Generalized Fractal Dimensions: Equivalences and Basic Properties

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Abstract

Given a positive probability Borel measure μ, we establish some basic properties of the associated functions $\tau_\mu^\pm(q)$ and of the generalized fractal dimensions $D_\mu^\pm(q)$ for $q \in \mathbb{R}$. We first give the connections between the generalized fractal dimensions, the Rényi dimensions and the mean-q dimensions when $q > 0$. We then use these relations to prove some regularity properties for $\tau_\mu^\pm(q)$ and $D_\mu^\pm(q)$; we also give some estimates for these functions as well as for their product of convolution. We finally present some calculations for specific examples.

1 Introduction

The fractal dimensions of measures arised in the 80's as quantities of the most relevant interest in the field of dynamical systems (*e.g.* [26] and ref. therein). For a given measure μ, the generalized fractal dimensions (or multifractal dimensions) $D_\mu^\pm(q)$ (see Section 2 for precise definitions) define a family of fractal dimensions, decreasing in q, with *a priori* $q \in \mathbb{R}$. The most famous elements of this family are the correlation dimension $D_\mu^\pm(2)$, the information dimension (or entropy dimension) $D_\mu^\pm(1)$ and the topogical entropy $D_\mu^\pm(0)$. The Hausdorff and packing dimensions of μ can also be connected to the $D_\mu^\pm(q)$'s, see Section 4. The family $D_\mu^\pm(q)$ not only interpolates between these particular dimensions but supplies a rich family

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that extends them. Although the first "observed" dimensions were the correlation dimension, information dimension and Hausdorff dimension, the whole family of dimensions $D_\mu^\pm(q)$ turns out to play a non trivial role in the setting of dynamical systems and chaos phenomena (e.g. [4], [8], [18], [19], [24], [26], [27], [29]).

In quantum dynamics (actually dynamics in quasi-periodic environnement), an important breakthrough has been achieved in the 90's by Guarneri [13], [14]. Guarneri's ideas, pushed further in [7, 22], allowed to relate the behaviour of a quantum system initially in a state ψ to the Hausdorff dimension of the spectral measure μ_ψ associated to ψ. More recently Barbaroux, Germinet and Tcheremchantsev [2, 3] on one hand and Guarneri and Schulz-Baldes [16] on the other hand proved a more refined lower bound on the dynamics, involving the generalized fractal dimensions $D_\mu^\pm(q)$ for $q<1$.

Although these generalized fractal dimensions $D_\mu^\pm(q)$ appear fairly frequently in physicists literature, the main knowledge about these dimensions is mostly heuristic. As pointed out in [30] in 1998, "they were never examined with real mathematical rigor". Actually rigorous general results on the generalized fractal dimensions $D_\mu^\pm(q)$ are quite few and very basic (see Theorem 3.1). Define for any $q \neq 1$, $\varepsilon > 0$, the integral

$$I_\mu(q,\varepsilon) = \int_{\mathrm{supp}\mu} \mu([x-\varepsilon, x+\varepsilon))^{q-1}\, \mathrm{d}\mu(x) \ ,$$

and then

$$\tau_\mu^+(q) = \limsup_{\varepsilon\downarrow 0} \frac{\log I_\mu(q,\varepsilon)}{-\log\varepsilon} \ , \quad \text{and} \quad D_\mu^+(q) = \limsup_{\varepsilon\downarrow 0} \frac{\log I_\mu(q,\varepsilon)}{(q-1)\log\varepsilon} \ .$$

The functions $\tau_\mu^-(q)$ and $D_\mu^-(q)$ are defined in the same way but with lower limits.

In physicists literature, it is for instance proved numerically, argued or commonly believed that $\lim_{q\to 0} D_\mu^\pm(q) = \dim_B^\pm(\mathrm{supp}\mu)$, where $\dim_B^\pm(\mathrm{supp}\mu)$ is the box dimension of the support of μ and that $\lim_{q\to 1} D_\mu^\pm(q) = D_\mu^\pm(1)$, where $D_\mu^\pm(1)$ is the information dimension (or entropy dimension, see (1.2) below). This also implicitely means that $D_\mu^\pm(q)$ should be somehow continuous at $q=1$. As well, it seems to be commonly accepted that the numbers $D_\mu^\pm(q)$ do not change if we replace $I_\mu(q,\varepsilon)$ by its discretized version $S_\mu(q,\varepsilon)$ defined below in (1.1).

Therefore, there are, at the first sight, two classes of questions that one may address:
(a) Show the equivalence between the different definitions used in the literature, and in particular between the generalized fractal dimensions and their discretized (and quite useful) version, *i.e.* the Rényi dimensions $RD_\mu^\pm(q)$.
(b) Show some basic properties on the families $\tau_\mu^\pm(q)$ and $D_\mu^\pm(q)$, like monotinicity, finiteness, regularity, behaviour at $\pm\infty$, or also: does one have $D_\mu^\pm(q) \in [0,1]$? Such properties have been observed by numerical computations in dynamical systems (*e.g.* [18]) and are part of a "folklore" knowledge in physicists literature.

We first achieve (a). The main result consists in getting a basic equivalence relation between the integrals $I_\mu(q,\varepsilon)$ and their discretized versions, namely, with

$$S_\mu(q,\varepsilon) = \sum_{j\in\mathbb{Z}} a_j^q \ , \quad a_j = \mu([j\varepsilon,(j+1)\varepsilon)) \tag{1.1}$$

$\varepsilon > 0$, whose increasing exponents are the Rényi dimensions $RD_\mu^\pm(q)$. In Lemma 2.1 we prove that for any positive probability measure μ, any $\varepsilon > 0$ and $q > 0$, $q \neq 1$,

$$C_1(q)\ S_\mu(q,\varepsilon)\ \leq\ I_\mu(q,\varepsilon)\ \leq\ C_2(q)\ S_\mu(q,\varepsilon)\ ,$$

for some finite constants $C_1(q), C_2(q) > 0$.
By definition of $D_\mu^\pm(q)$ and $RD_\mu^\pm(q)$ this yields directly the equality of these two dimensions, for $q > 0$, $q \neq 1$.
For $q = 1$, definitions are different. The continuous and discretized information dimensions (or entropies) are indeed defined as the upper or lower limits of respectively $I_\mu(1,\varepsilon)/\log\varepsilon$ and $S_\mu(1,\varepsilon)/\log\varepsilon$, where

$$I_\mu(1,\varepsilon)\ =\ -\int_{\mathrm{supp}\mu} \log(\mu[x-\varepsilon, x+\varepsilon))\,\mathrm{d}\mu(x)\ , \tag{1.2}$$

and

$$S_\mu(1,\varepsilon)\ =\ -\sum_{j\in\mathbb{Z},\ a_j\neq 0} a_j \log a_j,$$

whith again $a_j = \mu([j\varepsilon, (j+1)\varepsilon))$. We shall show that they give rise to the same dimensions as well. We note that for $q < 0$ the equivalence is not true. The region $q \leq 0$ seems to be a totally different (and interesting) regime. We refer the reader to Remarks 2.3 and 3.1.

One may think that these equivalences, at least for $q > 0$ and $q \neq 1$ are well known. It is actually perfectly known for $q > 1$ in full generality, and for $q \in \mathbb{R}\backslash\{1\}$ provided the measure satisfies to a "doubling condition" (with a slighlty different definition of $S_\mu(q,\varepsilon)$) [25] [27]. However in the general case, unlike it is stated in [29], the proof known for $q > 1$ (which is quite simple) is not at all "completely analogous" to the case $q \in (0,1)$, and the same tricks do not apply (to use the same approach one would need a "doubling condition"). As explained in Section 2 (Proof of Lemma 2.1), getting in full generality that equivalence for $q \in (0,1)$, and for $q = 1$ as well, requires a specific analysis of the situation, and a better understanding of the relation between $\mu((x-3\varepsilon, x+3\varepsilon))$ and $\mu((x-\varepsilon, x+\varepsilon))$. In a few words, the equivalence will be obtained by proving that the contribution to the discrete sum $S_\mu(q,\varepsilon)$ of intervals $I_j = [j\varepsilon, (j+1)\varepsilon)$ which have a weight "comparable" to its two direct neighboors I_{j-1} and I_{j+1}, is preponderant.

We point out, as a consequence of the equivalence results presented in this work, that for $q > 0$ it is thus sufficient to work with the particular family of grids $I_j = [j\varepsilon, (j+1)\varepsilon)$, $j \in \mathbb{Z}$, $\varepsilon > 0$ (with a natural extension to higher dimensions). That means two things: first using these sets I_j gives rise to a pertinent discretization that totally encodes the dimensions $D_\mu^\pm(q)$, and second, for $q > 0$, there is no need to consider abstract families of grids, coverings or packing and then to add to the definition of the fractal dimensions the infimum (or supremum) over those families, which pretty much complicates the situation [25] [26]. That is of importance if one wants to extract more informations on the $\tau_\mu^\pm(q)$ and $D_\mu^\pm(q)$, because it dramatically simplifies the calculation, as illustrated in Sections 3 and 5.
In addition, we remark in Theorem 2.2 that considering translated grids $t\varepsilon + I_j = [(j+t)\varepsilon, (j+t+1)\varepsilon)$, for some $t \in (0,1)$, also leads to the same dimensions $D_\mu^\pm(q)$, $q > 0$.

In Section 3 we shall turn to the issues listed in the class (b) above, that is: what can one say about these objects $I_\mu(q,\varepsilon)$, $\tau_\mu^\pm(q)$ and $D_\mu^\pm(q)$? After summurazing some well-known results (mainly Theorem 3.1), we study for which set of q's the functions $\tau_\mu^\pm(q)$, $D_\mu^\pm(q)$ are finite and $D_\mu^\pm(q)\in[0,1]$. We shall take the advantage of the equivalence between the integral $I_\mu(q,\varepsilon)$ and the sums $S_\mu(q,\varepsilon)$ for $q>0$. First, we provide (Proposition 3.1) a new and elementary proof of the fact that for any probability measure, $D_\mu^\pm(q)\in[0,1]$ for $q>1$. Next, we put forward the sums

$$\sum_{k\in\mathbb{Z}}\mu([k,k+1))^q,\quad q>0,$$

as relevant quantities. Among other things we show (Proposition 3.2) that for $q\in(0,1)$ these sums and $\tau_\mu^\pm(q)$ are both finite or infinite (and in the first case, always $D_\mu^\pm(q)\in[0,1]$). In particular, we obtain an alternative proof of the fact (first proven in [3]) that $D_\mu^\pm(q)\in[0,1]$, $q\in(0,1)$, for any compactly supported measure μ. The result of Proposition 3.2 also allows us to give a simple characterization of the important quantity q_μ^* defined as

$$q_\mu^* := \inf\{q \mid \tau_\mu^+(q)<+\infty\} \in [-\infty,1].$$

Next, we study the regularity of both the $D_\mu^\pm(q)$ and the related functions $\tau_\mu^\pm(q)$ on the set $(q_\mu^*,+\infty)$, where all these functions are finite. We show that on $(q_\mu^*,+\infty)$ both $\tau_\mu^+(q)$ and $\tau_\mu^-(q)$ are continuous. More precisely we show (Point (i) of Theorem 3.2) that for any $A>q_\mu^*$ there exists $K(A)$ such that

$$\forall\, q,r\in[A,+\infty),\quad |\tau^\pm(q)-\tau^\pm(r)|\le K(A)|q-r|.$$

This in turn implies the continuity of both the functions $D_\mu^+(q)$ and $D_\mu^-(q)$ on $(q_\mu^*,\,1)\cup(1,\,+\infty)$. Some consequences on the derivatives of $\tau_\mu^+(q)$ and $\tau_\mu^-(q)$ are then derived (Point (ii) and (iii) of Theorem 3.2, Corollary 3.3). Some regularity results (differentiability) were obtained for particular classes of measures that arise in dynamical systems, like quasi-Bernoulli measure [20], but not in full generality. Note that the Lipschitz continuity of $\tau_\mu^+(q)$ is already known as a direct consequence of its convexity. That the result remains true for $\tau_\mu^-(q)$ was not clear at all.

We then establish some inequalities that make more precise the behaviour of the functions $\tau_\mu^\pm(q)$ at $\pm\infty$ (Proposition 3.3). Finally, we discuss the relation between the local exponents of the measure μ and the behaviour of $D_\mu^\pm(q)$ at $\pm\infty$ (Propositions 3.4 and 3.5).

