CHAOTIC SIZE DEPENDENCE IN THE ISING MODEL WITH RANDOM BOUNDARY CONDITIONS

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Abstract. We study the nearest-neighbour Ising model with a class of random boundary conditions, chosen from a symmetric i.i.d. distribution. We show for dimensions 4 and higher that almost surely the only limit points for a sequence of increasing cubes are the plus and the minus state. For d=2 and d=3 we prove a similar result for sparse sequences of increasing cubes. This question was raised by Newman and Stein. Our results imply that the Newman-Stein metastate is concentrated on the plus and the minus state.

1. Introduction

In most studies of phase transitions, one considers boundary conditions which are either symmetric between the different possible phases (such as free, periodic or antiperiodic boundary conditions in low temperature Ising, Potts or vector models) or a priori known to be in some sense typical for one of the phases. The latter case in its best-known (and purest) form is realized by the choice of plus or minus boundary conditions for the Ising model, and in some sense by the wired or free boundary conditions for the random cluster model. Other examples of an unambiguous preference for one phase are the cases of uniform weak boundary conditions [16, 18] for low temperature Ising or Potts models, predominantly plus boundary conditions for the Ising model [11], and random or weak boundary conditions at the high-q Potts transition temperature [8, 18, 26]. Complementary results generalizing (deterministic) symmetric boundary conditions can be found in [2, 4, 3, 18], for instance.

A question which comes up naturally in the theory of spin-glasses, where the phases are unknown so that the choice of coherent boundary conditions is not available (although one can put the question in substantial greater generality), is what will happen if one chooses the

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boundary conditions randomly, without having a systematic preference for one of the phases.

This is the type of question we want to study here. To be more definite, we consider the standard Ising model at low temperature, with symmetric i.i.d. boundary conditions, c.f. [21], example iii.4. It is conjectured there (compare also [9, 25]) that when one would take an increasing sequence of volumes, one would oscillate randomly between being close to the plus and the minus phase. Heuristically, the fluctuations of the free energies of the plus and minus states should scale with the square root of the boundary, which diverges with increasing volumes. Thus, one would expect that weight of either the plus or the minus measure in a mixture will be of order $e^{-V^{(d-1)/2d}}$, which becomes negligible for sufficiently large volumes.

This non-convergence to a single thermodynamic limit measure is an example of what Newman and Stein call "chaotic size dependence". They have developed their ideas within the formalism of metastates [22, 23, 20]. Since then similar arguments have been made precise for a class of random mean-field models [5, 13, 15, 14, 6]. However, for short-range models hardly any precise result has been obtained.

Here we study a simple version of this problem for the standard Ising model. Our simplification is the condition that the bonds on the boundary are weaker than the bonds in the bulk. This removes "by hand" any interface (large contour) because these get attracted to the boundary. Our result is that the above heuristics is correct, and that a chaotic size dependence occurs indeed. The two possible limit states occurring with the same limit frequency are the plus and the minus states. In other words, the metastate is concentrated with equal weight on these two pure states. This can be shown for a sequence of strictly increasing cubes in high dimensions, in dimension 2 and 3 we need to impose an extra condition of "sparsity" on the sequence of increasing volumes.

The structure of the set of limit states can be inferred through a toy-model, where only two configurations are allowed (the zero-temperature approximation). Having in mind the Ising model coupled to a random environment, let $\Lambda_n \subset \mathbb{Z}^d$ be a cube with side n and $\sigma \in \{-1, 1\}$ the possible states. Considering a collection $\{\lambda_x\}_{x\in\mathbb{Z}^d}$ of identical independent variables with the distribution $\mathsf{P}(-1) = \mathsf{P}(1) = \frac{1}{2}$, the Hamiltonian of the toy-model is

$$H_n^\lambda(\sigma) = -\sigma \sum_{x\in\partial\Lambda_n} \lambda_x \ . \tag{1.1}$$

Using μ_n^λ to denote the corresponding finite-volume Gibbs measure, we immediately have $\mu_n^\lambda(\sigma) = \tanh S_n^\lambda$ with $S_n^\lambda = \sum_{x\in\partial\Lambda_n} \lambda_x$. Thus, in order to find the limit points of $\{\mu_n^\lambda\}$, it is sufficient to find the limit points of $\{S_n^\lambda\}$. However, the latter is a sequence of "essentially" independent functions, each of them being the sum of independent variables. Therefore, one can readily use the local-limit theorem to show that $\mathsf{P}(S_n^\lambda = k) \simeq n^{-\frac{d-1}{2}}$ for any $k \in 2\mathbb{Z}$. Then, according to the Borel-Cantelli lemmas, $k \in 2\mathbb{Z}$ is a limit point of $\{S_n^\lambda\}$ P-a.s. if and only if $\sum_n \mathsf{P}(S_n^\lambda = k) = \infty$. Realizing that $-\infty$ and ∞ are always limit points P-a.s. (due to the Borel-Cantelli argument and the symmetry of the distribution), we can conclude the following: for $d = 2, 3$ the set of limit points of $\{S_n^\lambda\}$ is $2\mathbb{Z} \cup \{-\infty, \infty\}$ P-a.s., while for $d > 3$ it is $\{-\infty, \infty\}$ P-a.s. As a consequence, the only limit points of the sequence of the Gibbs measures for $d > 3$ are δ_1 and δ_{-1} P-a.s. On the contrary, in dimensions $d = 2, 3$ we obtain an infinite set of limit measures. Note that this picture differs from the case of free boundary conditions, where there is only one limit state $\frac{1}{2}(\delta_1 + \delta_{-1})$.

In the sequel, we show that this behaviour is stable with respect to thermal fluctuations for $d > 3$. This is done by generalizing the above scenario and proving a weak variant of the local-limit theorem for the boundary term. A similar result is obtained in dimensions 2 and 3 for "sparse" sequences.

On the other hand, the full low-dimensional structure of limit points remains out of the scope of the present paper. Presumably mixed states can appear as limit points, but with disappearing probabilities.

The organization of the paper is as follows. In Section 2, we specify the studied model and state our main result. Section 3 is devoted to the contour analysis of our model, leading to its reformulation in terms of contour ensembles. The cluster-expansion control over these ensembles is proved in Section 4, using statements for the abstract polymer model in Appendix A. The proof is finished by the probabilistic arguments of Section 5. Finally, some comments and remarks are given in Section 6.

2. Model and Results

Let $\{\Lambda_n\}_{n\in\mathbb{N}}$ be the sequence of d-dimensional cubes on $\mathbb{Z}^d$, $d \geq 2$, given by[1]

$$\Lambda_n = \{x \in \mathbb{Z}^d : -\frac{n}{2} < x_i \leq \frac{n}{2} \quad \forall\, i = 1, \ldots, d\}. \tag{2.1}$$

[1]Notice that the cube Λ_n has side-length $n-1$ and is centred at the origin if n is odd and at $\frac{1}{2}\mathbb{I}$ if n is even, where $\mathbb{I} \in \mathbb{Z}^d$ is the unit vector.

In this paper, we study the ferromagnetic, nearest-neighbour Ising model in Λ_n exposed to random boundary fields $\lambda \in \mathbb{R}^{\mathbb{Z}^d}$ in the thermodynamic limit $n \to \infty$. Namely, using Ω to denote the set $\{-1,1\}^{\mathbb{Z}^d}$ of spin configurations on $\mathbb{Z}^d$ and Ω_n to denote the set $\{-1,1\}^{\Lambda_n}$ of spin configurations in Λ_n, we consider the *Hamiltonian*

$$H_n^\lambda(\sigma_n) = -\,\beta \sum_{\substack{\langle x,y\rangle \\ x,y\in\Lambda_n}} (\sigma_x\sigma_y - 1) \;-\; \sum_{x\in\partial\Lambda_n} \lambda_x\sigma_x, \qquad \sigma_n \in \Omega_n. \tag{2.2}$$

Here $\langle x,y\rangle$ stands for a pair of nearest-neighbour sites $x,y \in \mathbb{Z}^d$, the bulk coupling $\beta > 0$, the set $\partial\Lambda_n$ contains all $x \in \Lambda_n$ having at least one nearest-neighbour site in Λ_n^c, and $\{\lambda_x\}_{x\in\mathbb{Z}^d}$ are identical, independent, symmetrically distributed random variables with zero mean. The latter represent random boundary conditions with boundary terms of strength λ_x. We will write P for the (product) *probability law* of λ and E for the *expectation* with respect to P. Let $\varphi(t) = \mathsf{E}\, e^{it\lambda_0}$, $t \in \mathbb{R}$, be the *characteristic function* of λ_0. We restrict ourselves to distributions with bounded supports, precisely, we assume that $\mathsf{P}(|\lambda_0| > \lambda^*) = 0$ for a certain finite λ^*.

The finite-volume *Gibbs measure* μ_n^λ corresponding to the Hamiltonian (2.2) is defined as

$$\mu_n^\lambda(A) = \sum_{\sigma_n\in A} \frac{e^{-H_n^\lambda(\sigma_n)}}{Z_n^\lambda}, \quad A \subset \Omega_n, \tag{2.3}$$

where the normalizing constant $Z_n^\lambda = \sum_{\sigma_n\in\Omega_n} e^{-H_n^\lambda(\sigma_n)}$ is the *partition function*. Given β sufficiently large, our aim is the analysis of the set of limit points of the sequence of random measures $\{\mu_n^\lambda\}_{n\in\mathbb{N}}$. Let

$$\mu_n^\pm(A) = \sum_{\sigma_n\in A} \frac{e^{-H_n^\pm(\sigma_n)}}{Z_n^\pm}, \quad A \subset \Omega_n, \tag{2.4}$$

where

$$H_n^\pm(\sigma_n) = -\,\beta \sum_{\substack{\langle x,y\rangle \\ x,y\in\Lambda_n}} (\sigma_x\sigma_y - 1) \;\mp\; \beta \sum_{x\in\partial\Lambda_n} \sigma_x \tag{2.5}$$

and $Z_n^\pm = \sum_{\sigma_n\in\Omega_n} e^{-H_n^\pm(\sigma_n)}$. The weak limits $\mu^\pm$ of the sequences $\{\mu_n^\pm\}$ are the only extremal translation-invariant Gibbs measures of the Ising model and $\mu^+ \neq \mu^-$ for $d \geq 2$ and β large enough, see e.g. [10]. In this paper we prove the following theorem.

Theorem 2.1. *Given $d \geq 2$ and $0 < \lambda^* < \infty$, there exists a constant $\beta_0 = \beta_0(\lambda^*, d) < \infty$ such that for any $\beta \geq \beta_0$ and any symmetric distribution P of boundary fields with zero mean, strictly positive variance, and satisfying $\mathsf{P}(|\lambda_0| > \lambda^*) = 0$, one has:*

1. *If $d > 3$, then the set of limit points of $\{\mu_n^\lambda\}_{n\in\mathbb{N}}$ is $\{\mu^+, \mu^-\}$ P-a.s.*
2. *If $d \in \{2, 3\}$ and $\omega > 0$, then the set of limit points of the "sparse" sequence $\{\mu^\lambda_{[n^{4-d+\omega}]}\}_{n\in\mathbb{N}}$ is $\{\mu^+, \mu^-\}$ P-a.s.*

The conclusion of the theorem implies that the Newman-Stein metastate is $\frac{1}{2}(\delta_{\mu^+} + \delta_{\mu^-})$.

The proof of the theorem is carried out in two steps. First, we rewrite our model in terms of contours, using two auxiliary contour ensembles with the corresponding measures $\mu_n^{\lambda,+}$ and $\mu_n^{\lambda,-}$ and the partition functions $Z_n^{\lambda,+}$ and $Z_n^{\lambda,-}$. This enables us to express μ_n^λ through $\mu_n^{\lambda,+}$, $\mu_n^{\lambda,-}$, and $F_n^\lambda = \log Z_n^{\lambda,+} - \log Z_n^{\lambda,-}$. Establishing the relation between the limit points of $\{\mu_n^\lambda\}$ and the limits of $\{\mu_n^\pm\}$ in Corollary 4.7, the original problem gets reduced to the task of finding the limit points of $\{F_n^\lambda\}$. In the second step, we solve this task with the help of a weak variant of the local-limit theorem and the Borel-Cantelli lemmas, and state the result in Proposition 5.4.

