ZETA FUNCTIONS FOR THE SPECTRUM OF THE NON-COMMUTATIVE HARMONIC OSCILLATORS*

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Abstract. This paper investigates the spectral zeta function of the non-commutative harmonic oscillator studied in [PW1, 2]. It is shown, as one of the basic analytic properties, that the spectral zeta function is extended to a meromorphic function in the whole complex plane with a simple pole at $s = 1$, and further that it has a zero at all non-positive even integers, i.e. at $s = 0$ and at those negative even integers where the Riemann zeta function has the so-called trivial zeros. As a by-product of the study, both the upper and the lower bounds are also given for the first eigenvalue of the non-commutative harmonic oscillator.
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1. Introduction

When we try to study the so-called spectral zeta function associated with some given operator, basically it seems difficult to expect it to share with the Riemann zeta function too many properties such as a precise information of the location of the poles/zeros (apart from the so-called essential zeros in the strip $0 < \mathrm{Re}\ s < 1$), the functional equation, the Euler product and so forth. However, we might understand part of the information concerning the analytic continuation from the absolutely convergent region to the left, an exact knowledge of the first singularity, etc (see e.g. [MP]). Furthermore, once we get such information, it even allows us to show the so-called Weyl law which describes the number of eigenvalues of the operator less than x for $x \to \infty$.

The aim of the present paper is then to investigate the spectral zeta function of the non-commutative harmonic oscillators. It is defined via the spectrum of the following ordinary differential operator introduced in [PW1, 2]:

$$Q(x, D_x) = A\Big(-\frac{\partial_x^2}{2} + \frac{x^2}{2}\Big) + J\Big(x\partial_x + \frac{1}{2}\Big), \quad x \in \mathbb{R}, \quad \partial_x := \frac{d}{dx}, \tag{1.1}$$

where $A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ and $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We assume that $\alpha, \beta \in \mathbb{R}$ are positive and $\alpha\beta > 1$. Then it is known that Q defines a positive, self-adjoint operator in the Hilbert space $L^2(\mathbb{R}) \otimes \mathbb{C}^2$ which has only a discrete spectrum $(0 <)\, \lambda_1 \le \lambda_2 \le \ldots \le \lambda_n \le \ldots \to \infty$ with uniformly bounded multiplicity (see [PW2, 3]). Then the spectral zeta function $\zeta_Q(s)$ of the system is defined as

$$\zeta_Q(s) = \sum_{j=1}^{\infty} \frac{1}{\lambda_j^s} \tag{1.2}$$

for sufficiently large $\mathrm{Re}\ s > 0$. We can prove in fact ([IW]) that the series converges absolutely in $\mathrm{Re}\ s > 1$.

As described in [PW2], when $\alpha = \beta$ the system becomes unitarily equivalent to a couple of the usual quantum harmonic oscillators, whereas this cannot hold otherwise. In particular, if $\alpha = \beta = \sqrt{2}$ then one knows that $Q = Q_0 \cong \frac{1}{2}(-\partial_x^2 + x^2)I$, with I being the 2×2 identity matrix, where the intertwining unitary operator is also constructed (see Corollary 4.1 in [PW2]). Therefore its spectrum is known and actually

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given by $\{n+\frac{1}{2}\}$ $(n=0,1,2,\dots)$ with multiplicity two. This implies the spectral zeta function $\zeta_{Q_0}(s)$ is explicitly calculated as

$$\zeta_{Q_0}(s) = 2\sum_{n=0}^{\infty}\frac{1}{(n+\frac{1}{2})^s} = 2(2^s-1)\zeta(s), \tag{1.3}$$

where $\zeta(s)=\sum_{n=1}^{\infty} n^{-s}$ is the Riemann zeta function. From this expression, the zeta function $\zeta_Q(s)$ introduced above can be considered as a deformation of the Riemann zeta function (see Corollary 4.7). Though theoretically, the spectrum is described by using certain continued fractions (see [PW2, 4]) almost nothing is known in reality about the eigenvalues when $\alpha \neq \beta$ (see [NNW] for some numerical observation), since we cannot expect the existence of the annihilation and the creation operators which enable us to easily understand a structure of the system like the usual quantum harmonic oscillator. Thus the main concern of the study of $\zeta_Q(s)$ is to discuss the following questions:

(1) Does the zeta function $\zeta_Q(s)$ have an analytic continuation to the whole complex plane ?
(2) What can one say about a Weyl law for the eigenvalues ?
(3) Does one have an information about the location of zeros and poles ?
(4) Is it possible to calculate the special values, for instance, at the integer points ? etc.

As to questions (1), (2) and a part of (3), we have good answers. In fact, we first recall that the series (1.2) defining $\zeta_Q(s)$ converges absolutely in the region Re $s>1$, that is, $\zeta_Q(s)$ is holomorphic there (see Theorem 3.3 in [IW]), and, based on this result, prove that it has a simple pole at $s=1$ as in the case of $\zeta(s)$ (see §2). From this fact with the information about the residue at $s=1$ (see below), one can conclude that Weyl's law in the present case is stated as

$$\sum_{\lambda_n<x} 1 \sim \frac{\alpha+\beta}{\sqrt{\alpha\beta(\alpha\beta-1)}}x \qquad (x\to\infty).$$

Furthermore, studying the heat kernel of the operator we prove that $\zeta_Q(s)$ can be extended meromorphically to the whole complex plane $\mathbb{C}$. To be remarkable, we can show that $\zeta_Q(s)$ possesses a kind of "trivial zero" at each non-positive even integer point. In fact, the main theorem of the paper can be formulated as follows.

Main Theorem. *There exist constants $C_{Q,j}$ $(j=1,2,\dots)$ such that for every positive integer n one has*

$$\zeta_Q(s) = \frac{1}{\Gamma(s)}\left[\frac{\alpha+\beta}{\sqrt{\alpha\beta(\alpha\beta-1)}}\,\frac{1}{s-1} + \sum_{j=1}^{n}\frac{C_{Q,j}}{s+2j-1} + H_{Q,n}(s)\right], \tag{1.4}$$

where $H_{Q,n}(s)$ is a holomorphic function in Re $s>-2n$. *Consequently, the spectral zeta function $\zeta_Q(s)$ is meromorphic in the whole complex plane with a simple pole at $s=1$ and has zeros for s being non-positive even integers.*

Obviously the question (3) above should be also related to the question whether or not there exist a functional equation and/or an Euler product. However, in our case, it seems very hard to expect any functional equation or any Euler product expression. Hence the problem is still mysterious whether the "essential zeros" of $\zeta_Q(s)$ are all situated in the same critical strip $0<$ Re $s<1$ as those of $\zeta(s)$ or not. Actually, it is not yet known if $\zeta_Q(s)$ is free from zero in the half plane Re $s>1$, although in the case of the Riemann zeta function the corresponding fact immediately follows from its Euler product expression $\zeta(s)=\prod_{p:\text{primes}}(1-p^{-s})^{-1}$ for Re $s>1$. We only note (see Proposition 2.10) that $\zeta_Q(s)$ does not vanish in the region Re $s>\sigma_0$ with a sufficiently large $\sigma_0>1$. But still, in this connection, as a by-product of the study, we give the upper and lower bounds for the first eigenvalue of the operator Q (Theorem 2.9), which are best possible in the sense that both these bounds coincide when $\alpha=\beta$, i.e. when Q is essentially a couple of the harmonic oscillators.

We will start early the proof of the main theorem in §2 and finally complete it at the very end of §4, the last section. The method we develop here to prove the main theorem is based on the asymptotic expansion of the trace of the heat kernel, the integral kernel of the self-adjoint semigroup e^{-tQ} for $t\downarrow 0$. In this sense, the numbers $C_{Q,j}$ in the theorem are regarded as analogues of Bernoulli's numbers (see e.g. [E, T]). As to this point, we also refer to Remark 2 in the last section. Such an approach as made in the paper may be in a vein similar to the study [MP].

We have not treated here the question (4), which describes the special values of the spectral zeta function at the positive integral points. But in [IW] we have observed that these values are closely related to certain integrals which involve elliptic integrals and further, at least in case where n is small, there is a close connection between the special values and the solutions of certain singly confluent Heun's ordinary differential equations. Here the so-called Heun differential equation is a Fuchsian type ordinary differential equation with four regular singular points in a complex domain (see, e.g. [WW], [SL]). In fact, the values of $\zeta_Q(s)$ at $n=2,3$ are described in terms of the solutions of such confluent Heun's equations [IW]. In this sense, it is quite interesting to understand the relation between the values $\zeta_Q(-2m+1)=(2m-1)!\,C_{Q,m}$ and $\zeta_Q(2m)$ through Heun's equations.

2. Heat kernel and its expansion

Consider the self-adjoint operator [PW1] defined by

$$Q := Q_{(\alpha,\beta)}(x, D_x) = A\,\frac{1}{2}(-\partial_x^2 + x^2) + J(x\partial_x + \frac{1}{2}), \tag{2.1}$$

where $A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ with positive α, β and $\alpha\beta > 1$, and $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. It is known [PW1] that Q has only a discrete spectrum. Let $K(t,x,y)$ be the heat kernel for Q, i.e. the integral kernel of the self-adjoint semigroup $K(t) := e^{-tQ}$. Throughout this paper, Tr stands for the operator trace, while tr does for the 2×2-matrix trace.

We now use $\mathrm{Tr}\,K(t) = \int \mathrm{tr}K(t,x,x)dx$ to define the zeta function $\zeta_Q(s)$ for the operator Q through the Mellin transform

$$\zeta_Q(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}\mathrm{Tr}\,K(t)dt, \tag{2.2}$$

which makes sense for the moment at least for $\mathrm{Re}\,s$ sufficiently large. Now, let $K_1(t)$ be the operator with integral kernel $K_1(t,x,y)$ given by the pseudo-differential operator

$$\begin{aligned}(K_1(t)f)(x) &= \int K_1(t,x,y)f(y)dy \\ &= (2\pi)^{-1} \iint e^{i(x-y)\xi} \exp\Big[-t\Big(A(\xi^2+y^2)/2 + Jyi\xi\Big)\Big] f(y)dyd\xi,\end{aligned} \tag{2.3}$$

for $f \in \mathcal{S}(\mathbb{R}, \mathbb{C}^2) = \mathcal{S}(\mathbb{R}) \otimes \mathbb{C}^2$. Then we put

$$R_2(t) = K(t) - K_1(t) \quad \text{or} \quad R_2(t,x,y) = K(t,x,y) - K_1(t,x,y). \tag{2.4}$$

Since $K(t,x,y)$ satisfies the heat equation

$$0 = (\partial_t + Q)K(t,x,y) = (\partial_t + Q)K_1(t,x,y) + (\partial_t + Q)R_2(t,x,y), \quad t>0, \quad (x,y)\in\mathbb{R}^2,$$

we have

$$(\partial_t + Q)R_2(t,x,y) = -(\partial_t + Q)K_1(t,x,y) =: F(t,x,y), \tag{2.5}$$

and

$$R_2(t,x,y) \to 0I, \qquad t \downarrow 0,$$

because we can check that $K_1(t,x,y) \to \delta(x-y)I$ as $t \downarrow 0$. Therefore, by the definition of $F(t,x,y)$ in (2.5), we have by Duhamel's principle (see e.g., pp.202–204 in [CH]) that

$$R_2(t) = \int_0^t e^{-(t-u)Q} F(u)du,$$

where

$$(F(u)f)(x) = \int F(t,x,y)f(y)dy$$

and also

$$\begin{aligned}R_2(t,x,y) &= \int_0^t du \int e^{-(t-u)Q}(x,z)F(u,z,y)dz \\ &= \int_0^t du \int K(t-u,x,z)F(u,z,y)dz \\ &= \int_0^t du \int K_1(t-u,x,z)F(u,z,y)dz + \int_0^t du \int R_2(t-u,x,z)F(u,z,y)dz. \end{aligned} \tag{2.6}$$

In view of the definition of $F(t,x,y)$ in (2.5) again we have

$$\begin{aligned} &\int F(t,x,y)f(y)dy \\ =&\int -(\partial_t + Q)K_1(t,x,y)f(y)dy \\ =&\frac{1}{2\pi}\iint \Big[\Big(A\frac{\xi^2+y^2}{2} + Jyi\xi\Big) - \Big(A\frac{-\partial_x^2+x^2}{2} + J(x\partial_x + \frac{1}{2})\Big)\Big]\Big[e^{i(x-y)\xi}e^{-t[A\frac{\xi^2+y^2}{2}+Jyi\xi]}\Big]f(y)dyd\xi \\ =&\frac{1}{2\pi}\iint e^{i(x-y)\xi}\Bigg[\Big(A\frac{\xi^2+y^2}{2} + Jyi\xi\Big)e^{-t[A\frac{\xi^2+y^2}{2}+Jyi\xi]} \\ &\quad - \Big(A\frac{\xi^2+x^2}{2}e^{-t[A\frac{\xi^2+y^2}{2}+Jyi\xi]}\Big) - J\Big(xi\xi\, e^{-t[A\frac{\xi^2+y^2}{2}+Jyi\xi]} + \frac{1}{2}e^{-t[A\frac{\xi^2+y^2}{2}+Jyi\xi]}\Big)\Bigg]f(y)dyd\xi \\ =&\frac{1}{2\pi}\iint e^{i(x-y)\xi}\Big[A\frac{y^2-x^2}{2} + J(y-x)i\xi\Big]e^{-t[A\frac{\xi^2+y^2}{2}+Jyi\xi]}f(y)dyd\xi \\ &\quad - \frac{1}{2}(2\pi)^{-1}\iint e^{i(x-y)\xi}Je^{-t[A\frac{\xi^2+y^2}{2}+Jyi\xi]}f(y)dyd\xi. \end{aligned}$$

Hence we obtain

$$\begin{aligned} &F(t,x,y) \\ =&\frac{1}{2\pi}\int e^{i(x-y)\xi}\Big[A\frac{y^2-x^2}{2} + J(y-x)i\xi\Big]e^{-t[A\frac{\xi^2+y^2}{2}+Jyi\xi]}d\xi - \frac{1}{4\pi}\int e^{i(x-y)\xi}Je^{-t[A\frac{\xi^2+y^2}{2}+Jyi\xi]}d\xi \\ =&: F_1(t,x,y) + F_2(t,x,y). \end{aligned} \tag{2.7}$$

We now write $\zeta_Q(s)$ as

$$\begin{aligned}\zeta_Q(s) &= \frac{1}{\Gamma(s)}\int_0^1 t^{s-1}\mathrm{Tr}\,K(t)dt + \frac{1}{\Gamma(s)}\int_1^\infty t^{s-1}\mathrm{Tr}\,K(t)dt \\ &=: Z_0(s) + Z_\infty(s). \end{aligned} \tag{2.8}$$

We show first that $Z_\infty(s)$ is holomorphic, and study $Z_0(s)$ later. Actually, putting $\hat{Z}_\infty(s) := \Gamma(s)Z_\infty(s)$ we prove the following assertion by the aid of the result obtained in [IW].

Proposition 2.1. *The function $\hat{Z}_\infty(s) = \int_1^\infty t^{s-1}\mathrm{Tr}\,K(t)dt$ is holomorphic in the whole complex plane. As a result, it is also true for $Z_\infty(s)$.*

Proof. We have only to show that $\hat{Z}_\infty(s)$ is holomorphic, since $\frac{1}{\Gamma(s)}$ is holomorphic. Let $\{\lambda_n\}_{n=1}^\infty$ be the eigenvalues of Q. They are all positive. We have $\mathrm{Tr}\,K(t) = \sum_{n=1}^\infty e^{-\lambda_n t}$, so that

$$\hat{Z}_\infty(s) = \Gamma(s)Z_\infty(s) = \sum_{n=1}^\infty \int_1^\infty t^{s-1}e^{-\lambda_n t}dt.$$

We need to show that the last member above converges absolutely and locally uniformly in the complex plane. Note that $t^a e^{-t} \le (a/e)^a$ for all $t > 0$ and $a > 0$.

Suppose first that $\sigma = \mathrm{Re}\, s \le 1$. Then, since $\sum_{n=1}^{\infty} \lambda_n^{-2} < \infty$ (§3 in [IW] or by Lemma 2.8 below), we have

$$|\hat{Z}_\infty(s)| \le \sum_{n=1}^{\infty} \int_1^{\infty} e^{-\lambda_n t} dt = \sum_{n=1}^{\infty} \frac{e^{-\lambda_n}}{\lambda_n} \le \frac{1}{e} \sum_{n=1}^{\infty} \lambda_n^{-2} < \infty.$$

Next suppose that $\sigma = \mathrm{Re} s > 1$. Then

$$\begin{aligned}
|\hat{Z}_\infty(s)| &\le \sum_{n=1}^{\infty} \int_1^{\infty} (\lambda_n/2)^{-(\sigma-1)} \big((\lambda_n t/2)^{\sigma-1} e^{-\lambda_n t/2}\big) e^{-\lambda_n t/2} dt \\
&\le \Big(\frac{\sigma-1}{e}\Big)^{\sigma-1} \sum_{n=1}^{\infty} \int_1^{\infty} (\lambda_n/2)^{-(\sigma-1)} e^{-\lambda_n t/2} \\
&= \Big(\frac{\sigma-1}{e}\Big)^{\sigma-1} \sum_{n=1}^{\infty} (\lambda_n/2)^{-(\sigma-1)} \frac{2e^{-\lambda_n/2}}{\lambda_n} = \Big(\frac{\sigma-1}{e}\Big)^{\sigma-1} 2^\sigma \sum_{n=1}^{\infty} \frac{e^{-\lambda_n}/2}{\lambda_n^\sigma} < \infty.
\end{aligned}$$

This proves the assertion of Proposition 2.1. □

Before coming to the study of $Z_0(s)$ in (2.8), we explain briefly what we are going to do for our $\zeta_Q(s)$ from now on, by illustrating the Riemann zeta function case. It is easy to derive by the integral representation of the gamma function that

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{e^t - 1} dt.$$

This corresponds exactly to the equation (2.2) above for $\zeta_Q(s)$. Let $t/(e^t-1) = \sum_{n=0}^{\infty} (B_n/n!)\, t^n$ ($|t| < 2\pi$) be the Taylor expansion of $t/(e^t-1)$ at $t = 0$, where B_n are the Bernoulli numbers [E]. Using this expansion, we immediately get through the Mellin transform

$$\zeta(s) = \frac{1}{\Gamma(s)} \left[\sum_{n=0}^{\infty} \frac{B_n}{n!} \cdot \frac{1}{s+n-1} + \int_1^{\infty} \frac{t^{s-1}}{e^t-1} dt \right].$$

Obviously, the integral on the right-hand side defines an entire function, just by the same reasoning as in Proposition 2.1 above. Hence $\zeta(s)$ is meromorphically extended to the whole complex plane. In particular, since $B_{2m+1} = 0$ for $m = 1, 2, \ldots$, this shows $\zeta(s)$ has a zero at each negative integer (see Remark 2 in §4). However, to get a meromorphic extension of $\zeta(s)$ in this way, we notice that what one needs is only to find an asymptotic expansion of $t/(e^t-1)$ for small $t > 0$, since there is no problem for large $t > 0$, although, one has had actually the Taylor expansion of $t/(e^t-1)$.