Section 4 is devoted to the study of the critical point $q=1$. As proven in Section 3, $D_\mu^+(q)$ and $D_\mu^-(q)$ are continuous on $(q_\mu^*,+\infty)$ except, maybe, at the point $q=1$. We made the choice to define the information dimension (or entropy dimension) $D_\mu^\pm(1):=h_\mu^\pm$, as it is usually done in the literature [12], [30]. But there is no need for $D_\mu^\pm(1)$ to be equal to either $D_\mu^\pm(1-0)$ or $D_\mu^\pm(1+0)$. This is actually not true in full generality. A relation that certainly holds is the following (Proposition 4.2):

$$D_\mu^\pm(1+0)\le D_\mu^\pm(1)\le D_\mu^\pm(1-0).$$

In addition to that, as it is known, many other dimensions arise in the two intervals $[D_\mu^\pm(1+0),D_\mu^\pm(1-0)]$, and in particular the Hausdorff and packing dimensions. One shows (Proposition 4.1) that

$$D_\mu^\pm(1+0)\le \mu-\text{ess.inf}\,\gamma_\mu^\pm(x)\le\mu-\text{ess.sup}\,\gamma_\mu^\pm(x)\le D_\mu^\pm(1-0),$$

where $\gamma_\mu^\pm(x)$ are the lower and upper local exponents. It is well known that μess.sup $\gamma_\mu^\pm(x)$ are equal to respectively the Hausdorff and packing dimension of μ [11], [20]. Finally the picture is shown to be complete. Indeed one always has

$$\mu, \text{ess.inf}\, \gamma_\mu^-(x) \leq D_\mu^-(1), \quad \text{and} \quad D_\mu^+(1) \leq \mu- \text{ ess.sup}\, \gamma_\mu^+(x).$$

But $D_\mu^-(1)$ and ess.sup $\gamma_\mu^-(x)$ on the one hand and $D_\mu^+(1)$ and ess.inf $\gamma_\mu^+(x)$ on the other hand cannot be compared in full generality. We shall review the existing results, essentially coming (as far as we know) from [31], [11], [20], [5], and provide proofs of the inequalities that we did not manage to find in the literature.

Section 5 deals with convolution of measures. How $D_{\mu*\nu}^\pm(q)$ are related to $D_\mu^\pm(q)$ and $D_\nu^\pm(q)$? Some results are known for Hausdorff dimension [21] and found a theoretical application to quantum mechanics in [6]. We show that for any $q > 0$, $q \neq 1$,

$$\max(D_\mu^+(q),\, D_\nu^+(q)) \leq D_{\mu*\nu}^+(q) \leq D_\mu^+(q) + D_\nu^+(q),$$

and similar estimates for $D_{\mu*\nu}^-(q)$. The proofs we provide do take advantage of the equivalence between the integrals $I_\mu(q, \varepsilon)$ that leads to the $D_\mu^\pm(q)$ and their discretized equivalent $S_\mu(q, \varepsilon)$. These results may be seen as an illustration of the interest of the equivalence proved in Section 2 that enables one to work with simplified tools.

In Section 6, as an illustration of their behaviour we provide calculations of the generalized fractal dimensions $D_\mu^\pm(q)$ for some specific examples. The first example is a complete analysis of a discrete measure (with one accumulating point in its support) which possesses a non trivial spectrum of dimensions. The existence of such measures is of importance in quantum dynamics [2, 3]. Indeed, point measures having Hausdorff and packing dimension zero, it is of interest to know that nevertheless the generalized fractal dimensions $D_\mu^\pm(q)$ may be non trivial for $q < 1$ (which would lead to non trivial diffusion in quantum dynamics).

Other examples are useful illustrations concerning the results of Sections 2-4. In particular, Example 8 show that the discrete and continuous version of the generalized fractal dimensions $D_\mu^\pm(q)$ are not equivalent for negative q's.

Finally, in the Appendix, we compare, for $q \in (0, 1)$, the dimensions $D_\mu^\pm(q)$ and some slightly different dimensions, where $\mu(x - \varepsilon, x + \varepsilon) = \int \chi_{[-1,1]}((x - y)/\varepsilon)\mathrm{d}y$ is replaced by $\mu^{(R)}(x - \varepsilon, x + \varepsilon) = \int R((x - y)/\varepsilon)\mathrm{d}y$. Here $\chi_{[-1,1]}$ is the characteristic function of the interval $[-1, 1]$ and R is any positive function with fast decay at $\pm\infty$. We prove that $\mu(x - \varepsilon, x + \varepsilon)$ and $\mu^{(R)}(x - \varepsilon, x + \varepsilon)$ give rise to the same dimension numbers, namely $D_\mu^\pm(q)$. That was of technical importance in [2, 3] and [16]. In [2, 3] that equivalence was proved to hold in some particular cases (like compactly supported measures). In this appendix we use the ideas we developped in Section 2 to get the equivalence of these two dimensions in full generality.

2 Definitions of the generalized fractal dimensions and equivalence

The main result of this section is concerned with the equivalent definitions of the generalized fractal dimensions $D_\mu^\pm(q)$. As already mentionned in the introduction, the results presented here are already known for $q > 1$ (see e.g. [26]). However, to our best knowledge, for general probability measures there is no proof in the case $q \leq 1$, which is a quite different regime as explained in the proof of Lemma 2.1.

From now on, all considered measures, denoted by μ or ν, are positive probability Borel measures. We first define the quantities that are of interest for us.

Definition 2.1. *The functions* $\tau_\mu^\pm$.
For $q \in \mathbb{R}$ *and* $\varepsilon \in (0,1)$, *we consider the following functions with values in* $\mathbb{R} \cup \{+\infty\}$

$$I_\mu(q,\varepsilon) = \int_{\mathrm{supp}\mu} \mu([x-\varepsilon, x+\varepsilon))^{q-1} \mathrm{d}\mu(x) \ , \tag{2.1}$$

where $\mathrm{supp}\mu$ *is the support of the measure* μ. *We then define the two functions with values in* $\mathbb{R} \cup \{+\infty\}$

$$\tau_\mu^-(q) = \liminf_{\varepsilon \downarrow 0} \frac{\log I_\mu(q,\varepsilon)}{-\log \varepsilon} \ , \qquad \tau_\mu^+(q) = \limsup_{\varepsilon \downarrow 0} \frac{\log I_\mu(q,\varepsilon)}{-\log \varepsilon} \ ,$$

with the understanding that $\tau_\mu^\pm(q) = +\infty$ *if for some* $\varepsilon > 0$, $I_\mu(q,\varepsilon) = +\infty$.

Definition 2.2. *Generalized fractal dimensions - Entropy.*
The lower and upper generalized fractal dimensions of μ *are respectively defined for* $q \in \mathbb{R}\backslash\{1\}$ *as*

$$D_\mu^-(q) = \liminf_{\varepsilon \downarrow 0} \frac{\log I_\mu(q,\varepsilon)}{(q-1)\log \varepsilon} \ , \quad D_\mu^+(q) = \limsup_{\varepsilon \downarrow 0} \frac{\log I_\mu(q,\varepsilon)}{(q-1)\log \varepsilon} \ , \tag{2.2}$$

with values in $[0,+\infty]$. *The entropy dimensions are*

$$D_\mu^-(1) := \ h_\mu^- \ = \liminf_{\varepsilon \downarrow 0} \frac{\int_{\mathrm{supp}\mu} \log(\mu[x-\varepsilon, x+\varepsilon))\mathrm{d}\mu(x)}{\log \varepsilon} \ , \tag{2.3}$$

$$D_\mu^+(1) := \ h_\mu^+ \ = \limsup_{\varepsilon \downarrow 0} \frac{\int_{\mathrm{supp}\mu} \log(\mu[x-\varepsilon, x+\varepsilon))\mathrm{d}\mu(x)}{\log \varepsilon} \ . \tag{2.4}$$

Remark 2.1. (i) In the sequel, we mainly use the notation $h_\mu^\pm$ for the entropy dimensions. The notation $D_\mu^\pm(1)$ is used for convenience only in Proposition 4.2 and Theorem 3.1.
ii) Note that the function $x \mapsto \mu([x-\varepsilon, x+\varepsilon))$ is Borel measurable (by decomposing μ into continuous and discrete parts), thus the integrals $I_\mu(q,\varepsilon)$ make sense (and $L_\mu(q,\varepsilon)$ in Definition 2.4 below as well).
iii) In the equation (2.1), (2.3) and (2.4), changing the intervals $[x-\varepsilon, x+\varepsilon)$ into $(x-\varepsilon, x+\varepsilon)$, $(x-\varepsilon, x+\varepsilon]$ or $[x-\varepsilon, x+\varepsilon]$ does not change the values of the functions $\tau_\mu^\pm$ and $D_\mu^\pm$.
iv) Note that

$$D_\mu^\pm(q) = \frac{\tau_\mu^\pm(q)}{(1-q)}, \quad \forall\, q < 1,$$

whereas

$$D_\mu^\pm(q) = \frac{\tau_\mu^\mp(q)}{(1-q)}, \quad \forall\, q > 1,$$

i.e. in the latter the $+$ and the $-$ signs are reversed. The definitions are so that for any $q \in \mathbb{R}$, $D_\mu^+(q) \geq D_\mu^-(q)$ and $\tau_\mu^+(q) \geq \tau_\mu^-(q)$, which seems to us natural.
v) In the literature the above dimensions are often refered to as Hentschel-Procaccia generalized dimensions (see [19], [26]).
vi) The definitions of $h_\mu^\pm$ may differ from one author to another (see e.g. [26] or [28]). We will discuss in Section 3 the connection between $h_\mu^\pm$ and the right and left limits of $D_\mu^\pm(q)$ when q tends to 1.

We now define a discretized version of the previous dimensions. For a given $\varepsilon \in (0,1)$, we consider the intervals $I_j^{(\varepsilon)} = [j\varepsilon, (j+1)\varepsilon)$, $j \in \mathbb{Z}$.

Definition 2.3. *Generalized Rényi dimensions.*
We denote $a_j = \mu\left(I_j^{(\varepsilon)}\right)$ *and set, for* $q \in \mathbb{R}\backslash\{1\}$,

$$S_\mu(q,\varepsilon) = \sum_{a_j \neq 0} a_j^q \ , \tag{2.5}$$

which takes its values in $\mathbb{R} \cup \{+\infty\}$. *For* $q = 1$ *we define*

$$S_\mu(1,\varepsilon) = \sum_{a_j \neq 0} a_j \log(1/a_j). \tag{2.6}$$

For $q \in \mathbb{R} \setminus \{1\}$, *the lower and upper generalized Rényi dimensions of the measure* μ *are respectively given by*

$$RD_\mu^-(q) = \liminf_{\varepsilon \downarrow 0} \frac{\log S_\mu(q,\varepsilon)}{(q-1)\log\varepsilon} \ , \quad RD_\mu^+(q) = \limsup_{\varepsilon \downarrow 0} \frac{\log S_\mu(q,\varepsilon)}{(q-1)\log\varepsilon} \ .$$

and the Rényi entropies are

$$Rh_\mu^- = \liminf_{\varepsilon \downarrow 0} \frac{S_\mu(1,\varepsilon)}{-\log\varepsilon} \ , \quad Rh_\mu^+ = \limsup_{\varepsilon \downarrow 0} \frac{S_\mu(1,\varepsilon)}{-\log\varepsilon} \ .$$

Remark 2.2.
i) Notice that we have

$$RD_\mu^-(0) = \liminf_{\varepsilon \downarrow 0} \frac{\log\left(\mathrm{Card}\{j \in \mathbb{Z} \ : \ \mu(I_j^{(\varepsilon)}) \neq 0\}\right)}{-\log\varepsilon}$$

and the same for $RD_\mu^+(0)$. Since

$$\mathrm{Card}\{j \in \mathbb{Z} : \mu(I_j^{(\varepsilon)}) \neq 0\} \leq \mathrm{Card}\{j \in \mathbb{Z} : I_j^{(\varepsilon)} \cap \mathrm{supp}\mu \neq \emptyset\} \leq 2\,\mathrm{Card}\{j \in \mathbb{Z} : \mu(I_j^{(\varepsilon)}) \neq 0\},$$

the dimensions $RD_\mu^\pm(0)$ may be seen as the lower and upper box counting dimension of the support of μ [9], *i.e.*

$$RD_\mu^\pm(0) \ = \ \dim_B^\pm(\mathrm{supp}\mu)\,.$$

ii) In [26, §18], the generalized Rényi dimensions are defined for $q \geq 0$ by considering more general families of partitions of $\mathbb{R}^m$ (called (β, r) grids) instead of a single partition $\{[j\varepsilon, (j+1)\varepsilon)\}$. However, for $q > 1$ all these families were shown to lead to the same values of dimensions, namely $D_\mu^\pm(q)$, defined in $\mathbb{R}^m$ (see [26, Theorem 18.2]) . Our main result which states $Rh_\mu^\pm = h_\mu^\pm$ and $RD_\mu^\pm(q) = D_\mu^\pm(q)$ for $q \in (0,1)$ (Theorem 2.1) can be also generalized in this way, both to (β, r) grids and to higher dimensions. We also refer the reader to Theorem 2.2 for translated grids.
iii) A different definition of Rényi dimensions using centered packing of supp μ is adopted in [25].

We now finish this series of definition by introducing a continuous version of the Rényi dimensions different from (2.2) (see e.g. [23], [17], [1]).