3. Contour Representations

In this section, we introduce contour representations for our model given by the Hamiltonian H_n^λ as well as for the models corresponding to the Hamiltonians H_n^+ and H_n^-. In the former case, we are interested in boundary fields λ of small strength. Hence, we make use of contours suitable for the study of lattice models under free boundary conditions. These may be "open" and their definition is in the spirit of [3]. In the latter case, the standard, "closed" Ising contours are employed. It turns out that the difference between the two cases merely concerns "boundary contours", i.e. those containing the sites from $\partial\Lambda_n$, see below.

Nevertheless, we shall proceed in a slightly more general context, allowing at the same time to study the expectation of local observables in the above models as $n \to \infty$. By virtue of an "FKG-argument", see Corollary 4.7, it is sufficient to control the infinite-volume expectation of the spin at each site $x \in \mathbb{Z}^d$. Observing that there exists $n_x < \infty$ such that $x \in \Lambda_n$ for all $n \geq n_x$, we therefore introduce an external field $\eta \in \mathbb{R}$ at x, i.e. for each $x \in \mathbb{Z}^d$ we consider models with the perturbed Hamiltonians

$$H_n^{\lambda,x}(\sigma_n) = H_n^\lambda(\sigma_n) - \eta\,\sigma_x \quad \text{and} \quad H_n^{\pm,x}(\sigma_n) = H_n^\pm(\sigma_n) - \eta\,\sigma_x\,. \tag{3.1}$$

Here σ_x is the restriction of σ_n to x if $n \geq n_x$, while it is 0 otherwise. All other quantities associated with these new models will also have the additional superscript x. The superscript will be suppressed whenever we will be in the original situation, corresponding to $\eta = 0$.

Let $\square_x$, $x \in \mathbb{Z}^d$, be the closed unit cube in $\mathbb{R}^d$ whose centre is at x and let $V_n = \cup_{x\in\Lambda_n} \square_x$. Given $\sigma_n \in \Omega_n$, we define $V_n^{\pm}(\sigma_n) = \cup_{x\in\Lambda_n:\, \sigma_x=\pm 1} \square_x \subset V_n$ as the "$\pm$ regions" corresponding to σ_n and $\mathcal{D}(\sigma_n)$ as the set of connected components of $V_n^+(\sigma_n) \cap V_n^-(\sigma_n)$. Thus, the set $\mathcal{D}(\sigma_n)$ represents the connected and mutually disjoint boundaries separating $V_n^+(\sigma_n)$ from $V_n^-(\sigma_n)$. A *contour* is any element of the union $\mathcal{D}_n = \cup_{\sigma_n\in\Omega_n} \mathcal{D}(\sigma_n)$. We write $|\gamma|$ for the number of *plaquettes* (i.e. closed $(d-1)$-dimensional faces of the closed unit cubes) lying in the contour γ.

Next, we will define the *interior* and *exterior* of a contour γ. For each corner $k = [k_1, \ldots, k_d]$ of the box V_n,[2] let us introduce the "octant" associated with k as

$$\mathcal{O}_n(k) = \{x \in \mathbb{R}^d :\ x_i \geq k_i \ \text{ if } \ i \in I_-,\ x_i \leq k_i \ \text{ if } \ i \in I_+\}, \tag{3.2}$$

where $i \in I_-$ whenever $y_i \geq k_i$ for all $y \in V_n$, while $i \in I_+$ whenever $y_i \leq k_i$ for all $y \in V_n$. Notice that $V_n = \cap_k \mathcal{O}_n(k)$. Two possibilities arise:

1. There is a corner k of V_n such that $\gamma \cap \partial V_n \subset \partial \mathcal{O}_n(k)$. Then $\operatorname{Int}\gamma$ is the union of all finite components of $\mathcal{O}_n(k)\backslash\gamma$ and $\operatorname{Ext}\gamma$ is $V_n \setminus (\gamma \cup \operatorname{Int}\gamma)$.[3]
2. There is no corner k of V_n for which $\gamma \cap \partial V_n \subset \partial \mathcal{O}_n(k)$, i.e. there is a kind of interface. We then choose $\operatorname{Ext}\gamma$ to be the largest component of $V_n\backslash\gamma$ and $\operatorname{Int}\gamma$ to be the union of the remaining components of $V_n\backslash\gamma$.[4]

We point out that the joint exterior $\operatorname{Ext}(\partial) = \cap_{\gamma\in\partial} \operatorname{Ext}\gamma$ of any set of contours $\partial \in \mathcal{D}_n$ such that $\partial = \mathcal{D}(\sigma_n)$ is either a subset of $V_n^+(\sigma_n)$ or $V_n^-(\sigma_n)$. Hence, the set Ω_n may be written as a union $\Omega_n^+ \cup \Omega_n^-$ of disjoint subsets $\Omega_n^{\pm} = \{\sigma_n \in \Omega_n : \operatorname{Ext}(\mathcal{D}(\sigma_n)) \subset V_n^{\pm}(\sigma_n)\}$. Finally, let $\Lambda(\gamma) = \operatorname{Int}\gamma \cap \Lambda_n$ and $v(\gamma) = \max_{\gamma'\subset \operatorname{Int}\gamma} |\Lambda(\gamma')|$ for any contour $\gamma \in \mathcal{D}_n$.

We shall now rewrite the partition function $Z_n^{\lambda,x}$ in terms of contours. Given $\Theta \subset \mathbb{Z}^d$, let $S_\Theta^\lambda = \sum_{x\in\Theta\cap\partial\Lambda_n} \lambda_x$ and $E_\Theta^{\lambda,x,\pm} = \mp(S_\Theta^\lambda + \eta \mathbf{1}_{x\in\Theta})$,

[2] If n is odd, $n = 2m+1$, then one obviously has $|k_i| = m$, $i = 1, \ldots, d$, for the corner k. However, if n is even, $n = 2m$, then $|k_i|$ equals either m or $m-1$.

[3] Cf. [3]. This definition does not depend on the choice of k if more corners are possible. This case also covers the contours which do not touch the boundary ∂V_n.

[4] If there are several components of $V_n\backslash\gamma$ with the largest volume, we take the first one in some fixed (e.g. lexicographic) order.

$x \in \mathbb{Z}^d$. For simplicity, we write $S_n^\lambda = S_{\partial\Lambda_n}^\lambda$ and $E_n^{\lambda,x,\pm} = E_{\Lambda_n}^{\lambda,x,\pm}$. Let us introduce the quantities $Z_\gamma^{\lambda,x,\pm}$ and $K_\gamma^{\lambda,x,\pm}$ for any contour $\gamma \in \mathcal{D}_n$ in the following inductive manner:

1. We set

$$Z_\gamma^{\lambda,x,\pm} = e^{-E_{\Lambda(\gamma)}^{\lambda,x,\pm}} \quad \text{and} \quad K_\gamma^{\lambda,x,\pm} = e^{-2\beta|\gamma| + E_{\Lambda(\gamma)}^{\lambda,x,\pm} - E_{\Lambda(\gamma)}^{\lambda,x,\mp}} \tag{3.3}$$

for any contour γ with $v(\gamma) = 0$.

2. Assuming that $K_{\gamma'}^{\lambda,x,\pm}$ and $Z_{\gamma'}^{\lambda,x,\pm}$ have been defined for all contours having $v(\gamma') < N \leq |\Lambda_n|$, for any γ with $v(\gamma) = N$ we set

$$Z_\gamma^{\lambda,x,\pm} = e^{-E_{\Lambda(\gamma)}^{\lambda,x,\pm}} \sum_{\partial \sqsubset \operatorname{Int}\gamma} \prod_{\gamma' \in \partial} K_{\gamma'}^{\lambda,x,\pm} \tag{3.4}$$

and

$$K_\gamma^{\lambda,x,\pm} = e^{-2\beta|\gamma|} \, \frac{Z_\gamma^{\lambda,x,\mp}}{Z_\gamma^{\lambda,x,\pm}} \, . \tag{3.5}$$

Here the sum is over all families $\partial \subset \mathcal{D}_n$ of mutually disjoint contours which all lie in $\operatorname{Int}\gamma$; the term corresponding to $\partial = \emptyset$ is set equal to 1.

In the second step one uses the fact that $v(\gamma') < v(\gamma)$ for any γ' which lies in $\operatorname{Int}\gamma$ (notice that $\gamma \cap \operatorname{Int}\gamma = \emptyset$). Observing that

$$H_n^{\lambda,x}(\sigma_n) = 2\beta \sum_{\gamma \in \mathcal{D}(\sigma_n)} |\gamma| + E_{V_n^+(\sigma_n)\cap\Lambda_n}^{\lambda,x,+} + E_{V_n^-(\sigma_n)\cap\Lambda_n}^{\lambda,x,-} \tag{3.6}$$

for all $\sigma_n \in \Omega_n$, one may use standard arguments [27, 1, 3] to express $Z_n^{\lambda,x}$ as the sum of two partition functions of auxiliary contour ensembles. Namely, one has

$$Z_n^{\lambda,x} = Z_n^{\lambda,x,+} + Z_n^{\lambda,x,-} \qquad \text{with} \quad Z_n^{\lambda,x,\pm} = e^{-E_n^{\lambda,x,\pm}} \sum_{\partial \sqsubset V_n} \prod_{\gamma \in \partial} K_\gamma^{\lambda,x,\pm}, \tag{3.7}$$

where the summation is over all families $\partial \subset \mathcal{D}_n$ of mutually disjoint contours; the term corresponding to $\partial = \emptyset$ is set equal to 1. Each of the contour ensembles may be associated with a measure $\mu_n^{\lambda,x,\pm}$ given through the restricted sets of configurations $\Omega_n^\pm$,

$$\mu_n^{\lambda,x,\pm} = \begin{cases} \frac{e^{-H_n^{\lambda,x}(\sigma_n)}}{Z_n^{\lambda,x,\pm}} & \sigma_n \in \Omega_n^\pm, \\ 0 & \text{otherwise.} \end{cases} \tag{3.8}$$

These contour ensembles provide a suitable representation for the finite-volume Gibbs measure μ_n^λ,

$$\mu_n^\lambda = \frac{Z_n^{\lambda,+}\mu_n^{\lambda,+} + Z_n^{\lambda,-}\mu_n^{\lambda,-}}{Z_n^{\lambda,+} + Z_n^{\lambda,-}} = \frac{\mu_n^{\lambda,+}}{1+e^{-F_n^\lambda}} + \frac{\mu_n^{\lambda,-}}{1+e^{F_n^\lambda}}\ , \tag{3.9}$$

where

$$F_n^\lambda = \log Z_n^{\lambda,+} - \log Z_n^{\lambda,-}\ . \tag{3.10}$$

In the case of $\pm$ boundary conditions, one considers the set $\mathcal{D}^\pm(\sigma_n)$ of connected components of the boundary $\partial V_n^\mp(\sigma_n)$. The set of contours in V_n is then defined as $\hat{\mathcal{D}}_n = \cup_{\sigma_n\in\Omega_n}\mathcal{D}^+(\sigma_n) = \cup_{\sigma_n\in\Omega_n}\mathcal{D}^-(\sigma_n)$. Given $\hat\gamma \in \hat{\mathcal{D}}_n$, its interior and exterior are introduced naturally: $\operatorname{Int}\hat\gamma$ is the union of all finite components of $\mathbb{R}^d \setminus \hat\gamma$ and $\operatorname{Ext}\hat\gamma = V_n \setminus (\hat\gamma \cup \operatorname{Int}\hat\gamma)$. Again, we let $\Lambda(\hat\gamma) = \operatorname{Int}\hat\gamma \cap \Lambda_n$ and $v(\hat\gamma) = \max_{\hat\gamma'\subset\hat\gamma}|\Lambda(\hat\gamma')|$. Clearly, the sets $\mathcal{D}_n$ and $\hat{\mathcal{D}}_n$ only differ in "boundary contours" since $\Lambda(\gamma)\cap\partial\Lambda_n = \emptyset$ iff $\gamma \in \mathcal{D}_n \cap \hat{\mathcal{D}}_n$. Setting $E_\Theta^{\pm,x} = \mp\eta\,\mathbf{1}_{x\in\Theta}$ for any $\Theta \subset \mathbb{Z}^d$ and $x \in \mathbb{Z}^d$ and observing that

$$H_n^{\pm,x}(\sigma_n) = 2\beta\sum_{\gamma\in\mathcal{D}(\sigma_n)}|\gamma| + E^{+,x}_{V_n^+(\sigma_n)\cap\Lambda_n} + E^{-,x}_{V_n^-(\sigma_n)\cap\Lambda_n}, \tag{3.11}$$

the quantities $Z_{\hat\gamma}^{\pm,x}$ and $K_{\hat\gamma}^{\pm,x}$ are introduced in an inductive manner analogously to $Z_\gamma^{\lambda,x,\pm}$ and $K_\gamma^{\lambda,x,\pm}$. It then follows that

$$Z_n^{\pm,x} = e^{-E_n^{\pm,x}}\sum_{\hat\partial\sqsubset V_n}\prod_{\hat\gamma\in\hat\partial}K_{\hat\gamma}^{\pm,x}, \tag{3.12}$$

where the sum goes over all families $\hat\partial \subset \hat{\mathcal{D}}_n$ of mutually disjoint contours with the term corresponding to $\hat\partial = \emptyset$ being set equal to 1.