Thus, in the study of the property of $\zeta_Q(s)$, the main point is to investigate the behavior of $\mathrm{Tr}\, K(t)$ when $t \downarrow 0$, so that the problem is reduced to seeking an asymptotic expansion of $\mathrm{Tr}\, K(t)$. As in the case of the usual harmonic oscillator, we expect the expansion to start from the term for t^{-1} like

$$\mathrm{Tr}\, K(t) \sim c_{-1} t^{-1} + c_0 t^0 + c_1 t + c_2 t^2 + c_3 t^3 + \cdots. \tag{2.9}$$

In order to get this expansion (2.9), we now come back to study $Z_0(s)$ in (2.8):

$$\begin{aligned}
Z_0(s) &= \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} \mathrm{Tr}\, K_1(t)\, dt + \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} \mathrm{Tr}\, R_2(t)\, dt \\
&=: Z_{01}(s) + Z_{02}'(s).
\end{aligned} \tag{2.10}$$

The first task turns out to determine the very first coefficient c_{-1} in (2.9).

Proposition 2.2. *For the trace of $K_1(t)$ defined in (2.3), one has*

$$\operatorname{Tr} K_1(t) = \frac{\alpha+\beta}{\sqrt{\alpha\beta(\alpha\beta-1)}}\, t^{-1}, \tag{2.11a}$$

$$Z_{01}(s) = \frac{\alpha+\beta}{\sqrt{\alpha\beta(\alpha\beta-1)}} \frac{1}{\Gamma(s)} \cdot \frac{1}{s-1}. \tag{2.11b}$$

Proof. We have by (2.3) and by change of variables $\xi' = t^{1/2}\xi$, $x' = t^{1/2}x$,

$$\begin{aligned} \operatorname{Tr} K_1(t) &= \int \operatorname{tr} K_1(t,x,x)\, dx \\ &= \frac{1}{2\pi t} \iint \operatorname{tr}\, \exp\Big[-\Big(A(\xi^2+x^2)/2 + Jxi\xi\Big)\Big] dx d\xi. \end{aligned}$$

To calculate the last integral of the exponential or its matrix trace, we use the polar coordinates $\xi = \rho\cos\theta$, $x = \rho\sin\theta$, $0 \le \rho < \infty$, $0 \le \theta < 2\pi$. Then

$$\begin{aligned} \iint \exp\Big[-\Big(A(\xi^2+x^2)/2 + Jxi\xi\Big)\Big] dx d\xi &= \int_0^\infty \int_0^{2\pi} \exp\Big[-\rho^2/2 \begin{pmatrix} \alpha & -i\sin 2\theta \\ i\sin 2\theta & \beta \end{pmatrix}\Big] \rho d\rho d\theta \\ &= \int_0^\infty \int_0^{2\pi} \exp\Big[-\rho' \begin{pmatrix} \alpha & -i\sin\theta' \\ i\sin\theta' & \beta \end{pmatrix}\Big] d\rho' d\theta'. \end{aligned}$$

Integrating first in ρ' and next in θ', we see the last integral is equal to

$$\begin{aligned} &\int_0^{2\pi} d\theta' \Big[-\begin{pmatrix} \alpha & -i\sin\theta' \\ i\sin\theta' & \beta \end{pmatrix}^{-1} \exp\Big[-\rho \begin{pmatrix} \alpha & -i\sin\theta' \\ i\sin\theta' & \beta \end{pmatrix}\Big]\Big]_{\rho=0}^{\rho=\infty} \\ &= \int_0^{2\pi} d\theta' \begin{pmatrix} \alpha & -i\sin\theta' \\ i\sin\theta' & \beta \end{pmatrix}^{-1} = \int_0^{2\pi} d\theta \begin{pmatrix} \beta & i\sin\theta \\ -i\sin\theta & \alpha \end{pmatrix} /(\alpha\beta - \sin^2\theta). \end{aligned}$$

Therefore we have

$$\operatorname{Tr} K_1(t) = \frac{\alpha+\beta}{2\pi t} \int_0^{2\pi} \frac{d\theta}{\alpha\beta - \sin^2\theta} = \frac{\alpha+\beta}{\pi t} \int_0^{\pi} \frac{d\theta}{(\alpha\beta - \frac{1}{2}) + \frac{1}{2}\cos 2\theta} = \frac{1}{t} \frac{\alpha+\beta}{\sqrt{\alpha\beta(\alpha\beta-1)}}.$$

This proves the first assertion of Proposition 2.2. Here we have used the well known formula

$$\int_0^{\pi} \frac{d\theta}{a + b\cos\theta} = \frac{\pi}{\sqrt{a^2-b^2}}, \quad a > |b|. \tag{2.12}$$

For the second part, taking the Mellin transform of $\operatorname{Tr} K_1(t)$, we have for $\operatorname{Re} s > 1$,

$$Z_{01}(s) = \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} \operatorname{Tr} K_1(t)\, dt = \frac{1}{\Gamma(s)} \frac{\alpha+\beta}{\sqrt{\alpha\beta(\alpha\beta-1)}} \int_0^1 t^{s-1} t^{-1} dt = \frac{\alpha+\beta}{\sqrt{\alpha\beta(\alpha\beta-1)}} \frac{1}{\Gamma(s)(s-1)}.$$

This ends the proof of Proposition 2.2. □

We next study the trace of the remainder term $R_2(t)$ in (2.4).
Since by (2.6)

$$R_2(t) = \int_0^t K_1(t-u)F(u)\, du + \int_0^t R_2(t-u)F(u)\, du =: K_2(t) + R_3(t),$$

by iteration we get

$$
\begin{aligned}
R_3(t) &= \int_0^t du_1 \int_0^{t-u_1} K(t-u_1-u_2)F(u_2)F(u_1)\,du_2 \\
&= \int_0^t du_1 \int_0^{t-u_1} K_1(t-u_1-u_2)F(u_2)F(u_1)\,du_2 + \int_0^t du_1 \int_0^{t-u_1} R_2(t-u_1-u_2)F(u_2)F(u_1)\,du_2 \\
&=: K_3(t) + R_4(t).
\end{aligned}
$$

In this way we define $K_m(t)$, $1 \le m \le n$, and $R_{n+1}(t)$ successively by

$$
K(t) = e^{-tQ} = \sum_{m=1}^{n} K_m(t) + R_{n+1}(t), \tag{2.13a}
$$

$$
\begin{aligned}
K_m(t) = \int_0^t du_1 \int_0^{t-u_1} du_2 \int_0^{t-u_1-u_2} du_3 \cdots \int_0^{t-u_1-u_2-\cdots-u_{m-2}} du_{m-1} \\
\times K_1(t-u_1-u_2-\cdots-u_{m-1})F(u_{m-1})F(u_{m-2})\cdots F(u_2)F(u_1), \quad 1 \le m \le n,
\end{aligned} \tag{2.13b}
$$

$$
\begin{aligned}
R_{n+1}(t) = \int_0^t du_1 \int_0^{t-u_1} du_2 \int_0^{t-u_1-u_2} du_3 \cdots \int_0^{t-u_1-u_2-\cdots-u_{n-1}} \\
\times K(t-u_1-u_2-\cdots-u_n)F(u_n)F(u_{n-1})\cdots F(u_2)F(u_1)\,du_n.
\end{aligned} \tag{2.13c}
$$

Further, based on the decomposition $F(u) = F_1(u) + F_2(u)$ in (2.7), we introduce a way of decomposing $K_m(t)$ into the sum

$$
K_m(t) = \sum_{\varepsilon \in \mathbb{Z}_2^{m-1}} K_{m,\varepsilon}(t), \tag{2.14}
$$

where $\varepsilon = (\varepsilon_1, \dots, \varepsilon_{m-1}) \in \mathbb{Z}_2^{m-1} = \{\pm\}^{m-1}$ and each ε_j is so determined as to be $+/-$ according as, in the decomposition of $F(u_{m-1})F(u_{m-2})\cdots F(u_2)F(u_1)$ in the integrand of $K_m(t)$, one chooses $F_1(u_j)/F_2(u_j)$. For instance, we have

$$
K_{4,(+,-,+)}(t) = \int_0^t du_1 \int_0^{t-u_1} du_2 \int_0^{t-u_1-u_2} du_3 K_1(t-u_1-u_2-u_3)F_1(u_3)F_2(u_2)F_1(u_1).
$$

We first observe the asymptotic behavior of $R_n(t)$ when $t \downarrow 0$.

Proposition 2.3. *One has*

$$
\begin{aligned}
|\mathrm{Tr}\, R_2(t)| &\le C(\varepsilon) t^{-\varepsilon} \quad \text{for every } \varepsilon > 0, \\
|\mathrm{Tr}\, R_{n+1}(t)| &\le C^n \frac{\Gamma(1/2)^n}{\Gamma(1+n/2)} t^{n/2}, \quad n \ge 2.
\end{aligned} \tag{2.15}
$$

Here $C(\varepsilon)$ is a positive constant independent of t but dependent on $\varepsilon > 0$, and C a positive constant independent of t and n.

To prove this proposition, we provide the following lemma. If T is a compact operator on a Hilbert space with singular values $\{\mu_n\}_{n=1}^{\infty}$, we denote by $\|T\|_p$, for $p \ge 1$, the norm $\|T\|_p = (\sum_{n=1}^{\infty} \mu_n^p)^{1/p}$. For instance, $\|T\|_1$ is the trace norm and $\|T\|_2$ the Hilbert–Schmidt norm.

Lemma 2.4. *For small $t > 0$,*

$$
\|F(t)\|_2 = O(t^{-1/2}). \tag{2.16}
$$

Proof. With $F(t) = F_1(t) + F_2(t)$ in (2.7), we have only to show that

$$
\|F_1(t)\|_2 = O(t^{-1/2}), \tag{2.17a}
$$

$$
\|F_2(t)\|_2 = \left(\frac{\alpha+\beta}{8\sqrt{\alpha\beta(\alpha\beta-1)}} \right)^{1/2} t^{-1/2}. \tag{2.17b}
$$

First consider $F_2(t)$. Note that $\int e^{-ix\xi}dx = 2\pi\delta(\xi)$. Using this we can calculate $\|F_2(t)\|_2^2 = \mathrm{Tr}\,[F_2^*(t)F_2(t)]$ as

$$\begin{aligned}
\|F_2(t)\|_2^2 &= \frac{1}{(4\pi)^2}\mathrm{tr}\int\Big[\iiint e^{-i(y-x)\xi}e^{-t[A\frac{\xi^2+x^2}{2}+Jxi\xi]}(-J)e^{i(y-z)\eta}Je^{-t[A\frac{\eta^2+z^2}{2}+Jzi\eta]}\,d\xi d\eta dy\Big]\Big|_{z=x}dx\\
&= \frac{1}{(4\pi)^2}\mathrm{tr}\iiiint e^{iy(\eta-\xi)}e^{ix(\xi-\eta)}e^{-t[A\frac{\xi^2+x^2}{2}+Jxi\xi]}e^{-t[A\frac{\eta^2+x^2}{2}+Jxi\eta]}d\xi d\eta dy dx\\
&= \frac{1}{(4\pi)^2}\mathrm{tr}\iiint 2\pi\delta(\eta-\xi)e^{ix(\xi-\eta)}e^{-t[A\frac{\xi^2+x^2}{2}+Jxi\xi]}e^{-t[A\frac{\eta^2+x^2}{2}+Jxi\eta]}d\xi d\eta dx\\
&= \frac{1}{8\pi}\mathrm{tr}\iint e^{-2t[A\frac{\xi^2+x^2}{2}+Jxi\xi]}d\xi dx.
\end{aligned}$$

Let $\lambda^{\pm}(x,\xi)$ be the two eigenvalues of the matrix

$$q(x,\xi) := A\frac{\xi^2+x^2}{2}+Jxi\xi = \begin{pmatrix}\alpha\frac{\xi^2+x^2}{2} & -xi\xi\\ xi\xi & \beta\frac{\xi^2+x^2}{2}\end{pmatrix}. \tag{2.18a}$$

It is clear that

$$\lambda^{\pm}(x,\xi) = \frac{1}{4}\big[(\alpha+\beta)(\xi^2+x^2)\pm\sqrt{(\alpha-\beta)^2(\xi^2+x^2)^2+16x^2\xi^2}\big]. \tag{2.18b}$$

Then, from the calculation above we obtain

$$\begin{aligned}
\|F_2(t)\|_2^2 &= \frac{1}{8\pi}\iint[e^{-2t\lambda^+(x,\xi)}+e^{-2t\lambda^-(x,\xi)}]d\xi dx\\
&= \frac{1}{8\pi}\iint\Big[e^{-\frac{t}{2}[a(\xi^2+x^2)+\sqrt{b^2(\xi^2+x^2)^2+16x^2\xi^2}]}+e^{-\frac{t}{2}[a(\xi^2+x^2)-\sqrt{b^2(\xi^2+x^2)^2+16x^2\xi^2}]}\Big]d\xi dx.
\end{aligned}$$

where we put $a := \alpha+\beta$ and $b := \alpha-\beta$. Putting $\xi = \rho\cos\theta$, $x = \rho\sin\theta$, we have

$$\begin{aligned}
\|F_2(t)\|_2^2 &= \frac{1}{8\pi}\int_0^{2\pi}\int_0^{\infty}\Big[e^{-\frac{t\rho^2}{2}[a+\sqrt{b^2+16\cos^2\theta\sin^2\theta}]}+e^{-\frac{t\rho^2}{2}[a-\sqrt{b^2+16\cos^2\theta\sin^2\theta}]}\Big]\rho d\rho d\theta\\
&= \frac{1}{8\pi}\int_0^{2\pi}\Big[\frac{1}{t[a+\sqrt{b^2+4\sin^2\,2\theta}]}+\frac{1}{t[a-\sqrt{b^2+4\sin^2\,2\theta}]}\Big]d\theta\\
&= \frac{1}{8\pi t}\int_0^{2\pi}\frac{2a}{a^2-b^2-4\sin^2\,2\theta}d\theta = \frac{1}{8\pi t}\int_0^{2\pi}\frac{(\alpha+\beta)}{2\alpha\beta-1+\cos\theta}d\theta\\
&= \frac{\alpha+\beta}{8\sqrt{\alpha\beta(\alpha\beta-1)}}\frac{1}{t}.
\end{aligned}$$

Here in the last equality we have used the integral formula (2.12). This proves (2.17b).

We next consider $\|F_1(t)\|_2 = \mathrm{Tr}\,[F_1^*(t)F_1(t)]$. We have

$$\begin{aligned}
\|F_1(t)\|_2^2 &= \frac{1}{(2\pi)^2}\mathrm{tr}\int\Big[\iiint e^{-i(y-x)\xi}e^{-t[A\frac{\xi^2+x^2}{2}+(-J)x(-i)\xi]}\Big[A\frac{x^2-y^2}{2}+(-J)(x-y)(-i)\xi\Big]\\
&\qquad\times e^{i(y-z)\eta}\Big[A\frac{z^2-y^2}{2}+J(z-y)i\eta\Big]e^{-t[A\frac{\eta^2+z^2}{2}+Jzi\eta]}\,d\xi d\eta dy\Big]\Big|_{z=x}dx\\
&= \frac{1}{(2\pi)^2}\mathrm{tr}\iiiint e^{i(x-y)(\xi-\eta)}e^{-t[A\frac{\xi^2+x^2}{2}+Jxi\xi]}\Big[A\frac{x^2-y^2}{2}+J(x-y)i\xi\Big]\\
&\qquad\times\Big[A\frac{x^2-y^2}{2}+J(x-y)i\eta\Big]e^{-t[A\frac{\eta^2+x^2}{2}+Jxi\eta]}d\xi d\eta dy dx\\
&= \frac{1}{(2\pi)^2}\mathrm{tr}\iiiint e^{iz(\xi-\eta)}e^{-t[A\frac{\xi^2+x^2}{2}+Jxi\xi]}\Big[(Ax+Ji\xi)z+Az^2/2\Big]\\
&\qquad\times\Big[(Ax+Ji\eta)z+Az^2/2\Big]e^{-t[A\frac{\eta^2+x^2}{2}+Jxi\eta]}d\xi d\eta dz dx \qquad (z := x-y)\\
&= \frac{1}{(2\pi)^2}\mathrm{tr}\iiiint e^{-t[A\frac{\xi^2+x^2}{2}+Jxi\xi]}\Big(\Big[(Ax+Ji\xi)(-i\partial_\xi)+\frac{1}{2}A(-i\partial_\xi)^2\Big]e^{iz\xi}\Big)\\
&\qquad\times\Big(\Big[(Ax+Ji\eta)(i\partial_\eta)+\frac{1}{2}A(i\partial_\eta)^2\Big]e^{-iz\eta}\Big)e^{-t[A\frac{\eta^2+x^2}{2}+Jxi\eta]}d\xi d\eta dz dx,
\end{aligned}$$

where we write $\partial_\xi = \frac{\partial}{\partial\xi}$, $\partial_\eta = \frac{\partial}{\partial\eta}$. Then first, by integration by parts, we have

$$\begin{aligned}
&\|F_1(t)\|_2^2 \\
=&\frac{1}{(2\pi)^2}\mathrm{tr}\iiiint e^{iz(\xi-\eta)} \\
\times&\Big[-e^{-t[A\frac{\xi^2+x^2}{2}+Jxi\xi]}J-(-i\partial_\xi)e^{-t[A\frac{\xi^2+x^2}{2}+Jxi\xi]}(Ax+Ji\xi)+(-i\partial_\xi)^2e^{-t[A\frac{\xi^2+x^2}{2}+Jxi\xi]}(A/2)\Big] \\
\times&\Big[Je^{-t[A\frac{\eta^2+x^2}{2}+Jxi\eta]}+(Ax+Ji\eta)(-i\partial_\eta)e^{-t[A\frac{\eta^2+x^2}{2}+Jxi\eta]}+\frac{1}{2}A(-i\partial_\eta)^2e^{-t[A\frac{\eta^2+x^2}{2}+Jxi\eta]}\Big]d\xi d\eta dz dx \qquad (2.19)\\
=&\frac{2\pi}{(2\pi)^2}\mathrm{tr}\iint \\
\times&\Big[-e^{-t[A\frac{\xi^2+x^2}{2}+Jxi\xi]}J-(-i\partial_\xi)e^{-t[A\frac{\xi^2+x^2}{2}+Jxi\xi]}(Ax+Ji\xi)+(-i\partial_\xi)^2e^{-t[A\frac{\xi^2+x^2}{2}+Jxi\xi]}(A/2)\Big] \\
\times&\Big[Je^{-t[A\frac{\xi^2+x^2}{2}+Jxi\xi]}+(Ax+Ji\xi)(-i\partial_\xi)e^{-t[A\frac{\xi^2+x^2}{2}+Jxi\xi]}+\frac{1}{2}A(-i\partial_\xi)^2e^{-t[A\frac{\xi^2+x^2}{2}+Jxi\xi]}\Big]d\xi dx.
\end{aligned}$$