Definition 2.4. *Mean-q dimensions.*
For $q \in \mathbb{R}$, let

$$L_\mu(q, \varepsilon) = \frac{1}{\varepsilon} \int_{\mathbb{R}} \mu([x - \varepsilon, x + \varepsilon))^q \mathrm{d}x \ . \tag{2.7}$$

The lower and upper mean-q dimensions of μ are respectively defined for $q \in \mathbb{R} \setminus \{1\}$ as

$$d_\mu^-(q) = \liminf_{\varepsilon \downarrow 0} \frac{\log L_\mu(q, \varepsilon)}{(q-1)\log \varepsilon} \quad \text{and} \quad d_\mu^+(q) = \limsup_{\varepsilon \downarrow 0} \frac{\log L_\mu(q, \varepsilon)}{(q-1)\log \varepsilon}$$

We are now ready to state our first result, which is the main result of this section. It states the equivalence for $q > 0$ of all the generalized fractal dimensions defined above. That equivalence is argued and commonly accepted in the physics literature (see e.g. [18], [19]). In the specific case $q = 2$ a rigorous proof is available in [17], [1]. To our knowledge the best known result concerns Point (i) of Theorem 2.1 which has been proved to hold for $q > 1$ only (see e.g. [26, §18] and references therein). But as we shall discuss it in the proof of Theorem 2.1 the regime $q > 1$ is much different (and simpler) than the regime $q \in (0, 1]$ where subtle effects of intervals of small weight may happen. From this point of view the case $q = 1$ is close to the regime $q < 1$. We stress that it was not completely clear *a priori* that Point (ii) should hold (whereas it seems the equivalence was expected for $q \in (0,1)$).
In [25], one can find some results connecting generalized fractal dimensions and Rényi dimensions (defined in a different way) for any $q \in \mathbb{R} \setminus \{1\}$ in the case of measures verifying a "doubling condition" (see [25, §2.1] or [26, §8] for details). See also *e.g.* [27] for a similar statement in the case of Gibbs measures (which are shown to verify the "doubling condition").

Theorem 2.1.
(i) For any $q > 0$, $q \neq 1$, one has

$$D_\mu^-(q) = RD_\mu^-(q) \ , \quad D_\mu^+(q) = RD_\mu^+(q) \ , \tag{2.8}$$

and for any $q \leq 0$

$$D_\mu^-(q) \leq RD_\mu^-(q) \ , \quad D_\mu^+(q) \leq RD_\mu^+(q) \ . \tag{2.9}$$

(ii) The entropy and the Rényi entropy dimensions are equal

$$h_\mu^\pm = Rh_\mu^\pm \ .$$

(iii) For any $q > 0$, $q \neq 1$

$$D_\mu^-(q) = d_\mu^-(q) \ , \quad D_\mu^+(q) = d_\mu^+(q) \ . \tag{2.10}$$

Remark 2.3. *For $q < 0$, (2.9) may turn into strict inequalities (see Example 8 of Section 6). This actually means that considering the basic grid $[j\varepsilon, (j+1)\varepsilon)$, $j \in \mathbb{Z}$ is too crude to encodes totally the $D_\mu(q)$ for $q < 0$. In the example 8 of Section 6 we shall see that boundary effects may appear. However the situation is not clear for $q = 0$: could one get $D^\pm(0) = RD^\pm(0)$ in great generality or not ?*

We considered above the particular grid $I_j = [j\varepsilon, (j+1)\varepsilon)$, $j \in \mathbb{Z}$. It is quite clear that the same results hold for translated grids $t\varepsilon + I_j = [(j+t)\varepsilon, (j+t+1)\varepsilon)$, $j \in \mathbb{Z}$ with any $t \in [0,1)$. With obvious notations we define new discrete sums $S_\mu^{\{t\}}(q,\varepsilon)$. In that paper we shall take advantage of the observation that whatever $t \in (0,1)$ is, the corresponding sums $S_\mu^{\{t\}}(q,\varepsilon)$ do lead to the same multifractal dimensions $RD_\mu^\pm(q)$, for $q \geq 0$, *i.e.* the growth exponents of $S_\mu^{\{0\}}(q,\varepsilon) = S_\mu(q,\varepsilon)$. Our general philosophy is then that in order to prove properties of the generalized Rényi dimensions for positive q's, one can restrict oneself to a particular grid, which dramatically simplifies the proofs.

This is summed up in the following theorem that we state for the reader's convenience, since we shall use it later on.

Theorem 2.2. *For any $q \geq 0$, $t, t' \in [0,1)$ and $\varepsilon > 0$, one has, for $q \neq 1$,*

$$S_\mu^{\{t\}}(q,\varepsilon) \leq C(q) S_\mu^{\{t'\}}(q,\varepsilon), \tag{2.11}$$

where $C(q) = 1$ if $q \in [0,1)$ and $C(q) = 2^{q+1}$ if $q > 1$; and for $q = 1$, one has,

$$S_\mu^{\{t\}}(1,\varepsilon) \leq S_\mu^{\{t'\}}(1,\varepsilon) + 2 \tag{2.12}$$

In particular for any $q \geq 0$, the translated grids lead to the same multifractal dimensions defined for $t = 0$, namely the numbers $RD_\mu^\pm(q)$.

Proof of Theorem 2.2.
The equivalence of the generalized Rényi dimensions $RD_\mu^\pm(q)$, $q \geq 0$, for translated grids may be seen as a direct consequence of Lemmas 2.1 and 2.2 below, for the proofs of the latter are not sensible to the particular grid $t\varepsilon + I_j$, $t \in [0,1)$ that is used (in the proof $t = 0$).

However, this is not the most natural way of looking at the question! Indeed, Estimate (2.11) can easily be seen directly since for any $t \in [0,1)$, any interval $t\varepsilon + I_j$ can be covered by two intervals of the second grid (with parameter t'). That leads, for any $q > 0$, $q \neq 1$, to the factor $C(q) = 1$ for $q \in [0,1)$ and $C(q) = 2^{q+1}$ for $q > 1$: $S_\mu^{\{t\}}(q,\varepsilon) \leq C(q) S_\mu^{\{t'\}}(q,\varepsilon)$. Unfortunately for $q = 1$ the same rough estimate leads to an extra multiplicative factor 2, *i.e.* $S_\mu^{\{t\}}(1,\varepsilon) \leq 2S_\mu^{\{t'\}}(1,\varepsilon)$ which destroys the equivalence. Hence a more carefull treatment is required.

Fix $t \in (0,1)$, and without loss of generality, let us take for the second grid the basic one, *i.e.* $t' = 0$. We show that $S_\mu^{\{t\}}(1,\varepsilon) \leq S_\mu(1,\varepsilon) + 2$. Define by f the function $f(x) = x\log(1/x)$,

$x>0$, and $f(0)=0$. Hence $S_\mu^{\{t\}}(1,\varepsilon)=\sum_{j\in\mathbb{Z}} f(\mu(t\varepsilon+I_j))$. Any interval $t\varepsilon+I_j$ intersects the two intervals I_j and I_{j+1}, or conversely any interval I_j of the basic grid splits in two parts, one in $t\varepsilon+I_{j-1}$ and the second in $t\varepsilon+I_j$. Let us denote by $\beta_j^{(1)}$, $\beta_j^{(2)}$ the relative weight of I_j on these two intervals, namely $\beta_j^{(1)}=\mu(I_j\cap(t\varepsilon+I_{j-1}))/\mu(I_j)$, and $\beta_j^{(2)}=\mu(I_j\cap(t\varepsilon+I_j))/\mu(I_j)$ (with the convention that $\beta_j^{(1,2)}=0$ if $\mu(I_j)=0$). So that one has

$$\beta_j^{(1)}+\beta_j^{(2)}=1,$$

and

$$\mu(t\varepsilon+I_j)=\beta_j^{(2)}\mu(I_j)+\beta_{j+1}^{(1)}\mu(I_{j+1}).$$

Finally, note that for any $a,b>0$, one has $f(a+b)\leq f(a)+f(b)$. Using the previous notations and remarks, and the fact that $f(x)\leq 1$, one has

$$\begin{aligned}
S_\mu^{\{t\}}(1,\varepsilon) &= \sum_{j\in\mathbb{Z}} f(\beta_j^{(2)}\mu(I_j)+\beta_{j+1}^{(1)}\mu(I_{j+1}))\\
&\leq \sum_{j\in\mathbb{Z}} \beta_j^{(2)}\mu(I_j)\left(\log\frac{1}{\mu(I_j)}+\log\frac{1}{\beta_j^{(2)}}\right)+\beta_{j+1}^{(1)}\mu(I_{j+1})\left(\log\frac{1}{\mu(I_{j+1})}+\log\frac{1}{\beta_{j+1}^{(1)}}\right)\\
&\leq \sum_{j\in\mathbb{Z}}(\beta_j^{(1)}+\beta_j^{(2)})\mu(I_j)\log\frac{1}{\mu(I_j)}+2\sum_{j\in\mathbb{Z}}\mu(I_j)\\
&= S_\mu(1,\varepsilon)+2\,.
\end{aligned}$$

□

We turn to the main result of this section, that is Theorem 2.1, which relies on the following three lemmas. They constitute the proof of Theorem 2.1 (for $q\leq 0$ the announced inequalities are trivial).

Lemma 2.1. *For any $q>0$, $q\neq 1$, $S_\mu(q,\varepsilon)$ and $I_\mu(q,\varepsilon)$ are either both converging or both diverging, and there exists some finite constants $C_1(q)$, $C_2(q)>0$ so that for all $\varepsilon\in(0,1)$*

$$C_1(q)S_\mu(q,\varepsilon)\leq I_\mu(q,\varepsilon)\leq C_2(q)S_\mu(q,\varepsilon).$$

In particular, $S_\mu(q,\varepsilon)$ and $I_\mu(q,\varepsilon)$ have the same growth exponents $D_\mu^{\pm}(q)$, $q>0$, $q\neq 1$.

Lemma 2.2. *For any $K>2e$, one has*

$$S_\mu(1,\varepsilon)\geq\int_{\mathrm{supp}\mu}\log\frac{1}{\mu([x-\varepsilon,x+\varepsilon])}\mathrm{d}\mu(x)\geq\left(1-\frac{2}{K}\right)S_\mu(1,\varepsilon)-\frac{2\log K}{K}-\log(K+1) \quad (2.13)$$

Thus the following equalities hold for the entropy dimensions

$$h_\mu^-=Rh_\mu^- \quad \text{and} \quad h_\mu^+=Rh_\mu^+\ . \tag{2.14}$$

Lemma 2.3. *For any $q>0$, $S_\mu(q,\varepsilon)$ and $L_\mu(q,\varepsilon)$ are either both converging or both diverging. Moreover, for any $\varepsilon\in(0,1)$ we have*

$$S_\mu(q,\varepsilon)\leq L_\mu(q,\varepsilon)\leq 3^{q+1}S_\mu(q,\varepsilon).$$

In particular, $S_\mu(q,\varepsilon)$ and $L_\mu(q,\varepsilon)$ have the same growth exponents $D_\mu^{\pm}(q)$, $q>0$, $q\neq 1$.

For the proof of the above lemmas, we adopt the following notations: for simplicity, we drop the indices (ε) from $I_j^{(\varepsilon)}$; furthermore, we define $a_j = \mu\left(I_j^{(\varepsilon)}\right)$ and $b_j = a_{j-1}+a_{j+1}$, where $a_j, b_j \in (0,1)$ for any j and $\sum_j a_j = 1$. The basic estimate that allows one to compare integrals and discrete sums comes from the following simple observation. For any $x \in I_j = [j\varepsilon, (j+1)\varepsilon)$,

$$I_j \subset [x-\varepsilon, x+\varepsilon) \subset I_{j-1} \cup I_j \cup I_{j+1} \ ,$$

therefore

$$a_j \le \mu([x-\varepsilon, x+\varepsilon)) \le a_{j-1} + a_j + a_{j+1} = a_j + b_j \ . \tag{2.15}$$

Proof of Lemma 2.1.
The integral $I_\mu(q,\varepsilon)$ can be written as

$$I_\mu(q,\varepsilon) = \sum_{j\in\mathbb{Z}} \int_{I_j \cap \operatorname{supp}\mu} \mu[x-\varepsilon, x+\varepsilon)^{q-1}\,\mathrm{d}\mu(x) = \sum_{j\in\mathbb{Z},\, a_j\neq 0} \int_{I_j} \mu[x-\varepsilon, x+\varepsilon)^{q-1}\,\mathrm{d}\mu(x) \ . \tag{2.16}$$

Consider first the case $q>1$. From (2.16) and (2.15) we get

$$\begin{aligned} S_\mu(q,\varepsilon) = \sum_{j\in\mathbb{Z},\, a_j\neq 0} a_j^q \le I_\mu(q,\varepsilon) \ &\le \ \sum_{j\in\mathbb{Z},\, a_j\neq 0} a_j(a_{j-1}+a_j+a_{j+1})^{q-1} && (2.17)\\ &\le \ \sum_{j\in\mathbb{Z},\, a_j\neq 0} (a_{j-1}+a_j+a_{j+1})^{q} && (2.18)\\ &\le \ 3^q \sum_{j\in\mathbb{Z},\, a_j\neq 0} (a_j^q + a_{j-1}^q + a_{j+1}^q) \\ &\le \ 3^{q+1} S_\mu(q,\varepsilon) \ . \end{aligned}$$

The above proof is trivial and well-known. We refer the reader to [26], where a proof in the same spirit is given (in a more general case of measures on $\mathbb{R}^m$ and (β, r) grids instead of our simple partition $\{[j\varepsilon,(j+1)\varepsilon)\}$). What makes things simpler if $q>1$ is that there is no need to compare a_j to its neighboors a_{j-1} and a_{j+1}. One gets (2.18) from (2.17) by simply majorating a_j by $a_{j-1}+a_j+a_{j+1}$. If now $q<1$ then the inequalities in (2.17) are reversed and that trivial majoration does not help anymore.