4. Perturbative Control of the Contour Ensembles

Let us define the relation ι of incompatibility on the sets $\mathcal{D}_n$ and $\hat{\mathcal{D}}_n$ of contours in V_n as disjointness (for instance, for any $\gamma,\gamma' \in \mathcal{D}_n$ one has $\gamma\iota\gamma'$ iff $\gamma\cap\gamma' \neq \emptyset$). Then the logarithms of the partition functions $Z_n^{\lambda,x,\pm}$ and $Z_n^{\pm,x}$ as well as those of $Z_\gamma^{\lambda,x,\pm}$ and $Z_{\hat\gamma}^{\pm,x}$ can be expressed in the form of cluster expansions, see Appendix A; we write $\mathcal{C}_n$ and $\hat{\mathcal{C}}_n$ for the set of all clusters $C \subset \mathcal{D}_n$ and $\hat C \subset \hat{\mathcal{D}}_n$, respectively. Here we prove the convergence of these expansions with the help of the Kotecký-Preiss criterion (A.5), see [12] and also [7, 19, 24], for instance.

We begin with a geometrical lemma giving an estimate on the size of the set $\mathcal{P}(\gamma)$ of all plaquettes of the boundary $\partial\,\overline{\operatorname{Int}\gamma}$ lying on ∂V_n.

Lemma 4.1. *Let $d \geq 2$ and $n \in \mathbb{N}$. For an arbitrary contour $\gamma \in \mathcal{D}_n$ the estimate $|\mathcal{P}(\gamma)| \leq \theta\,|\gamma|$ with $\theta = \frac{2^{1/d}+1}{2^{1/d}-1}$ holds true.*

Proof. See Lemma B.3 in [3]. □

Writing the cluster expansions

$$\log Z_n^{\lambda,x,\pm} = -E_n^{\lambda,x,\pm} + \sum_{C\in\mathcal{C}_n} \Phi_C^{\lambda,x,\pm}\ , \tag{4.1}$$

the next lemma provides a control over these series and their derivatives with respect to η.

Lemma 4.2. *Let $d \geq 2$ and $\lambda^*, \eta^* \geq 0$. There exist $c_1, c_2 < \tau < \infty$ such that for any $\beta \geq \tau$ and $x_0 \in \mathbb{Z}^d$ one has*

$$\sum_{C:\,\Lambda(C)\ni x} e^{2(\beta-c_1)|C|}|\Phi_C^{\lambda,x_0,\pm}| \leq 1 \tag{4.2}$$

and

$$\sum_{C:\,\Lambda(C)\ni x} e^{2(\beta-c_2)|C|}\left|\frac{\partial\Phi_C^{\lambda,x_0,\pm}}{\partial\eta}\right| \leq 1 \tag{4.3}$$

for all $x \in \Lambda_n$ and $n \in \mathbb{N}$ whenever $|\lambda_y| \leq \lambda^*$ for all $y \in \mathbb{Z}^d$ and $|\eta| \leq \eta^*$. Here $|C| = \sum_{\gamma\in C}|\gamma|$ and $\Lambda(C) = \cup_{\gamma\in C}\Lambda(\gamma)$.

Remark 4.3. It will turn out in the proof that the dependence of the constants c_1, c_2, and τ on d, λ^*, and η^* is of the form $\theta\lambda^*+\eta^*+\text{const}(d)$. Similar dependencies will also occur in the subsequent statements.

Proof of Lemma 4.2. Let $\beta \geq \tau$ with τ to be specified later. Let $x_0 \in \mathbb{Z}^d$ and $x \in \Lambda_n$ be given. Together with the inequalities (4.2) and (4.3), we simultaneously prove that

$$\sum_{\gamma:\,\Lambda(\gamma)\ni x} e^{2(\beta-c_1')|\gamma|}K_\gamma^{\lambda,x,\pm} \leq 1 \tag{4.4}$$

and

$$\sum_{\gamma:\,\Lambda(\gamma)\ni x} e^{2(\beta-c_2')|\gamma|}\left|\frac{\partial K_\gamma^{\lambda,x,\pm}}{\partial\eta}\right| \leq 1 \tag{4.5}$$

for some constants $c_1', c_2' < \tau$ (depending on d, λ^*, η^*). We shall proceed by induction on the size of the volumes $v(\gamma)$ and $v(C) = \max_{\gamma\in C} v(\gamma)$.

First, let us consider only contours and clusters with $v(\gamma) = 0$ and $v(C) = 0$, respectively. From (3.3) it follows that

$$K_\gamma^{\lambda,x_0,\pm} \leq e^{-2\left(\beta|\gamma|-|S^\lambda_{\Lambda(\gamma)}|-\eta^*\right)} \leq e^{-2(\beta-\theta\lambda^*-\eta^*)|\gamma|}, \tag{4.6}$$

where we also used Lemma 4.1. Since $|\gamma| \geq d$ and since there exists a constant $\varkappa = \varkappa(d) < \infty$ such that the number of all contours $\gamma \in \mathcal{D}_n$ with $|\gamma| = \ell$ and such that $\Lambda(\gamma)$ contains a given site from Λ_n can be bounded by $\varkappa^\ell$, the last estimate implies

$$\sum_{\substack{\gamma:\,\Lambda(\gamma)\ni x\\ v(\gamma)=0}} e^{2(\beta-c_1')|\gamma|} K_\gamma^{\lambda,x_0,\pm} \leq \sum_{\ell=d}^{\infty} (\varkappa e^{-2(c_1'-\theta\lambda^*-\eta^*)})^\ell \leq \frac{1}{2}, \tag{4.7}$$

provided $c_1' - \theta\lambda^* - \eta^* \geq \frac{1}{2}\log 2\varkappa$, say, which in its turn yields

$$\sum_{\substack{\gamma'\iota\gamma:\\ v(\gamma')=0}} e^{2(\beta-c_1')|\gamma'|} K_{\gamma'}^{\lambda,x_0,\pm} \leq |\gamma| \max_{p\subset V_n} |\{x \in \Lambda_n : \square_x \cap p \neq \emptyset\}| \leq 3^d|\gamma|. \tag{4.8}$$

The condition (A.5) is thus satisfied in our case with $a(\gamma) = |\gamma|$ and $b(\gamma) = [2(\beta - c_1') - d\log 3 - 1]\,|\gamma|$. Hence, in view of Proposition A.1, we have

$$\begin{aligned}\sum_{\substack{C:\,\Lambda(C)\ni x\\ v(C)=0}} e^{2(\beta-c_1)|C|}|\Phi_C^{\lambda,x_0,\pm}| &\leq \sum_{\substack{\gamma:\,\Lambda(\gamma)\ni x\\ v(\gamma)=0}} \sum_{\substack{C:\,C\ni\gamma\\ v(C)=0}} e^{2(\beta-c_1)|C|}|\Phi_C^{\lambda,x_0,\pm}| \leq \\ \leq \sum_{\substack{\gamma:\,\Lambda(\gamma)\ni x\\ v(\gamma)=0}} e^{-[2(c_1-c_1')-d\log 3-1]\,|\gamma|} &\sum_{\substack{C:\,C\iota\gamma\\ v(C)=0}} e^{[2(\beta-c_1')-d\log 3-1]\,|C|}|\Phi_C^{\lambda,x_0,\pm}| \leq \\ \leq \sum_{\substack{\gamma:\,\Lambda(\gamma)\ni x\\ v(\gamma)=0}} e^{-[2(c_1-c_1')-d\log 3-2]\,|\gamma|} &\leq 2(3^d\varkappa e^{-2(c_1-c_1'-1)})^d \leq 1\end{aligned} \tag{4.9}$$

if $c_1 - c_1' - 1 \geq \frac{1}{2}\log(2.3^d\varkappa)$. By virtue of (3.3), one has $\left|\frac{\partial}{\partial\eta} K_\gamma^{\lambda,x_0,\pm}\right| \leq 2K_\gamma^{\lambda,x_0,\pm}$. Combined with the above, we therefore verified the inequalities (4.2) to (4.5) for the considered contours and clusters, providing that $c_1' \geq \theta\lambda^* + \eta^* + \frac{1}{2}\log 2\varkappa$, $c_1 \geq c_1' + 1 + \frac{1}{2}\log(2.3^d\varkappa)$, $c_2' \geq \theta\lambda^* + \eta^* + \frac{1}{2}\log 2\varkappa$, and $c_2 \geq c_2' + 1 + \frac{1}{2}\log(2.3^d\varkappa)$.

Next, let us prove these inequalities for any contours and clusters with $v(\gamma) = N$ and $v(C) = N$, respectively, assuming that they have already been proved for all contours and clusters with their volumes smaller than N. Recalling that for any $\gamma' \subset \operatorname{Int}\gamma$ one necessarily has $v(\gamma') < v(\gamma)$, from the inductive assumption it follows that $\log Z_\gamma^{\lambda,x_0,\pm}$ can be controlled by convergent cluster expansions. In view of (3.4) and (4.1), we thus have

$$\log Z_\gamma^{\lambda,x_0,+} - \log Z_\gamma^{\lambda,x_0,-} = 2S_{\Lambda(\gamma)}^\lambda + 2\eta\mathbf{1}_{x_0\in\Lambda(\gamma)} + \sum_{C:\,C\sqsubset\operatorname{Int}\gamma} \Delta\Phi_C^{\lambda,x_0}, \tag{4.10}$$

where $\Delta\Phi_C^{\lambda,x_0} = \Phi_C^{\lambda,x_0,+} - \Phi_C^{\lambda,x_0,-}$ and the sum runs only over the clusters C such that all of its contours are in $\operatorname{Int}\gamma$. Observing that $\Delta\Phi_C^{\lambda,x_0}$ vanishes whenever $\Lambda(C)\cap\partial\Lambda_n = \emptyset$ or $\Lambda(C)\not\ni x_0$, we get

$$\sum_{C\colon C\sqsubset\operatorname{Int}\gamma} |\Delta\Phi_C^{\lambda,x_0}| \le \sum_{y\in\partial_n(\gamma)} \sum_{\substack{C\colon C\sqsubset\operatorname{Int}\gamma\\ \Lambda(C)\ni y}} |\Delta\Phi_C^{\lambda,x_0}| + \sum_{\substack{C\colon C\sqsubset\operatorname{Int}\gamma\\ \Lambda(C)\ni x_0}} |\Delta\Phi_C^{\lambda,x_0}|, \quad (4.11)$$

where $\partial_n(\gamma) = \Lambda(\gamma)\cap\partial\Lambda_n$. Using the inductive assumption (4.2) and Lemma 4.1, the former sum may be estimated by

$$\sum_{y\in\partial_n(\gamma)} \sum_{C\colon\Lambda(C)\ni y} |\Delta\Phi_C^{\lambda,x_0}| \le 2e^{-2(\beta-c_1)}|\partial_n(\gamma)| \le$$
$$\le 2e^{-2(\beta-c_1)}|\mathcal{P}(\gamma)| \le 2de^{-2(\beta-c_1)}|\gamma| \le |\gamma| \quad (4.12)$$

once $\tau\ge\tau_1 = c_1 + \frac12\log 2d$, while the latter sum is smaller than 1 if $\tau \ge c_1 + \frac12\log 2$. Combining these bounds with the definition (3.5) of $K_\gamma^{\lambda,x_0,\pm}$, we therefore find

$$K_\gamma^{\lambda,x_0,\pm} \le e^{-2\left(\beta|\gamma| - |S^\lambda_{\Lambda(\gamma)}| - \eta^* - |\gamma| - 1\right)} \le e^{-2(\beta-\theta\lambda^*-\eta^*-2)|\gamma|} \quad (4.13)$$

as long as $\tau\ge\tau_1$. Moreover,

$$\Bigl|\frac{\partial K_\gamma^{\lambda,x_0,\pm}}{\partial\eta}\Bigr| \le 4K_\gamma^{\lambda,x_0,\pm} \quad (4.14)$$

for any $\tau\ge c_2$. To see this, it suffices to combine (3.5) with the bound

$$\Bigl|\frac{\partial}{\partial\eta}(\log Z_\gamma^{\lambda,x_0,\mp} - \log Z_\gamma^{\lambda,x_0,\pm})\Bigr| \le 2 + \sum_{\substack{C\colon C\sqsubset\operatorname{Int}\gamma\\ \Lambda(C)\ni 0}} \Bigl|\frac{\partial\Delta\Phi_C^{\lambda,x_0}}{\partial\eta}\Bigr| \le 4 \quad (4.15)$$

following from the cluster expansion (4.10) and the inductive assumption (4.3). Using (4.13) and (4.14), the arguments from the case $N=0$ readily yield the bounds (4.2) to (4.5) if $c_1'\ge\theta\lambda^*+\eta^*+2+\frac12\log 2\varkappa$, $c_1 \ge c_1'+1+\frac12\log(2.3^d\varkappa)$, $c_2'\ge\theta\lambda^*+\eta^*+2+\frac12\log 2\varkappa$, $c_2\ge c_2'+1+\frac12\log(4.3^d\varkappa)$, and $\tau\ge\max\{c_1+\log 2d,\, c_2\}$. □