By integrating in z again with use of $\int e^{iz(\xi-\eta)}dz = 2\pi\delta(\xi-\eta)$, we have

$$\begin{aligned}
\|F_1(t)\|_2^2 = \frac{1}{2\pi}\iint \mathrm{tr}\Big[&e^{-2t[A\frac{\xi^2+x^2}{2}+Jxi\xi]} && \text{(FI1)}\\
&-e^{-t[A\frac{\xi^2+x^2}{2}+Jxi\xi]}J(Ax+Ji\xi)(-i\partial_\xi)e^{-t[A\frac{\xi^2+x^2}{2}+Jxi\xi]} && \text{(FI2)}\\
&-e^{-t[A\frac{\xi^2+x^2}{2}+Jxi\xi]}J(A/2)(i\partial_\xi)^2e^{-t[A\frac{\xi^2+x^2}{2}+Jxi\xi]} && \text{(FI3)}\\
&-(-i\partial_\xi)e^{-t[A\frac{\xi^2+x^2}{2}+Jxi\xi]}(Ax+Ji\xi)Je^{-t[A\frac{\xi^2+x^2}{2}+Jxi\xi]} && \text{(FI4)}\\
&-(-i\partial_\xi)e^{-t[A\frac{\xi^2+x^2}{2}+Jxi\xi]}(Ax+Ji\xi)^2(-i\partial_\xi)e^{-t[A\frac{\xi^2+x^2}{2}+Jxi\xi]} && \text{(FI5)}\\
&-(-i\partial_\xi)e^{-t[A\frac{\xi^2+x^2}{2}+Jxi\xi]}(Ax+Ji\xi)(A/2)(i\partial_\xi)^2e^{-t[A\frac{\xi^2+x^2}{2}+Jxi\xi]} && \text{(FI6)}\\
&+(i\partial_\xi)^2e^{-t[A\frac{\xi^2+x^2}{2}+Jxi\xi]}(A/2)Je^{-t[A\frac{\xi^2+x^2}{2}+Jxi\xi]} && \text{(FI7)}\\
&+(i\partial_\xi)^2e^{-t[A\frac{\xi^2+x^2}{2}+Jxi\xi]}(A/2)(Ax+Ji\xi)(-i\partial_\xi)e^{-t[A\frac{\xi^2+x^2}{2}+Jxi\xi]} && \text{(FI8)}\\
&+(i\partial_\xi)^2e^{-t[A\frac{\xi^2+x^2}{2}+Jxi\xi]}(A^2/4)(i\partial_\xi)^2e^{-t[A\frac{\xi^2+x^2}{2}+Jxi\xi]}\Big]d\xi dx && \text{(FI9)}
\end{aligned}$$

Among the integrals (FI1)–(FI9), we see easily that (FI1) + (FI2) + (FI4) = 0. In fact, by integration by parts, we have

$$\begin{aligned}
\text{(FI2)} = \text{(FI4)} &= -\frac{1}{2}\frac{1}{2\pi}\iint \mathrm{tr}\Big[J(Ax+Ji\xi)(-i\partial_\xi)e^{-2t[A\frac{\xi^2+x^2}{2}+Jxi\xi]}\Big]d\xi dx \\
&= \frac{1}{2}\frac{1}{2\pi}\iint \mathrm{tr}\Big[J^2e^{-2t[A\frac{\xi^2+x^2}{2}+Jxi\xi]}\Big]d\xi dx = -\frac{1}{2}\text{(FI1)}.
\end{aligned}$$

So by cancelling out these three integrals, we have by change of variables $\xi' = \sqrt{t}\xi$, $x' = \sqrt{t}x$,

$$\|F_1(t)\|_2^2 = \frac{1}{2\pi t}\iint \operatorname{tr}\Big[- te^{-[A\frac{\xi^2+x^2}{2}+Jxi\xi]}J(A/2)(i\partial_\xi)^2 e^{-[A\frac{\xi^2+x^2}{2}+Jxi\xi]} \tag{FI3}$$

$$- (-i\partial_\xi)e^{-[A\frac{\xi^2+x^2}{2}+Jxi\xi]}(Ax+Ji\xi)^2(-i\partial_\xi)e^{-[A\frac{\xi^2+x^2}{2}+Jxi\xi]} \tag{FI5}$$

$$- t(-i\partial_\xi)e^{-[A\frac{\xi^2+x^2}{2}+Jxi\xi]}(Ax+Ji\xi)(A/2)(i\partial_\xi)^2 e^{-[A\frac{\xi^2+x^2}{2}+Jxi\xi]} \tag{FI6}$$

$$+ t(i\partial_\xi)^2 e^{-[A\frac{\xi^2+x^2}{2}+Jxi\xi]}(A/2)Je^{-[A\frac{\xi^2+x^2}{2}+Jxi\xi]} \tag{FI7}$$

$$+ t(i\partial_\xi)^2 e^{-[A\frac{\xi^2+x^2}{2}+Jxi\xi]}(A/2)(Ax+Ji\xi)(-i\partial_\xi)e^{-[A\frac{\xi^2+x^2}{2}+Jxi\xi]} \tag{FI8}$$

$$+ t^2(i\partial_\xi)^2 e^{-[A\frac{\xi^2+x^2}{2}+Jxi\xi]}(A^2/4)(i\partial_\xi)^2 e^{-[A\frac{\xi^2+x^2}{2}+Jxi\xi]}\Big] d\xi dx \tag{FI9}$$

It follows that $\|F_1(t)\|_2 = O(t^{-1/2})$. This proves (2.17a). Thus we can conclude that $\|F(t)\|_2 \le \|F_1(t)\|_2 + \|F_2(t)\|_2 = O(t^{-1/2})$. This shows (2.16). This completes the proof of Lemma 2.4. □

Proof of Proposition 2.3. First we treat the case $n = 2$, i.e. consider $R_2(t)$. Since

$$\operatorname{Tr} R_2(t) = \int_0^t \operatorname{Tr}[e^{-(t-u)Q}F(u)]du, \tag{2.20}$$

we get

$$|\operatorname{Tr}[e^{-(t-u)Q}F(u)]| \le \|e^{-(t-u)Q}F(u)\|_1 \le \|e^{-(t-u)Q}\|_2\|F(u)\|_2. \tag{2.21}$$

We notice here that

$$\|e^{-(t-u)Q}\|_2 \le C_2(\varepsilon)(t-u)^{-(1/2+\varepsilon)}, \tag{2.22}$$

with an arbitrary $\varepsilon > 0$ and a constant $C_2(\varepsilon) > 0$ dependent on ε. Indeed, if $\{\lambda_n\}_{n=1}^\infty$ is the set of the eigenvalues of Q, since $\lambda_n \to +\infty$ we have

$$\begin{aligned}\|e^{-(t-u)Q}\|_2 = \Big(\sum_{n=1}^\infty e^{-2(t-u)\lambda_n}\Big)^{1/2} &= \Big(\sum_{n=1}^\infty (2(t-u)\lambda_n)^{-(1+\varepsilon)}\big\{(2(t-u)\lambda_n)^{1+\varepsilon}e^{-2(t-u)\lambda_n}\big\}\Big)^{1/2} \\ &\le \Big(\frac{1+\varepsilon}{2e}\Big)^{(1+\varepsilon)/2}\Big(\sum_{n=1}^\infty \lambda_n^{-(1+\varepsilon)}\Big)^{1/2}(t-u)^{-(1+\varepsilon)/2},\end{aligned}$$

whence the bound (2.22) follows from the fact that $\zeta_Q(s) = \sum_{n=1}^\infty \lambda^{-s}$ is bounded in Re $s \ge 1+\varepsilon$ for every $\varepsilon > 0$ (see Theorem 3.3 in [IW] or by Lemma 2.8 below). Hence by (2.21) we obtain that $|\operatorname{Tr}[e^{-(t-u)Q}F(u)]| \le CC(\varepsilon)(t-u)^{-(\frac{1}{2}+\varepsilon)}u^{-\frac{1}{2}}$ by use of $\|F(u)\|_2 \le Cu^{-1/2}$ in Lemma 2.4. It follows from (2.20) that $|\operatorname{Tr} R_2(t)| \le C(\varepsilon)t^{-\varepsilon}$ for every $\varepsilon > 0$.

We next study the case $n \ge 2$. Since by Lemma 2.4 we have $\|F(u)\|_{2(n-1)} \le \|F(u)\|_2 \le Cu^{-1/2}$ with a constant $C > 0$, and

$$\|F(u_{n-1})\cdots F(u_2)F(u_1)\|_2 \le \|F(u_{n-1})\|_{2(n-1)}\cdots\|F(u_2)\|_{2(n-1)}\|F(u_1)\|_{2(n-1)},$$

we obtain for $n \geq 2$,

$$
\begin{aligned}
|\operatorname{Tr} R_{n+1}(t)| &\leq \|\operatorname{Tr} R_{n+1}(t)\|_1 \\
&\leq \int_0^t du_1 \int_0^{t-u_1} du_2 \int_0^{t-u_1-u_2} du_3 \cdots \int_0^{t-u_1-u_2-\cdots-u_{n-1}} du_n \\
&\qquad\qquad \times \|e^{-(t-u_1-u_2-\cdots-u_n)Q} F(u_n)\|_2 \|F(u_{n-1}) \cdots F(u_2)F(u_1)\|_2 \\
&\leq \int_0^t du_1 \int_0^{t-u_1} du_2 \int_0^{t-u_1-u_2} du_3 \cdots \int_0^{t-u_1-u_2-\cdots-u_{n-1}} du_n \\
&\qquad\qquad \times \|e^{-(t-u_1-u_2-\cdots-u_n)Q}\| \|F(u_n)\|_2 \|F(u_{n-1}) \cdots F(u_2)F(u_1)\|_2 \\
&\leq C^n \int_0^t du_1 \int_0^{t-u_1} du_2 \int_0^{t-u_1-u_2} du_3 \cdots \int_0^{t-u_1-u_2-\cdots-u_{n-1}} (u_n u_{n-1} \cdots u_2 u_1)^{-1/2} du_n \\
&= C^n t^{n/2} \int_0^1 du_1 \int_0^{1-u_1} du_2 \int_0^{1-u_1-u_2} du_3 \cdots \int_0^{1-u_1-u_2-\cdots-u_{n-1}} (u_n u_{n-1} \cdots u_2 u_1)^{-1/2} du_n \\
&= C^n \frac{\Gamma(1/2)^n \Gamma(1)}{\Gamma(1+n/2)} t^{n/2} = C^n \frac{\Gamma(1/2)^n}{\Gamma(1+n/2)} t^{n/2}.
\end{aligned}
$$

Here in the last second equality we have made the change of variables $u_j' = u_j/t, j = 1, 2, \ldots, n$, and then rewritten the new u_j' as the u_j again. This shows (2.15), ending the proof of Proposition 2.3. □

To proceed further, we now recall (2.2) for $\zeta_Q(s)$, (2.8), (2.10) for $Z_0(s) = \Gamma(s)^{-1}\hat{Z}_0(s)$:

$$
\begin{aligned}
\zeta_Q(s) &= Z_0(s) + Z_\infty(s), && \text{(2.23a)} \\
Z_0(s) &= Z_{01}(s) + Z_{02}'(s), && \text{(2.23b)}
\end{aligned}
$$

and Proposition 2.1 for $Z_\infty(s) = \Gamma(s)^{-1}\hat{Z}_\infty(s)$ and Proposition 2.2 for $Z_{01}(s) =: \Gamma(s)^{-1}\hat{Z}_{01}(s)$.

We perform now analytic continuation of $\zeta_Q(s)$ one step to the left from the region $\operatorname{Re} s > 1$.

Proposition 2.5. *$\zeta_Q(s)$ is holomorphic in $\sigma = \operatorname{Re} s > 0$, except at $s = 1$, and*

$$
\begin{aligned}
\zeta_Q(s) &= \frac{\alpha+\beta}{\sqrt{\alpha\beta(\alpha\beta-1)}} \frac{1}{\Gamma(s)} \frac{1}{s-1} + \frac{1}{\Gamma(s)} \hat{h}(s), \\
\hat{h}(s) &:= \hat{Z}_{02}'(s) + \hat{Z}_\infty(s).
\end{aligned} \tag{2.24}
$$

Here $\hat{Z}_\infty(s)$ is holomorphic in the whole complex plane, while $\hat{h}(s)$ is holomorphic in $\operatorname{Re} s > 0$ and uniformly bounded in $\operatorname{Re} s \geq \varepsilon$ for every $\varepsilon > 0$.

Proof. Putting $\hat{Z}_{02}'(s) = \Gamma(s) Z_{02}'(s)$, we have only to show $\hat{Z}_{02}'(s)$ is holomprphic $\sigma = \operatorname{Re} s > 0$, because $\frac{1}{\Gamma(s)}$ is holomorphic in the whole complex plane. We have by Proposition 2.3 with $n = 1$ that

$$
|\hat{Z}_{02}'(s)| \leq \int_0^1 t^{\sigma-1} |\operatorname{Tr} R_2(t)| dt \leq C(\varepsilon) \int_0^1 t^{\sigma-\varepsilon-1} dt = \frac{C(\varepsilon)}{\sigma-\varepsilon}
$$

for any $\varepsilon > 0$ with a constant $C(\varepsilon) > 0$, so that $\hat{Z}_{02}'(s)$ is holomorphic in $\sigma = \operatorname{Re} s > 0$. This together with the previous observation shows the assertion of Proposition 2.5. □

As an application of the proposition we now show the so-called Weyl law for the spectrum of our Q. Note that each eigenvalue λ_j is positive. To count the number of the eigenvalues of Q less than a given $T > 0$, we define the counting function of eigenvalues by

$$
N_Q(T) = \#\{\lambda_j \in \operatorname{Spec} Q\,;\ \lambda_j < T\}.
$$

As a corollary of Proposition 2.5 we have the following estimate of $N_Q(T)$.

Corollary 2.6. *One has*

$$N_Q(T) \sim \frac{\alpha+\beta}{\sqrt{\alpha\beta(\alpha\beta-1)}}T \quad (T \to \infty).$$

Proof. Since for $a > 0$ we have

$$e^{-as} = \int_0^\infty e^{-st}\delta(t-a)dt,$$

it follows that, if $\lambda_j > 1$,

$$\lambda_j^{-s} = e^{-s\log\lambda_j} = \int_0^\infty e^{-st}\delta(t-\log\lambda_j)dt. \tag{2.25}$$

Since we can write $N_Q(T) = \sum_{\lambda_j<T} 1$ we have

$$\sum_{\lambda_j>1} \delta(t-\log\lambda_j) = N_Q(e^t) - \sum_{\lambda_j\le 1} 1.$$

Note that the last sum is finite. Hence by the formula (2.25) we obtain

$$\begin{aligned} \zeta_Q(s) - \sum_{\lambda_j\le 1}\lambda_j^{-s} &= \int_0^\infty e^{-st}\Big(N_Q(e^t) - \sum_{\lambda_j\le 1} 1\Big)dt \\ &= \int_0^\infty e^{-st}N_Q(e^t)dt - \frac{1}{s}\sum_{\lambda_j\le 1} 1, \end{aligned} \tag{2.26}$$

for Re $s > 0$. By Proposition 2.5 we know that $\zeta_Q(s)$ can be written as

$$\zeta_Q(s) = \frac{\alpha+\beta}{\sqrt{\alpha\beta(\alpha\beta-1)}}\frac{1}{s-1} + h(s),$$

where $h(s)$ is holomorphic in Re $s > 0$. Hence by (2.26) we have

$$\int_0^\infty e^{-st}N_Q(e^t)dt = \frac{\alpha+\beta}{\sqrt{\alpha\beta(\alpha\beta-1)}}\frac{1}{s-1} + f(s) \tag{2.27}$$

for some function $f(s)$ which is holomorphic in Re $s > 0$. We now recall the following Tauberian theorem due to Wiener–Ikehara (see e.g. [Wi]).

Lemma 2.7. *Let $g(t)$ be a non-decreasing and positive function defined on $t \ge 0$. Suppose that the integral $\int_0^\infty e^{-st}g(t)dt$ is expressed as*

$$\int_0^\infty e^{-st}g(t)dt = \frac{1}{s-1} + f(s)$$

in a domain containing Re $s > 1$ *with some continuous function $f(s)$ in* Re $s \ge 1$*. Then we have*

$$g(t) \sim e^t \quad (t \to \infty). \quad \square$$

By the expression (2.27) it immediately follows that

$$\frac{\sqrt{\alpha\beta(\alpha\beta-1)}}{\alpha+\beta}N_Q(e^t) \sim e^t \quad (t \to \infty).$$

This actually shows the assertion of the corollary with $T = e^t$. $\square$

In order to describe a zero free region of $\zeta_Q(s)$ we need the following result.

Lemma 2.8. *Let $Q' = A^{-1/2}QA^{-1/2} = \frac{1}{2}(-\partial_x^2 + x^2) + \gamma J(x\partial_x + \frac{1}{2})$ where $\gamma := (\alpha\beta)^{-1/2}$. Then for real s satisfying $s > 1$ it holds that*

$$(\max\{\alpha,\beta\})^{-s}\mathrm{Tr}\, Q'^{-s} \leq \mathrm{Tr}\, Q^{-s} \leq (\min\{\alpha,\beta\})^{-s}\mathrm{Tr}\, Q'^{-s}. \tag{2.28}$$

In other words, for $s > 1$ one has

$$(\max\{\alpha,\beta\})^{-s}2(1-\gamma^2)^{-s/2}(2^s-1)\zeta(s) \leq \zeta_Q(s) \leq (\min\{\alpha,\beta\})^{-s}2(1-\gamma^2)^{-s/2}(2^s-1)\zeta(s). \tag{2.29}$$

Proof. The proofs of the left and right inequalities of (2.28) are similar, where we use the Lieb-Thirring inequality ([LT], [Ar]). We have given a proof to the right one in [IW] as Eq.(2.28). Instead of repeating it, we show here only the left one. Since ${Q'}^{-1} = A^{1/2}Q^{-1}A^{1/2}$, we have for $s > 1$,

$$\begin{aligned}
\mathrm{Tr}\, Q'^{-s} &= \mathrm{Tr}\,(A^{1/2}Q^{-1}A^{1/2})^s \\
&\leq \mathrm{Tr}\, A^{s/2}Q^{-s}A^{s/2} \\
&= \mathrm{Tr}\, A^sQ^{-s} = \mathrm{Tr}\, Q^{-s/2}A^sQ^{-s/2} \\
&\leq (\max\{\alpha,\beta\})^s\mathrm{Tr}\, Q^{-s}.
\end{aligned}$$

This proves (2.28). To show (2.29), it is enough to recall the following formula (see Eq.(3.16) in [IW]):

$$\zeta_{Q'}(s) = \mathrm{Tr}\, Q'^{-s} = 2(1-\gamma^2)^{-s/2}(2^s-1)\zeta(s).$$

Hence the lemma follows. □

Remark 1. The operator Q' above is unitarily equivalent to $(\alpha\beta)^{-1/2}$ times a couple of the usual harmonic oscillators with mass 1 and classical oscillator frequency $\sqrt{\alpha\beta - 1}$ (see Corollaries 4.5 and 4.1 in [PW2]), i.e.

$$Q' \cong (\alpha\beta)^{-1/2}\Big[-\frac{\partial_x^2}{2} + \frac{(\alpha\beta-1)}{2}x^2\Big]I. \qquad \square$$

Using the inequality (2.29) we can give the bounds of the first eigenvalue λ_1 of the operator Q.