We turn to the case $q\in(0,1)$. Using (2.15) in (2.16) yields

$$\sum_{j\in\mathbb{Z},\, a_j\neq 0} \frac{a_j}{(a_j+b_j)^{1-q}} \le I_\mu(q,\varepsilon) \le \sum_{j\in\mathbb{Z},\, a_j\neq 0} a_j^q \ . \tag{2.19}$$

As one can see, if $q<1$ one has to find a lower bound for $\frac{a_j}{(a_j+b_j)^{1-q}}$ and immediately arises the issue of the size of a_j compared to its neighboors $b_j = a_{j-1}+a_{j+1}$. One would like to know that a_j and b_j have comparable sizes so that $\frac{a_j}{(a_j+b_j)^{1-q}}$ would be transformed in a_j^q; otherwise one would end up with a_j/b_j^{1-q}, which may be significantly smaller than a_j^q. But in full generality a_j cannot be compared to b_j. Of course if one imposes that μ be such that $a_j \ge C(a_j+b_j)$ for some uniform constant C then one recovers a_j^q and the proof functions in the same way as before ($q>1$). This is close to the philosophy of the proof of Olsen

[25], where the doubling condition is assumed to hold (with a different definition of Rényi dimension though).

To get, in full generality, a lower bound in terms of $\sum_{j\in\mathbb{Z}} a_j^q$, and thus prove the equivalence, we shall control the part of the "bad" a_j's, *i.e.* those which are too small with respect to b_j (see the set B below). This will be achieved Line (2.21).

Fix $K = 2^{(q+2)/q}$ and define the sets A and B as follows

$$A = \{j \in \mathbb{Z} \mid 0 \leq b_j/K < a_j\} \ , \qquad B = \{j \in \mathbb{Z} \mid 0 < a_j \leq b_j/K\} \ .$$

Then we have

$$\begin{aligned} \sum_{j\in\mathbb{Z},\, a_j\neq 0} \frac{a_j}{(a_j+b_j)^{1-q}} &\geq \sum_{j\in A} \frac{a_j}{(a_j+b_j)^{1-q}} \\ &\geq \frac{1}{(K+1)^{1-q}} \sum_{j\in A} a_j^q \ . \end{aligned} \tag{2.20}$$

We now show that the major contribution to the sum $\sum_{j\in\mathbb{Z}} a_j^q$ comes from the j's in A . Indeed one has, by the choice of $K = 2^{(q+2)/q}$,

$$\begin{aligned} \sum_{j\in B} a_j^q &\leq K^{-q} \sum_{j\in B} b_j^q \leq K^{-q} 2^q \sum_{j\in B} (a_{j-1}^q + a_{j+1}^q) \\ &\leq K^{-q} 2^{q+1} \sum_{j\in\mathbb{Z},\, a_j\neq 0} a_j^q = \frac{1}{2} \sum_{j\in\mathbb{Z},\, a_j\neq 0} a_j^q. \end{aligned} \tag{2.21}$$

It obviously implies

$$\sum_{j\in A} a_j^q = \sum_{j\in\mathbb{Z},\, a_j\neq 0} a_j^q - \sum_{j\in B} a_j^q \geq \frac{1}{2} \sum_{j\in\mathbb{Z},\, a_j\neq 0} a_j^q \ . \tag{2.22}$$

Now, (2.22) together with (2.20) and (2.19) implies

$$\frac{1}{2}\left(1 + 2^{\frac{q+2}{q}}\right)^{q-1} S_\mu(q,\varepsilon) \leq I_\mu(q,\varepsilon) \leq S_\mu(q,\varepsilon) \ .$$

That ends the proof of Lemma 2.1

□

Proof of Lemma 2.2.

Although it is more technical, the spirit of the proof of Lemma 2.2 is close to the case $q \in (0,1)$ above. It is because of the concavity of the function $x \to x\log(1/x)$. The issue is still to get a control on the contribution of the "bad" j's for which a_j is too small with respect to b_j. Discretizing, as at the beginning of Lemma 2.1, the integral that enters into account in $h_\mu^\pm$ leads to (using (2.15))

$$\sum_{j\in\mathbb{Z},\, a_j\neq 0} a_j \log \frac{1}{a_{j-1}+a_j+a_{j+1}} \leq \int_{\mathrm{supp}\mu} \log \frac{1}{\mu([x-\varepsilon, x+\varepsilon])} \mathrm{d}\mu(x) \leq \sum_{j\in\mathbb{Z},\, a_j\neq 0} a_j \log \frac{1}{a_j} \tag{2.23}$$

As previoulsy, we should obtain a lower bound for the l.h.s. of (2.23) in terms of $\sum_{a_j\neq 0} a_j \log(1/a_j)$. Let $K>0$ to be precised later. Consider $A=\{j|\ 0\le b_j/K<a_j\}$ and $B=\{j|\ 0<a_j\le b_j/K\}$. Then

$$\begin{aligned}\sum_{j\in\mathbb{Z},\, a_j\neq 0} a_j \log\frac{1}{a_j+b_j} &\ge \sum_{j\in A} a_j\log\frac{1}{a_j+b_j}\ge\sum_{j\in A} a_j\log\frac{1}{(K+1)a_j}\\ &\ge \sum_{j\in A} a_j\log\frac{1}{a_j}-\log(K+1). \qquad (2.24)\end{aligned}$$

One then has to estimate the remaining part of the sum, that is $-\sum_B a_j\log a_j$. Notice that the function $x\to x\log(1/x)$ (defined to be zero for $x=0$) increases on $]0,1/e[$ and then decreases. Hence, by choosing K so that $K>2e$, one makes sure that for any j, $b_j/K\le 1/e$, which implies that for any $j\in B$

$$a_j\log\frac{1}{a_j}\le\frac{b_j}{K}\log\frac{K}{b_j}=\frac{b_j}{K}\left(\log K+\log\frac{1}{b_j}\right). \qquad (2.25)$$

As a second remark, note that for any $a,\, b>0$, one checks that

$$(a+b)\log\frac{1}{a+b}\le a\log\frac{1}{a}+b\log\frac{1}{b} \qquad (2.26)$$

Using successively (2.25), (2.26) and the fact that $\sum_j a_j=1$ leads to

$$\begin{aligned}\sum_{j\in B} a_j\log\frac{1}{a_j} &\le \frac{1}{K}\sum_{j\in B}(a_{j-1}+a_{j+1})\left(\log K+\log\frac{1}{a_{j-1}+a_{j+1}}\right)\\ &\le \frac{2\log K}{K}+\frac{2}{K}\sum_{j\in\mathbb{Z},\, a_j\neq 0} a_j\log\frac{1}{a_j} \qquad (2.27)\end{aligned}$$

(recall we imposed that $a_j=0$ implies $a_j\log(1/a_j)=0$). Putting together (2.24) and (2.27) yields

$$\begin{aligned}\sum_{j\in\mathbb{Z},\, a_j\neq 0} a_j\log\frac{1}{a_j+b_j} &\ge \sum_{j\in\mathbb{Z},\, a_j\neq 0} a_j\log\frac{1}{a_j}-\left(\sum_{j\in B} a_j\log\frac{1}{a_j}+\log(K+1)\right)\\ &\ge \left(1-\frac{2}{K}\right)\sum_{j\in\mathbb{Z},\, a_j\neq 0} a_j\log\frac{1}{a_j}-\frac{2\log K}{K}-\log(K+1). \quad (2.28)\end{aligned}$$

Dividing by $-\log\varepsilon$ and taking the limit (inf or sup), (2.23) together with the last estimate (2.28) implies that, for any $K>2e$,

$$\left(1-\frac{2}{K}\right)Rh_\mu^\pm\le h_\mu^\pm\le Rh_\mu^\pm.$$

The latter clearly implies that $Rh_\mu^\pm=h_\mu^\pm$. $\square$

Proof of Lemma 2.3.

The integral $L_\mu(q,\varepsilon)$ can be written as

$$L_\mu(q,\varepsilon) = \sum_{j\in\mathbb{Z}} L_j(q,\varepsilon), \quad L_j(q,\varepsilon) = \frac{1}{\varepsilon}\int_{I_j} \mu([x-\varepsilon, x+\varepsilon))^q \mathrm{d}x. \tag{2.29}$$

For any $q>0$ the key observation (2.15) yields $a_j^q \le \mu([x-\varepsilon,x+\varepsilon))^q \le (a_{j-1}+a_j+a_{j+1})^q$, for any $x\in I_j$. Integrating this inequality over I_j, we obtain for any $j\in\mathbb{Z}$, since $|I_j|=\varepsilon$,

$$\begin{aligned} a_j^q \le L_j(q,\varepsilon) &\le (a_{j-1}+a_j+a_{j+1})^q \\ &\le 3^q\left(a_{j-1}^q + a_j^q + a_{j+1}^q\right) . \end{aligned} \tag{2.30}$$

The result of the Lemma follows directly from (2.29) -(2.30). □

Remark 2.4. The result is not true if one integrates not over $\mathbb{R}$ but over $\mathrm{supp}(\mu)$ in the definition of $L_\mu(q,\varepsilon)$. For example, for a measure $\mu=\delta_0$ that would give $L_\mu(q,\varepsilon)=0$, but $S_\mu(q,\varepsilon)=1$. On the other hand, integrating over $\mathbb{R}$ for $q<0$ yields $L_\mu(q,\varepsilon)=+\infty$ for any compactly supported measure. Therefore, the result of the Lemma can hardly be extended to the case $q<0$ in any reasonable sense.

3 Properties of the generalized fractal dimensions

In this section we list and prove some basic properties of the functions $\tau_\mu^\pm(q)$ and the generalized fractal dimensions $D_\mu^\pm(q)$ introduced in Section 2, Definitions 2.1 and 2.2. Some of them are well-known, see Theorem 3.1. Some others are part of a "common" knowledge, but, as far as we know, with no proofs.
The results contained in this section use two different crucial tools, which leads to two series of properties: the "convexity" inequality (3.1) and the equivalence proved in Section 2 that allows one to work with the more convenient sums $\sum_j a_j^q$ for $q>0$.

Theorem 3.1. *Let μ be a positive probability Borel measure.*
(i) $\tau_\mu^-(q)$, $\tau_\mu^+(q)$, $D_\mu^-(q)$ and $D_\mu^+(q)$ are nonincreasing functions of $q\in\mathbb{R}$.
(ii) $\tau_\mu^\pm(1)=0$.
(iii) $\tau_\mu^+(q)$ is a convex function.

Proof of Theorem 3.1. Part (i) is already known (see e.g. [8], [28]), and is a straightforward consequence of Jensen inequalities (for $q=1$ see Proposition 4.2).
Property (ii) is immediate from the definition of $\tau_\mu^\pm(1)$.
Point (iii) goes as follows. Let t be in $(0,1)$ and r, s be two real numbers. Using Hölder inequality with $p=1/t$ and $p'=1/(1-t)$ gives

$$\begin{aligned} I_\mu(tr+(1-t)s,\varepsilon) &= \int_{\mathrm{supp}\mu} \mu([x-\varepsilon,x+\varepsilon))^{tr+(1-t)s-1}\mathrm{d}\mu(x) \\ &= \int_{\mathrm{supp}\mu} \mu([x-\varepsilon,x+\varepsilon))^{t(r-1)+(1-t)(s-1)}\mathrm{d}\mu(x) \\ &\le (I_\mu(r,\varepsilon))^t (I_\mu(s,\varepsilon))^{1-t} , \end{aligned} \tag{3.1}$$

and thus (iii) holds true. A similar result was obtained in [28] using the same convexity inequality (3.1). □

As it follows directly from the definition of $\tau_\mu^\pm(q)$, we have $\tau_\mu^\pm(q) \geq 0$ for $q < 1$ and $\tau_\mu^\pm(q) \leq 0$ for $q > 1$. As to $D_\mu^\pm(q)$, they are defined so that always $D_\mu^\pm(q) \geq 0$. The first problem we shall consider is the characterisation of the set of q's where $\tau_\mu^\pm(q)$ and $D_\mu^\pm(q)$ are finite. We shall discuss also the validity of the common belief that $D_\mu^\pm(q)$ are smaller than 1. The result of the following proposition is well-known, *e.g.* [1], [28]. However, the proof we provide here is new and quite elementary. In particular, it does not require the use of the local exponents $\gamma_\mu^\pm(x)$. This result implies that $\tau_\mu^\pm(q) > -\infty$ for any $q \in \mathbb{R}$.