The convergence of the cluster expansion

$$\log Z_n^{\pm,x} = -E_n^{\pm,x} + \sum_{\hat C\in\hat{\mathcal{C}}_n} \Phi_{\hat C}^{\pm,x}, \quad x\in\mathbb{Z}^d, \quad (4.16)$$

and its derivative with respect to η is guaranteed by this lemma.

Lemma 4.4. *There exist constants* $\hat{c}_1, \hat{c}_2 < \hat{\tau} \le \tau$ *depending on* d *and* η^* *such that for any* $\beta \ge \hat{\tau}$ *and* $x_0 \in \mathbb{Z}^d$ *one has*

$$\sum_{\hat{C}:\,\Lambda(\hat{C})\ni x} e^{2(\beta-\hat{c}_1)|\hat{C}|}|\Phi_{\hat{C}}^{\pm,x_0}| \le 1 \tag{4.17}$$

and

$$\sum_{\hat{C}:\,\Lambda(\hat{C})\ni x} e^{2(\beta-\hat{c}_2)|\hat{C}|}\left|\frac{\partial\Phi_{\hat{C}}^{\pm,x_0}}{\partial\eta}\right| \le 1 \tag{4.18}$$

for all $x \in \Lambda_n$ *and* $n \in \mathbb{N}$ *whenever* $|\eta| \le \eta^*$*. Here* τ *is the constant from Lemma 4.2.*

Proof. The proof goes along the same lines as that of Lemma 4.2. It should be clear that the constant $\hat{\tau}$ can be chosen in such a way that $\hat{\tau} \le \tau - \theta\lambda^*$. □

In order to prove the uniformness in the distribution of the boundary fields λ, we will need the following lemma.

Lemma 4.5. *There exist constants* $c_3 < \tilde{\tau} < \infty$ *depending on* d *and* λ^* *such that for any* $\beta \ge \tilde{\tau}$ *one has*

$$\sum_{C:\,\Lambda(C)\ni x} e^{2(\beta-c_3)|C|}\frac{|\Delta\Phi_C^\lambda|}{\sinh\bigl(4\sum_{y\in\partial_n(C)}|\lambda_y|\bigr)} \le 1 \tag{4.19}$$

for any $x \in \Lambda_n$ *and* $n \in \mathbb{N}$*, provided* $|\lambda_x| \le \lambda^*$ *for all* $x \in \mathbb{Z}^d$*. Here* $\Delta\Phi_C^\lambda = \Phi_C^{\lambda,+} - \Phi_C^{\lambda,-}$*,* $\partial_n(C) = \cup_{\gamma\in C}\partial_n(\gamma)$ *with* $\partial_n(\gamma) = \Lambda(\gamma)\cap\partial\Lambda_n$*, and in the summation we adopt the convention that* $0.\infty = 0$*.*

Proof. Let $\beta \ge \tilde{\tau}$ with $\tilde{\tau}$ to be determined and $x \in \Lambda_n$. Proceeding by induction on $N \in \mathbb{N}$, where N is the maximal size of $v(\gamma)$ and $v(C)$ for γ and C under consideration, along with the estimate (4.19) we shall also prove that

$$\sum_{\gamma:\,\Lambda(\gamma)\ni x} e^{2(\beta-c_3')|\gamma|}\frac{|\Delta K_\gamma^\lambda|}{\sinh\bigl(4\sum_{y\in\partial_n(\gamma)}|\lambda_y|\bigr)} \le 1 \tag{4.20}$$

for some $c_3' < \tilde{\tau}$ (depending on d and λ^*), where $\Delta K_\gamma^\lambda = K_\gamma^{\lambda,+} - K_\gamma^{\lambda,-}$.

First, let $N = 0$. Then (3.3) yields

$$|\Delta K_\gamma^\lambda| \le 2e^{-2\beta|\gamma|}\sinh\Bigl(2\sum_{x\in\partial_n(\gamma)}|\lambda_x|\Bigr). \tag{4.21}$$

As a consequence,

$$\sum_{\substack{\gamma:\,\Lambda(\gamma)\ni x\\ v(\gamma)=0}} \frac{e^{2(\beta-c_3')|\gamma|}\,|\Delta K_\gamma^\lambda|}{\sinh\bigl(4\sum_{y\in\partial_n(\gamma)}|\lambda_y|\bigr)} \le 2\sum_{\substack{\gamma:\,\Lambda(\gamma)\ni x\\ v(\gamma)=0}} e^{-2c_3'|\gamma|} \le 4(\varkappa e^{-2c_3'})^d \le 1 \tag{4.22}$$

once $c_3' \ge \frac{1}{2}\log 2\varkappa$, which verifies (4.20) and leads to the estimate

$$\sum_{\gamma'\iota\gamma:\,v(\gamma')=0} \frac{e^{2(\beta-c_3')|\gamma'|}\,|\Delta K_{\gamma'}^\lambda|}{\sinh\bigl(4\sum_{x\in\partial_n(\gamma')}|\lambda_x|\bigr)} \le 3^d|\gamma|. \tag{4.23}$$

Using Corollary A.2, the latter implies (c.f. (4.9))

$$\sum_{\substack{C:\,\Lambda(C)\ni x\\ v(C)=0}} \frac{e^{2(\beta-c_3)|C|}\,|\Delta\Phi_C^\lambda|}{\sinh\bigl(4\sum_{y\in\partial_n(C)}|\lambda_y|\bigr)} \le 2\sum_{\substack{\gamma:\,\Lambda(\gamma)\ni x\\ v(\gamma)=0}} e^{-[2(c_3-c_3'-2)-d\log 3]\,|\gamma|} \le 1 \tag{4.24}$$

if $c_3' \ge c_1'$, $\tilde\tau \ge \tilde\tau_1 = c_3'+1+\frac{d}{2}\log 3$, and $c_3 \ge c_3'+2+\frac{1}{2}\log(2.3^d\varkappa)$ (here c_1' is the constant from (4.4)), which proves (4.19) in the case $N=0$.

Supposing now that the estimates (4.19) and (4.20) have been proved for all integers smaller than N, let us prove them for N. The relations (3.5) and (4.10) with $\eta=0$ yield

$$|\Delta K_\gamma^\lambda| \le 2e^{-2\beta|\gamma|}\sinh\Bigl(2\sum_{x\in\partial_n(\gamma)}|\lambda_x| + \sum_{\substack{C:\,C\sqsubset\mathrm{Int}\,\gamma\\ \partial_n(C)\neq\emptyset}}|\Delta\Phi_C^\lambda|\Bigr), \tag{4.25}$$

where in the second sum only the clusters C such that all of their contours lie in $\mathrm{Int}\,\gamma$ are considered. Using the inductive assumption (4.19), let us first show that

$$\sum_{\substack{C:\,C\sqsubset\mathrm{Int}\,\gamma\\ \partial_n(C)\neq\emptyset}}|\Delta\Phi_C^\lambda| \le 2\sum_{x\in\partial_n(\gamma)}|\lambda_x| \tag{4.26}$$

whenever $\tilde\tau \ge \tilde\tau_2 = c_3+2\theta\lambda^*+1+\frac{1}{2}\log 4\theta$. Since $\sinh x \le xe^x$ for any $x\ge 0$, the left-hand side of the last inequality can be bounded by

$$\begin{aligned}4e^{-2(\beta-c_3-2\theta\lambda^*-1)}\sum_{x\in\partial_n(\gamma)}\sum_{\substack{C:\,C\sqsubset\mathrm{Int}\,\gamma\\ \Lambda(C)\ni x}}\frac{e^{2(\beta-c_3-2\theta\lambda^*-1)|C|}\,|\Delta\Phi_C^\lambda|}{\sinh\bigl(4\sum_{y\in\partial_n(C)}|\lambda_y|\bigr)}\times\\ \times\; e^{4\theta\lambda^*|C|}\sum_{z\in\partial_n(C)}|\lambda_z| \le\\ \le \frac{1}{\theta}\sum_{z\in\partial_n(\gamma)}|\lambda_z|\sum_{C:\,\Lambda(C)\ni z}\;\sum_{x\in\partial_n(C)}\frac{e^{2(\beta-c_3-1)|C|}\,|\Delta\Phi_C^\lambda|}{\sinh\bigl(4\sum_{y\in\partial_n(C)}|\lambda_y|\bigr)}.\end{aligned} \tag{4.27}$$

Realizing that the last summand is independent of x and that one has $|\partial_n(C)| \leq \sum_{\gamma' \in C} |\partial_n(\gamma')| \leq \theta|C|$, we obtain (4.26). Combining this estimate and (4.25) with the arguments from the case $N = 0$, one readily arrives at (4.19) and (4.20) for any $c_3' \geq \max\{\frac{1}{2} \log 2\varkappa,\, c_1'\}$, $c_3 \geq c_3' + 2 + \frac{1}{2} \log(2.3^d \varkappa)$, and $\tilde{\tau} \geq \max\{\tilde{\tau}_1,\, \tilde{\tau}_2\} = \tilde{\tau}_2$. □

In the following proposition we prove that the limits of $\{\mu_n^{\lambda,\pm}\}$ and $\{\mu_n^{\pm}\}$ coincide on the level of "magnetizations". Instead of proving that the limits are actually identical, in Corollary 4.7 we use an abstract argument to show that the limit points of $\{\mu_n^{\lambda}\}$ coincide with $\mu^{\pm}$ whenever they correspond to infinite limit points of F_n^{λ}.

Proposition 4.6. *Let $\beta \geq \tau$, where τ is the constant from Lemma 4.2. Then for every $x \in \mathbb{Z}^d$ one has $\lim_{n\to\infty} \mu_n^{\lambda,\pm}(\sigma_x) = \mu^{\pm}(\sigma_x)$.*

Proof. Let $\beta \geq \tau$ and $x \in \mathbb{Z}^d$. Taking into account the convergent cluster expansions (4.1) and (4.16), we have

$$\mu_n^{\lambda,\pm}(\sigma_x) = \frac{\partial \log Z_n^{\lambda,x,\pm}}{\partial \eta}\Big|_{\eta=0} = \pm 1 + \sum_{C:\,\Lambda(C)\ni x} \frac{\partial \Phi_C^{\lambda,x,\pm}}{\partial \eta}\Big|_{\eta=0} \tag{4.28}$$

and

$$\mu_n^{\pm}(\sigma_x) = \frac{\partial \log Z_n^{\pm,x}}{\partial \eta}\Big|_{\eta=0} = \pm 1 + \sum_{\hat{C}:\,\Lambda(\hat{C})\ni x} \frac{\partial \Phi_{\hat{C}}^{\pm,x}}{\partial \eta}\Big|_{\eta=0} \tag{4.29}$$

for all $n \geq n_x$.[5] Since any contour γ from $\mathcal{D}_n$ whose volume $\Lambda(\gamma)$ does not intersect $\partial\Lambda_n$ is necessarily in $\hat{\mathcal{D}}_n$ and vice versa and since $K_\gamma^{\lambda,x,\pm} = K_\gamma^{\pm,x}$ for such γ, it follows that

$$\mu_n^{\lambda,\pm}(\sigma_x) - \mu_n^{\pm}(\sigma_x) = \sum_{\substack{C:\,\Lambda(C)\ni x\\ \Lambda(C)\cap\partial\Lambda_n\neq\emptyset}} \frac{\partial \Phi_C^{\lambda,x,\pm}}{\partial \eta}\Big|_{\eta=0} - \sum_{\substack{\hat{C}:\,\Lambda(\hat{C})\ni x\\ \Lambda(\hat{C})\cap\partial\Lambda_n\neq\emptyset}} \frac{\partial \Phi_{\hat{C}}^{\pm,x}}{\partial \eta}\Big|_{\eta=0} \tag{4.30}$$

for all $n \geq n_x$. Realizing that a cluster contributing to any of the last two sums must necessarily satisfy $|C| \geq \frac{n}{4}$ whenever $n \geq 2n_x$, Lemma

[5] Recall that $x \in \Lambda_n$ for all $n \geq n_x$.