Theorem 2.9. *The first eigenvalue λ_1 of the operator Q satisfies*

$$\min\{\alpha,\beta\}\sqrt{1-1/(\alpha\beta)} \leq 2\lambda_1 \leq \max\{\alpha,\beta\}\sqrt{1-1/(\alpha\beta)}. \tag{2.30}$$

Moreover, let m_Q be the multiplicity of the first eigenvalue λ_1. Then,

$$\begin{aligned}
m_Q \leq 2 \quad &\textit{when} \quad \lambda_1 = \frac{1}{2}\min\{\alpha,\beta\}\sqrt{1-1/(\alpha\beta)}, \\
m_Q \geq 2 \quad &\textit{when} \quad \lambda_1 = \frac{1}{2}\max\{\alpha,\beta\}\sqrt{1-1/(\alpha\beta)}.
\end{aligned}$$

Proof. Though there is a simpler second proof of (2.30), as in Remark 3 below, which is based on the fact noted in Remark 1 above, we will give here a direct proof in the course. Let m_Q be the multiplicity of the first eigenvalue λ_1. Since

$$\zeta_Q(\sigma) = m_Q\lambda_1^{-\sigma} + \sum_{\lambda_n>\lambda_1}\lambda_n^{-\sigma} \leq 2\{\min\{\alpha,\beta\}\sqrt{1-1/(\alpha\beta)}\}^{-\sigma}(2^\sigma-1)\zeta(\sigma)$$

by the right inequality of (2.29), we have

$$m_Q\left(\frac{\min\{\alpha,\beta\}\sqrt{1-1/(\alpha\beta)}}{2\lambda_1}\right)^\sigma + \sum_{\lambda_n>\lambda_1}\left(\frac{\min\{\alpha,\beta\}\sqrt{1-1/(\alpha\beta)}}{2\lambda_n}\right)^\sigma \leq 2\frac{2^\sigma-1}{2^\sigma}\zeta(\sigma) \to 2 \quad (\sigma\to+\infty), \tag{2.31}$$

because $\zeta(\sigma) \to 1$. Here we used the fact that $\zeta(\sigma) < 1 + \int_1^\infty \frac{dx}{x^\sigma} = \frac{\sigma}{\sigma-1}$ for $\sigma > 1$. This implies in particular the inequality

$$\frac{\min\{\alpha,\beta\}\sqrt{1-1/(\alpha\beta)}}{2\lambda_1} \leq 1.$$

Further, if the equality holds above, we obviously have $m_Q \leq 2$. Similarly we have from the left of (2.29) that

$$\begin{aligned}
2\frac{2^\sigma - 1}{2^\sigma}\zeta(\sigma) \leq & m_Q \left(\frac{\max\{\alpha,\beta\}\sqrt{1-1/(\alpha\beta)}}{2\lambda_1}\right)^\sigma + \sum_{\lambda_n > \lambda_1} \left(\frac{\max\{\alpha,\beta\}\sqrt{1-1/(\alpha\beta)}}{2\lambda_n}\right)^\sigma \\
\leq & m_Q \left(\frac{\max\{\alpha,\beta\}\sqrt{1-1/(\alpha\beta)}}{2\lambda_1}\right)^\sigma \\
+ & \left(\frac{\max\{\alpha,\beta\}\sqrt{1-1/(\alpha\beta)}}{2\lambda_1}\right)^{\sigma-2} \sum_{\lambda_n > \lambda_1} \left(\frac{\max\{\alpha,\beta\}\sqrt{1-1/(\alpha\beta)}}{2\lambda_n}\right)^2 \\
\leq & m_Q \left(\frac{\max\{\alpha,\beta\}\sqrt{1-1/(\alpha\beta)}}{2\lambda_1}\right)^\sigma \\
+ & \frac{1}{4} \left(\frac{\max\{\alpha,\beta\}\sqrt{1-1/(\alpha\beta)}}{2\lambda_1}\right)^{\sigma-2} \Big\{\max\{\alpha,\beta\}\sqrt{1-1/(\alpha\beta)}\Big\}^2 \zeta_Q(2)
\end{aligned} \tag{2.32}$$

for $\sigma > 2$, whence letting $\sigma \to +\infty$ we obtain the inequality

$$1 \leq \frac{\max\{\alpha,\beta\}\sqrt{1-1/(\alpha\beta)}}{2\lambda_1}.$$

Otherwise, the last member of (3.32) should go to 0, contradicting the fact that $\zeta(\sigma) \to 1$ as $\sigma \to +\infty$. If the equality holds above, it is also clear that $2 \leq m_Q$. Hence the assertion follows. □

Remark 2. Suppose $\alpha \neq \beta$. It is known [NNW] that $m_Q = 1$ when α and β are large enough. □

Remark 3. We have given above a direct proof to the bounds (2.30) of the first eigenvalue λ_1 of Q in Theorem 2.9. However, we can give a simpler proof, appealing to the non-trivial fact on Q' noted in Remark 1 to Lemma 2.8. Indeed, this implies that the first eigenvalue of Q' is $(\alpha\beta)^{-1/2}\frac{1}{2}\sqrt{\alpha\beta - 1} = \frac{1}{2}\sqrt{1-1/(\alpha\beta)}$. Therefore, for the lower bound, since $Q = A^{1/2}Q'A^{1/2}$, we have for $u \in \mathcal{S}(\mathbb{R}) \otimes \mathbb{C}^2$

$$(Qu,u) \geq \frac{1}{2}\sqrt{1-1/(\alpha\beta)}(Au,u) \geq \frac{1}{2}\min\{\alpha,\beta\}\sqrt{1-1/(\alpha\beta)}(u,u).$$

It follows that $\lambda_1 \geq \frac{1}{2}\min\{\alpha,\beta\}\sqrt{1-1/(\alpha\beta)}$. This lower bound coincides with the one in (2.30). On the other hand, for the upper bound, since $Q' = A^{-1/2}QA^{-1/2}$ in turn, and since $(Qu,u) = (Q'A^{-1/2}u, A^{-1/2}u) \geq \lambda_1(A^{-1}u,u)$ for $u \in \mathcal{S}(\mathbb{R}) \otimes \mathbb{C}^2$, we obtain

$$\frac{1}{2}\sqrt{1-1/(\alpha\beta)} \geq \lambda_1 \min\{\alpha^{-1},\beta^{-1}\} = \frac{\lambda_1}{\max\{\alpha,\beta\}}.$$

Hence $\lambda_1 \leq \frac{1}{2}\max\{\alpha,\beta\}\sqrt{1-1/(\alpha\beta)}$. This upper bound coincides with the one in (2.30).

We note also that our result (2.30) is explicitly refining an assertion, Corollary 7.11, p.596, in [PW2], that the first eigenvalue λ_1 of Q is in an unspecified neighborhood of the point $\mu_0^*(\alpha,\beta) = \sqrt{\alpha\beta}\sqrt{\alpha\beta - 1}/(\alpha+\beta)$, because this point lies between our two bounds obtained in (2.30). □

Related to these bounds of the first eigenvalue of Q, one can show the following

Proposition 2.10. *There exists a $\sigma_0 > 1$ large enough such that the zeta function $\zeta_Q(s)$ does not vanish in* $\mathrm{Re}\ s \geq \sigma_0$.

Proof. Since

$$\zeta_Q(s) = \lambda_1^{-s}\Big\{m_Q + \sum_{\lambda_n > \lambda_1}\Big(\frac{\lambda_n}{\lambda_1}\Big)^{-s}\Big\}, \tag{2.33}$$

if $\sigma = \mathrm{Re}\ s$ satisfies the condition

$$\sum_{\lambda_n > \lambda_1}\Big|\frac{\lambda_n}{\lambda_1}\Big|^{-\sigma} < m_Q, \tag{2.34}$$

we have $\zeta_Q(s) \neq 0$. Obviously this can be achieved if we take σ sufficiently large. This proves the proposition. $\square$

Remark 4. We try to find σ_0 in Proposition 2.10 as small as possible. First note that (2.34) is equivalent to

$$\zeta_Q(\sigma)\lambda_1^{\sigma} < 2m_Q. \tag{2.34'}$$

So we need to let σ_0 satisfy (2.34'). Indeed, it does by the right inequality of (2.29), so long as σ_0 satisfies

$$(\min\{\alpha,\beta\})^{-\sigma_0}2(1-\gamma^2)^{-\sigma_0/2}(2^{\sigma_0}-1)\zeta(\sigma_0) < 2m_Q\lambda_1^{-\sigma_0}$$

or

$$\frac{2^{\sigma_0}-1}{2^{\sigma_0}}\zeta(\sigma_0) < m_Q\left(\frac{\min\{\alpha,\beta\}(1-\gamma^2)^{1/2}}{2\lambda_1}\right)^{\sigma_0}. \tag{2.35}$$

If $\lambda_1 = \frac{1}{2}\min\{\alpha,\beta\}\sqrt{1-1/(\alpha\beta)}$, we see, since $\zeta(\sigma) < \frac{\sigma}{\sigma-1}$ for $\sigma > 1$, that there exists a $\sigma_0 \geq \frac{m_Q}{m_Q-1}$ which satisfies (2.35), so that $\zeta_Q(s) \neq 0$ when $\mathrm{Re}\ s \geq \sigma_0$. However, if $\lambda_1 > \frac{1}{2}\min\{\alpha,\beta\}\sqrt{1-1/(\alpha\beta)}$, there may necessarily exists no $\sigma_0 > 1$ which satisfies (2.35), since, as $\sigma_0 \to \infty$, the right-hand side of (2.35) tends to 0, while the left-hand side of (2.35) tends to 1, again because $1 \leq \zeta(\sigma) < \frac{\sigma}{\sigma-1}$ for $\sigma > 1$.

In particular, when $\alpha = \beta = \sqrt{2}$, i.e. in the case of a couple of the harmonic oscillators $Q = Q_0$, the right-hand side of (2.35) is equal to 2 because $\lambda_1 = \frac{1}{2}$ and $m_{Q_0} = 2$, so that $\frac{2^{\sigma_0}-1}{2^{\sigma_0}}\zeta(\sigma_0) < 2$. Therefore, applying the above analysis to the Riemann zeta function case can only give the result that there exists σ_0 with $1 < \sigma_0 < \frac{3}{2}$ such that $\zeta(s)$ does not vanish for $\mathrm{Re}\ s \geq \sigma_0$, though $\zeta(s)$ does not vanish in fact in $\mathrm{Re}\ s > 1$, what can be indeed assured by the Euler product. $\square$

3. Asymptotic behavior of $\mathrm{Tr}\, K_2(t)$

In this section, we establish the asymptotic expansion of $\mathrm{Tr}\ K_2(t)$ for $t \downarrow 0$. This is a preparation to learn how the general case will go in the subsequent section. We shall present necessary ideas to provide lemmas which enable us to develop the arguments in the general case, that is, the asymptotic expansion for $\mathrm{Tr}\ K_m(t)$.

The main purpose of this section is then to show the following proposition.

Proposition 3.1. *For small $t > 0$,*

$$\mathrm{Tr}\, K_2(t) \sim \sum_{j=0}^{\infty} c_{2,j}t^j, \tag{3.1}$$

with $c_{2,j} = 0$ for $j = 2\ell$ being nonnegative even integers.

Proof. Let $F = F_1 + F_2$ be in (2.7). Then, by (2.14) we may write $K_2(t)$ as

$$K_2(t) = K_{2,+}(t) + K_{2,-}(t),$$

where from $K_i(t)$ in (2.13b) we have

$$K_{2,+}(t) = \int_0^t K_1(t-u)F_1(u)du,$$
$$K_{2,-}(t) = \int_0^t K_1(t-u)F_2(u)du.$$

Then, for instance, for $K_{2,+}(t)$ we have

$$\begin{aligned}\int K_{2,+}(t,x,x)dx &= \frac{1}{(2\pi)^2}\int_0^t du \iiiint e^{i(x-z)\eta}e^{-(t-u)[A\frac{\eta^2+z^2}{2}+Jzi\eta]}\\ &\qquad\times\Big[A\frac{x^2-z^2}{2}+J(x-z)i\xi\Big]e^{i(z-x)\xi}e^{-u[A\frac{\xi^2+x^2}{2}+Jxi\xi]}d\eta dz d\xi dx\\ &= \frac{1}{(2\pi)^2}\int_0^t du \iiiint e^{i(x-z)(\eta-\xi)}e^{-(t-u)[A\frac{\eta^2+z^2}{2}+Jzi\eta]}\\ &\qquad\times\Big[A\frac{x^2-z^2}{2}+J(x-z)i\xi\Big]e^{-u[A\frac{\xi^2+x^2}{2}+Jxi\xi]}d\eta dz d\xi dx.\end{aligned}$$

We hence get by change of variables $u'=tu$, $x'=\sqrt{t}x$, $z'=\sqrt{t}z$, $\xi'=\sqrt{t}\xi$, $\eta'=\sqrt{t}\eta$,

$$\begin{aligned}\int K_{2,+}(t,x,x)dx =&\frac{1}{(2\pi)^2 t}\int_0^1 du \iiiint e^{i(x-z)(\eta-\xi)/t}\\ &\times e^{-(1-u)[A\frac{\eta^2+z^2}{2}+Jzi\eta]}\Big[A\frac{x^2-z^2}{2}+J(x-z)i\xi\Big]e^{-u[A\frac{\xi^2+x^2}{2}+Jxi\xi]}d\eta dz d\xi dx\end{aligned}\tag{3.2}$$

Similarly we have

$$\begin{aligned}&\int K_{2,-}(t,x,x)dx\\ &=-\frac{1}{2(2\pi)^2 t}\int_0^1 du \iiiint e^{i(x-z)(\eta-\xi)/t}e^{-(1-u)[A\frac{\eta^2+z^2}{2}+Jzi\eta]}Je^{-u[A\frac{\xi^2+x^2}{2}+Jxi\xi]}d\eta dz d\xi dx.\end{aligned}\tag{3.3}$$

For the traces of both $K_{2,+}(t)$ and $K_{2,-}(t)$, we are going to show the following lemma.

Lemma 3.2. *For $t\downarrow 0$, one has*

(1) $\operatorname{Tr} K_{2,+}(t)\sim 0$.
(2) $\operatorname{Tr} K_{2,-}(t)\sim \sum_{j=0}^\infty c_j^{(2,-)}t^j$, *with* $c_j^{(2,-)}=0$ *for* $j=2\ell$ *being nonnegative even integers.*

Proof of Proposition 3.1. Since $\operatorname{Tr} K_2(t)=\operatorname{Tr} K_{2,+}(t)+\operatorname{Tr} K_{2,-}(t)$, it is clear that the assertion of Proposition 3.1 immediately follows from this lemma by taking $c_{2,j}=c_j^{(2,-)}$. □

Now we give a proof of Lemma 3.2, which is a little lengthy. First we prove (1).

Proof of Lemma 3.2 (1). Write $\operatorname{Tr} K_{2,+}(t)=T_1(t)+T_2(t)$. Here we put

$$\begin{aligned}T_1(t)=&\frac{1}{(2\pi)^2 t}\int_0^1 du\,\mathrm{tr}\iiiint e^{i(x-z)(\eta-\xi)/t}e^{-(1-u)[A\frac{\eta^2+z^2}{2}+Jzi\eta]}\Big[A\frac{x^2-z^2}{2}\Big]e^{-u[A\frac{\xi^2+x^2}{2}+Jxi\xi]}d\eta dz d\xi dx,\\ T_2(t)=&\frac{1}{(2\pi)^2 t}\int_0^1 du\,\mathrm{tr}\iiiint e^{i(x-z)(\eta-\xi)/t}e^{-(1-u)[A\frac{\eta^2+z^2}{2}+Jzi\eta]}[J(x-z)i\xi]e^{-u[A\frac{\xi^2+x^2}{2}+Jxi\xi]}d\eta dz d\xi dx.\end{aligned}$$

We show that $\operatorname{Tr} K_{2,+}(t)$ is real and $T_1(t)=0$. For $T_1(t)$ we have

$$\begin{aligned}&(2\pi)^2 t\,\overline{T_1(t)}\\ =&\int_0^1 du\,\mathrm{tr}\iiiint e^{-i(x-z)(\eta-\xi)/t}e^{-(1-u)[A\frac{\eta^2+z^2}{2}-Jzi\eta]}\Big[A\frac{x^2-z^2}{2}\Big]e^{-u[A\frac{\xi^2+x^2}{2}-Jxi\xi]}d\eta dz d\xi dx\\ =&\int_0^1 du\,\mathrm{tr}\iiiint e^{i(x-z)(\eta-\xi)/t}e^{-(1-u)[A\frac{\eta^2+z^2}{2}+Jzi\eta]}\Big[A\frac{x^2-z^2}{2}\Big]e^{-u[A\frac{\xi^2+x^2}{2}+Jxi\xi]}d\eta dz d\xi dx \quad (\eta\to-\eta,\ \xi\to-\xi)\\ =&\int_0^1 du\,\mathrm{tr}\iiiint e^{i(x-z)(\eta-\xi)/t}e^{-(1-u)[A\frac{\eta^2+z^2}{2}+Jzi\eta]}\Big[A\frac{x^2-z^2}{2}\Big]e^{-u[A\frac{\xi^2+x^2}{2}+Jxi\xi]}d\eta dz d\xi dx\\ =&(2\pi)^2 t\,T_1(t).\end{aligned}$$

Hence we have $\overline{T_1(t)} = T_1(t)$. In the same way we have $\overline{T_2(t)} = T_2(t)$. This proves $\mathrm{Tr}\, K_{2,+}(t)$ is real.

Next, we show $T_1(t) = 0$. This is seen, because

$$\begin{aligned}
&\int_0^1 du\, \mathrm{tr} \iiiint e^{i(x-z)(\eta-\xi)/t} e^{-(1-u)[A\frac{\eta^2+z^2}{2}+Jzi\eta]} A\frac{x^2}{2} e^{-u[A\frac{\xi^2+x^2}{2}+Jxi\xi]} d\eta dz d\xi dx \\
=&\int_0^1 du\, \mathrm{tr} \iiiint e^{i(x-z)(\eta-\xi)/t} e^{-(1-u)[A\frac{\eta^2+z^2}{2}-Jzi\eta]} A\frac{z^2}{2} e^{-u[A\frac{\xi^2+x^2}{2}-Jxi\xi]} d\eta dz d\xi dx \quad (x \leftrightarrow z,\, \xi \leftrightarrow \eta, 1-u \leftrightarrow u) \\
=&\int_0^1 du\, \mathrm{tr} \iiiint e^{i(x-z)(\eta-\xi)/t} e^{-(1-u)[A\frac{\eta^2+z^2}{2}+Jzi\eta]} A\frac{z^2}{2} e^{-u[A\frac{\xi^2+x^2}{2}+Jxi\xi]} d\eta dz d\xi dx.
\end{aligned}$$

where we used the relation $\mathrm{tr}\,(ABC) = \overline{\mathrm{tr}\,(C^*B^*A^*)}$ at the last equality.

Thus, in order to show that $\mathrm{Tr}\, K_{2,+}(t) \sim 0$, it suffices to prove that $T_2(t) \sim 0$. We need the following lemma.

Lemma 3.3. *(Asymptotic Formula). The asymptotic expansion holds:*

$$e^{i\lambda xy} \sim 2\pi \sum_{k=0}^{\infty} i^k \frac{\partial_x^k \delta(x) \partial_y^k \delta(y)}{k!\, \lambda^{k+1}}, \quad \lambda \to \infty, \tag{3.4}$$

in the sense of tempered distributions in $\mathbb{R}^2$, i.e. in $\mathcal{S}'(\mathbb{R}^2)$.

The statement of this lemma means that for all $f \in \mathcal{S}(\mathbb{R}^2)$, we have, for every positive integer m,

$$\begin{aligned}
&\langle e^{i\lambda xy} - 2\pi \sum_{k=0}^{m-1} i^k \frac{\partial_x^k \delta(x) \partial_y^k \delta(y)}{k!\, \lambda^{k+1}}, f\rangle \\
&= \iint e^{i\lambda xy} f(x,y) dxdy - 2\pi \sum_{k=0}^{m-1} \frac{i^k}{k!\, \lambda^{k+1}} \partial_x^k \partial_y^k f(0,0) \\
&= O(\lambda^{-(m+1)}), \quad \lambda \to \infty.
\end{aligned} \tag{3.5}$$

Note that the lemma is stated in [EK], p. 225, but seems rather involved. So we will give a direct proof.