Proposition 3.1. *Let μ be a positive probability Borel measure. For any $q > 1$,*

$$1 - q \leq \tau_\mu^-(q) \leq \tau_\mu^+(q) \leq 0 \quad \text{and} \quad 0 \leq D_\mu^-(q) \leq D_\mu^+(q) \leq 1.$$

Proof of Proposition 3.1. Let $I_j = [j\varepsilon, (j+1)\varepsilon), \varepsilon < 1/2,\ j \in \mathbb{Z}$ and $a_j = \mu(I_j)$. For any $k \in \mathbb{Z}$, we have

$$[k, k+1) \subset \bigcup_{j=p}^{r} I_j \subset [k-1, k+2), \tag{3.2}$$

where $p = [\frac{k}{\varepsilon}] - 1,\ r = [\frac{k+1}{\varepsilon}] + 1$ ($[x]$ denotes the integer part of x). Remark that $r - p < 2/\varepsilon$ for ε small enough. It follows from (3.2) that

$$d_k \equiv \mu([k, k+1)) \leq \sum_{j=p}^{r} a_j.$$

With Hölder inequality we estimate for any $q > 1,\ k \in \mathbb{Z}$:

$$d_k \leq \left(\sum_{j=p}^{r} a_j^q\right)^{1/q} \left(\sum_{j=p}^{r} 1\right)^{(q-1)/q} \leq (\varepsilon/2)^{(1-q)/q} \left(\sum_{j=p}^{r} a_j^q\right)^{1/q}.$$

Therefore,

$$S_\mu(q, \varepsilon) \equiv \sum_{j\in\mathbb{Z}} a_j^q \geq \sum_{j=p}^{r} a_j^q \geq C(q)\varepsilon^{q-1} d_k^q \tag{3.3}$$

with some positive constant $C(q)$. As $\mu(\mathbb{R}) = \sum_{k\in\mathbb{Z}} d_k = 1$, there exists k such that $d_k > 0$, so (3.3) yields $S_\mu(q, \varepsilon) \geq C\varepsilon^{q-1}$. The result follows now from Lemma 2.1 and the definition of $\tau_\mu^\pm(q), D_\mu^\pm(q)$. □

In the next proposition, we study the case $q \in (0, 1)$.

Proposition 3.2. *Let $d_k = \mu([k, k+1))$, $k \in \mathbb{Z}$.*
(i) If $\sum_k d_k^q = +\infty$ for some $q \in (0, 1)$ then $\tau_\mu^\pm(q) = +\infty$ (and so $D_\mu^\pm(q) = +\infty$).
(ii) If $\sum_k d_k^q < +\infty$ for some $q \in (0, 1)$ then $0 \leq \tau_\mu^-(q) \leq \tau_\mu^+(q) \leq 1 - q$ and $0 \leq D_\mu^-(q) \leq D_\mu^+(q) \leq 1$.

Proof of Proposition 3.2.
With the same notations as in the proof of Proposition 3.1, (3.2) implies

$$d_k \equiv \mu([k,k+1)) \le \sum_{j=p}^{r} a_j \le d_{k-1} + d_k + d_{k+1} \ . \tag{3.4}$$

Using the elementary inequality $(\sum a_j)^q \le \sum a_j^q$, where $0<q<1$ and $a_j \ge 0$, we obtain

$$d_k^q \le \sum_{j=p}^{r} a_j^q$$

Therefore,

$$\sum_{k\in\mathbb{Z}} d_k^q \le \sum_{k\in\mathbb{Z}} \sum_{j=p}^{r} a_j^q \le 2\sum_{j\in\mathbb{Z}} a_j^q =: 2S_\mu(q,\varepsilon) \tag{3.5}$$

(the factor 2 comes from the fact that some a_j^q are counted twice if $j\varepsilon$ or $(j+1)\varepsilon$ is close to an integer). If $\sum_{k\in\mathbb{Z}} d_k^q = +\infty$, the bound (3.5) yields $S_\mu(q,\varepsilon) = +\infty$, so $\tau_\mu^{\pm}(q) = +\infty$.
To prove the second part of the proposition, we apply the Hölder inequality with $t = 1/q, t' = 1/(1-q)$. Using the r.h.s. of (3.4), we obtain

$$\begin{aligned}
\sum_{j=p}^{r} a_j^q &\le \left(\sum_{j=p}^{r} a_j\right)^q \left(\sum_{j=p}^{r} 1\right)^{1-q} \\
&\le C(q)\varepsilon^{q-1}(d_{k-1}+d_k+d_{k+1})^q \\
&\le C(q)\varepsilon^{q-1}(d_{k-1}^q+d_k^q+d_{k+1}^q) \ ,
\end{aligned}$$

where we have used the fact that $p = [k/\varepsilon]-1$, $r = [(k+1)/\varepsilon]+1$, $r-p<2/\varepsilon$. Therefore,

$$S_\mu(q,\varepsilon) = \sum_{j\in\mathbb{Z}} a_j^q \le \sum_{k\in\mathbb{Z}} \sum_{j=p}^{r} a_j^q \le 3C(q)\varepsilon^{q-1}\sum_{k\in\mathbb{Z}} d_k^q. \tag{3.6}$$

As clearly $S_\mu(q,\varepsilon) \ge 1$, (3.6) gives the second statement of the Proposition.

□

One can now define the following natural quantity associated to the measure μ.

$$q_\mu^* := \inf\{q \ : \ \tau_\mu^+(q) < +\infty\} \ . \tag{3.7}$$

It is clear that $q_\mu^* \le 1$. As a straightforward consequence of the two previous propositions, we get:

Corollary 3.1. *Let $d_k = \mu([k,k+1))$, $s_\mu^* = \inf\{q>0 \mid \sum_k d_k^q < +\infty\}$, $s_\mu^* \in [0,1]$.*
(i) $s_\mu^* = 0$ iff $q_\mu^* \le 0$.
(ii) If $s_\mu^* > 0$, then $q_\mu^* = s_\mu^*$.
(iii) If $0 \le s_\mu^* < 1$, then for any $q \in (s_\mu^*, 1)$, we have

$$0 \le \tau_\mu^-(q) \le \tau_\mu^+(q) \le 1-q,$$

and for any $q > s^*_\mu$,

$$0 \le D^-_\mu(q) \le D^+_\mu(q) \le 1.$$

(iv) If $s^*_\mu > 0$*, then*

$$q^*_\mu = s^*_\mu = \inf\{q \mid \tau^+_\mu(q) < +\infty\} = \inf\{q \mid \tau^-_\mu(q) < +\infty\}.$$

(v) If μ *has compact support, then* $q^*_\mu \le 0$*, and therefore, for any* $q > 0$*,* $\tau^\pm_\mu(q)$ *are finite and* $D^\pm_\mu(q) \in [0,1]$.

As it is shown by the Examples 3 and 7 of Section 6, q^*_μ may take any values in $[0,1]$. The results of the Corollary generalize the partial informations obtained in [3] for measures with compact support or with finite moments. The above results confirm the common belief that $D^\pm_\mu(q)$ should be smaller than 1. That commonly accepted picture turns out to be true for $q > 0$ in great generality (roughly speaking, if $D^\pm_\mu(q)$ is finite, $q > 0$, then $D^\pm_\mu(q) \in [0,1]$).

Remark 3.1. *The domain* $q \le 0$ *is much more difficult to study because in this case the result of Lemma 2.1 does not hold. If* μ *has a noncompact support, then in all examples we dispose,* $q^*_\mu = 0$*, i.e.* $\tau^\pm_\mu(q) = +\infty$ *for any* $q < 0$*. We don't know whether there exist measures with non compact support such that* $q^*_\mu < 0$.

Consider now the case of compactly supported measures. The examples of Section 6 show that there exist measures such that $\tau^\pm_\mu(q) = +\infty$ *for any* $q < 0$ *(*$q^*_\mu = 0$*), as well as measures such that* $\tau^\pm_\mu(q) < +\infty$ *for any* $q < 0$ *(*$q^*_\mu = -\infty$*). Moreover, in the latter case it is possible that for negative q's* $D^\pm_\mu(q) > 1$*, being finite. In Example 3 of Section 6, we note that the result* $q^*_\mu = 0$ *comes from the existence of a point* $x \in \operatorname{supp} \mu$ *(in the example:* $x = 0$*) with an infinite local exponent (see Definition 3.1).*

We do not have any examples where $q^*_\mu \in (-\infty, 0)$*. Another interesting open problem is the following: is it possible that*

$$\inf\{q \mid \tau^+_\mu(q) < +\infty\} > \inf\{q \mid \tau^-_\mu(q) < +\infty\}?$$

Which can only happen in the region $q < 0$*, according to Corollary 3.1. Finally, some partial results in the domain* $q < 0$ *can be found at the end of the present Section (Proposition 3.3 and Proposition 3.4).*

Now we shall study the regularity properties of $\tau^\pm_\mu(q)$ and of $D^\pm_\mu(q)$ on the set $(q^*_\mu, +\infty)$, where all these functions are finite. We have the following result on the regularity of $\tau^\pm_\mu$.

Theorem 3.2. *Let* μ *be a positive probability Borel measure.*
i) For all $A > q^*_\mu$*, there exists a constant* $K(A)$ *such that*

$$\forall\, q, r \in [A, +\infty), \quad |\tau^\pm(q) - \tau^\pm(r)| \le K(A)|q - r|\,. \tag{3.8}$$

Hence the functions $\tau^\pm_\mu(q)$ *are Lipschitz continuous functions on* $[A, +\infty)$ *and continuous on* $(q^*_\mu, +\infty)$.
ii) Furthermore $\frac{\mathrm{d}}{\mathrm{d}q}\tau^+_\mu(q)$ *exists everywhere on* $(q^*_\mu, +\infty)$ *except, at most, on a countable set of points, and* $\frac{\mathrm{d}}{\mathrm{d}q}\tau^-_\mu(q)$ *exists Lebesgue almost everywhere on* $(q^*_\mu, +\infty)$.

iii) If $\frac{\mathrm{d}}{\mathrm{d}q}\tau_\mu^+(q)$ *exists for some* $q > q_\mu^*$, *then for all* s *such that* $\tau_\mu^+(s) < +\infty$ *and* $s < q$ *we have*

$$\left|\frac{\mathrm{d}}{\mathrm{d}q}\tau_\mu^+(q)\right| \le \left|\frac{\tau_\mu^+(s) - \tau_\mu^+(q)}{s - q}\right| \tag{3.9}$$

and if $\frac{\mathrm{d}}{\mathrm{d}q}\tau_\mu^-(q)$ *exists for some* $q > q_\mu^*$, *then for all* s *such that* $\tau_\mu^+(s) < +\infty$ *and* $s < q$,

$$\left|\frac{\mathrm{d}}{\mathrm{d}q}\tau_\mu^-(q)\right| \le \left|\frac{\tau_\mu^+(s) - \tau_\mu^-(q)}{s - q}\right| . \tag{3.10}$$

Remark 3.2. i) Since $\tau_\mu^\pm(q_\mu^*)$ may be $+\infty$ (see Example 7 of Section 6), the domain of continuity for $\tau_\mu^\pm$ is optimal.
ii) The functions $\tau_\mu^\pm(q)$ and $D_\mu^\pm(q)$ may or may not be right continous at the point q_μ^* as shown by Examples 3, 4 and 7 of Section 6.

We have the two following immediate corollaries:

Corollary 3.2. *The functions* $D_\mu^\pm(q)$ *are continuous on* $(q_\mu^*, 1) \cup (1, +\infty)$.

Corollary 3.3.
(i) For all $q > 1$, *we have*

$$\left|\frac{\mathrm{d}}{\mathrm{d}q}\tau_\mu^\pm(q)\right| \le \frac{\tau_\mu^\pm(q)}{1 - q} = D_\mu^\mp(q) \tag{3.11}$$

(ii) If $q_\mu^* \le 0$, *then for all* $q \in (0, 1)$

$$\left|\frac{\mathrm{d}}{\mathrm{d}q}\tau_\mu^\pm(q)\right| \le \frac{\tau_\mu^+(0+0) - \tau_\mu^\pm(q)}{q} \le \frac{\tau_\mu^+(0+0)}{q} = \frac{D^+(0+0)}{q} \le \frac{1}{q}, \tag{3.12}$$

where $\tau_\mu^\pm(0+0) = \lim_{s\downarrow 0,\, s>0} \tau_\mu^\pm(s)$

The results of Corollary 3.3 follow directly from the bounds (3.9) and (3.10), with $s = 1$ and $s = 0+0$ respectively. The last inequality in (3.12) comes from Corollary 3.1, Point (iii).

Remark 3.3. The inequality (3.11) and the first inequality in (3.12) turn into an equality for all measures μ such that $D_\mu^\pm(q) = D$ are constant. As to the second ineqiality in (3.12), it is saturated when $q \to 1/\lambda$, $q < 1/\lambda$, in the Example 1 of Section 6. In particular, (3.12) implies that if $q_\mu^* \le 0$, then the derivates of $\tau^\pm(q)$ may be big only for small values of $q > 0$ (which turns to happen in Example 1 of Section 6).

The proof of the results contained in Theorem 3.2, Corollary 3.2 and Corollary 3.3 are based on the crucial inequality (3.1). We first exploit this inequality (3.1) in the following technical lemma.