4.2 and 4.4 yield

$$|\mu_n^{\lambda,\pm}(\sigma_x) - \mu_n^{\pm}(\sigma_x)| \le e^{-(\beta-\max\{c_2,\hat{c}_2\})\frac{n}{2}} \times$$
$$\times \Big(\sum_{C:\Lambda(C)\ni x} e^{2(\beta-c_2)|C|} \Big| \frac{\partial \Phi_C^{\lambda,x,\pm}}{\partial \eta} \Big| + \sum_{\hat{C}:\Lambda(\hat{C})\ni x} e^{2(\beta-\hat{c}_2)|\hat{C}|} \Big| \frac{\partial \Phi_{\hat{C}}^{x,\pm}}{\partial \eta} \Big| \Big)_{\eta=0} \le$$
$$\le 2e^{-(\beta-\max\{c_2,\hat{c}_2\})\frac{n}{2}}. \quad (4.31)$$

As a result, we have $\lim_{n\to\infty} |\mu_n^{\lambda,\pm}(\sigma_x) - \mu_n^{\pm}(\sigma_x)| = 0$. □

Corollary 4.7. *Let $\beta \ge \tau$, where τ is the constant from Lemma 4.2. If k_n is an increasing sequence of integers such that $\lim_n F_{k_n} = \pm\infty$, then $\mu_{k_n}^{\lambda} \to \mu^{\pm}$ weakly.*

Proof. Let $\lim_n F_{k_n} = \infty$. Since $\lim_n \mu_{k_n}^{\lambda,+}(\sigma_x) = \mu^+(\sigma_x)$ for all $x \in \mathbb{Z}^d$ due to Proposition 4.6, it follows by (3.9) that also $\lim_n \mu_{k_n}^{\lambda}(\sigma_x) = \mu^+(\sigma_x)$. Using a compactness argument, the sequence $\mu_{k_n}^{\lambda}$ has a limit point in the weak topology. If ν^λ is any such limit point, then it is FKG-dominated by μ^+ and satisfies $\nu^\lambda(\sigma_x) = \mu^+(\sigma_x)$ for all x. As a consequence, see [17], Corollary II.2.8, one has $\nu^\lambda = \mu^+$, implying $\lim_n \mu_{k_n}^{\lambda} = \mu^+$. The case $\lim_n F_{k_n} = -\infty$ then immediately follows by the spin-flip symmetry. □

5. Probabilistic Analysis

In view of Corollary 4.7, the study of the limit points of the sequence $\{\mu_n^\lambda\}$ boils down to the analysis of the sequence of random functions $\{F_n^\lambda\}$. Using (4.1) with $\eta = 0$, they have the form

$$F_n^\lambda = 2S_n^\lambda + \sum_{C\in\partial\mathcal{C}_n} \Delta\Phi_C^\lambda \quad (5.1)$$

with $\partial\mathcal{C}_n$ being the set of clusters from $\mathcal{C}_n$ for which $\mathcal{P}(C) = \cup_{\gamma\in C}\mathcal{P}(\gamma)$ is not empty. Notice that $\Delta\Phi_C^\lambda$ only depends on λ_x iff $x \in \partial_n(C)$.

Let us consider the *characteristic function*

$$\psi_n(t) = \mathsf{E}\, e^{itF_n^\lambda}, \quad t \in \mathbb{R}. \quad (5.2)$$

In order to control $\psi_n(t)$ for small values of t, we rewrite it as the partition function of a polymer model with complex weights as follows. Realizing that

$$\prod_{C\in\partial\mathcal{C}_n} e^{it\Delta\Phi_C^\lambda} = \sum_{\mathfrak{C}\subset\partial\mathcal{C}_n} \prod_{C\in\mathfrak{C}} \big(e^{it\Delta\Phi_C^\lambda} - 1\big), \quad (5.3)$$

where the term corresponding to $\mathfrak{C} = \emptyset$ is set equal to 1, and using $\partial_n(\mathfrak{C})$ to denote $\cup_{C\in\mathfrak{C}}\partial_n(C)$, it readily follows that

$$\psi_n(t) = \sum_{\mathfrak{C}\subset\partial\mathcal{C}_n} \Big(\mathsf{E}\, e^{2itS^\lambda_{\partial_n(\mathfrak{C})}} \prod_{C\in\mathfrak{C}} \big(e^{it\Delta\Phi^\lambda_C} - 1\big)\Big)\, \mathsf{E}\, e^{2itS^\lambda_{\partial\Lambda_n\setminus\partial_n(\mathfrak{C})}}. \tag{5.4}$$

Since $\mathsf{E}\, e^{2itS^\lambda_{\partial\Lambda_n\setminus\partial_n(\mathfrak{C})}} = (\varphi(2t))^{|\partial\Lambda_n\setminus\partial_n(\mathfrak{C})|}$, we therefore have

$$\psi_n(t) = (\varphi(2t))^{|\partial\Lambda_n|} \sum_{\mathfrak{C}\subset\partial\mathcal{C}_n} \varrho_{\mathfrak{C}}(t) \tag{5.5}$$

with

$$\varrho_{\mathfrak{C}}(t) = (\varphi(2t))^{-|\partial_n(\mathfrak{C})|}\, \mathsf{E}\, e^{2itS^\lambda_{\partial_n(\mathfrak{C})}} \prod_{C\in\mathfrak{C}} \big(e^{it\Delta\Phi^\lambda_C} - 1\big) \tag{5.6}$$

for all $t \in \mathbb{R}$ for which $\varphi(2t) \neq 0$; we only consider such t in the sequel.

Let $\mathcal{G}(\mathfrak{C})$ be the graph on the vertices of all clusters in $\mathfrak{C} \subset \mathcal{C}_n$ such that $C_1, C_2 \in \mathfrak{C}$ are connected by the edge iff $\mathcal{P}(C_1) \cap \mathcal{P}(C_2) \neq \emptyset$ (sharing of at least one plaquette of ∂V_n). We say that $\mathfrak{C}$ is a *connected family of clusters* whenever the corresponding graph $\mathcal{G}(\mathfrak{C})$ is connected. Clearly,

$$\varrho_{\mathfrak{C}}(t) = \prod_{\mathfrak{C}_{\mathrm{con}}\subset\mathfrak{C}} \varrho_{\mathfrak{C}_{\mathrm{con}}}(t), \tag{5.7}$$

where the product runs over all connected components of the family $\mathfrak{C}$. Writing $\mathcal{P}(\mathfrak{C}) = \cup_{C\in\mathfrak{C}}\mathcal{P}(C)$ and collecting all the connected families $\mathfrak{C}_{\mathrm{con}}$ of clusters with the same set $\mathcal{P}(\mathfrak{C}_{\mathrm{con}})$, we get

$$\sum_{\mathfrak{C}\subset\partial\mathcal{C}_n} \varrho_{\mathfrak{C}}(t) = \sum_{\{P_k\}} \sum_{\substack{\mathfrak{C}^{(1)}_{\mathrm{con}}:\,\mathcal{P}(\mathfrak{C}^{(1)}_{\mathrm{con}})=P_1,\\ \vdots}} \prod_k \varrho_{\mathfrak{C}^{(k)}_{\mathrm{con}}}(t) =$$

$$= \sum_{\{P_k\}} \prod_k \sum_{\mathfrak{C}_{\mathrm{con}}:\,\mathcal{P}(\mathfrak{C}_{\mathrm{con}})=P_k} \varrho_{\mathfrak{C}_{\mathrm{con}}}(t). \tag{5.8}$$

In view of (5.5), we have thus rewritten ψ_n as the partition function of a polymer model,

$$\psi_n(t) = (\varphi(2t))^{|\partial\Lambda_n|} \sum_{\{P_k\}} \prod_k w_{P_k}(t), \tag{5.9}$$

in which *polymers* are any (not necessarily connected) sets of plaquettes on ∂V_n, the polymer weights are

$$w_P(t) = \sum_{\mathfrak{C}_{\mathrm{con}}:\,\mathcal{P}(\mathfrak{C}_{\mathrm{con}})=P} \varrho_{\mathfrak{C}_{\mathrm{con}}}(t), \tag{5.10}$$

and *incompatibility* of two distinct polymers means sharing of at least one of their plaquettes. Denoting by $\mathfrak{Y}_n$ the set of all clusters of polymers in V_n, the characteristic function is given by the cluster expansion

$$\log \psi_n(t) = |\partial\Lambda_n| \log \varphi(2t) + \sum_{\mathcal{Y}\in\mathfrak{Y}_n} w^T_{\mathcal{Y}}(t) \ . \tag{5.11}$$

Lemma 5.1. *There exist constants $\beta_0 = \beta_0(d,\lambda^*) < \infty$ and $\epsilon > 0$ such that $\beta \geq \beta_0$ implies the inequality*

$$\sum_{\mathcal{Y}\in\mathfrak{Y}_n} |w^T_{\mathcal{Y}}(t)| \leq \frac{1}{2}\,\sigma^2 t^2\, |\partial\Lambda_n| \tag{5.12}$$

for all n and $|t| \leq \epsilon$. Here $\sigma^2 = \mathsf{E}\lambda_0^2$ is the variance of the distribution of the boundary fields.

Proof. Let $\beta \geq \beta_0$ and $|t| \leq \epsilon$, both the constants being specified in the course of the proof. For any set P of plaquettes in V_n, let

$$|P|_{\mathrm{con}} = \inf_{\substack{P'\supset P\\ \text{connected}}} |P'| \ , \tag{5.13}$$

where the infimum is taken over all connected sets of plaquettes containing P. It suffices to prove that

$$\sum_{P\ni p} e^{\frac{\sigma^2 t^2}{4d}|P|_{\mathrm{con}}} |w_P(t)| \leq \frac{\sigma^2 t^2}{4d} \tag{5.14}$$

holds for any plaquette $p \in \partial V_n$. Indeed, the last inequality implies that the condition (A.5) is satisfied with $a(P) = \frac{\sigma^2 t^2}{4d}|P|_{\mathrm{con}}$ and $b(P) = 0$. Considering the polymer ∂V_n and realizing that $|\partial V_n|_{\mathrm{con}} \leq 2d|\partial\Lambda_n|$, Proposition A.1 immediately yields (5.12).