Proof of Lemma 3.3. We show (3.5). Since $\mathcal{S}(\mathbb{R}^2) = \mathcal{S}(\mathbb{R}) \hat{\otimes} \mathcal{S}(\mathbb{R})$, where the tensor product is completed in the π- and/or ε-tensor product topology, because the these spaces are nuclear spaces, we have only to show it for $f(x,y) = \varphi(x)\psi(y)$ with φ, $\psi \in \mathcal{S}(\mathbb{R})$.

By Taylor's theorem

$$\varphi(x) = \sum_{k=0}^{m-1} \frac{\varphi^{(k)}(0)}{k\,!} x^k + \frac{x^m}{(m-1)\,!} \int_0^1 (1-\theta)^{m-1} \varphi^{(m)}(\theta x) d\theta, \tag{3.6}$$

$$\psi(y) = \sum_{k=0}^{m-1} \frac{\psi^{(k)}(0)}{k\,!} y^k + \frac{y^m}{(m-1)\,!} \int_0^1 (1-\theta)^{m-1} \psi^{(m)}(\theta y) d\theta. \tag{3.7}$$

Then we have by (3.6)

$$\begin{aligned}
&\iint e^{i\lambda xy} \varphi(x)\psi(y) dxdy \\
=&\frac{1}{\lambda} \iint e^{ixy} \varphi(x/\lambda)\psi(y)\, dxdy \quad (x' := \lambda x) \\
=&\frac{\sqrt{2\pi}}{\lambda} \int \varphi(x/\lambda) \hat{\psi}(-x)\, dx \\
=&\frac{\sqrt{2\pi}}{\lambda} \sum_{k=0}^{m-1} \frac{\varphi^{(k)}(0)}{k\,!\, \lambda^k} \int x^k \hat{\psi}(-x)\, dx + \frac{\sqrt{2\pi}}{(m-1)\,!\, \lambda^{m+1}} \int x^m \int_0^1 (1-\theta)^{m-1} \varphi^{(m)}(\theta x/\lambda) d\theta\, \hat{\psi}(-x)\, dx \\
=&:I_m(\lambda) + R_m(\lambda),
\end{aligned} \tag{3.8}$$

where $\hat{\psi}$ is the Fourier transform of ψ.

To calculate $I_m(\lambda)$, we see that

$$\begin{aligned}\int x^k\hat{\psi}(-x)\,dx &= (-1)^k\int x^k\hat{\psi}(x)\,dx = (-1)^k\int e^{ix\xi}x^k\hat{\psi}(x)\,dx\Big|_{\xi=0}\\ &= (-1)^k\sqrt{2\pi}(-i\partial_\xi)^k\psi(\xi)\,dx\Big|_{\xi=0} = \sqrt{2\pi}i^k\psi^{(k)}(0).\end{aligned}\tag{3.9}$$

Therefore

$$I_m(\lambda) = 2\pi\sum_{k=0}^{m-1} i^k\frac{\varphi^{(k)}(0)\psi^{(k)}(0)}{k\,!\,\lambda^{k+1}}.\tag{3.10}$$

To estimate $R_m(\lambda)$, we have

$$\begin{aligned}&\left|\int x^m\int_0^1(1-\theta)^{m-1}\varphi^{(m)}(\theta x/\lambda)d\theta\,\hat{\psi}(-x)\,dx\right|\\ &=\left|(-1)^m\int x^m\int_0^1(1-\theta)^{m-1}\varphi^{(m)}(\theta x/\lambda)d\theta\,\hat{\psi}(x)\,dx\right|\\ &=\left|\int x^m\int_0^1(1-\theta)^{m-1}\varphi^{(m)}(\theta x/\lambda)d\theta\,\hat{\psi}(x)(1+x^2)(1+x^2)^{-1}\,dx\right|\\ &\le\pi\sup_x|\varphi^{(m)}(x)|\,\sup_x|x^m(1+x^2)\hat{\psi}(x)| =: {C_m}',\end{aligned}$$

because $\int(1+x^2)^{-1}dx=\pi$. Hence we obtain

$$|R_m(\lambda)| \le \sqrt{2\pi}\frac{1}{(m-1)\,!\,\lambda^{m+1}}{C_m}' \le C_m\lambda^{-(m+1)}.\tag{3.11}$$

Thus with (3.10) and (3.11) we have proved Lemma 3.3. □

Let us return to the proof of $T_2(t)\sim 0$.

In the following, we shall abuse the notation to write the distributional inner product like the first member of (3.5) as the integral

$$\iint\Big(e^{i\lambda xy} - 2\pi\sum_{k=0}^{m-1} i^k\frac{\partial_x^k\delta(x)\partial_y^k\delta(y)}{k!\lambda^{k+1}}\Big)f(x,y)dxdy.$$

Then by Lemma 3.3 above we have for small $t>0$,

$$\begin{aligned}T_2(t) &\sim \sum_{j=0}^\infty\frac{1}{(2\pi)^2t}\int_0^1 du\,\mathrm{tr}\iiiint(2\pi)t^{j+1}\frac{i^j\,\partial_x^j\delta(x-z)\partial_\eta^j\delta(\eta-\xi)}{j!}\\ &\qquad\times e^{-(1-u)[A\frac{\eta^2+z^2}{2}+Jzi\eta]}J(x-z)i\xi e^{-u[A\frac{\xi^2+x^2}{2}+Jxi\xi]}\Big)d\eta dzd\xi dx\\ &\sim\frac{1}{2\pi}\sum_{j=0}^\infty t^j\int_0^1 du\,\mathrm{tr}\iiiint\delta(x-z)\delta(\eta-\xi)\partial_\eta^j\Big(e^{-(1-u)[A\frac{\eta^2+z^2}{2}+Jzi\eta]}\Big)\\ &\qquad\times\partial_x^j\Big(J(x-z)i\xi e^{-u[A\frac{\xi^2+x^2}{2}+Jxi\xi]}\Big)d\eta dzd\xi dx.\end{aligned}\tag{3.12}$$

In other words, if we write $T_2(t)\sim\sum_{j=0}^\infty c_j^{(2,+)}t^j$, by the Leibniz formula we obtain

$$\begin{aligned}c_j^{(2,+)} &= \frac{1}{2\pi}\frac{i^j}{j!}\int_0^1 du\,\mathrm{tr}\iiint\delta(x-z)\partial_\xi^j\Big(e^{-(1-u)[A\frac{\xi^2+z^2}{2}+Jzi\xi]}\Big)\Big[jJi\xi(\partial_x^{j-1}e^{-u[A\frac{\xi^2+x^2}{2}+Jxi\xi]})\\ &\qquad + J(x-z)i\xi(\partial_x^j e^{-u[A\frac{\xi^2+x^2}{2}+Jxi\xi]})\Big]dzd\xi dx\\ &= \frac{1}{2\pi}\frac{i^j}{j!}\int_0^1 du\,\mathrm{tr}\iint\Big(\partial_\xi^j e^{-(1-u)[A\frac{\xi^2+z^2}{2}+Jzi\xi]}\Big)jJi\xi\Big(\partial_x^{j-1}e^{-u[A\frac{\xi^2+x^2}{2}+Jxi\xi]}\Big)d\xi dx.\end{aligned}\tag{3.13}$$

We are hence going to show all the $c_j^{(2,+)}$ vanish. To estimate the integrand of the last integral in (3.13), we use the Taylor expansion of $e^{-t[A\frac{\xi^2+x^2}{2}+Jxi\xi]}$. First, note here that if $\lambda^-(x,\xi)$ is the smaller one of the two positive eigenvalues (see (2.18ab)) of the matrix $[A\frac{\xi^2+x^2}{2}+Jxi\xi]$, then its matrix norm obeys:

$$\|e^{-t[A\frac{\xi^2+x^2}{2}+Jxi\xi]}\| = e^{-t\lambda^-(x,\xi)} \leq e^{-ct(\xi^2+x^2)} \tag{3.14}$$

for all (x,ξ) with $c := c(\alpha,\beta) = \frac{\alpha\beta-1}{(\alpha+\beta)+\sqrt{(\alpha-\beta)^2+4}} > 0$. Indeed, to see this, we use the polar coordinates $\xi = r\cos\theta$, $x = r\sin\theta$ to get

$$\lambda^-(x,\xi) = \frac{r^2}{4}\Big[(\alpha+\beta) - \sqrt{(\alpha-\beta)^2+4\sin^2 2\theta}\Big] \geq \frac{\alpha\beta-1}{(\alpha+\beta)+\sqrt{(\alpha-\beta)^2+4}}r^2,$$

when $\alpha\beta > 1$.

We now recall the Taylor theorem for a matrix M:

$$e^{-tM} = \sum_{j=0}^{n}\frac{(-t)^j}{j!}M^j + \frac{(-t)^{n+1}}{n!}M^{n+1}\int_0^1 (1-\tau)^n e^{-\tau t M}d\tau.$$

Then

$$\begin{aligned} e^{-t[A\frac{\xi^2+x^2}{2}+Jxi\xi]} &= \sum_{p=0}^{n}\frac{(-t)^p}{p!}[A\frac{\xi^2+x^2}{2}+Jxi\xi]^p \\ &\quad + \frac{(-t)^{n+1}}{n!}[A\frac{\xi^2+x^2}{2}+Jxi\xi]^{n+1}\int_0^1 (1-\tau)^n e^{-\tau t[A\frac{\xi^2+x^2}{2}+Jxi\xi]}d\tau. \end{aligned} \tag{3.15}$$

Taking the x-derivatives ∂_x^j, we have

$$\begin{aligned} \partial_x^j e^{-t[A\frac{\xi^2+x^2}{2}+Jxi\xi]} &= \sum_{p=[j/2]+1}^{n}\frac{(-t)^p}{p!}\partial_x^j[A\frac{\xi^2+x^2}{2}+Jxi\xi]^p \\ &\quad + \frac{(-t)^{n+1}}{n!}\partial_x^j\Big([A\frac{\xi^2+x^2}{2}+Jxi\xi]^{n+1}\int_0^1 (1-\tau)^n e^{-\tau t[A\frac{\xi^2+x^2}{2}+Jxi\xi]}d\tau\Big). \end{aligned} \tag{3.16}$$

Hence, by virtue of the estimate (3.14) we obtain

$$\partial_x^j e^{-t[A\frac{\xi^2+x^2}{2}+Jxi\xi]} = \sum_{p=[j/2]+1}^{n}\frac{(-t)^p}{p!}\partial_x^j[A\frac{\xi^2+x^2}{2}+Jxi\xi]^p + T_{n+1}^{(j)}(t,x,\xi), \tag{3.17}$$

where the matrix norm of $T_{n+1}^{(j)}(t,x,\xi)$ satisfies

$$\|T_{n+1}^{(j)}(t,x,\xi)\| \leq C\frac{t^{n+1}}{n!}[R^{2n+2-j} + t^j R^{2n+2+j}], \tag{3.18}$$

with $\xi^2+x^2 \leq R^2$. The same is valid for the ξ-derivatives. Thus we see that the expansion

$$e^{-t[A\frac{\xi^2+x^2}{2}+Jxi\xi]} = \sum_{p=0}^{\infty}\frac{(-t)^p}{p!}[A\frac{\xi^2+x^2}{2}+Jxi\xi]^p \tag{3.19}$$

is, together with all its x- and ξ-derivatives, convergent in the matrix norm uniformly on each closed disc, $\xi^2+x^2 \leq R^2$, with radius $R > 0$.

We introduce a radially symmetric cutoff function $\chi_R(x,\xi)$ for $R>0$. Let $\rho(r)$ be a nonnegative C^∞-function in $r\geq 0$ with $\rho(r)=1$ for $r\leq 1/2$ and $=0$ for $r\geq 1$. Put $\chi_R(x,\xi)=\rho((\xi^2+x^2)^{1/2}/R)$. Then, from (3.13) we see that

$$2\pi\frac{j!}{i^j}c_j^{(2,+)}=\lim_{R\to\infty}\int_0^1 du\,\mathrm{tr}\iint \chi_R(x,\xi)\Big(\partial_\xi^j e^{-(1-u)[A\frac{\xi^2+x^2}{2}+Jxi\xi]}\Big)jJi\xi\Big(\partial_x^{j-1}e^{-u[A\frac{\xi^2+x^2}{2}+Jxi\xi]}\Big)d\xi dx. \tag{3.20}$$

Now using (3.17) we see for the (ξ,x)-integral in (3.20) that

$$\begin{aligned}
&\mathrm{tr}\iint \chi_R(x,\xi)\Big(\partial_\xi^j e^{-(1-u)[A\frac{\xi^2+x^2}{2}+Jxi\xi]}\Big)jJi\xi(\partial_x^{j-1}e^{-u[A\frac{\xi^2+x^2}{2}+Jxi\xi]})\Big]d\xi dx\\
=&\mathrm{tr}\iint \chi_R(x,\xi)\Big[\sum_{p=[j/2]+1}^{n}\frac{(-(1-u))^p}{p!}\partial_\xi^j[A\frac{\xi^2+x^2}{2}+Jxi\xi]^p+T_{n+1}^{(j)}(1-u,x,\xi)\Big]\\
&\times\Big[jJi\xi\Big(\sum_{q=[j/2]}^{n}\frac{(-u)^q}{q!}\partial_x^{j-1}[A\frac{\xi^2+x^2}{2}+Jxi\xi]^q+T_{n+1}^{(j-1)}(u,x,\xi)\Big)\Big]d\xi dx.
\end{aligned}$$

The integrals on the right-hand side above except the ones involving the remainders $T_{n+1}^{(j)}(1-u,x,\xi)$ or $T_{n+1}^{(j-1)}(u,x,\xi)$ vanish, because the integrands of these integrals are odd in x or ξ, or by taking the matrix trace. Thus we arrive at the estimate

$$\begin{aligned}
&\Big|\mathrm{tr}\iint \chi_R(x,\xi)\cdots\Big|\\
\leq&\iint d\xi dx\,\chi_R(x,\xi)\,C\Big[\sum_{p=[j/2]+1}^{n}\frac{(1-u)^p}{p!}R^{2p}\|T_{n+1}^{(j)}(1-u,x,\xi)\|\\
&\quad+\sum_{q=[j/2]}^{n}\frac{u^q}{q!}R^{2q}\|T_{n+1}^{(j)}(1-u,x,\xi)\|+\|T_{n+1}^{(j-1)}(1-u,x,\xi)\|\|T_{n+1}^{(j)}(u,x,\xi)\|\Big]\\
\leq&C\,\pi\Big[e^{R^2}\frac{R^{2n+4+j-1}+R^{2n+4+j}}{n!}+\frac{R^{4n+6+2j-1}}{(n!)^2}\Big]\ \longrightarrow 0,\quad n\to\infty,
\end{aligned}$$

for fixed $R>0$. This shows the desired assertion $c_j^{(2,+)}=0$. $\square$

We come to now the proof of (2) of Lemma 3.2.

Proof of Lemma 3.2 (2). From the expression (3.3) of $\mathrm{Tr}\,K_{2,-}(t)\sim\sum_{j=0}^\infty c_j^{(2,-)}t^j$, we have with the aid of Lemma 3.3

$$c_j^{(2,-)}=-\frac{1}{2(2\pi)}\frac{i^j}{j!}\int_0^1 du\,\mathrm{tr}\iiiint \delta(x-z)\delta(\eta-\xi)\partial_\eta^j\Big(e^{-(1-u)[A\frac{\eta^2+z^2}{2}+Jzi\eta]}\Big)J\partial_x^j\Big(e^{-u[A\frac{\xi^2+x^2}{2}+Jxi\xi]}\Big)d\eta dzd\xi dx. \tag{3.21}$$

Now we show that the $c_j^{(2,-)}$ vanish with $j=2\ell$ being non-negative even integers. We have from (3.21) that

$$\begin{aligned}
c_{2\ell}^{(2,-)}&=-\frac{1}{2(2\pi)}\frac{(-1)^\ell}{(2\ell)!}\int_0^1 du\,\mathrm{tr}\iiiint\delta(x-z)\delta(\eta-\xi)\\
&\quad\times\partial_\eta^{2\ell}\Big(e^{-(1-u)[A\frac{\eta^2+z^2}{2}+Jzi\eta]}\Big)J\partial_x^{2\ell}\Big(e^{-u[A\frac{\xi^2+x^2}{2}+Jxi\xi]}\Big)d\eta dzd\xi dx\\
&=-\frac{1}{2(2\pi)}\frac{(-1)^\ell}{(2\ell)!}\int_0^1 du\,\mathrm{tr}\iint J\partial_x^{2\ell}\Big(e^{-u[A\frac{\xi^2+x^2}{2}+Jxi\xi]}\Big)\partial_\xi^{2\ell}\Big(e^{-(1-u)[A\frac{\xi^2+x^2}{2}+Jxi\xi]}\Big)d\xi dx \qquad (3.22)\\
&=-\lim_{R\to\infty}\frac{1}{2(2\pi)}\frac{(-1)^\ell}{(2\ell)!}\int_0^1 du\,\mathrm{tr}\iint\chi_R(x,\xi)J\partial_x^{2\ell}\Big(e^{-u[A\frac{\xi^2+x^2}{2}+Jxi\xi]}\Big)\partial_\xi^{2\ell}\Big(e^{-(1-u)[A\frac{\xi^2+x^2}{2}+Jxi\xi]}\Big)d\xi dx
\end{aligned}$$

In the same reasoning as used before, we have only to show that for every $R > 0$,

$$\operatorname{tr} \iint \chi_R(x,\xi) J \partial_x^{2\ell} \Big(e^{-u[A\frac{\xi^2+x^2}{2}+Jxi\xi]} \Big) \partial_\xi^{2\ell} \Big(e^{-(1-u)[A\frac{\xi^2+x^2}{2}+Jxi\xi]} \Big) d\xi dx = 0. \tag{3.23}$$

Using (3.17) and its counterparts for the ξ-derivative, one finds the integrand in the last member of (3.22) turns to be

$$\begin{aligned} \operatorname{tr} \iint d\xi dx\, \chi_R(x,\xi) J \Big(\sum_{p=\ell+1}^{n} \frac{(-u)^p}{p!} \partial_x^{2\ell} [A\frac{\xi^2+x^2}{2} + Jxi\xi]^p + T_{n+1}^{(2\ell)}(u,x,\xi) \Big) \\ \times \Big(\sum_{q=\ell+1}^{n} \frac{(-(1-u))^q}{q!} \partial_\xi^{2\ell} [A\frac{\xi^2+x^2}{2} + Jxi\xi]^q + T_{n+1}^{(2\ell)}(1-u,x,\xi) \Big). \end{aligned}$$

Then, by analogous arguments used before, we see that the integrals except the ones involving the remainder terms $T_{n+1}^{(2\ell)}(u,x,\xi)$ and $T_{n+1}^{(2\ell)}(1-u,x,\xi)$ vanish, by taking the matrix trace or because the integrands are odd in x or ξ. Therefore, for fixed $R > 0$, the left-hand side of (3.23) obeys

$$\begin{aligned} &\Big| \operatorname{tr} \iint \chi_R(x,\xi) \cdots \Big| \\ &\le \operatorname{tr} \iint d\xi dx \chi_R(x,\xi) \Big[\sum_{p=\ell+1}^{n} \frac{u^p}{p!} R^{2p-2\ell} \| T_{n+1}^{(2\ell)}(1-u,x,\xi) \| \\ &\quad + \sum_{q=\ell+1}^{n} \frac{(1-u)^q}{q!} R^{2q-2\ell} \| T_{n+1}^{(2\ell)}(u,x,\xi) \| + \| T_{n+1}^{(2\ell)}(1-u,x,\xi) \| \| T_{n+1}^{(2\ell)}(u,x,\xi) \| \Big] \\ &\le C\pi \Big[e^{R^2} \frac{R^{2n+4} + R^{2n+4}}{n!} + \frac{R^{4n+6+4\ell}}{(n!)^2} \Big] \quad \longrightarrow 0, \quad n \to \infty. \end{aligned}$$

This yields (3.23) and hence $c_j^{(2,-)} = 0$ when $j = 2\ell$ is a non-negative even integer. This completes the proof of (2) of Lemma 3.2. $\square$

Corollary 3.4. *With $\hat{Z}_\infty(s)$ in Proposition 2.1,*

$$\zeta_Q(s) = \frac{\alpha+\beta}{\sqrt{\alpha\beta(\alpha\beta-1)}} \frac{1}{(s-1)\Gamma(s)} + \frac{1}{\Gamma(s)} \Big[\sum_{j=1}^{k} \frac{c_{2,2j-1}}{s+2j-1} + \hat{h}_1(s) + \hat{h}_2(s) + \hat{Z}_\infty(s) \Big], \tag{3.24}$$

where $\hat{h}_1(s)$ is holomorphic in $\operatorname{Re} s > -2k-1$, having a bound $|\hat{h}_1(s)| \le C_1(k)/(\operatorname{Re} s + 2k+1)$ for every positive integer k with a positive constant $C_1(k)$ dependent on k, and $\hat{h}_2(s)$ is holomorphic in $\operatorname{Re} s > -1$, having a bound $|\hat{h}_2(s)| \le C_2/(\operatorname{Re} s + 1)$ with a positive constant C_2.