Lemma 3.1. *Let* $s < q < r$. *The following inequalities hold*

$$\tau_\mu^+(q) \le \frac{q-s}{r-s}\tau_\mu^+(r) + \frac{r-q}{r-s}\tau_\mu^+(s) , \tag{3.13}$$

$$\tau_\mu^-(q) \le \min\left(\frac{q-s}{r-s}\tau_\mu^-(r) + \frac{r-q}{r-s}\tau_\mu^+(s),\ \frac{q-s}{r-s}\tau_\mu^+(r) + \frac{r-q}{r-s}\tau_\mu^-(s)\right) . \tag{3.14}$$

Proof of Lemma 3.1.
Taking $t = \frac{q-s}{r-s}$ in inequality (3.1) and taking the log on both sides gives for $f_\varepsilon(q) := \frac{\log I_\mu(q,\varepsilon)}{-\log \varepsilon}$

$$f_\varepsilon(q) \leq \frac{q-s}{r-s} f_\varepsilon(r) + \frac{r-q}{r-s} f_\varepsilon(s) \ . \tag{3.15}$$

But for two given functions g and h one has

$$\begin{aligned} &\limsup(g+h) \leq \limsup g + \limsup h, \\ &\liminf(g+h) \leq \limsup g + \liminf h. \end{aligned}$$

It allows us to conclude the proof. $\square$

Proof of Theorem 3.2.
Equation (3.13) implies that for any $s < q < r$ we have

$$\left|\tau_\mu^+(q) - \tau_\mu^+(r)\right| \leq \left|\frac{q-r}{r-s}\right| \left|\tau_\mu^+(r) - \tau_\mu^+(s)\right| \ . \tag{3.16}$$

Suppose that q_μ^* is finite and let $\delta > 0$ (if $q_\mu^* = -\infty$ then pick A finite and work in $[A, +\infty)$: the proof is the same). Let $q, r \in [q_\mu^* + \delta, +\infty)$. Taking $s = q_\mu^* + \delta/2$ in (3.16), we obtain

$$\begin{aligned} |\tau_\mu^+(q) - \tau_\mu^+(r)| &\leq |r-q| \left(\left|\frac{\tau_\mu^+(s)}{r-s}\right| + \left|\frac{\tau_\mu^+(r)}{r-s}\right| \right) \\ &\leq |r-q| \left(\left|\frac{\tau_\mu^+(q_\mu^* + \delta/2)}{\delta/2}\right| + \left|\frac{\tau_\mu^+(r)}{r-s}\right| \right) \\ &= |r-q| \left(K_1(\delta) + \left|\frac{\tau_\mu^+(r)}{r-s}\right| \right) . \end{aligned} \tag{3.17}$$

Thus to prove the Lipschitz continuity of $\tau_\mu^+(q)$, it is sufficient to estimate $\left|\frac{\tau^+(r)}{r-s}\right|$, where $s = q_\mu^* + \delta/2$ and $r \geq q_\mu^* + \delta$. If $r = 1$, then $\tau_\mu^+(r) = 0$ and there is nothing to prove. Suppose that $r \neq 1$. Then

$$\begin{aligned} \left|\frac{\tau_\mu^+(r)}{r-s}\right| &= \left|\frac{r-1}{r-s}\right| \left|\frac{\tau_\mu^+(r)}{r-1}\right| = \left|\frac{r-1}{r-s}\right| |D_\mu^\pm(r)| \qquad (3.18) \\ &\leq C(\delta) |D_\mu^\pm(q_\mu^* + \delta)| := K_2(\delta). \qquad (3.19) \end{aligned}$$

The choice of D_μ^+ or D_μ^- in (3.19) depends on whether $r < 1$ or $r > 1$. We have also used the fact that $D_\mu^\pm(q)$ are non-increasing. It follows from (3.17)-(3.19) that (3.8) is proved with $K(A) = K_1(\delta) + K_2(\delta)$, where $A = q_\mu^* + \delta$, and thus $\tau_\mu^+(q)$ is Lipschitz continuous on $[A, +\infty)$ for any $A > q_\mu^*$ and thus continuous on $(q_\mu^*, +\infty)$. Using (3.14) in a similar manner yields the Lipschitz continuity for τ_μ^- again on $[A, +\infty)$ and the continuity on $(q_\mu^*, +\infty)$.

The Lipschitz continuity of τ_μ^+ and its convexity implies that it is derivable except at most on a countable set of points. Since τ_μ^- is Lipschitz, it is derivable Lebesgue almost everywhere. From (3.16), we obtain for $q_\mu^* \leq s < q < r$

$$\left|\frac{\tau_\mu^+(q) - \tau_\mu^+(r)}{q-r}\right| \leq \left|\frac{\tau_\mu^+(r) - \tau_\mu^+(s)}{r-s}\right| \ .$$

Taking the limit when $r \to q,\ r > q$, assuming that $\frac{\mathrm{d}}{\mathrm{d}q}\tau_\mu^+(q)$ exists and using the continuity of τ_μ^+ at the point q, we get (3.9). Inequality (3.10) is derived similarly by using (3.14). $\square$

We end this section with a series of properties that precise the behaviour of the functions $\tau_\mu^\pm(q)$ in the two regions $q \le 0$, $q > 1$, and in particular their behaviour at $\pm\infty$. First, as the functions $D_\mu^\pm(q) \in [0,1]$ are decreasing in q, the limits at $\pm\infty$ always exist:

$$D_\mu^\pm(-\infty) := \lim_{q\to-\infty} D_\mu^\pm(q) = \lim_{q\to-\infty} \frac{\tau^\pm(q)}{-q} \in [0,+\infty]\,,$$

and

$$D_\mu^\pm(+\infty) := \lim_{q\to+\infty} D_\mu^\pm(q) = \lim_{q\to+\infty} \frac{\tau^\mp(q)}{-q} \in [0,1]\,.$$

Proposition 3.3. *Let μ be a positive probability Borel measure.*
(i) Assume that $D_\mu^-(-\infty) < +\infty$. Then then for all $q < 0$ we have

$$0 \le \tau_\mu^-(q) \le \tau_\mu^+(0+0) - qD_\mu^-(-\infty). \tag{3.20}$$

(ii) Assume that $D_\mu^+(-\infty) < +\infty$ (so that $q_\mu^* = -\infty$). Then for any $q < 0$ we have

$$0 \le \tau_\mu^\pm(q) \le \tau_\mu^+(0) - qD_\mu^\pm(-\infty) \tag{3.21}$$

As a consequence, for any $q < 0$,

$$(1-q)D_\mu^\pm(0) \le \tau^\pm(q) \le D_\mu^+(0) - qD_\mu^\pm(-\infty). \tag{3.22}$$

(iii) For any $q > 1$,

$$-qD_\mu^\pm(+\infty) \le \tau_\mu^\mp(q) \le (1-q)D_\mu^\pm(+\infty) \tag{3.23}$$

and in particular, $D_\mu^\pm(q) \le q/(q-1)D_\mu^\pm(+\infty)$

If one considers the Example 1 of Section 6 for $\lambda < \alpha+1$ and $q < 0$, we get $\tau_\mu^\pm(q) = \frac{1-\lambda q}{\alpha+1}$, $D_\mu^\pm(0) = \tau_\mu^\pm(0) = \frac{1}{\alpha+1}$ and $D_\mu^\pm(-\infty) = \frac{\lambda}{\alpha+1}$. Therefore, the second inequality in (3.22) turns into equalitity for all $q \le 0$.
As to the first inequality in (3.23), it turns into equality for the measure of the Example 5 of Section 6 for any $q > 1/a$.
We state separately the following lemma that we need to prove Point (iii) of Proposition 3.3, but it may be of independent interest.

Lemma 3.2. *For any $0 < q < r$, we have*

$$\frac{\tau_\mu^\pm(r)}{r} \le \frac{\tau_\mu^\pm(q)}{q}.$$

Proof of Lemma 3.2
Since $q/r < 1$, one has

$$S_\mu(r,\varepsilon)^{q/r} = \left(\sum_j a_j^r\right)^{q/r} \le \sum_j a_j^q = S_\mu(q,\varepsilon)$$

We get $S_\mu(r,\varepsilon)^{1/r} \le (S_\mu(q,\varepsilon))^{1/q}$, and the result follows, thanks to the equivalence given by Lemma 2.1.

□

Proof of Proposition 3.3

The first inequality in (3.22) and the second ineqality in (3.23) are trivially true because $D_\mu^\pm(q)$ are non-increasing. They turn into equalities, for example, if $D_\mu^\pm(q)$ are constant on $(-\infty,0]$ and on $(1,+\infty)$ respectively.

To prove the inequality for $\tau_\mu^+(q)$ in (ii), take inequality (3.13) for fixed $q<0$ and $r=0$ and pass to the limit $s\to-\infty$. The both inequalities for $\tau_\mu^-(q)$ in (i) and (ii) are proved in the same way by considering inequality (3.14). Point (iii) is a consequence of Lemma 3.2. Indeed one has, for any $r>q>1$,

$$\frac{\tau_\mu^\pm(q)}{q} \ge \frac{r-1}{r}\frac{\tau_\mu^\pm(r)}{(r-1)},$$

and the latter quantity tends to $-D_\mu^\mp(+\infty)$ as $r\to+\infty$, which gives the result. □

Finally, we shall discuss in this Section the relation between local exponents of the measure and its multifractal dimensions $D_\mu^\pm(q)$ for $q<0$ and $q>1$.

Definition 3.1. *For all $x\in\mathbb{R}$, we define the upper and lower local exponents of the measure μ at point x as*

$$\begin{aligned}
\gamma_\mu^+(x) &= \limsup_{\varepsilon\downarrow 0}\frac{\log\mu((x-\varepsilon,x+\varepsilon))}{\log\varepsilon},\\
\gamma_\mu^-(x) &= \liminf_{\varepsilon\downarrow 0}\frac{\log\mu((x-\varepsilon,x+\varepsilon))}{\log\varepsilon},
\end{aligned}$$

with the understanding that $\gamma_\mu^\pm(x)=+\infty$ if for some $\varepsilon>0$, $\mu(x-\varepsilon,x+\varepsilon)=0$.

Define now the following numbers:

$$L_\mu^\pm := \inf_{x\in\operatorname{supp}\mu}\gamma_\mu^\pm(x),$$

$$U_\mu^\pm := \sup_{x\in A}\gamma_\mu^\pm(x), \text{ where } \quad A=\{x\in\operatorname{supp}\mu \mid \gamma_\mu^+(x)<+\infty\}.$$

Clearly, $L_\mu^\pm \le \inf_{x\in A}\gamma_\mu^\pm(x)\le U_\mu^\pm$. One should stress that unlike in the next Section, it is the supremum or infimum and not $\mu-$ ess.inf or $\mu-$ ess.sup which are taken. It is well known that $\gamma_\mu^\pm(x)\le 1$ for $\mu-$a.e. x (*e.g.* [1]). Therefore, $\mu(A)=1$, and

$$L_\mu^\pm \le \mu-\text{ess.inf }\gamma_\mu^\pm(x)\le 1,$$

$$U_\mu^\pm \ge \mu-\text{ess.sup }\gamma_\mu^\pm(x).$$

It is clear that strict inequalities are quite possible. The results of the next two propositions should be compared with the inequalities

$$D_\mu^\pm(q)\le \mu-\text{ess.inf }\gamma_\mu^\pm(x),\quad q>1,$$

$$D_\mu^\pm(q)\ge \mu-\text{ess.sup }\gamma_\mu^\pm(x),\quad q<1,$$

of Proposition 4.1 of the next Section.

We begin with the case $q < 0$.