So, let us prove (5.14). Recalling that $\Delta\Phi^\lambda_C$ is an odd function of λ, we may use the symmetry of the distribution P to cast (5.6) into a more suitable form, namely,

$$\varrho_{\mathfrak{C}}(t) = (\varphi(2t))^{-|\partial_n(\mathfrak{C})|}\, \mathsf{E} \left\{ \begin{array}{c} \sin \\ \cos \end{array} \right\} \Big[t\Big(2S^\lambda_{\partial_n(\mathfrak{C})} + \frac{1}{2}\sum_{C\in\mathfrak{C}} \Delta\Phi^\lambda_C \Big)\Big] \times$$
$$\times \prod_{C\in\mathfrak{C}} 2\sin\Big(\frac{t\Delta\Phi^\lambda_C}{2}\Big) \ . \tag{5.15}$$

Here sin is to be taken iff the cardinality of the set of clusters card$(\mathfrak{C})$ is odd and cos whenever the cardinality is even; to distinguish both cases, we will use the notation $r_{\mathfrak{C}} = 1$ and $r_{\mathfrak{C}} = 0$, respectively. Taking now ϵ such that $\epsilon\lambda^* \leq \frac{1}{2}$, one has $\varphi(2t) = \mathsf{E}\,\cos(2t\lambda_0) \geq \frac{1}{2}$ whenever

$|t| \leq \epsilon$, and we can estimate

$$|\varrho_{\mathfrak{C}}(t)| \leq 2^{|\partial_n(\mathfrak{C})|} t^{\mathrm{card}(\mathfrak{C}) + r_{\mathfrak{C}}} \, \mathsf{E} \left(2|S_{\partial_n(\mathfrak{C})}| + \frac{1}{2} \sum_{C \in \mathfrak{C}} |\Delta\Phi_C^\lambda| \right)^{r_{\mathfrak{C}}} \prod_{C \in \mathfrak{C}} |\Delta\Phi_C^\lambda| \, . \tag{5.16}$$

Since $|\lambda_x| \leq \lambda^*$ and $\sinh x \leq e^x \min\{1, x\}$ for any $x \geq 0$, with the help of Lemma 4.1 we have the inequality

$$|\Delta\Phi_C^\lambda| \leq z_C^\lambda \min\{1, \, 4 \sum_{y \in \partial_n(C)} |\lambda_y|\} \, , \tag{5.17}$$

where we introduced the shorthand

$$z_C^\lambda = \frac{|\Delta\Phi_C^\lambda|}{\sinh(4 \sum_{x \in \partial_n(C)} |\lambda_x|)} \, e^{4\theta\lambda^*|C|} \, . \tag{5.18}$$

Lemma 4.5 then gives the estimate

$$\begin{aligned} \frac{1}{2} \sum_{C \in \mathfrak{C}} |\Delta\Phi_C^\lambda| &\leq 2 \sum_{y \in \partial_n(\mathfrak{C})} |\lambda_y| \sum_{\substack{C \in \mathfrak{C} \\ y \in \partial_n(C)}} z_C^\lambda \leq \\ &\leq 2 \sum_{y \in \partial_n(\mathfrak{C})} |\lambda_y| \sum_{C \colon y \in \Lambda(C)} z_C^\lambda \leq 2 \sum_{x \in \partial_n(\mathfrak{C})} |\lambda_x| \end{aligned} \tag{5.19}$$

provided $\beta_0 \geq 2\theta\lambda^* + c_3$, where c_3 is the constant from Lemma 4.5. Using the bounds (5.17) and (5.19), we obtain

$$\begin{aligned} |\varrho_{\mathfrak{C}}(t)| &\leq 2^{|\partial_n(\mathfrak{C})|+1} \, |t|^{\mathrm{card}(\mathfrak{C}) + r_{\mathfrak{C}}} \, \mathsf{E} \left(4 \sum_{x \in \partial_n(\mathfrak{C})} |\lambda_x| \right) \times \\ &\quad \times \prod_{C \in \mathfrak{C}} z_C^\lambda \min\{1, 4 \sum_{x \in \partial_n(C)} |\lambda_x|\} \leq \\ &\leq 2^{|\partial_n(\mathfrak{C})|+1} \, t^2 \, \mathsf{E} \left(4 \sum_{x \in \partial_n(\mathfrak{C})} |\lambda_x| \right)^2 \prod_{C \in \mathfrak{C}} z_C^\lambda \end{aligned} \tag{5.20}$$

once $\epsilon \leq 1$. Since $\mathsf{E}\,|\lambda_x| \leq (\mathsf{E}\,\lambda_x^2)^{1/2}$ by the Cauchy-Schwartz inequality, we have

$$\mathsf{E} \left(\sum_{x \in \partial_n(\mathfrak{C})} |\lambda_x| \right)^2 = \sum_{x \in \partial_n(\mathfrak{C})} \mathsf{E}\,\lambda_x^2 + \sum_{\substack{x, y \in \partial_n(\mathfrak{C}) \\ x \neq y}} \mathsf{E}\,|\lambda_x| \, \mathsf{E}|\lambda_y| \leq \sigma^2 |\partial_n(\mathfrak{C})|^2 \, , \tag{5.21}$$

yielding the bound

$$|\varrho_{\mathfrak{C}}(t)| \leq 32\sigma^2 t^2 \, 2^{|\partial_n(\mathfrak{C})|} |\partial_n(\mathfrak{C})|^2 \prod_{C \in \mathfrak{C}} z_C^\lambda \leq 32\sigma^2 t^2 \prod_{C \in \mathfrak{C}} (2e^2)^{\theta|C|} z_C^\lambda \, . \tag{5.22}$$

Observing that, in view of Lemma 4.1,

$$|P|_{\text{con}} \le |P \cup (\cup_{C\in\mathfrak{C}_{\text{con}}} C)| \le (d+1) \sum_{C\in\mathfrak{C}_{\text{con}}} |C| \tag{5.23}$$

for any $\mathfrak{C}_{\text{con}}$ with $\mathcal{P}(\mathfrak{C}_{\text{con}}) = P$, we thus get

$$\begin{aligned} |w_P(t)| &\le 32\sigma^2 t^2 e^{-2\frac{\beta-\vartheta}{d+1}|P|_{\text{con}}} \sum_{\substack{\mathfrak{C}_{\text{con}}:\\ \mathcal{P}(\mathfrak{C}_{\text{con}})=P}} \prod_{C\in\mathfrak{C}} \tilde{z}_C^\lambda \le \\ &\le 32\sigma^2 t^2 e^{-2\frac{\beta-\vartheta}{d+1}|P|_{\text{con}}} \sum_{\substack{\mathfrak{C}:\, \mathcal{P}(C)\cap P\neq\emptyset\\ \text{for all } C\in\mathfrak{C}}} \prod_{C\in\mathfrak{C}} \tilde{z}_C^\lambda \end{aligned} \tag{5.24}$$

for any $\vartheta > 0$, where $\tilde{z}_C^\lambda = (2e^2)^{\theta|C|} e^{2(\beta-\vartheta)|C|} z_C^\lambda$. With the help of Lemma 4.5, the last sum may be further estimated by

$$\prod_{C:\,\mathcal{P}(C)\cap P\neq\emptyset} (1+\tilde{z}_C^\lambda) \le e^{\sum_{C:\,\mathcal{P}(C)\cap P\neq\emptyset} \tilde{z}_C^\lambda} \le e^{|P| \inf_x \sum_{C:\, x\in\Lambda(C)} \tilde{z}_C^\lambda} \le e^{|P|} \tag{5.25}$$

whenever $\vartheta \ge 2\theta\lambda^* + c_3 + d(1+\frac{1}{2}\ln 2)$. Realizing that $\sigma^2 t^2 \le (\lambda^*\epsilon)^2 \le 1$ and that the number of connected polymers containing a given plaquette and having the size $|P| = l$ may be estimated by κ^l with a constant $\kappa > 0$, we finally get

$$\begin{aligned} \sum_{P\ni p} e^{\frac{\sigma^2 t^2}{4d}|P|_{\text{con}}} |w_P(t)| &\le 32\sigma^2 t^2 \sum_{P\ni p} \sum_{\substack{P'\supset P\\ \text{connected}}} e^{-2\left(\frac{\beta-\vartheta}{d+1}-1\right)|P'|} \le \\ &\le 32\sigma^2 t^2 \sum_{\substack{P'\ni p\\ \text{connected}}} \left[2e^{-2\left(\frac{\beta-\vartheta}{d+1}-1\right)}\right]^{|P'|} \le \\ &\le 32\sigma^2 t^2 \sum_{l=1}^{\infty} \left[2\kappa e^{-2\left(\frac{\beta-\vartheta}{d+1}-1\right)}\right]^{l} \le \frac{\sigma^2 t^2}{4d}\ , \end{aligned} \tag{5.26}$$

providing that $\beta_0 \ge \vartheta + (d+1)[1 + 4\ln 2 + \frac{1}{2}\log(2d\kappa)]$, say. ☐

Corollary 5.2. *Let $\sigma^2 > 0$ and $\beta \ge \beta_0$ with β_0 being the constant from Lemma 5.1. Then there exists $t_0 > 0$ such that*

$$|\psi_n(t)| \le \exp\Bigl(-\frac{1}{2}\sigma^2 t^2 |\partial\Lambda_n|\Bigr) \tag{5.27}$$

holds true for any n and $|t| \le t_0$.

Proof. Since $\log\varphi(t) = -\frac{1}{2}\sigma^2 t^2 + o(t^2)$ and since $\sigma^2 > 0$, there exists $t_1 > 0$ such that $|\log\varphi(t) + \frac{1}{2}\sigma^2 t^2| \le \frac{1}{4}\sigma^2 t^2$ whenever $|t| \le t_1$, yielding

$$|\varphi(t)| \le e^{-\frac{1}{2}\sigma^2 t^2} e^{|\log\varphi(t) + \frac{1}{2}\sigma^2 t^2|} \le e^{-\frac{1}{4}\sigma^2 t^2}\ . \tag{5.28}$$

Using the cluster expansion (5.11) and Lemma 5.1, we immediately get the above statement with $t_0 = \min\{t_1, \epsilon\}$. □

Having established a control over the behaviour of the characteristic function in a neighbourhood of the origin, we prove the following weak variant of the local-limit theorem.

Lemma 5.3. *Let the assumptions of Corollary 5.2 be satisfied. Then, for any finite interval $\mathcal{I} \subset \mathbb{R}$ whose end-points are a and b and any $\zeta > 0$, we have*

$$\limsup_{n\to\infty} n^{\frac{d-1}{2}-\zeta}\, \mathsf{P}\left(F_n^\lambda \in n^\zeta \mathcal{I}\right) < \infty \,. \tag{5.29}$$

Here $n^\zeta \mathcal{I}$ is the interval with the end-points an^ζ and bn^ζ.

Proof. The idea of the proof is to "blur" the probabilistic distribution of the random variable F_n^λ by convoluting it with a smooth function without changing the inequality (5.29). This trick will enable us to obtain a sufficient control over the asymptotic behaviour of the characteristic function outside the regime where the cluster expansions hold.

Let $g \in C^\infty$ be a positive function with a compact support in $[-1, 1]$ and satisfying the normalization condition $\int_{\mathbb{R}} g(x)\, dx = 1$. Further, we use F_n to denote the distribution function of F_n^λ and define

$$\tilde{\mathsf{F}}_n(z) = \int_{-\infty}^{z} dx \int_{\mathbb{R}} g_n(x-y)\, d\mathsf{F}_n(y) \,, \tag{5.30}$$

where

$$g_n(x) = n^{-\zeta} g(xn^{-\zeta}) \tag{5.31}$$

with $\zeta > 0$. The function $\tilde{\mathsf{F}}_n$ is clearly a distribution function due to the properties of g. Given an interval $\mathcal{I} \subset \mathbb{R}$ with the end-points $a \le b$, the lemma will be proved once we show that

$$\limsup_{n\to\infty} n^{\frac{d-1}{2}-\zeta} \int_{n^\zeta \tilde{\mathcal{I}}} d\tilde{\mathsf{F}}_n < \infty, \tag{5.32}$$

where $\tilde{\mathcal{I}} = [a-1, b+1]$. Indeed, since

$$\begin{aligned}\int_{n^\zeta\tilde{\mathcal{I}}} d\tilde{\mathsf{F}}_n &= \int_{n^\zeta\tilde{\mathcal{I}}} dx \int_{\mathbb{R}} g_n(x-y)\, d\mathsf{F}_n(y) \ge \\ &\ge \int_{n^\zeta\tilde{\mathcal{I}}} dx \int_{n^\zeta\mathcal{I}} g_n(x-y)\, d\mathsf{F}_n(y) = \\ &= \int_{n^\zeta\mathcal{I}} d\mathsf{F}_n(y) \int_{\mathbb{R}} g_n(x-y)\, dx = \int_{n^\zeta\mathcal{I}} d\mathsf{F}_n\end{aligned} \tag{5.33}$$

by Fubini's theorem and the normalization condition $\int_{\mathbb{R}} g_n(x)dx = 1$, this estimate combined with (5.33) immediately yields the lemma.