Proof. We have by (2.2)/(2.13a)

$$\begin{aligned} \zeta_Q(s) &= \frac{1}{\Gamma(s)} \Big[\int_1^\infty t^{s-1} \operatorname{Tr} e^{-tQ}\, dt + \int_0^1 t^{s-1} \operatorname{Tr} [K_1(t) + K_2(t) + R_3(t)] dt \Big] \\ &= \frac{1}{\Gamma(s)} \Big[\int_1^\infty t^{s-1} \operatorname{Tr} e^{-tQ}\, dt + \int_0^1 t^{s-1} \operatorname{Tr} [K_1(t) + \sum_{j=1}^{k} c_{2,2j-1} t^{2j-1} + \{K_2(t) - \sum_{j=1}^{k} c_{2,2j-1} t^{2j-1}\} + R_3(t)] dt \Big] \\ &= Z_\infty(s) + \frac{\alpha+\beta}{\sqrt{\alpha\beta(\alpha\beta-1)}} \frac{1}{(s-1)\Gamma(s)} + \frac{1}{\Gamma(s)} \sum_{j=1}^{k} c_{2,2j-1} \frac{1}{s+2j-1} + h_1(s) + h_2(s), \end{aligned}$$

where we have by Proposition 3.1 and by (2.15)

$$\begin{aligned} \hat{h}_1(s) &:= \Gamma(s) h_1(s) = \int_0^1 t^{s-1} \operatorname{Tr} \big\{ K_2(t) - \sum_{j=1}^{k} c_{2,2j-1} t^{2j-1} \big\}\, dt = \int_0^1 t^{s-1} O(t^{2k+1})\, dt, \\ \hat{h}_2(s) &:= \Gamma(s) h_2(s) = \int_0^1 t^{s-1} \operatorname{Tr} R_3(t)\, dt = \int_0^1 t^{s-1} O(t)\, dt. \end{aligned}$$

From these expressions, it is easy to verify that $\hat{h}_1(s)$ is holomorphic in Re $s > -2k-1$ and $\hat{h}_2(s)$ is holomorphic in Re $s > -1$, respectively, with their bounds mentioned in the assertion. This proves Corollary 3.4. $\square$

4. Asymptotic behavior of $\operatorname{Tr} K_m(t)$ and the main theorem

In this section, we study the trace of $K_m(t)$, $m = 3, 4, \ldots$, in general. Actually, we show the following asymptotic expansion of Tr $K_m(t)$ for small $t > 0$ by developing the idea used in the previous section. Using this result we will obtain the asymptotic expansion of Tr $K(t)$ and hence the main theorem of the present paper.

Theorem 4.1. *For $m = 2, 3, \ldots$, one has for $t \downarrow 0$,*

$$\operatorname{Tr} K_m(t) \sim \sum_{j=0}^{\infty} c_{m,j} t^j, \tag{4.1}$$

with $c_{m,j} = 0$ for $0 \le j < m-2$ and $j = 2\ell$ being positive even integers.

To save the space and argument we shall use the following notations.

Let $V(x)$ be a linear space of formal power series in x. We denote by $V^{even}(x)$ (resp. $V^{odd}(x)$) the subspace of $V(x)$ consisting of the even (resp. odd) power series. Let D^+ (resp. D^-) be the space of all diagonal (resp. anti-diagonal) 2×2 matrices with entries in $\mathbb{C}$.

Since $D^\pm D^\pm \subset D^+$, $D^\pm D^\mp \subset D^-$, $D^\mp D^\pm \subset D^-$, we can see from the Taylor expansion, already given in (3.19),

$$e^{-u[A\frac{\xi^2+x^2}{2}+J\xi ix]} = \sum_{p=0}^{\infty} \frac{(-u)^p}{p!}\Big(A\frac{\xi^2+x^2}{2} + Jxi\xi\Big)^p$$

the following

Lemma 4.2.

$$e^{-u[A\frac{\xi^2+x^2}{2}+J\xi ix]} \in V^{even}(x) \otimes V^{even}(\xi) \otimes D^+ + V^{odd}(x) \otimes V^{odd}(\xi) \otimes D^-. \tag{4.2}$$

Here the parameter u is regarded as a positive number and each tensor product $\otimes$ is understood to be commutative. $\square$

From this fact, for each positive integer j, it immediately follows that

$$\partial_x^{2j-1}\, e^{-u[A\frac{\xi^2+x^2}{2}+J\xi ix]} \in V^{odd}(x) \otimes V^{even}(\xi) \otimes D^+ + V^{even}(x) \otimes V^{odd}(\xi) \otimes D^-, \tag{4.3}$$

$$\partial_x^{2j}\, e^{-u[A\frac{\xi^2+x^2}{2}+J\xi ix]} \in V^{even}(x) \otimes V^{even}(\xi) \otimes D^+ + V^{odd}(x) \otimes V^{odd}(\xi) \otimes D^-. \tag{4.4}$$

The above formulas obviously hold also for the differentiation ∂_ξ in place of ∂_x. (For explicit calculation of the derivatives of $e^{-u[A\frac{\xi^2+x^2}{2}+J\xi ix]}$ by x and ξ, see Lemma 4.9 below, though we don't use them in the discussion in the sequel.)

We are now trying to illustrate with the present notations how to recover our result obtained in Proposition 3.1 by means of the following

Example. In Proposition 3.1/Lemma 3.2, we have shown that in the expansion $\operatorname{Tr} K_{(2,-)}(t) \sim \sum_{j=0}^{\infty} c_j^{(2,-)} t^j$, all the coefficients $c_{2j}^{(2,-)}$ of the even powers of t vanish. In the proof of Lemma 3.2, we have used the Taylor theorem to estimate the remainder terms. As a result, we have shown these remainder terms did not give any effect on the evaluation of the values of the coefficients $c_{2j}^{(2,-)}$. Therefore, since the integrals like (3.22) are guaranteed to converge because they are essentially Gaussian integrals, it turns out that only what we have to perform is termwise integration for the terms coming from the Taylor expansion (3.19), by taking account of the parity in ξ or x, or the matrix trace.

Thus, by the expression (3.22) we have

$$
\begin{aligned}
(-1)^{j+1}4\pi(2j)!c_{2j}^{(2,-)} &= \int_0^1 du\,\mathrm{tr}\Big[J\iint \partial_\xi^{2j}e^{-u[A\frac{\xi^2+x^2}{2}+Jxi\xi]}\partial_x^{2j}e^{-(1-u)[A\frac{\xi^2+x^2}{2}+Jxi\xi]}d\xi dx\Big]\\
&\in \mathrm{tr}\iint\Big[J\Big\{V^{even}(x)\otimes V^{even}(\xi)\otimes D^+ + V^{odd}(x)\otimes V^{odd}(\xi)\otimes D^-\Big\}\\
&\times\Big\{V^{even}(x)\otimes V^{even}(\xi)\otimes D^+ + V^{odd}(x)\otimes V^{odd}(\xi)\otimes D^-\Big\}\Big]d\xi dx\\
&\subset \mathrm{tr}\iint\Big[J\Big\{V^{even}(x)\otimes V^{even}(\xi)\otimes D^+ + V^{odd}(x)\otimes V^{odd}(\xi)\otimes D^-\Big\}\Big]d\xi dx\\
&=\mathrm{tr}D^+\times\iint V^{odd}(x)\otimes V^{odd}(\xi)d\xi dx=\{0\},
\end{aligned}
$$

which reproduces the desired assertion in Lemma 3.2 (2). □

Now we deal with the general case. Let

$$\Delta^{m-1}=\{u=(u_1,\dots,u_{m-1})\in\mathbb{R}^{m-1}\ ;\ u_j\geq 0,\ u_1+\cdots+u_{m-1}\leq 1\}$$

be the simplex in $\mathbb{R}^{m-1}$, $du=du_1\cdots du_{m-1}$ and denote by $\theta_{m-1}(u)$ the characteristic function of the simplex Δ^{m-1}. Namely, for instance, $\theta_1(u)=\theta(u)\theta(1-u)$ if $m=2$, and $\theta_2(u_1,u_2)=\theta(u_1)\theta(1-u_1)\theta(u_2)\theta(1-u_2)\theta(1-u_1-u_2)$ if $m=3$, etc., where $\theta(u)$ is the Heaviside function. Put

$$
\begin{aligned}
T_+(x,y,\xi)&=A\frac{x^2-y^2}{2}+J(x-y)i\xi,\\
T_-(x,y,\xi)&=-\frac{1}{2}J.
\end{aligned}
$$

For $\varepsilon=(\varepsilon_1,\dots,\varepsilon_{m-1})\in\mathbb{Z}_2^{m-1}$ where $\varepsilon_j=\pm$, we denote by $\ell(\varepsilon)=\#\{j\ ;\ \varepsilon_j=+,\ (1\leq j\leq m-1)\}$ the number of the $+$ in ε. Note that the function T_+ is homogeneous; $T_+(t^{\frac12}x,t^{\frac12}y,t^{\frac12}\xi)=tT_+(x,y,\xi)$ for $t>0$. Recall the decomposition (2.14) of $K_m(t)$. Then it is not hard to see from (2.13b) with (2.3) and (2.7) that $\mathrm{Tr}\,K_{m,\varepsilon}(t)$ can be represented as

$$
\begin{aligned}
&\mathrm{Tr}\,K_{m,\varepsilon}(t)\\
&=t^{-\ell(\varepsilon)-1}\int_{\Delta^{m-1}}du\,\theta_{m-1}(u)\int^{\otimes 2m}\prod_{j=1}^{m}d\xi_j\prod_{j=0}^{m-1}dz_j\,\frac{1}{(2\pi)^m}e^{i[(z_0-z_{m-1})\xi_m+(z_{m-1}-z_{m-2})\xi_{m-1}+\cdots+(z_1-z_0)\xi_1]/t} \qquad (4.5)\\
&\times\mathrm{tr}\Big\{e^{-(1-u_1-\cdots-u_{m-1})\Big[A\frac{\xi_m^2+z_{m-1}^2}{2}+J\xi_m iz_{m-1}\Big]}\overleftarrow{\prod_{j=1}^{m-1}}T_{\varepsilon_j}(z_{j-1},z_j,\xi_j)e^{-u_j\Big[A\frac{\xi_j^2+z_{j-1}^2}{2}+J\xi_j iz_{j-1}\Big]}\Big\}\\
&=t^{-\ell(\varepsilon)-1}\int_{\Delta^{m-1}}du\,\theta_{m-1}(u)\int^{\otimes 2m}\prod_{j=1}^{m}d\xi_j\prod_{j=0}^{m-1}dz_j\,\frac{1}{(2\pi)^m}\prod_{j=1}^{m-1}e^{i(z_j-z_{j-1})(\xi_j-\xi_m)/t}\\
&\times\mathrm{tr}\Big\{e^{-(1-u_1-\cdots-u_{m-1})\Big[A\frac{\xi_m^2+z_{m-1}^2}{2}+J\xi_m iz_{m-1}\Big]}\overleftarrow{\prod_{j=1}^{m-1}}T_{\varepsilon_j}(z_{j-1},z_j,\xi_j)e^{-u_j\Big[A\frac{\xi_j^2+z_{j-1}^2}{2}+J\xi_j iz_{j-1}\Big]}\Big\}.
\end{aligned}
$$

Here we note that

$$(z_0-z_{m-1})\xi_m+(z_{m-1}-z_{m-2})\xi_{m-1}+\cdots+(z_1-z_0)\xi_1=\sum_{j=1}^{m-1}(z_j-z_{j-1})(\xi_j-\xi_m)$$

and use the convention

$$\overleftarrow{\prod_{j=1}^{m-1}}B_j=B_{m-1}\cdots B_1,\qquad \overrightarrow{\prod_{j=1}^{m-1}}B_j=B_1\cdots B_{m-1} \qquad (4.6)$$

for matrices B_j.

Using the asymptotic expansion formula described in Lemma 3.3 we see that

$$\begin{aligned}&\prod_{j=1}^{m-1} e^{i(z_j-z_{j-1})(\xi_j-\xi_m)/t}\\ \sim&(2\pi)^{m-1}\sum_{\ell_1=0}^{\infty}\cdots\sum_{\ell_{m-1}=0}^{\infty}\frac{i^{\ell_1+\cdots+\ell_{m-1}}t^{\ell_1+\cdots+\ell_{m-1}+m-1}}{\ell_1!\cdots\ell_{m-1}!}\prod_{j=1}^{m-1}\partial_{z_j}^{\ell_j}\delta(z_j-z_{j-1})\partial_{\xi_j}^{\ell_j}\delta(\xi_j-\xi_m).\end{aligned} \tag{4.7}$$

Integration by parts for each ξ_j-variable therefore yields

$$\begin{aligned}\operatorname{Tr} K_{m,\varepsilon}(t) \sim&\frac{1}{2\pi}\sum_{\ell_1=0}^{\infty}\cdots\sum_{\ell_{m-1}=0}^{\infty}\frac{i^{\ell_1+\cdots+\ell_{m-1}}}{\ell_1!\cdots\ell_{m-1}!}t^{\ell_1+\cdots+\ell_{m-1}+m-\ell(\varepsilon)-2}\\ &\times\int_{\Delta^{m-1}}du\,\theta_{m-1}(u)\int^{\otimes 2m}\prod_{j=1}^{m}d\xi_j\prod_{j=0}^{m-1}dz_j\\ &\times\prod_{j=1}^{m-1}(-1)^{\ell_j}\partial_{z_{j-1}}^{\ell_j}\delta(z_j-z_{j-1})\prod_{j=1}^{m-1}(-1)^{\ell_j}\delta(\xi_j-\xi_m)\\ &\times\operatorname{tr}\Bigg[\Big\{e^{-(1-u_1-\cdots-u_{m-1})\left[A\frac{\xi_m^2+z_{m-1}^2}{2}+J\xi_m i z_{m-1}\right]}\Big\}\\ &\times\overleftarrow{\prod_{j=1}^{m-1}}\partial_{\xi_j}^{\ell_j}\Big\{T_{\varepsilon_j}(z_{j-1},z_j,\xi_j)e^{-u_j\left[A\frac{\xi_j^2+z_{j-1}^2}{2}+J\xi_j i z_{j-1}\right]}\Big\}\Bigg].\end{aligned} \tag{4.8}$$

Here we note that there are no terms with respect to t^k with k negative integers in the asymptotic expansion (4.8) for small $t>0$, because we see by (2.13abc) and (2.15) that $K_m(t)$ is part of $R_m(t)$ and $|\operatorname{Tr} R_m(t)| \le C_m t^{(m-1)/2}$ with a constant $C_m>0$ dependent on m for m large.

Now we introduce the following convention: Assign a pair of integers k, j to each of the four cases

$$f(x,\xi)\in\begin{cases}V^{even}(x)\otimes V^{even}(\xi)\otimes D^{\pm} & : \ k\equiv j\equiv 0 \mod 2,\\ V^{even}(x)\otimes V^{odd}(\xi)\otimes D^{\pm} & : \ k\equiv 0,\ j\equiv 1 \mod 2,\\ V^{odd}(x)\otimes V^{even}(\xi)\otimes D^{\pm} & : \ k\equiv 1,\ j\equiv 0 \mod 2,\\ V^{odd}(x)\otimes V^{odd}(\xi)\otimes D^{\pm} & : \ k\equiv j\equiv 1 \mod 2,\end{cases}$$

to write

$$f(x,\xi)=x(k)\xi(j)D^{\pm}\,.$$

The idea of the following lemma is useful.

Lemma 4.3. *For each non-negative integer L one has*

$$\int\delta(x-z)\,\partial_x^L\Big\{T_-(x,z,\xi)e^{-u\left[A\frac{\xi^2+x^2}{2}+J\xi ix\right]}\Big\}dz = x(L)\xi(0)D^- + x(1+L)\xi(1)D^+, \tag{4.9}$$

$$\int\delta(x-z)\,\partial_\xi^L\Big\{T_-(x,z,\xi)e^{-u\left[A\frac{\xi^2+x^2}{2}+J\xi ix\right]}\Big\}dz = x(0)\xi(L)D^- + x(1)\xi(1+L)D^+, \tag{4.10}$$

$$\int\delta(x-z)\,\partial_x^L\Big\{T_+(x,z,\xi)e^{-u\left[A\frac{\xi^2+x^2}{2}+J\xi ix\right]}\Big\}dz = \begin{cases}x(L)\xi(0)D^+ + x(L+1)\xi(1)D^- & (L\ge 1),\\ 0 & (L=0),\end{cases} \tag{4.11}$$

$$\int\delta(x-z)\,\partial_\xi^L\Big\{T_+(x,z,\xi)e^{-u\left[A\frac{\xi^2+x^2}{2}+J\xi ix\right]}\Big\}dz = 0. \tag{4.12}$$

Proof. By the formulas (4.2) and (4.3) after Lemma 4.2, note first that

$$\partial_x^L\, e^{-u\left[A\frac{\xi^2+x^2}{2}+J\xi ix\right]} = x(L)\xi(0)D^+ + x(1+L)\xi(1)D^-. \tag{4.13}$$

Since $JD^{\pm} \subset D^{\mp}$, we have the first two assertions (4.9)/(4.10) immediately.