Proposition 3.4. *The following statements hold:*
(i)

$$D_\mu^\pm(-\infty) \geq U_\mu^\pm \tag{3.24}$$

(ii) For any $q < 0$,

$$D_\mu^+(q) \geq \frac{q}{q-1} U_\mu^+ . \tag{3.25}$$

(iii) If $\gamma_\mu^+(x) = \gamma_\mu^-(x)$ *for any* $x \in A$, *so that* $U_\mu^+ = U_\mu^- := U_\mu$, *then for any* $q < 0$

$$D_\mu^-(q) \geq \frac{q}{q-1} U_\mu .$$

Proof of Proposition 3.4 . Let $x_0 \in A, q < 0, \varepsilon \in (0,1)$. As it was mentioned in the Introduction, one can take $\mu((x-\varepsilon, x+\varepsilon))$ instead of $\mu([x-\varepsilon, x+\varepsilon))$ in the definition of $I(q,\varepsilon)$ without changing the dimensions $D_\mu^\pm(q)$. One can estimate

$$I_\mu(q,\varepsilon) \geq \int_{(x_0-\varepsilon,x_0+\varepsilon)\cap \operatorname{supp}\mu} \mu((x-\varepsilon,x+\varepsilon))^{q-1} d\mu(x) \geq \mu((x_0-2\varepsilon,x_0+2\varepsilon))^{q-1} \mu((x_0-\varepsilon,x_0+\varepsilon)).$$

Therefore,

$$\frac{\log I_\mu(q,\varepsilon)}{(q-1)\log\varepsilon} \geq \frac{\log \mu((x_0-2\varepsilon, x_0+2\varepsilon))}{\log\varepsilon} + \frac{1}{q-1}\frac{\log\mu((x_0-\varepsilon,x_0+\varepsilon))}{\log\varepsilon}.$$

Taking the liminf or limsup, we obtain (remember that $\gamma_\mu^+(x_0) < +\infty$)

$$D_\mu^\pm(q) \geq \gamma_\mu^\pm(x_0) + \frac{1}{q-1}\gamma_\mu^+(x_0) . \tag{3.26}$$

Taking the limit $q \to -\infty$ in (3.26) yields

$$D_\mu^\pm(-\infty) \geq \gamma_\mu^\pm(x_0). \tag{3.27}$$

The statements of the Proposition follow directly from (3.26), (3.27). □

In the first Example of Section 6, $\mu -$ ess.sup $\gamma_\mu^\pm(x) = 0$, as the measure is pure point. However, $U_\mu^\pm = \gamma_\mu^\pm(0) = \frac{\lambda-1}{\alpha} > 0$. In the case $\lambda < 1+\alpha$,

$$D_\mu^\pm(-\infty) = \frac{\lambda}{1+\alpha} > U_\mu^\pm,$$

and in the case $\lambda > 1+\alpha$,

$$D_\mu^\pm(-\infty) = U_\mu^\pm \quad \text{and} \quad D_\mu^\pm(q) = \frac{q}{q-1}U_\mu^\pm$$

for any $q < -\frac{\alpha}{\lambda-\alpha-1}$. So, the inequalities (3.24) and (3.25) may turn into equalities for some μ and q. One can note also that a single point ($x = 0$) from $\operatorname{supp}\mu$ such that $\mu(\{x\}) = 0$ may determine the behaviour of $D_\mu^\pm(q)$ as $q \to -\infty$.

A similar result can be proved for $q \to +\infty$.

Proposition 3.5. *For any $q > 1$,*

$$D_\mu^\pm(+\infty) \le L_\mu^\pm. \tag{3.28}$$

Proof of Proposition 3.5 . As in the proof of Proposition 3.4, we easily see that for any $x_0 \in \mathrm{supp}\mu, q > 1$,

$$I_\mu(q, 2\varepsilon) \ge \int_{(x_0-\varepsilon, x_0+\varepsilon)\cap \mathrm{supp}\mu} \mu((x-2\varepsilon, x+2\varepsilon))^{q-1} d\mu(x) \ge \mu((x_0-\varepsilon, x_0+\varepsilon))^q.$$

Therefore,

$$D_\mu^\pm(q) \le \frac{q}{q-1}\gamma_\mu^\pm(x_0).$$

Taking the limit $q \to +\infty$ and then sup over x_0, we obtain the result of the Proposition. □

In the Example 5 of Section 6, $\mu -$ ess.inf $\gamma_\mu^\pm(x) = 1$ and $L_\mu^\pm = \gamma_\mu^\pm(0) = 1 - a < 1$. For $q > 1/a$,

$$D_\mu^\pm(q) = \frac{q}{q-1} L_\mu^\pm,$$

so the inequality (3.28) may turn into equality. Again, a single point $x = 0 \in \mathrm{supp}\ \mu$ such that $\mu(\{x\}) = 0$, determines the behaviour of $D_\mu^\pm(q)$ as $q \to +\infty$.

4 Around q=1

In all this section, we will denote by $D_\mu^\pm(1+0)$ the quantities $\lim_{q\to 1,\ q>1} D_\mu^\pm(q)$, which ly in $[0, 1]$ according to Proposition 3.1, and by $D_\mu^\pm(1-0)$ the quantities $\lim_{q\to 1,\ q<1} D_\mu^\pm(q)$, which ly in $\mathbb{R}_+ \cup \{+\infty\}$. Notice that as $\tau_\mu^\pm(1) = 0$,

$$D_\mu^\pm(1-0) = -(\tau_\mu^\pm)'_l(1) \text{ and } D_\mu^\pm(1+0) = -(\tau_\mu^\mp)'_r(1),$$

where f'_l and f'_r denote the left and right derivates of a function f.

Proposition 4.1. *We have*

$$D_\mu^-(1+0) \le \mu\,\mathrm{ess.inf}\,\gamma_\mu^-(x) \le \mu\,\mathrm{ess.sup}\,\gamma_\mu^-(x) = \dim_H(\mu) \le D_\mu^-(1-0) \tag{4.1}$$
$$D_\mu^+(1+0) \le \mu\,\mathrm{ess.inf}\,\gamma_\mu^+(x) \le \mu\,\mathrm{ess.sup}\,\gamma_\mu^+(x) = \dim_P(\mu) \le D_\mu^+(1-0) \tag{4.2}$$

where $\dim_H(\mu)$ *and* $\dim_P(\mu)$ *are respectively the Hausdorff dimension and packing dimension of the measure* μ.

Remark 4.1. i) The definition of $\dim_H(\mu)$ and $\dim_P(\mu)$ we use are the following:

$$\dim_H(\mu) = \inf\{dim_H(E) \mid \mu(E) = 1\}$$

and similarly for $\dim_P(\mu)$.
ii) In [20] it was proved in the case of measures with compact support that

$$RD_\mu^-(1+0) \le \mu - \mathrm{ess.inf}\ \gamma_\mu^-(x) \le Rh_\mu^- \tag{4.3}$$

and

$$Rh_\mu^+ \le \mu - \mathrm{ess.sup}\ \gamma_\mu^+(x) \le RD_\mu^+(1-0). \tag{4.4}$$

Due to our equivalence Lemmas, (4.3), (4.4) imply the first inequality in (4.1) and the last inequality in (4.2).

Proof of Proposition 4.1: In (4.1) and (4.2), the second inequalities are obvious. The equalities are well known [20] [11]. The last inequalities in (4.1) and (4.2) were proved in [3]. The proof of the first two inequalities can be done in the same spirit. For the sake of completeness, we shall provide below the proofs of all of them.

For any $\nu \in (0,1)$ consider the sets

$$A_\nu(\varepsilon) = \{x \in \mathbb{R} \mid \mu(x-\varepsilon, x+\varepsilon) \geq \varepsilon^{a+\nu}\},$$

where $\varepsilon > 0,\ a = \mu\,\text{ess.inf}\ \gamma_\mu^+(x)$. It is easy to check that

$$A_\nu := \liminf_{\varepsilon \downarrow 0} A_\nu(\varepsilon) \supset B_\nu := \{x \in \mathbb{R} \mid \gamma_\mu^+(x) \leq a + \nu/2\}$$

The definition of $\mu\,$ess.inf implies that $\mu(A_\nu) \geq \mu(B_\nu) > 0$ for any $\nu > 0$. Now for any $q > 1$ one can estimate:

$$\begin{aligned} I_\mu(q,\varepsilon) &= \int_{\mathbb{R}} \mu((x-\varepsilon, x+\varepsilon))^{q-1} d\mu(x) \geq \int_{A_\nu} \mu((x-\varepsilon, x+\varepsilon))^{q-1} d\mu(x) \\ &\geq \varepsilon^{(q-1)(a+\nu)} \mu(A_\nu), \end{aligned}$$

which yields

$$D_\mu^+(q) \leq a + \nu$$

for any $q > 1, \nu > 0$. Since $\nu > 0$ can be arbitrarily small, we obtain the first inequality in (4.2).

Similarly, considering

$$A_\nu(\varepsilon) = \{x \in \mathbb{R} \mid \mu(x-\varepsilon, x+\varepsilon) \leq \varepsilon^{a-\nu}\},$$

where $a = \mu\,\text{ess.sup}\ \gamma_\mu^-(x)$, leads to

$$A_\nu := \liminf_{\varepsilon \downarrow 0} A_\nu(\varepsilon) \supset B_\nu := \{x \in \mathbb{R} \mid \gamma_\mu^-(x) \geq a - \nu/2\},$$

where $\mu(A_\nu) \geq \mu(B_\nu) > 0$. Therefore, for any $q < 1$,

$$I_\mu(q,\varepsilon) \geq \int_{A_\nu} \mu((x-\varepsilon, x+\varepsilon))^{q-1} d\mu(x) \geq \varepsilon^{(q-1)(a-\nu)} \mu(A_\nu),$$

and we obtain

$$D_\mu^-(q) \geq a - \nu,$$

which yields the last inequality in (4.1).

To prove the first inequality in (4.1), consider the sets

$$A_\nu(\varepsilon_k) = \{x \in \mathbb{R} \mid \mu(x-\varepsilon_k, x+\varepsilon_k) \geq \varepsilon_k^{a+\nu}\},$$

where $\varepsilon_k = \exp(-k),\ k \in \mathbb{N},\ a = \mu\,\text{ess.inf}\ \gamma_\mu^-(x)$. Since $\lim_{k\to\infty} \log \varepsilon_k / \log \varepsilon_{k+1} = 1$, we have

$$\limsup_{k\to\infty} \frac{\log \mu((x-\varepsilon_k, x+\varepsilon_k))}{\log \varepsilon_k} = \limsup_{\varepsilon \downarrow 0} \frac{\log \mu((x-\varepsilon, x+\varepsilon))}{\log \varepsilon}.$$

Using this equality, one cheks that

$$A_\nu := \limsup_{k\to\infty} A_\nu(\varepsilon_k) \supset B_\nu := \{x \in \mathbb{R} \mid \gamma_\mu^-(x) \le a + \nu/2\}.$$

Again, we have $\mu(A_\nu) \ge \mu(B_\nu) > 0$ for any $\nu > 0$ as a direct consequence of definition of μ ess.inf. Using the Borel-Cantelli Lemma (as done in [15]), we obtain that $\sum_k \mu(A_\nu(\varepsilon_k)) = +\infty$ and thus, there exists a subsequence $\varepsilon_{k(n)}$ of ε_k such that

$$\mu(A_\nu(\varepsilon_{k(n)})) \ge (k(n))^{-2} = |\log \varepsilon_{k(n)}|^{-2}.$$

Letting $A = A_\nu(\varepsilon_{k(n)})$, we obtain for any $q > 1$:

$$I_\mu(q, \varepsilon_{k(n)}) \ge \int_A \mu((x - \varepsilon_{k(n)}, x + \varepsilon_{k(n)}))^{q-1} d\mu(x) \ge \varepsilon_{k(n)}^{(q-1)(a+\nu)} |\log \varepsilon_{k(n)}|^{-2}.$$

This inequality implies that

$$D_\mu^-(q) \le a + \nu$$

and we obtain the first inequality in (4.1).

Similarly, considering the sets

$$A_\nu(\varepsilon_k) = \{x \in \mathbb{R} \mid \mu(x - \varepsilon_k, x + \varepsilon_k) \le \varepsilon_k^{a-\nu}\}$$

with the same ε_k and $a = \mu -$ ess.sup $\gamma_\mu^+(x)$, one can prove the last inequality in (4.2). □

Quite often, the first and last inequalities in (4.1) and (4.2) turn into equalities (for example, this is the case if $D_\mu^\pm(q) =$ const or in the examples 1 and 5 of Section 6). However, as show the examples 2 and 6, strict inequalities may occur.

As it follows from the above results, if μ ess.inf$\gamma_\mu^-(x) < \mu$ ess.sup$\gamma_\mu^-(x)$, then $D_\mu^-(q)$ is discontinuous at $q = 1$ (the same for γ_μ^+ and D_μ^+).

On the other hand, if $D_\mu^\pm(1-0) = D_\mu^\pm(1+0) = D$, then (4.1) and (4.2) imply

$$\gamma_\mu^+(x) = \gamma_\mu^-(x) = D, \quad \text{for } \mu - \text{a.e. } x \ , \tag{4.5}$$

i.e. the measure μ is "exact dimensional" [26]. One should stress that the converse is not true: there are measures such that (4.5) holds, but $D_\mu^\pm(q) \ne$ const (most of examples of Section 6). This is not in general true even for $q > 1$ (Example 5), unlike the result of Theorem 9.2 in [26] (see also p.183 in [26]). The point is that there the definition of multifractal dimensions (HPM -spectrum in notations of [26]) is different from our's (HP-spectrum in notations of [26]). The same comment should be done about the result of Theorem 18.1 in [26] which states that the HPM-multifractal dimensions of equivalent measures are identical for $q > 1$. In our case (HP dimensions) this is not true (consider the Example 1 with the same α and different λ's (in the case $q < 1$) or the Example 5 with different a's (in the case $q > 1$)).

Proposition 4.2. *We have*

$$\mu\,\text{ess.inf}\,\gamma_\mu^-(x) \le h_\mu^- \ , \tag{4.6}$$

$$h_\mu^+ \le \mu\,\text{ess.sup}\,\gamma_\mu^+(x) \ , \tag{4.7}$$

$$D_\mu^\pm(1+0) \le h_\mu^\pm = D_\mu^\pm(1) \le D_\mu^\pm(1-0) \ . \tag{4.8}$$

Remark 4.2. *Again, due to our equivalence Lemmas, the inequalities (4.3), (4.4) of [20] imply (4.6), (4.7) and two of four inequalities in (4.8).*

Proof of Proposition 4.2: As it was mentioned above, the inequality (4.6) as well as the inequality (4.7) is a consequence of our Lemma 2.2 and Theorem 4.1 in [20]. A direct proof that do not use the Rényi entropies $Rh_\mu^\pm$ can easily be done by using Fatou's lemma.