Turning now to the proof of (5.32), we first introduce the functions

$$\hat{g}(t) = \int_{\mathbb{R}} e^{ixt} g(x)\, dx \tag{5.34}$$

and $\hat{g}_n(t) = \hat{g}(tn^{\zeta})$. Moreover, since $g \in C^{\infty}$ and has a compact support, for all $k = 0, 1, 2, \ldots$ and $t \in \mathbb{R} \setminus \{0\}$ the bounds $|\hat{g}(t)| \le c_k |t|^{-k}$ are true, where $c_0 = 1$ and $c_k < \infty$ for $k \ge 1$, implying

$$|\hat{g}_n(t)| \le c_k n^{-k\zeta} |t|^{-k} \tag{5.35}$$

for all $n \in \mathbb{N}$. It also immediately follows that the modified characteristic function

$$\tilde{\psi}_n(t) = \int_{\mathbb{R}} e^{ixt}\, d\tilde{\mathsf{F}}_n(x) = \psi_n(t)\hat{g}_n(t) \tag{5.36}$$

satisfies the condition $\int_{\mathbb{R}} |\tilde{\psi}_n(t)|\, dx < \infty$ and, therefore, $\tilde{\mathsf{F}}_n$ is given by the inversion formula

$$\mathsf{F}_n(z) = \int_{-\infty}^{z} \frac{dx}{2\pi} \int_{\mathbb{R}} e^{-itx} \tilde{\psi}_n(t) dt\ . \tag{5.37}$$

Using (5.36) and the fact that $|\psi_n(t)| \le 1$, we obtain the estimate

$$\begin{aligned} n^{\frac{d-1}{2}-\zeta} \int_{\tilde{\mathcal{I}}n^{\zeta}} d\tilde{\mathsf{F}}_n &= n^{\frac{d-1}{2}-\zeta} \int_{\tilde{\mathcal{I}}n^{\zeta}} \frac{dx}{2\pi} \int_{\mathbb{R}} e^{-itx}\, \tilde{\psi}_n(t)\, dt \le \\ &\le n^{\frac{d-1}{2}} \int_{\tilde{\mathcal{I}}} \frac{dx}{2\pi} \int_{\mathbb{R}} |\tilde{\psi}_n(t)|\, dt \le (I_n^1 + I_n^2) \int_{\tilde{\mathcal{I}}} \frac{dx}{2\pi} \end{aligned} \tag{5.38}$$

for all $n \in \mathbb{N}$. Here

$$I_n^1 = n^{\frac{d-1}{2}} \int_{|t| \le t_0} |\psi_n(t)|\, dt \quad \text{and} \quad I_n^2 = n^{\frac{d-1}{2}} \int_{|t| > t_0} |\hat{g}_n(t)|\, dt \tag{5.39}$$

with t_0 being the constant from Corollary 5.2. The interval $\tilde{\mathcal{I}}$ being finite, it now suffices to show that the integrals I_n^1 and I_n^2 are uniformly bounded if $n \to \infty$. First, in view of (5.35), one can conclude that

$$I_n^2 \le c_k n^{\frac{d-1}{2}-k\zeta} \int_{|t| > t_0} |t|^{-k}\, dt. \tag{5.40}$$

Hence, choosing an integer $k > \max\{1, \frac{d-1}{2\zeta}\}$, we get $\lim_{n\to\infty} I_n^2 = 0$. In order to estimate the integral I_n^1, we make use of Corollary 5.2 to obtain

$$\limsup_{n\to\infty} I_n^1 \le \limsup_{n\to\infty} n^{\frac{d-1}{2}} \int_{\mathbb{R}} \exp\Bigl(-\frac{1}{2}\,\sigma^2 t^2 |\partial\Lambda_n|\Bigr)\, dt = \frac{1}{\sigma}\left(\frac{\pi}{d}\right)^{1/2}, \tag{5.41}$$

which finishes the proof. □

The proof of Theorem 2.1 will be finished once we prove the following lemma, yielding the structure of limit points of the sequence F_n^λ. For convenience, we use $\mathfrak{L}^\lambda$ to denote the (random) set of all limit points of the sequence $\{F_n^\lambda\}$ and $\mathfrak{L}^\lambda_{d,\omega}$ for the set of limit points of the "sparse" sequence $\{F^\lambda_{n^{[4-d+\omega]}}\}$.

Proposition 5.4. *Let $\sigma^2 > 0$ and $\beta \geq \beta_0$ with β_0 from Lemma 5.1.*

1. *If $d > 3$, then $\mathfrak{L}^\lambda = \{\infty, -\infty\}$ P-a.s.*
2. *If $d \in \{2,3\}$ and $\omega > 0$, then $\mathfrak{L}^\lambda_{d,\omega} = \{\infty, -\infty\}$ P-a.s.*

Proof. (1) Let $d > 3$. First, we shall show that $\mathfrak{L}^\lambda \cap \mathbb{R} = \emptyset$ a.s. Defining the events

$$\mathcal{E}^\zeta_{n,k} = \{\lambda :\, -kn^\zeta < F_n^\lambda < kn^\zeta\} \tag{5.42}$$

for all $k \in \mathbb{N}$ and $\zeta \geq 0$, Lemma 5.3 implies that there are constants $c_k(\zeta), n_k(\zeta) < \infty$ such that

$$\mathsf{P}(\mathcal{E}^\zeta_{n,k}) \leq c_k(\zeta)\, n^{-\frac{d-1}{2}+\zeta} \tag{5.43}$$

for any $\zeta > 0$ whenever $n \geq n_k(\zeta)$. Choosing $0 < \zeta < \frac{d-3}{2}$, this yields

$$\sum_n \mathsf{P}(\mathcal{E}^0_{n,k}) \leq \sum_n \mathsf{P}(\mathcal{E}^\zeta_{n,k}) < \infty\ . \tag{5.44}$$

Using the Borel-Cantelli lemma, it follows that $\mathsf{P}(\limsup_n \mathcal{E}^0_{n,k}) = 0$, where $\limsup_n \mathcal{E}^0_{n,k} = \cap_n \cup_{m=n}^\infty \mathcal{E}^0_{m,k}$ is the event that infinitely many events $\mathcal{E}^0_{n,k}$ occur. As a consequence, we have $\mathsf{P}(\mathfrak{L}^\lambda \cap (-k,k) \neq \emptyset) = 0$. Hence,

$$\mathsf{P}(\mathfrak{L}^\lambda \cap \mathbb{R} \neq \emptyset) \leq \sum_k \mathsf{P}(\mathfrak{L}^\lambda \cap (-k,k) \neq \emptyset) = 0\ . \tag{5.45}$$

Further, since the events $\mathcal{A}_n^+ = \{\lambda : F_n^\lambda \geq 0\}$ and $\mathcal{A}_n^- = \{\lambda : F_n^\lambda \leq 0\}$ satisfy $\mathsf{P}(\mathcal{A}_n^\pm) = \frac{1}{2}$ due to the symmetry of the distribution P and since $\{\mathcal{A}^\pm_{2n}\}$ are the subsequences of independent events, one gets $\mathsf{P}(\limsup_n \mathcal{A}_n^\pm) = 1$ by the (second) Borel-Cantelli lemma. Combined with the compactness of the intervals $[0,\infty]$ and $[-\infty,0]$, we obtain that $\mathsf{P}(\mathfrak{L}^\lambda \cap [0,\infty] \neq \emptyset) = 1$ as well as $\mathsf{P}(\mathfrak{L}^\lambda \cap [-\infty,0] \neq \emptyset) = 1$.

(2) Let $d \in \{2,3\}$. Recalling the definition (5.42) of the events $\mathcal{E}^\zeta_{n,k}$, this time one arrives at the inequality

$$\sum_n \mathsf{P}(\mathcal{E}^0_{[n^{4-d+\omega}],\,k}) < \infty \tag{5.46}$$

whenever choosing $0 < \zeta < \frac{d-1}{2} - \frac{1}{4-d+\omega}$. The rest of the proof runs along the same lines as for $d > 3$. □

6. Concluding Remarks

In the introduction, we have concluded that the set of limit points of $\{\mu_n^\lambda\}$ for $d = 2, 3$ and $\beta = \infty$ is countable a.s., containing all the convex combinations $\frac{1+\alpha}{2}\,\delta_1 + \frac{1-\alpha}{2}\,\delta_{-1}$ whenever $\alpha = \tanh\alpha'$ for some $\alpha' \in 2\mathbb{Z}\cup\{-\infty,\infty\}$. Nevertheless, once $\beta_0 \le \beta < \infty$, our conjecture is that any convex combination of μ^+ and μ^- is a limit point of $\{\mu_n^\lambda\}$ a.s. In order to verify this, one would need to show that

$$\liminf_{n\to\infty}\, n^{\frac{d-1}{2}+\zeta}\,\mathsf{P}(F_n^\lambda \in (a,b)) > 0$$

for a sufficiently small $\zeta \ge 0$ whenever $a < b$. In the context of our perturbation scheme, this would require a variant of Lemma 4.5, yielding a lower bound on the cluster-weight differences $\Delta\Phi_C^\lambda$. The problem of lower bounds on cluster weights is highly non-trivial, however.

Presumably, Theorem 2.1 remains valid for all symmetric distributions with zero mean and a positive variance, provided β is large enough (depending only on d). It is the uniformity of cluster expansions in realizations of the boundary fields λ why we restrict ourselves to λ of small strength in the paper. As a result, one has an extra attraction of contours to the boundary (suppressing interfaces even for the Dobrushin boundary field realizations) which always leads to a convex combination of the two translation-invariant extremal states. In order to prove the theorem for all distributions of λ, one should perhaps refine the strategy by replacing the uniformity with the "typicality" and by using a coarse-graining argument to show that large-scale contours are typically suppressed or attracted to the boundary even without the above-mentioned extra attraction. We defer the details to a later investigation.

It is interesting to consider the situation of asymmetric distributions of λ (keeping the mean zero). Clearly, whenever $\liminf_n \frac{1}{|\partial\Lambda_n|}\,\mathsf{E}\,F_n^\lambda > 0$ or $\limsup_n \frac{1}{|\partial\Lambda_n|}\,\mathsf{E}\,F_n^\lambda < 0$, then $\mu_n \to \mu^+$ a.s. or $\mu_n \to \mu^-$ a.s., respectively. By adding a boundary "magnetic-field" term to the Hamiltonian, one can find a transition between these two regimes. Heuristically, taking into account only the clusters surrounding a single site, the leading asymptotics of the transition point is $h \approx e^{-2(2d-1)\beta}\,\mathsf{E}\sinh\lambda_0$. Yet, this time one cannot conclude whether the chaotic-size dependence actually occurs at the transition point because Lemma 5.4, where the symmetry of the distribution plays a crucial role, is not valid any more.

Appendix A. Cluster expansions

In this appendix we summarize statements on the convergence of cluster expansions for the abstract polymer model in the context of the

Kotecký-Preiss formalism [12]. Besides the standard result concerning the exponential decay of cluster weights, we consider a model with parameter-dependent weights and prove estimates on their derivatives. The proof proposed here is entirely in the spirit of [12].

Let $\mathcal{K}$ be a countable set and let us call its elements *polymers.* Given a reflexive and symmetric relation $\iota \subset \mathcal{K} \times \mathcal{K}$, two polymers $\Gamma_1, \Gamma_2 \in \mathcal{K}$ are *incompatible* if $\Gamma_1 \iota \Gamma_2$; otherwise they are *compatible.* A finite set $\Delta \subset \mathcal{K}$ is called compatible whenever all polymers from Δ are pairwise compatible. If Δ is not a union of two disjoint, non-empty sets Δ_1 and Δ_2 such that Γ_1 is compatible with Γ_2 for any $\Gamma_1 \in \Delta_1$ and $\Gamma_2 \in \Delta_2$, then Δ is a *cluster.* We write $\Delta \iota \Gamma$ whenever there exists $\Gamma' \in \Delta$ such that $\Gamma' \iota \Gamma$.