Now we prove the third, (4.11). It is clear in the case $L = 0$, so we may assume $L \geq 1$. Then by (4.13) we see that

$$\begin{aligned}
&\partial_x^L\Big\{T_+(x,z,\xi)e^{-u\left[A\frac{\xi^2+x^2}{2}+J\xi ix\right]}\Big\}\\
=&\partial_x^L\Big\{\Big(A\frac{x^2-z^2}{2}+J(x-z)i\xi\Big)e^{-u\left[A\frac{\xi^2+x^2}{2}+J\xi ix\right]}\Big\}\\
=&\frac{L(L-1)}{2}A\partial_x^{L-2}\, e^{-u\left[A\frac{\xi^2+x^2}{2}+J\xi ix\right]} + L(Ax+Ji\xi)\partial_x^{L-1}\, e^{-u\left[A\frac{\xi^2+x^2}{2}+J\xi ix\right]}\\
&\quad+\Big(A\frac{x^2-z^2}{2}+J(x-z)i\xi\Big)\partial_x^L\, e^{-u\left[A\frac{\xi^2+x^2}{2}+J\xi ix\right]}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\int \delta(x-z)\,\partial_x^L\Big\{T_+(x,z,\xi)e^{-u\left[A\frac{\xi^2+x^2}{2}+J\xi ix\right]}\Big\}dz\\
=&L(L-1)\{x(L-2)\xi(0)AD^+ + x(L-1)\xi(1)AD^-\} + Lx\{x(L-1)\xi(0)AD^+ + x(L)\xi(1)AD^-\}\\
&\quad+L\xi\{x(L-1)\xi(0)JD^+ + x(L)\xi(1)JD^-\}\\
=&L(L-1)\{x(L-2)\xi(0)D^+ + x(L-1)\xi(1)D^-\} + L\{x(L)\xi(0)D^+ + x(L+1)\xi(1)D^-\}\\
&\quad+L\{x(L-1)\xi(1)D^- + x(L)\xi(0)D^+\}\\
=&L(L-1)\{x(L)\xi(0)D^+ + x(L+1)\xi(1)D^-\} + L\{x(L)\xi(0)D^+ + x(L+1)\xi(1)D^-\}\\
=&L\{x(L)\xi(0)D^+ + x(L+1)\xi(1)D^-\}.
\end{aligned}$$

This proves the third assertion. The last assertion (4.12) is clear because there is always a factor $x - z$ and it holds that $x\delta(x) = 0$. This completes the proof of the lemma. ☐

In order to prove the terms of t^{2k} with integers $k \geq 0$ are absent from the asymptotic expansion of $\operatorname{Tr} K_m(t)$ for small t, we need a little more preparation. First, by analogous arguments made in the proof of Lemma 4.3 above, we see also that

$$\begin{aligned}
&\partial_\xi^L\Big\{T_+(x,z,\xi)e^{-u\left[A\frac{\xi^2+x^2}{2}+J\xi ix\right]}\Big\}\\
\in&D^-(x-z)\partial_\xi^{L-1}\, e^{-u\left[A\frac{\xi^2+x^2}{2}+J\xi ix\right]} + \Big(A\frac{x^2-z^2}{2}+J(x-z)i\xi\Big)\partial_\xi^L\, e^{-u\left[A\frac{\xi^2+x^2}{2}+J\xi ix\right]}\\
\in&D^-(x-z)\Big(x(0)\xi(L-1)D^+ + x(1)\xi(L)D^-\Big) + \Big((x^2-z^2)D^+ + (x-z)\xi D^-\Big)\Big(x(0)\xi(L)D^+ + x(1)\xi(L+1)D^-\Big).
\end{aligned}$$

We may regard

$$D^-(x-z)\Big(x(0)\xi(L-1)D^+ + x(1)\xi(L)D^-\Big) = D^-(x-z)\xi\Big(x(0)\xi(L)D^+ + x(1)\xi(L+1)D^-\Big),$$

since we have only to keep the parity of functions. Hence we obtain

$$\begin{aligned}
&\partial_\xi^L\Big\{T_+(x,z,\xi)e^{-u\left[A\frac{\xi^2+x^2}{2}+J\xi ix\right]}\Big\}\\
&\in\Big((x^2-z^2)D^+ + (x-z)\xi D^-\Big)\Big(x(0)\xi(L)D^+ + x(1)\xi(L+1)D^-\Big).
\end{aligned} \tag{4.14}$$

A relation with $T_-(x,z,\xi)$ in place of $T_+(x,z,\xi)$ also is easily obtained, rather as a special case of (4.14).

Thus, defining

$$\kappa(\epsilon) = \begin{cases} 1 & \text{if} \quad \varepsilon = +, \\ 0 & \text{if} \quad \varepsilon = -, \end{cases}$$

we have shown by (4.14) the following

Lemma 4.4.

$$\begin{aligned}
&\partial_\xi^L\Big\{T_\varepsilon(x,z,\xi)e^{-u\left[A\frac{\xi^2+x^2}{2}+J\xi ix\right]}\Big\}\\
&\in\Big\{\kappa(\varepsilon)A(x,z,\xi)+(1-\kappa(\varepsilon))D^-\Big\}\Big(x(0)\xi(L)D^++x(1)\xi(L+1)D^-\Big),
\end{aligned}\tag{4.15}$$

where

$$A(x,z,\xi):=(x^2-z^2)D^++(x-z)\xi D^-.\quad\square$$

We are now in a position to show that the coefficients of t^{2k} with k non-negative integers vanish in the asymptotic expansion of $\operatorname{Tr}K_m(t)$ for $t\downarrow 0$.

Proposition 4.5. *For $m\geq 2$, one has $c_{m,2\ell}=0$ for every integer $\ell\geq 0$.*

Proof. Since $K_m(t)=\sum_{\varepsilon\in\mathbb{Z}_2^{m-1}}K_{m,\varepsilon}(t)$, we have

$$c_{m,k}=\sum_{\varepsilon\in\mathbb{Z}_2^{m-1}}c_k^{(m,\varepsilon)},\tag{4.16}$$

so that the assertion immediately follows if we prove $c_{2\ell}^{(m,\varepsilon)}=0$. Note by (4.8) that the coefficient $c_k^{(m,\varepsilon)}$ of t^k in the asymptotic expansion $\operatorname{Tr}K_{m,\varepsilon}(t)\sim\sum_{k=0}^\infty c_k^{(m,\varepsilon)}t^k$ is given by

$$c_k^{(m,\varepsilon)}=\sum_{\substack{\ell_1,\dots,\ell_{m-1}\geq 0;\\ \ell_1+\cdots+\ell_{m-1}+m-\ell(\varepsilon)-2=k}}c_{\ell_1,\cdots,\ell_{m-1}}^{(m,\varepsilon)},\tag{4.17a}$$

$$\begin{aligned}
&c_{\ell_1,\cdots,\ell_{m-1}}^{(m,\varepsilon)}\\
=&\frac{i^{\ell_1+\cdots+\ell_{m-1}}}{2\pi\ell_1!\cdots\ell_{m-1}!}\int_{\Delta^{m-1}}du\,\theta_{m-1}(u)\int^{\otimes 2m}\prod_{j=1}^m d\xi_j\prod_{j=0}^{m-1}dz_j\prod_{j=1}^{m-1}\partial_{z_{j-1}}^{\ell_j}\delta(z_j-z_{j-1})\prod_{j=1}^{m-1}\delta(\xi_j-\xi_m)\\
&\times\operatorname{tr}\left[\Big\{e^{-(1-u_1-\cdots-u_{m-1})\left[A\frac{\xi_m^2+z_{m-1}^2}{2}+J\xi_m iz_{m-1}\right]}\Big\}\overleftarrow{\prod_{j=1}^{m-1}}\partial_{\xi_j}^{\ell_j}\Big\{T_{\varepsilon_j}(z_{j-1},z_j,\xi_j)e^{-u_j\left[A\frac{\xi_j^2+z_{j-1}^2}{2}+J\xi_j iz_{j-1}\right]}\Big\}\right].
\end{aligned}\tag{4.17b}$$

Hence by Lemmas 4.2 and 4.4 it is easy to verify with $\ell_m\equiv 0$ that

$$\begin{aligned}
&c_{\ell_1,\cdots,\ell_{m-1}}^{(m,\varepsilon)}\in\operatorname{tr}\int_{\Delta^{m-1}}du\,\theta_{m-1}(u)\int d\xi\int^{\otimes m}\prod_{j=0}^{m-1}dz_j\prod_{j=1}^{m-1}\partial_{z_{j-1}}^{\ell_j}\delta(z_j-z_{j-1})\\
&\times\left[\prod_{j=1}^m\Big\{z_{j-1}(0)\xi(\ell_j)D^++z_{j-1}(1)\xi(\ell_j+1)D^-\Big\}\prod_{j=1}^{m-1}\Big\{\kappa(\varepsilon_j)A(z_{j-1},z_j,\xi)+(1-\kappa(\varepsilon_j))D^-\Big\}\right].
\end{aligned}\tag{4.18}$$

Note here that it is legitimate to change the order of the products in the integrand above because we have the relation $D^+D^-=D^-D^+=D^-$ etc.

Integration by parts with respect to z_0 therefore yields

$$\begin{aligned}
&c_{\ell_1,\cdots,\ell_{m-1}}^{(m,\varepsilon)}\in\operatorname{tr}\int_{\Delta^{m-1}}du\,\theta_{m-1}(u)\int d\xi\int^{\otimes m}\prod_{j=0}^{m-1}dz_j\prod_{j=2}^{m-1}\partial_{z_{j-1}}^{\ell_j}\delta(z_j-z_{j-1})\\
&\times\Bigg[\prod_{j=2}^m\Big\{z_{j-1}(0)\xi(\ell_j)D^++z_{j-1}(1)\xi(\ell_j+1)D^-\Big\}\prod_{j=2}^{m-1}\Big\{\kappa(\varepsilon_j)A(z_{j-1},z_j,\xi)+(1-\kappa(\varepsilon_j))D^-\Big\}\\
&\times\delta(z_1-z_0)\partial_{z_0}^{\ell_1}\Big\{\Big(z_0(0)\xi(\ell_1)D^++z_0(1)\xi(\ell_1+1)D^-\Big)\Big(\kappa(\varepsilon_1)A(z_0,z_1,\xi)+(1-\kappa(\varepsilon_1))D^-\Big)\Big\}\Bigg].
\end{aligned}\tag{4.19}$$

Here we use the Leibniz formula to calculate the $\partial_{z_0}^{\ell_1}$ derivative as

$$\begin{aligned}
&\partial_{z_0}^{\ell_1}\Big\{\Big(z_0(0)\xi(\ell_1)D^+ + z_0(1)\xi(\ell_1+1)D^-\Big)\Big(\kappa(\varepsilon_1)A(z_0,z_1,\xi)+(1-\kappa(\varepsilon_1))D^-\Big)\Big\}\\
&=\sum_{k=0}^{2}\binom{\ell_1}{k}\Big\{\partial_{z_0}^{\ell_1-k}\Big(z_0(0)\xi(\ell_1)D^+ + z_0(1)\xi(\ell_1+1)D^-\Big)\Big\}\Big\{\partial_{z_0}^{k}\Big(\kappa(\varepsilon_1)A(z_0,z_1,\xi)+(1-\kappa(\varepsilon_1))D^-\Big)\Big\}\\
&\in\Big(z_0(\ell_1)\xi(\ell_1)D^+ + z_0(\ell_1+1)\xi(\ell_1+1)D^-\Big)\Big(\kappa(\varepsilon_1)A(z_0,z_1,\xi)+(1-\kappa(\varepsilon_1))D^-\Big)\\
&\quad+\Big(z_0(\ell_1-1)\xi(\ell_1)D^+ + z_0(\ell_1)\xi(\ell_1+1)D^-\Big)\kappa(\varepsilon_1)(z_0D^+ + \xi D^-)\\
&\quad+\Big(z_0(\ell_1-2)\xi(\ell_1)D^+ + z_0(\ell_1-1)\xi(\ell_1+1)D^-\Big)\kappa(\varepsilon_1)D^+\\
&\subset\Big(z_0(\ell_1)\xi(\ell_1)D^+ + z_0(\ell_1+1)\xi(\ell_1+1)D^-\Big)\Big(\kappa(\varepsilon_1)\big(D^+ + A(z_0,z_1,\xi)\big)+(1-\kappa(\varepsilon_1))\big)D^-\Big).
\end{aligned}\tag{4.20}$$

Thus by (4.20) we see that (4.19) becomes

$$\begin{aligned}
&c^{(m,\varepsilon)}_{\ell_1,\cdots,\ell_{m-1}}\in\operatorname{tr}\int_{\Delta^{m-1}}du\,\theta_{m-1}(u)\int d\xi\int^{\otimes m-1}\prod_{j=1}^{m-1}dz_j\prod_{j=2}^{m-1}\partial_{z_{j-1}}^{\ell_j}\delta(z_j-z_{j-1})\\
&\times\Bigg[\prod_{j=2}^{m}\Big\{z_{j-1}(0)\xi(\ell_j)D^+ + z_{j-1}(1)\xi(\ell_j+1)D^-\Big\}\prod_{j=2}^{m-1}\Big\{\kappa(\varepsilon_j)A(z_{j-1},z_j,\xi)+(1-\kappa(\varepsilon_j))D^-\Big\}\\
&\times\Big\{z_1(\ell_1)\xi(\ell_1)D^+ + z_1(\ell_1+1)\xi(\ell_1+1)D^-\Big\}\Bigg]\Big(\kappa(\varepsilon_1)D^+ + (1-\kappa(\varepsilon_1))D^-\Big).
\end{aligned}\tag{4.21}$$

By integration by parts with respect to the next variable z_1, we see by (4.20) that

$$\begin{aligned}
&c^{(m,\varepsilon)}_{\ell_1,\cdots,\ell_{m-1}}\in\operatorname{tr}\int_{\Delta^{m-1}}du\,\theta_{m-1}(u)\int d\xi\int^{\otimes m-1}\prod_{j=1}^{m-1}dz_j\prod_{j=3}^{m-1}\partial_{z_{j-1}}^{\ell_j}\delta(z_j-z_{j-1})\\
&\times\Bigg[\prod_{j=3}^{m}\Big\{z_{j-1}(0)\xi(\ell_j)D^+ + z_{j-1}(1)\xi(\ell_j+1)D^-\Big\}\prod_{j=3}^{m-1}\Big\{\kappa(\varepsilon_j)A(z_{j-1},z_j,\xi)+(1-\kappa(\varepsilon_j))D^-\Big\}\\
&\times\delta(z_2-z_1)\partial_{z_1}^{\ell_2}\Big\{\Big(z_1(\ell_1)\xi(\ell_1+\ell_2)D^+ + z_1(\ell_1+1)\xi(\ell_1+\ell_2+1)D^-\Big)\Big(\kappa(\varepsilon_2)A(z_1,z_2,\xi)+(1-\kappa(\varepsilon_2))D^-\Big)\Big\}\Bigg]\\
&\times\Big(\kappa(\varepsilon_1)D^+ + (1-\kappa(\varepsilon_1))D^-\Big).
\end{aligned}\tag{4.22}$$

Calculation of the $\partial_{z_1}^{\ell_2}$ derivative, similar to (4.20), gives

$$\begin{aligned}
&\partial_{z_1}^{\ell_2}\Big\{\Big(z_1(\ell_1)\xi(\ell_1+\ell_2)D^+ + z_1(\ell_1+1)\xi(\ell_1+\ell_2+1)D^-\Big)\Big(\kappa(\varepsilon_2)A(z_1,z_2,\xi)+(1-\kappa(\varepsilon_2))D^-\Big)\Big\}\\
&\in\Big(z_1(\ell_1+\ell_2)\xi(\ell_1+\ell_2)D^+ + z_1(\ell_1+\ell_2+1)\xi(\ell_1+\ell_2+1)D^-\Big)\\
&\qquad\qquad\times\Big(\kappa(\varepsilon_2)(D^+ + A(z_1,z_2,\xi))+(1-\kappa(\varepsilon_2))D^-\Big).
\end{aligned}\tag{4.22}$$

Hence it follows from (4.21) that

$$\begin{aligned}
&c^{(m,\varepsilon)}_{\ell_1,\cdots,\ell_{m-1}}\in\operatorname{tr}\int_{\Delta^{m-1}}du\,\theta_{m-1}(u)\int d\xi\int^{\otimes m-2}\prod_{j=2}^{m-1}dz_j\prod_{j=3}^{m-1}\partial_{z_{j-1}}^{\ell_j}\delta(z_j-z_{j-1})\\
&\times\Bigg[\prod_{j=3}^{m}\Big\{z_{j-1}(0)\xi(\ell_j)D^+ + z_{j-1}(1)\xi(\ell_j+1)D^-\Big\}\prod_{j=3}^{m-1}\Big\{\kappa(\varepsilon_j)A(z_{j-1},z_j,\xi)+(1-\kappa(\varepsilon_j))D^-\Big\}\\
&\times\Big\{z_2(\ell_1+\ell_2)\xi(\ell_1+\ell_2)D^+ + z_2(\ell_1+\ell_2+1)\xi(\ell_1+\ell_2+1)D^-\Big\}\Bigg]\prod_{k=1}^{2}\Big(\kappa(\varepsilon_k)D^+ + (1-\kappa(\varepsilon_k))D^-\Big).
\end{aligned}\tag{4.23}$$

Continuing this procedure successively and recalling $\ell_m=0$ we are now arriving at the following result.

Lemma 4.6.

$$c^{(m,\varepsilon)}_{\ell_1,\cdots,\ell_{m-1}} \in \operatorname{tr}\Bigg[\int_{\Delta^{m-1}} du\,\theta_{m-1}(u)\int\int d\xi dz_{m-1}$$
$$\times\Big\{z_{m-1}(\ell_1+\cdots+\ell_{m-1})\xi(\ell_1+\cdots+\ell_{m-1})D^+ + z_{m-1}(\ell_1+\cdots+\ell_{m-1}+1)\xi(\ell_1+\cdots+\ell_{m-1}+1)D^-\Big\}$$
$$\times\prod_{j=1}^{m-1}\Big(\kappa(\varepsilon_j)D^+ + (1-\kappa(\varepsilon_j))D^-\Big)\Bigg]. \quad \square \tag{4.24}$$

Proof of Proposition 4.5 (continuation). It follows from Lemma 4.6 that

$$c^{(m,\varepsilon)}_{\ell_1,\cdots,\ell_{m-1}} \in \operatorname{tr}\Bigg[\int_{\Delta^{m-1}} du\,\theta_{m-1}(u)\int d\xi$$
$$\times\Big\{\xi(\ell_1+\cdots+\ell_{m-1})D^+ + \xi(\ell_1+\cdots+\ell_{m-1}+1)D^-\Big\}(D^+)^{\ell(\varepsilon)}(D^-)^{m-1-\ell(\varepsilon)}$$
$$\subset \operatorname{tr}\Bigg[\int d\xi\Big\{\xi(\ell_1+\cdots+\ell_{m-1})(D^-)^{m-1-\ell(\varepsilon)} + \xi(\ell_1+\cdots+\ell_{m-1}+1)(D^-)^{m-\ell(\varepsilon)}\Big\}\Bigg] = \{0\}, \tag{4.25}$$

whenever $\ell_1+\cdots+\ell_{m-1}+m-\ell(\varepsilon)$ is even. Thus the coefficient $c^{(m,\varepsilon)}_{\ell_1,\cdots,\ell_{m-1}}$ vanishes when $\ell_1+\cdots+\ell_{m-1}+m-\ell(\varepsilon)$ is even. This completes the proof of the proposition. $\square$

What remains for the proof of Theorem 4.1 is to show the following proposition.

