ be a bounded measurable function and let, for any $t > 0$,*

$$\Gamma_t := \sup spec_{L^2(\nu)}\{\mathfrak{L} + V(t,\cdot)\}.$$

Then

$$\mathbb{E}_\nu\Big[\exp\Big\{\int_0^t V(s, x_s)ds\Big\}\Big] \leq \exp\Big\{\int_0^t \Gamma_s\, ds\Big\} \quad \forall t > 0.$$

A.7. **Two Blocks Estimate.** For a treatment of the Two Blocks estimate in non disordered systems see [23] and reference therein. Let us state and prove a generalized version.

Proposition A.9. *Given $\gamma > 0$, for almost any disorder configuration α*

$$\limsup_{a\downarrow 0, k\uparrow\infty, \epsilon\downarrow 0}\ \sup_{w:|w|\leq\frac{a}{\epsilon}}\ \sup spec_{L^2(\mu_\epsilon)}\{\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}|m_{x,k} - m_{x+w,k}| + \gamma\epsilon^{d-2}\mathcal{L}_\epsilon\} \leq 0. \tag{A.15}$$

Proof. We extend to the disordered case the proof of the Two Blocks estimate of [37] thanks to the ergodicity of the random field α. To this aim let us introduce the scale parameter ℓ with $\ell \uparrow \infty$ after $k \uparrow \infty$. Then, with a negligible error of order $O(\ell/k)$, for any $x \in \mathbb{T}^d_\epsilon$ we can substitute $m_{x,k}$ with $\mathrm{Av}_{y\in\Lambda_k} m_{x+y,\ell}$. Therefore, thanks to the subadditivy of $\sup spec$, the l.h.s. of (A.15) can be bounded from above (with an error $O(\delta)$) by

$$\sup_{w:|w|\leq\frac{a}{\epsilon}} \mathrm{Av}_{y\in\Lambda_k}\mathrm{Av}_{y'\in\Lambda_k\,:\,|w+y'-y|>2\ell}\ \sup_{L^2(\mu_\epsilon)} spec\{\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}|m_{x+y,\ell} - m_{x+w+y',\ell}| + \gamma\epsilon^{d-2}\mathcal{L}_\epsilon\}$$

where the additional restriction $|w + y' - y| > 2\ell$ is painless. By renaming the index variables, it is enough to show that given $\gamma > 0$, for almost any disorder configuration α,

$$\limsup_{\ell\uparrow\infty, a\downarrow 0, \epsilon\downarrow 0}\ \sup_{w:2\ell<|w|\leq\frac{a}{\epsilon}}\ \sup_{L^2(\mu_\epsilon)} spec\,\{\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}|m_{x,\ell} - m_{x+w,\ell}| + \gamma\epsilon^{d-2}\mathcal{L}_\epsilon\} \leq 0. \tag{A.16}$$

For any $u, v \in \mathbb{Z}^d$ let us define $\hat{\mathcal{L}}_{u,v} = (1 + e^{-(\alpha_u - \alpha_v)(\eta_u - \eta_v)})\nabla_{u,v}$. It is simple to check that $\hat{\mathcal{L}}_{u,v}$ is self-adjoint w.r.t. Gibbs measures. Then, given w as above, thanks to the Moving Particle lemma (see lemma A.7) and the properties of the transition rates, it is simple to prove that

$$\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\mathrm{Av}_{u\in\Lambda_{x,\ell}}\mathrm{Av}_{v\in\Lambda_{x+w,\ell}}(-\hat{\mathcal{L}}_{u,v}) \le c\, a^2\epsilon^{d-2}(-\mathcal{L}_\epsilon). \tag{A.17}$$

Therefore, by localizing as in (4.12), the $supspec$ in (A.16) is bounded by

$$\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\sup_\nu\ \sup spec_{L^2(\nu)}\{|m_{x,\ell} - m_{x+w,\ell}| + c\,\gamma a^{-2}\,\mathrm{Av}_{u\in\Lambda_{x,\ell}}\mathrm{Av}_{v\in\Lambda_{x+w,\ell}}(-\hat{\mathcal{L}}_{u,v})\}$$

where ν varies in $\mathcal{M}(\Lambda_{x,\ell}\cup\Lambda_{x+w,\ell})$. Thanks to perturbation theory (see proposition 4.2) we only need to prove that, for almost any disorder configuration α,

$$\limsup_{\ell\uparrow\infty, a\downarrow 0,\epsilon\downarrow 0}\ \sup_{w:2\ell<|w|\le\frac{a}{\epsilon}}\ \mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\sup_\nu\ \nu(|m_{x,\ell} - m_{x+w,\ell}|). \tag{A.18}$$

We observe that by lemma A.5 in the above expression we can
substitute ν with the grand canonical measure μ such that $\mu(m_\Lambda) = \nu(m_\Lambda)$ where $\Lambda := \Lambda_{x,\ell}\cup\Lambda_{x+w,\ell}$.
Let us introduce the scale parameter s with $s\uparrow\infty$ after $\ell\uparrow\infty$. Then, by approximating $m_{x,\ell}$ with $\mathrm{Av}_{y\in\Lambda_{x,\ell}}m_{y,s}$ and thanks to lemma A.3

$$\mu(|m_{x,\ell} - m_{x+w,\ell}|) \le c\,\mathrm{Av}_{y\in\Lambda_{x,\ell}}\mu^{\lambda_0(m)}(|m_{y,s} - m_{y+w,s}|) + c\,s^d\big|m - \mu^{\lambda_0(m)}(m_\Lambda)\big| + O(s/\ell).$$

where $m = \mu(m_\Lambda) = \nu(m_\Lambda)$ and Λ is defined as above. Therefore, it is enough to prove that for almost all disorder configuration α

$$\limsup_{s\uparrow\infty,\ell\uparrow\infty,\epsilon\downarrow 0}\ \mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\sup_m\mathrm{Av}_{y\in\Lambda_\ell}\mu^{\lambda_0(m)}(|m_{x+y,s} - m|) = 0,$$

$$\limsup_{\ell\uparrow\infty,\epsilon\downarrow 0}\mathrm{Av}_{x\in\mathbb{T}^d_\epsilon}\sup_m|m - \mu^{\lambda_0(m)}(m_{x,\ell})| = 0.$$

Since $\mathbb{E}\mu^{\lambda_0(m)}(m_{x,n}) = m$ for any integer n and any site x, the above limits follow by straightforward arguments from the ergodicity of the random field α and the technical estimate (A.3). $\square$

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