Let us consider a function $w : \mathcal{K} \to \mathbb{C}$ called *weight.* For any finite set $\Delta \subset \mathcal{K}$, we let

$$w(\Delta) = \begin{cases} \prod_{\Gamma \in \Delta} w(\Gamma) & \text{if } \Delta \text{ is compatible,} \\ 0 & \text{otherwise.} \end{cases} \tag{A.1}$$

The *partition function* in a finite set $\Lambda \subset \mathcal{K}$ is defined as

$$\mathcal{Z}(\Lambda) = \sum_{\Delta \subset \Lambda} w(\Delta) \ . \tag{A.2}$$

Its logarithm may be formally written in the form

$$\log \mathcal{Z}(\Lambda) = \sum_{\Delta \subset \Lambda} w^T(\Delta) \ , \tag{A.3}$$

where the weights w^T are unique and given by the Möbius inversion formula

$$w^T(\Delta) = \sum_{\Lambda \subset \Delta} (-1)^{|\Delta \setminus \Lambda|} \log \mathcal{Z}(\Lambda) \tag{A.4}$$

for any finite $\Delta \subset \mathcal{K}$. As a consequence, one has $w^T(\Delta) = 0$ whenever Δ is not a cluster.

Proposition A.1. *Given functions $a, b : \mathcal{K} \to [0, \infty)$, let the condition*

$$\sum_{\Gamma \iota \Gamma_0} e^{(a+b)(\Gamma)} |w(\Gamma)| \leq a(\Gamma_0) \tag{A.5}$$

be satisfied for every $\Gamma_0 \in \mathcal{K}$. Then

$$\sum_{\Delta \iota \Gamma_0} e^{b(\Delta)} |w^T(\Delta)| \leq a(\Gamma_0) \tag{A.6}$$

with $b(\Delta) = \sum_{\Gamma \in \Delta} b(\Gamma)$. Moreover, let the weights w be differentiable functions in an open interval $\mathcal{I} \subset \mathbb{R}$. If the condition (A.5) *is true* *uniformly in $\mathcal{I}$ with $a \leq b$ and if $c : \mathcal{K} \to [0, \infty]$ is a function such that*[6]

$$\sum_{\Gamma \iota \Gamma_0} c(\Gamma)\, e^{(a+b)(\Gamma)} \Big| \frac{dw(\Gamma)}{d\eta} \Big| \leq a(\Gamma_0) \ , \tag{A.7}$$

then

$$\sum_{\Delta \iota \Gamma_0} c(\Delta)\, e^{(b-a)(\Delta)} \Big| \frac{dw^T(\Delta;\eta)}{d\eta} \Big| \leq 2a(\Gamma_0) \tag{A.8}$$

for any $\Gamma_0 \in \mathcal{K}$ and $\eta \in \mathcal{I}$. Here, $c(\Delta) = \min_{\Gamma \in \Delta} c(\Gamma)$.

Proof. The bound (A.6) is proved in [12]. The proof of (A.8) goes along the same lines as follows.

Let $\Gamma_0 \in \mathcal{K}$ be fixed. Let $w_s(\Gamma)$ be the s-dependent weight defined by $sw(\Gamma)$ if $\Gamma \iota \Gamma_0$ and by $w(\Gamma)$ otherwise. Using the Möbius formula (A.4), we obtain

$$\begin{aligned} \frac{dw_s^T(\Delta;\eta)}{d\eta} &= \sum_{\Lambda \subset \Delta} (-1)^{|\Delta \setminus \Lambda|} \sum_{\Gamma \in \Lambda} \frac{d \log \mathcal{Z}_s(\Lambda)}{dw_s(\Gamma)} \, \frac{dw_s(\Gamma;\eta)}{d\eta} = \\ &= \sum_{\Gamma \in \Delta} \frac{dw_s(\Gamma;\eta)}{d\eta} \sum_{\Lambda :\, \Gamma \in \Lambda \subset \Delta} (-1)^{|\Delta \setminus \Lambda|} \frac{\mathcal{Z}_s(\Lambda \setminus [\Gamma])}{\mathcal{Z}_s(\Lambda)} . \end{aligned} \tag{A.9}$$

Here $[\Gamma]$ is the set of all polymers incompatible with Γ and, due to the formula (A.3), we have the cluster expansion

$$\frac{\mathcal{Z}_s(\Lambda \setminus [\Gamma])}{\mathcal{Z}_s(\Lambda)} = \exp\Big[- \sum_{\substack{\Delta' \subset \Lambda \\ \Delta' \iota \Gamma}} w_s^T(\Delta') \Big] . \tag{A.10}$$

Following the strategy of [12], we write

$$\frac{d}{ds} \sum_{\Delta \iota \Gamma_0} c(\Delta)\, e^{(b-a)(\Delta)} \Big| \frac{dw_s^T(\Delta)}{d\eta} \Big| = X_1 + X_2 \ , \tag{A.11}$$

[6] We allow $c(\Gamma)$ to be ∞ and use the convention $0 \cdot \infty = 0$ in order to cover the contour models introduced in Section 3.

where, in view of (A.10) and (A.11),

$$X_1 = \sum_{\Delta \iota \Gamma_0} \varepsilon(\Delta) c(\Delta)\, e^{(b-a)(\Delta)} \sum_{\substack{\Gamma \in \Delta \\ \Gamma \iota \Gamma_0}} \frac{dw(\Gamma)}{d\eta} \times$$

$$\times \sum_{\Lambda\colon \Gamma \in \Lambda \subset \Delta} (-1)^{|\Delta \setminus \Lambda|} \exp\Big[- \sum_{\substack{\Delta' \subset \Lambda \\ \Delta' \iota \Gamma}} w_s^T(\Delta')\Big] =$$

$$= \sum_{\Gamma \iota \Gamma_0} \frac{dw(\Gamma)}{d\eta} \sum_{\Delta \ni \Gamma} \varepsilon(\Delta) c(\Delta)\, e^{(b-a)(\Delta)} \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\substack{\Delta_1,\dots,\Delta_n \subset \Delta \\ \Delta_k \iota \Gamma,\, k=1,\dots,n}} \prod_{k=1}^{n} \big[-w_s^T(\Delta_k)\big] \times$$

$$\times \sum_{\Lambda\colon \{\Gamma\} \cup \bigcup_k \Delta_k \subset \Lambda \subset \Delta} (-1)^{|\Delta \setminus \Lambda|}\,, \quad \text{(A.12)}$$

and

$$X_2 = - \sum_{\Delta \iota \Gamma_0} \varepsilon(\Delta) c(\Delta)\, e^{(b-a)(\Delta)} \sum_{\Gamma \in \Delta} \frac{dw_s(\Gamma)}{d\eta} \times$$

$$\times \sum_{\Lambda\colon \Gamma \in \Lambda \subset \Delta} (-1)^{|\Delta \setminus \Lambda|} \sum_{\substack{\Delta' \subset \Lambda \\ \Delta' \iota \Gamma, \Gamma_0}} \frac{dw_s^T(\Delta')}{ds} \exp\Big[- \sum_{\substack{\Delta'' \subset \Lambda \\ \Delta'' \iota \Gamma}} w_s^T(\Delta'')\Big] =$$

$$= \sum_{\Delta' \iota \Gamma_0} \frac{dw_s^T(\Delta')}{ds} \sum_{\Gamma \iota \Delta'} \frac{dw_s(\Gamma)}{d\eta} \sum_{\substack{\Delta \ni \Gamma \\ \Delta \supset \Delta'}} \varepsilon(\Delta) c(\Delta)\, e^{(b-a)(\Delta)} \times$$

$$\times \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\substack{\Delta_1,\dots,\Delta_n \subset \Delta \\ \Delta_k \iota \Gamma,\, k=1,\dots,n}} \prod_{k=1}^{n} \big[-w_s^T(\Delta_k)\big] \sum_{\Lambda\colon \Delta' \cup \{\Gamma\} \cup \bigcup_k \Delta_k \subset \Lambda \subset \Delta} (-1)^{|\Delta \setminus \Lambda|}\,. \quad \text{(A.13)}$$

Here $\varepsilon(\Delta)$ is such that $\left|\frac{dw^T(\Delta)}{d\eta}\right| = \varepsilon(\Delta)\, \frac{dw^T(\Delta)}{d\eta}$. Since $\sum_{B \subset A} (-1)^{|A \setminus B|}$ equals 1 if $A = \emptyset$ and 0 otherwise, one can use the obvious estimates $b(\cup_k A_k) \le \sum_k b(A_k)$ and $c(\Delta) \le c(\Gamma)$ for any $\Gamma \in \Delta$ to get

$$|X_1| \le \sum_{\Gamma \iota \Gamma_0} c(\Gamma)\, e^{(b-a)(\Gamma)} \left|\frac{dw(\Gamma)}{d\eta}\right| \exp\Big[\sum_{\Delta' \iota \Gamma} e^{(b-a)(\Delta')} |w_s^T(\Delta')|\Big] \le$$

$$\le \sum_{\Gamma \iota \Gamma_0} c(\Gamma)\, e^{b(\Gamma)} \left|\frac{dw(\Gamma)}{d\eta}\right| \le a(\Gamma_0) \quad \text{(A.14)}$$

for any $s \in (0,1)$ due to (A.6) and (A.7). Similarly,

$$|X_2| \leq \sum_{\Delta' \iota \Gamma_0} e^{(b-a)(\Delta')} \Big| \frac{dw_s^T(\Delta')}{ds} \Big| \sum_{\Gamma \iota \Delta'} \Big| \frac{dw_s(\Gamma)}{d\eta} \Big| \times$$

$$\times \exp\Big[\sum_{\Delta' \iota \Gamma} c(\Gamma)\, e^{(b-a)(\Delta')} |w_s^T(\Delta')| \Big] \leq$$

$$\leq \sum_{\Delta' \iota \Gamma_0} e^{(b-a)(\Delta')} \Big| \frac{dw_s^T(\Delta')}{ds} \Big| \sum_{\Gamma' \in \Delta'} \sum_{\Gamma \iota \Gamma'} c(\Gamma)\, e^{a(\Gamma)} \Big| \frac{dw_s(\Gamma)}{d\eta} \Big| \leq$$

$$\leq \sum_{\Delta \iota \Gamma_0} a(\Delta) e^{(b-a)(\Delta)} \Big| \frac{dw_s^T(\Delta)}{ds} \Big|. \tag{A.15}$$

Using the inequality

$$\frac{d}{ds} \sum_{\Delta \iota \Gamma_0} e^{b(\Delta)} |w_s^T(\Delta)| \leq a(\Gamma_0) \tag{A.16}$$

for any $s \in (0,1)$ proved in [12], we may conclude that $|X_2| \leq a(\Gamma_0)$. To finish the proof, it now suffices to realize that

$$\sum_{\Delta \iota \Gamma_0} e^{(b-a)(\Delta)} \Big| \frac{dw^T(\Delta)}{d\eta} \Big| \leq \sup_{s \in (0,1)} \frac{d}{ds} \sum_{\Delta \iota \Gamma_0} e^{(b-a)(\Delta)} \Big| \frac{dw_s^T(\Delta)}{d\eta} \Big| \, . \tag{A.17}$$

□

As an application of the above proposition, let us consider a couple of weight functions $w_{1,2} : \mathcal{K} \to \mathbb{C}$. Then the above proposition implies an estimate on the difference of the corresponding cluster weights w_1^T and w_2^T.

Corollary A.2. *Let $a, b : \mathcal{K} \to [0, \infty)$, $a \leq b$, and let the condition* (A.5) *be satisfied for both polymer weights $w_{1,2}$. If there is a function $c : \mathcal{K} \to [0, \infty]$ such that*

$$\sum_{\Gamma \iota \Gamma_0} c(\Gamma)\, e^{(a+b)(\Gamma)} |(w_2 - w_1)(\Gamma)| \leq a(\Gamma_0) \tag{A.18}$$

is true for all $\Gamma_0 \in \mathcal{K}$, then

$$\sum_{\Delta \iota \Gamma_0} c(\Delta)\, e^{(b-a)(\Delta)} |(w_2^T - w_1^T)(\Delta)| \leq 2a(\Gamma_0) \, . \tag{A.19}$$

Proof. The parameter-dependent weight $w(\eta) = \eta w_2 + (1-\eta) w_1$ satisfies the conditions (A.6) and (A.7) uniformly in the interval $[0,1]$.

Therefore, using the inequality

$$\sum_{\Delta \iota \Gamma_0} c(\Delta)\, e^{(b-a)(\Delta)} |(w_2^T - w_1^T)(\Delta)| \leq \sup_{\eta \in (0,1)} \sum_{\Delta \iota \Gamma_0} c(\Delta)\, e^{(b-a)(\Delta)} \left| \frac{dw^T(\eta)}{d\eta} \right| \tag{A.20}$$

and (A.8), one immediately obtains (A.19). □

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