Proposition 4.7. *Suppose $m\geq 2$. The coefficients $c_{m,k}$ vanish for $k<m-2$.*

Proof. Recall the coefficients $c^{(m,\varepsilon)}_k$ and $c^{(m,\varepsilon)}_{\ell_1,\cdots,\ell_{m-1}}$ in (4.17ab) with respect to t^k in the asymptotic expansion $\operatorname{Tr}K_{m,\varepsilon}(t)\sim\sum_{k=0}^\infty c^{(m,\varepsilon)}_k t^k$. Therefore, to show the assertion, it suffices to check that $c^{(m,\varepsilon)}_{\ell_1,\cdots,\ell_{m-1}}=0$ when $\ell_1+\cdots+\ell_{m-1}<\ell(\varepsilon)$. In this case, since $\ell(\varepsilon)\leq m-1$, there exists some i with $1\leq i\leq m-1$ such that $\ell_i=0$ and $\varepsilon_i=+$. Hence, by analogous arguments used to derive (4.19) from (4.18), we can see that there is a factor $\delta(z_{i+1}-z_i)A(z_i,z_{i+1},\xi)$ in the integrand. Since $\delta(x)x=0$ we have the assertion. This proves the proposition. $\square$

It is clear that Theorem 4.1 immediately follows from Proposition 4.5 and Proposition 4.7. In particular, the fact we have shown, that those coefficients $\{c_{m,j}\}$ in the asymptotic expansion (4.1) are arranged in an (almost) triangular array, is highly non-trivial and is quite important. As a result, we can show that the spectral zeta function $\zeta_Q(s)$ has a zero at each non-positive even integer, i.e. at $s=0$ and at the same point as the Riemann zeta function has. In fact, we have the following theorem.

Theorem 4.8. *One has*

$$\zeta_Q(s)=\frac{\alpha+\beta}{\sqrt{\alpha\beta(\alpha\beta-1)}}\frac{1}{\Gamma(s)}\frac{1}{s-1}+\frac{1}{\Gamma(s)}\Bigg[\sum_{j=1}^n\frac{c_{2j-1}}{s+2j-1}+\hat h_1(s)+\hat h_2(s)+\hat Z_\infty(s)\Bigg], \tag{4.26}$$

where $\hat h_1(s)$ is holomorphic in $\sigma=\operatorname{Re}s>-2n-1$, having a bound $|\hat h_1(s)|\leq C_1(n)/(\operatorname{Re}s+2n+1)$, and $\hat h_2(s)$ holomorphic in $\sigma=\operatorname{Re}s>-n/2$, having a bound $|\hat h_2(s)|\leq C_2(n)/(\operatorname{Re}s+n/2)$, for every positive integer n with positive constants $C_1(n)$ and $C_2(n)$ dependent on n. Consequently, $\zeta_Q(s)$ is meromorphic in the whole complex plane with a single pole at $s=1$, and has zeros for s being non-positive even integers.

Proof. Note that $\operatorname{Tr}K(t)=\sum_{m=1}^n\operatorname{Tr}K_m(t)+\operatorname{Tr}R_{n+1}(t)$ where $|\operatorname{Tr}R_{n+1}(t)|\leq C^n\frac{\Gamma(1/2)^n}{\Gamma(1+n/2)}t^{n/2}$ with a constant $C > 0$, by (2.13) and Proposition 2.3. Hence by Theorem 4.1 together with Proposition 2.2 we have

$$\begin{aligned}\operatorname{Tr} K(t) \sim \frac{\alpha+\beta}{\sqrt{\alpha\beta(\alpha\beta-1)}} t^{-1} &+ c_{2,1}t + c_{2,3}t^3 + c_{2,5}t^5 + c_{2,7}t^7 + c_{2,9}t^9 + \cdots \\ &+ c_{3,1}t + c_{3,3}t^3 + c_{3,5}t^5 + c_{3,7}t^7 + c_{3,9}t^9 + \cdots \\ &+ c_{4,3}t^3 + c_{4,5}t^5 + c_{4,7}t^7 + c_{4,9}t^9 + \cdots \\ &+ c_{5,3}t^3 + c_{5,5}t^5 + c_{5,7}t^7 + c_{5,9}t^9 + \cdots \\ &+ c_{6,5}t^5 + c_{6,7}t^7 + c_{6,9}t^9 + \cdots \\ &\cdots\cdots .\end{aligned}$$

Thus, putting $c_{2j} = 0$ and $c_{2j-1} = \sum_{\ell=2}^{2j+2} c_{\ell,2j-1}$, we have

$$\operatorname{Tr} K(t) \sim \frac{\alpha+\beta}{\sqrt{\alpha\beta(\alpha\beta-1)}} t^{-1} + \sum_{j=1}^{n} c_{2j-1}t^{2j-1} + \Big\{ \sum_{m=2}^{n} \operatorname{Tr} K_m(t) - \sum_{j=1}^{n} c_{2j-1}t^{2j-1} \Big\} + \frac{C^n\Gamma(1/2)^n}{\Gamma(1+n/2)} O(t^{n/2}).$$

Noting this fact, to show the remaining part of the assertion we may use exactly the same argument as in the proof of Corollary 3.4. Here we have for $\hat{h}_1(s)$ and $\hat{h}_2(s)$ in (4.26)

$$|\hat{h}_1(s)| = \left| \int_0^1 t^{s-1} \Big\{ \sum_{m=2}^{n} \operatorname{Tr} K_m(t) - \sum_{j=1}^{n} c_{2j-1}t^{2j-1} \Big\} dt \right| \le C_1(n) \int_0^1 t^{\operatorname{Re} s+2n} dt = \frac{C_1(n)}{\operatorname{Re} s + 2n + 1},$$

and

$$|\hat{h}_2(s)| \le \left| \int_0^1 t^{s-1} \operatorname{Tr} R_{n+1}(t) dt \right| \le C^n \frac{\Gamma(1/2)^n}{\Gamma(1+n/2)} \int_0^1 t^{\operatorname{Re} s-1+n/2} dt = \frac{C^n\Gamma(1/2)^n}{\Gamma(1+n/2)} \frac{1}{\operatorname{Re} s + n/2}.$$

Thus the theorem has been shown. □

Putting $C_{Q,j} := c_{2j-1}$ the main theorem in the Introduction follows immediately from Theorem 4.8.

For $\operatorname{Re} s > 1$, it is easy to verify that in the classical limit, i.e. the limit when $q := \alpha/\beta$ approaches 1, $\zeta_Q(s)$ yields $(\alpha^2-1)^{-s/2} \cdot 2(2^s-1)\zeta(s)$ (see [PW1] or Lemma 2.8). Moreover, since the theorem above is true for all positive α, β with $\alpha\beta > 1$, we conclude that the classical limit of $\zeta_Q(s)$ essentially becomes the Riemann zeta function.

Corollary 4.9. *As $\alpha/\beta \to 1$, $\zeta_Q(s)$ converges to $(\alpha^2-1)^{-s/2} \cdot 2(2^s-1)\zeta(s)$ as meromorphic functions. Of course, this agrees with the well-known fact that $\zeta(s)$ has a simple pole at $s = 1$ with residue 1.*

Remark 1. It is furthermore interesting to study the situation in the limit when the ratio $q = \alpha/\beta$ tends to 0 (or $+\infty$) with the value of the product $\alpha\beta$ kept fixed, and to compare it with a so-called q-analogue [KKW] of the Riemann zeta function as well as the corresponding crystal zeta function [KWY]. See also [P] for a study of a perturbation of the spectrum of the non-commutative harmonic oscillator which may provide some idea to this direction. □

As a concluding remark of the paper, we discuss whether or not one can take the limit $n \to \infty$ in (4.26) in Theorem 4.8, namely, (1.4) in the main theorem. First we note that though we have $C_2(n) = \frac{C^n\Gamma(1/2)^n}{\Gamma(1+n/2)} \to 0$ as $n \to \infty$, we cannot conclude that the $C_1(n)$ tend to 0 nor are even bounded. In fact, it is not easy to get an effective estimate which allows us to conclude that the series $\sum_{j=1}^{n} \frac{c_{2j-1}}{s+2j-1}$ converges as $n \to \infty$. To explain the situation we try to give some estimation of $\partial_x^L \partial_\xi^M e^{-uq(x,\xi)}$ which is necessary to have a good bound of $c_k^{(m,\varepsilon)}$. Here recall (2.18a), namely,

$$q(x,\xi) = A\frac{\xi^2+x^2}{2} + Jxi\xi = \begin{pmatrix} \alpha\frac{\xi^2+x^2}{2} & -xi\xi \\ xi\xi & \beta\frac{\xi^2+x^2}{2} \end{pmatrix}.$$

Lemma 4.10. *The matrix $\partial_x^L \partial_\xi^M e^{-uq(x,\xi)}$ is hermitian for any non-negative integers L and M when $x,\, \xi \in \mathbb{R}$. One has*

$$\partial_\xi e^{-uq(x,\xi)} = P(x,\xi)\partial_\xi e^{-uq(x,\xi)} = \partial_\xi e^{-uq(x,\xi)}\, P(x,\xi)^*, \tag{4.27}$$

with $P(x,\xi) = P_1(x,\xi) + iP_2(x,\xi)$, where $P_j(x,\xi)$ are hermitian matrices given by

$$\begin{aligned} P_1(x,\xi) &= -\partial_\xi q(x,\xi)u = -(A\xi + Jix)u, \\ P_2(x,\xi) &= i\big[\partial_\xi q(x,\xi),\, q(x,\xi)\big] \\ &= -\frac{x^2-\xi^2}{4}x[J,\, A]u^2 = -(\alpha-\beta)\frac{x^2-\xi^2}{4}xu^2K. \end{aligned}$$

Here the commutator of matrices M_1 and M_2 is denoted by $[M_1, M_2] = M_1M_2 - M_2M_1$, and $K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. A similar equation also holds for the differentiation with respect to x in place of ξ. Moreover, for higher order derivatives, it holds that for any non-negative integers $L,\, M$ there is a matrix-valued polynomial $f_{L,M}(x,\xi)$ of degree $3(L+M)$ such that

$$\partial_x^L \partial_\xi^M e^{-uq(x,\xi)} = f_{L,M}(x,\xi)e^{-uq(x,\xi)} = e^{-uq(x,\xi)} f_{L,M}(x,\xi)^*.$$

Proof. The first assertion is obvious. For the second, note that

$$\begin{aligned} q(x,\xi+h) &= A\frac{x^2+(\xi+h)^2}{2} + Jix(\xi+h) = q(x,\xi) + A\xi h + A\frac{h^2}{2} + Jixh \\ &= q(x,\xi) + (A\xi + Jix)h + A\frac{h^2}{2} = q(x,\xi) + \partial_\xi q(x,\xi)h + O(h^2). \end{aligned}$$

We employ the Campbell-Hausdorff formula (see, e.g. [H]) which says that $\exp(tA)\exp(tB) = \exp\big(tA + tB + \frac{t^2}{2}[A,\, B] + O(t^3)\big)$. Then we have

$$\begin{aligned} \frac{1}{h}\Big(e^{-uq(x,\xi+h)} - e^{-uq(x,\xi)}\Big) =& \frac{1}{h}\Big(e^{-uq(x,\xi+h)}e^{uq(x,\xi)} - 1\Big)\cdot e^{-uq(x,\xi)} \\ =& \frac{1}{h}\Big[\exp\Big\{-u\partial_\xi q(x,\xi)h - \frac{u^2}{2}\big[\partial_\xi q(x,\xi),\, q(x,\xi)\big]h + O(h^2)\Big\} - 1\Big]\cdot e^{-uq(x,\xi)}. \end{aligned}$$

Thus taking the limit $h \to 0$ we see that

$$\partial_\xi e^{-uq(x,\xi)} = -\Big(u\partial_\xi q(x,\xi) + \frac{u^2}{2}\big[\partial_\xi q(x,\xi),\, q(x,\xi)\big]\Big)e^{-uq(x,\xi)}.$$

Then we have

$$\partial_\xi e^{-uq(x,\xi)} = -\big((A\xi + Jix)u + i(\alpha-\beta)\frac{x^2-\xi^2}{4}xKu^2\big)e^{-uq(x,\xi)},$$

since

$$\begin{aligned} \big[\partial_\xi q(x,\xi),\, q(x,\xi)\big] &= \big[A\xi + Jix,\, A\frac{x^2+\xi^2}{2} + Jix\xi\big] \\ &= \frac{x^2+\xi^2}{2}ix[J,\, A] + \xi ix\xi[A,\, J] \\ &= \frac{x^2-\xi^2}{2}ix[J,\, A] = i(\alpha-\beta)\frac{x^2-\xi^2}{2}xK. \end{aligned}$$

This proves the first equality of (4.27). The second one follows by taking the adjoint. The last assertion follows from the first formula. □

From this lemma we easily see that the hermitian matrix $\partial_x^L \partial_\xi^M e^{-uq(x,\xi)}$ obeys e.g.

$$\begin{aligned} \|\partial_x^L \partial_\xi^M e^{-uq(x,\xi)}\|_2 &= \Big[\operatorname{tr}\big(f_{L,M}(x,\xi)f_{L,M}(x,\xi)^* e^{-2uq(x,\xi)}\big)\Big]^{1/2} \\ &\le \|f_{L,M}(x,\xi)\|_2 \|e^{-uq(x,\xi)}\|. \end{aligned}$$

Here we denote the norm and the Hilbert-Schmidt norm for a matrix by the same notations $\|\cdot\|$ and $\|\cdot\|_2$, respectively, as used for an operator in §2. Obviously, $\|e^{-uq(x,\xi)}\| = e^{-u\lambda^+(x,\xi)}$, and $\|f_{L,M}(x,\xi)\|_2$ is a polynomial in x and ξ of degree $3(L+M)$, though of degree $L+M$ when $\alpha=\beta$ because the term $P_2(x,\xi)$ disappears. Therefore, for instance, if $L=0$, $M=1$, we have

$$\begin{aligned}\|f_{0,1}(x,\xi)\|_2^2 = \|P(x,\xi)\|_2^2 &= \operatorname{tr}\big((P_1(x,\xi)-iP_2(x,\xi))(P_1(x,\xi)+iP_2(x,\xi))\big)\\ &= u^2 \operatorname{tr}\big((A\xi+Jix)^2\big) + u^4 \operatorname{tr}\Big(\Big((\alpha-\beta)\frac{x^2-\xi^2}{4}xK\Big)^2\Big)\\ &= u^2(\alpha^2+\beta^2)\xi^2 + 2u^2x^2 + \frac{u^4}{8}(\alpha-\beta)^2(x^2-\xi^2)^2x^2.\end{aligned}$$

Accordingly, the observation above together with Lemma 4.10 will suggest us only to hold that the absolute values of the coefficients c_{2j-1} are dominated by $j^{3j/2}/j!$, which is clearly insufficient when $\alpha\neq\beta$ to prove the convergence of the series "$\sum_{j=1}^{\infty}\frac{c_{2j-1}}{s+2j-1}$". It would be desirable to elucidate whether this estimate $|c_{2j-1}|\le$ constant $\cdot\, j^{3j/2}/j!$ for all sufficiently large j is best possible or the same estimate $|c_{2j-1}|\le$ constant $\cdot\, j^{j/2}/j!$ holds as in the case where $\alpha=\beta$ (see Remark 3 below). In the latter case we may let $n\to\infty$ in Theorem 4.8, but not in the former case.

Remark 2. Note the zero of $\zeta_Q(s)$ at $s=0$ is simple. We conjecture also that the zeros of $\zeta_Q(s)$ (which are at least produced by $\Gamma(s)^{-1}$) at the negative even integer $s=-2j$ are all simple. □

Remark 3. Recall Bernoulli's numbers B_n (see e.g. [E], p.11) are defined by

$$\frac{t}{e^t-1} = \sum_{n=0}^{\infty}\frac{B_n}{n!}t^n = 1-\frac{1}{2}t+\frac{1}{6}\frac{t^2}{2!}-\frac{1}{30}\frac{t^4}{4!}+\frac{1}{42}\frac{t^6}{6!}-\cdots \quad (|t|<2\pi).$$

Note that $B_{2m+1}=0$ for $m=1,2,\dots$. (Notice that the definition of the Bernoulli's number in [T] is different from the present one.) Then it is well-known that

$$\zeta(0)=-\frac{1}{2},\quad \zeta(-2m)=0,\quad \zeta(1-2m)=-\frac{B_{2m}}{2m}\qquad (m=1,2,\dots). \tag{4.27}$$

Since $Q=Q_0\cong\frac{1}{2}(-\partial_x^2+x^2)I$ when $\alpha=\beta=\sqrt{2}$, the trace of the heat kernel is given by

$$\begin{aligned}&\operatorname{Tr}K(t)=\operatorname{Tr}e^{-Q_0t}\\ =&2\sum_{n=0}^{\infty}e^{-(n+\frac{1}{2})t} = 2e^{\frac{1}{2}t}\cdot\frac{1}{e^t-1}\\ =&2\Big\{1+\frac{1}{2}t+\frac{1}{2!}\Big(\frac{1}{2}t\Big)^2+\frac{1}{3!}\Big(\frac{1}{2}t\Big)^3+\frac{1}{4!}\Big(\frac{1}{2}t\Big)^4+\cdots\Big\}\Big\{\frac{1}{t}-\frac{1}{2}+B_2\frac{t}{2!}-B_4\frac{t^3}{4!}+\cdots\Big\}\\ =&2\Big[t^{-1}-\frac{1}{8}t+\Big(\frac{1}{2}\cdot\frac{1}{2!}B_2-\frac{1}{2}\cdot\frac{1}{2!2^2}+\frac{1}{3!2^3}\Big)t^2+\Big(-\frac{1}{2}\cdot\frac{1}{3!2^3}+\frac{1}{2!2^2}\cdot\frac{1}{2!}B_2+\frac{1}{4!2^4}-\frac{1}{4!}B_4\Big)t^3+\cdots\Big]\\ =&2t^{-1}-\frac{1}{4}t+\frac{14}{4!\,5!}t^3+\cdots,\end{aligned}$$

because $B_2=\frac{1}{6}$, $B_4=\frac{1}{30}$, etc. Note here that the equation above is the Laurent expansion of $\operatorname{Tr}K(t)=(\sinh\frac{t}{2})^{-1}$ at $t=0$ $(0<|t|<2\pi)$. Since the coefficients of this expansion are closely involved with Bernoulli's numbers, we are suggested to consider the constants $C_{Q,j}$ defined in the proof of the main theorem above as analogues of Bernoulli's numbers.

Notice also that when $\alpha=\beta$ we may take the limit $n\to\infty$ in Theorem 4.8 (hence in the main theorem) because $|c_{2j-1}|$ are dominated by $j^{j/2}/j!$. This is compatible with the fact that $\frac{B_n}{n!}\to 0$ (see §2). Actually, since $\frac{B_{2n}}{(2n)!}=2(-1)^{n-1}(2\pi)^{-2n}\zeta(2n)$ and $|\zeta(s)|\to 0$ when Re $s\to\infty$, from the Euler product expression we have $\frac{B_{2n}}{(2n)!}\to 0$ and $B_{2n+1}\equiv 0$ for $n\ge 1$.

Moreover, by the partial fraction expansion of $\sec z$, the functional equation $\zeta(s) = \Gamma(s)^{-1}2^{s-1}\pi^s\Big(\cos\frac{s\pi}{2}\Big)^{-1}\zeta(1-s)$ of $\zeta(s)$ ([E], [T]) yields

$$\zeta_{Q_0}(s) = 2(2^s-1)\zeta(s) = \frac{1}{\Gamma(s)}\left[2^{s+1}(2^s-1)\pi^{s-1}\sum_{j=-\infty}^{\infty}\frac{(-1)^{j+1}}{s-2j+1}\right]\zeta(1-s).$$

Hence, this equation together with the above interpretation of $C_{Q,j}$ shows the main theorem may be considered to give a quasi-functional equation of $\zeta_Q(s)$. $\square$

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