Two inequalities in (4.8, namely, $D^-(1+0) \leq h^-$ and $h^+ \leq D^+(1-0)$, follows directly from (4.3), (4.4) due to our equivalence Lemmas. Anyway, a simple proof we present below gives all of them, including these two. Assume first that $q < 1$. Using Jensen inequality with the convex function $g(y) = \frac{1}{q-1} \log y$ and $y(x) = \mu((x-\varepsilon, x+\varepsilon))^{q-1}$, we get, as $\log \varepsilon < 0$,

$$\frac{\log \left(\int_{\operatorname{supp}\mu} \mu((x-\varepsilon,\, x+\varepsilon))^{q-1} \mathrm{d}\mu(x) \right)}{(q-1)\log \varepsilon} \geq \frac{\int_{\operatorname{supp}\mu} \log\left(\mu([x-\varepsilon,\, x+\varepsilon))\right) \mathrm{d}\mu(x)}{\log \varepsilon} \,.$$

Taking the $\liminf_{\varepsilon\downarrow 0}$ or $\limsup_{\varepsilon \downarrow 0}$ we show that $D_\mu^\pm(q) \geq h_\mu^\pm$ for any $q < 1$. If $q > 1$, the function $g(y)$ is concave, and in the same manner we obtain $D_\mu^\pm(q) \leq h_\mu^\pm$ for any $q > 1$ which concludes the proof.

Remark 4.3. From Proposition 4.1 and 4.2, it is natural to ask if this is possible to locate, in great generality, h_μ^- with respect to $\mu\,$ess.sup $\gamma_\mu^-(x)$ and h_μ^+ with respect to $\mu\,$ess.inf $\gamma_\mu^+(x)$. The answer is no. It is already known from the example in [5, §4] that it may happen that $h_\mu^- > \mu\,$ess.sup $\gamma_\mu^-(x)$ and $h_\mu^+ < \mu\,$ess.inf $\gamma_\mu^+(x)$. On the other hand if one considers the measure $\mu = \frac{1}{2}\delta_0 + \frac{1}{2}\lambda_{[1,2]}$, where $\lambda_{[1,2]}$ is the Lebesgue measure on $[1,2]$, a straightforward calculation yields that

$$0 = \mu\,\text{ess.inf }\gamma_\mu^+(x) < h_\mu^+ = \frac{1}{2} = h_\mu^- < \mu\,\text{ess.sup }\gamma_\mu^-(x) = 1\,.$$

5 Convolution of measures

How do the above fractal dimensions of measures behave under product of convolution is a basic question that one can address. Concerning the Hausdorff dimension it is known that for "exact scaling" (or "exact dimensional") measures $\dim_H(\mu * \nu) = \min(1, \dim_H(\mu) + \dim_H(\nu))$ [21]. As an application, in the setting of quantum diffusion, this remark has been used in [6] to get some subdiffusive transport with absolutely continuous spectrum. We note that to the best of our knowledge, we do not know whether bounds like $\dim_H(\mu*\nu) \leq \dim_H(\mu) + \dim_H(\nu)$ and $\dim_P(\mu * \nu) \leq \dim_P(\mu) + \dim_P(\nu)$ could hold in full generality or not.

In this section, we show that the equivalence between the multifractal dimensions $D^\pm(q)$ and the Rényi dimensions (with the sums) is a useful tool to get quite easily some interesting informations on the dimensions of a product of convolution of two measures.

The result is the following.

Proposition 5.1. *Let ν and μ be two positive probability Borel measures on $\mathbb{R}$, and denote by $\mu * \nu$ their convolution. For all $q > 0$, $q \neq 1$, one has*

$$D_{\mu*\nu}^-(q) \;\geq\; \max(D_\mu^-(q), D_\nu^-(q)) \tag{5.1}$$
$$D_{\mu*\nu}^+(q) \;\geq\; \max(D_\mu^+(q), D_\nu^+(q)) \;, \tag{5.2}$$

and

$$\begin{aligned} D^-_{\mu*\nu}(q) &\leq \min\left(D^-_\mu(q)+D^+_\nu(q), D^+_\mu(q)+D^-_\nu(q)\right) && (5.3)\\ D^+_{\mu*\nu}(q) &\leq D^+_\mu(q)+D^+_\nu(q)\ . && (5.4)\end{aligned}$$

Remark 5.1. For $q=1$, *i.e.* for the entropy dimensions, the situation is not clear. The lower bound, $h^\pm_{\mu*\nu}\geq \max(h^\pm_\mu, h^\pm_\nu)$ is easy to get using Theorem 2.2 that deals with translated grids. Indeed, with the usual notations of this paper $\mu*\nu(I_j)=\int \mathrm{d}\mu(x)\nu(-x+I_j)$, and then with f the concave function defined as $f(x)=x\log(1/x)$ for $x>0$ and $f(0)=0$:

$$\begin{aligned}\sum_{j\in\mathbb{Z}} f(\mu*\nu(I_j)) &\geq \sum_{j\in\mathbb{Z}}\int \mathrm{d}\mu(x) f(\nu(-x+I_j)) = \int \mathrm{d}\mu(x) h_\nu^{\{-x\}}(\varepsilon)\\ &\geq \int \mathrm{d}\mu(x)(S_\nu(1,\varepsilon)-2) = S_\nu(1,\varepsilon)-2,\end{aligned}$$

where $h_\nu^{\{-x\}}(\varepsilon)=\sum_j f(\nu(-x+I_j)) = S_\nu^{\{\mathrm{Frac}(-x/\varepsilon)\}}(1,\varepsilon)$ with the notations of Theorem 2.2 ($\mathrm{Frac}(y)$ stands for the fractional part of y). Dividing by $-\log\varepsilon$ and taking the liminf, limsup yields the result.
For the upper bound Estimate (5.6) leads to $h^+_{\mu*\nu}\leq 2(h^+_\mu+h^+_\nu)$ (and to a similar bound in the spirit of (5.3) for $h^-_{\mu*\nu}$). That means that one gets an extra multiplicative factor 2. We met the same issue in the proof of Theorem 2.2. Unfortunately the same analysis we performed there does not apply here. It is not clear to us whether a general upper bound in the spirit of (5.3) and (5.4) could be obtained for the entropy dimensions.

Proof of Proposition 5.1. Since, thanks to Theorem 2.1, the different definitions of the multifractal dimensions of a measure turn out to supply the same number, we shall use, in each case the most convenient definitions. We first prove (5.1)-(5.2).
For $q\neq 1$ the fastest way consists in using the mean-q dimensions. If $q>1$, we have

$$\begin{aligned} L_{\nu*\mu}(q,\varepsilon) &= \frac{1}{\varepsilon}\int_{\mathbb{R}} (\nu*\mu)^q([u-\varepsilon,u+\varepsilon))du\\ &= \frac{1}{\varepsilon}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\mu([u-x-\varepsilon,u-x+\varepsilon))d\nu(x)\right)^q du\\ &\leq \frac{1}{\varepsilon}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}(\mu([u-x-\varepsilon,u-x+\varepsilon)))^q d\nu(x)\right) du\\ &= \frac{1}{\varepsilon}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}(\mu([u-x-\varepsilon,u-x+\varepsilon)))^q du\right) d\nu(x)\\ &= \frac{1}{\varepsilon}\int_{\mathbb{R}}(\mu([u-\varepsilon,u+\varepsilon)))^q du = L_\mu(q,\varepsilon)\ . && (5.5)\end{aligned}$$

This gives, together with Lemma 2.3 that $D^\pm_{\nu*\mu}(q)\geq D^\pm_\nu(q)$. Reversing the role of ν and μ in the above inequalities gives the expected result for q larger than 1.
In the case $q\in(0,1)$, the inequalities (5.5) are reversed. Going back to the definition of the generalized fractal dimensions of a measure for $q\in(0,1)$, leads also to inequalities (5.1)-(5.2).

We turn to the proof of (5.3) and (5.4), and we shall use the definition of the multifractal dimensions that involves the sum over the intervals I_j. We adopt the following simplified notations:

$$I_j^{(\varepsilon)} = I_j = [j\varepsilon,(j+1)\varepsilon),\quad a_j=\mu(I_j)\quad \text{and}\quad b_j=\nu(I_j)\ .$$

We first treat the case $q \in (0,1)$. Note that for any $x \in I_k$, the set $-x + I_j$ is included in $I_{j-k-1} \cup I_{j-k}$. Thus

$$\mu * \nu(I_j) = \int \mathrm{d}\mu(x)\nu(-x + I_j) \leq \sum_k a_k(b_{j-k-1} + b_{j-k}). \tag{5.6}$$

Therefore, since $q \in (0,1)$

$$\begin{aligned}\sum_j (\mu * \nu(I_j))^q &\leq \sum_j \left(\sum_k a_k(b_{j-k-1} + b_{j-k})\right)^q \\ &\leq \sum_j \sum_k a_k^q (b_{j-k-1}^q + b_{j-k}^q).\end{aligned}$$

It follows that

$$\sum_j (\mu * \nu(I_j))^q \leq \sum_k a_k^q \sum_j (b_{j-k-1}^q + b_{j-k}^q) \leq 2\left(\sum_k a_k^q\right)\left(\sum_j b_j^q\right).$$

Thus, this inequality together with Lemma 2.1 gives $\tau_{\mu*\nu}(q) \leq \tau_\mu(q) + \tau_\mu(q)$.

We now turn to case $q > 1$. We shall need here two grids of two different sizes: ε and 2ε. In order to stress the dependency of I_j in ε, we shall use the notation $I_j^{(\varepsilon)} = [j\varepsilon, (j+1)\varepsilon)$.
Note that if $x = (k+1)\varepsilon$ then $-x + I_j^{(2\varepsilon)} = [(2j-k-1)\varepsilon, (2j-k+1)\varepsilon)$ and if $x = k\varepsilon$ then $-x + I_j^{(2\varepsilon)} = [(2j-k)\varepsilon, (2j-k+2)\varepsilon)$. Therefore one checks that

$$\forall\, x \in I_k^{(\varepsilon)}, \quad I_{2j-k}^{(\varepsilon)} = [(2j-k)\varepsilon, (2j-k+1)\varepsilon) \subset \left(-x + I_j^{(2\varepsilon)}\right).$$

It follows that

$$\mu * \nu\left(I_j^{(2\varepsilon)}\right) = \sum_k \int_{I_k^{(\varepsilon)}} \mathrm{d}\mu(x)\ \nu\left(-x + I_j^{(2\varepsilon)}\right) \geq \sum_k a_k b_{2j-k}.$$

Where, again, $a_j = \mu\left(I_j^{(\varepsilon)}\right)$ and $b_j = \nu\left(I_j^{(\varepsilon)}\right)$ (to alleviate the notations we drop the dependency of a_j and b_j on the size ε of the grid). Then, since $q > 1$,

$$\begin{aligned}\sum_j \mu * \nu\left(I_j^{(2\varepsilon)}\right)^q &\geq \sum_j \left(\sum_k a_k b_{2j-k}\right)^q \\ &\geq \sum_j \sum_k a_k^q b_{2j-k}^q = \sum_k a_k^q \sum_j b_{2j-k}^q\end{aligned}$$

The latter estimate can be written

$$\sum_j \mu * \nu\left(I_j^{(2\varepsilon)}\right)^q \geq \sum_k a_{2k}^q \sum_j b_{2j-2k}^q + \sum_k a_{2k+1}^q \sum_j b_{2j-2k-1}^q. \tag{5.7}$$

Without loss of generality, assume that $\sum_j b_{2j}^q \geq \sum_j b_{2j+1}^q$, that is $\sum_j b_{2j}^q \geq \frac{1}{2}\sum_j b_j^q$. Then

$$\sum_j \mu * \nu \left(I_j^{(2\varepsilon)}\right)^q \geq \frac{1}{2}\left(\sum_k a_{2k}^q\right)\left(\sum_j b_j^q\right).$$

If again $\sum_k a_{2k}^q \geq \sum_k a_{2k+1}^q$ then we are done since

$$S_{\mu*\nu}(q, 2\varepsilon) = \sum_j \mu * \nu \left(I_j^{(2\varepsilon)}\right)^q \geq \frac{1}{4}\left(\sum_k a_k^q\right)\left(\sum_j b_j^q\right). \tag{5.8}$$

If not, that is if $\sum_k a_{2k+1}^q \geq \sum_k a_{2k}^q$, one then works with the family of intervals $\tilde{I}_j^{(2\varepsilon)} = \varepsilon + I_j^{(2\varepsilon)}$ instead of the $I_j^{(2\varepsilon)}$'s. Instead of (5.7), this leads to

$$\sum_j \mu * \nu \left(\tilde{I}_j^{(2\varepsilon)}\right)^q \geq \sum_k a_{2k}^q \sum_j b_{2j-2k+1}^q + \sum_k a_{2k+1}^q \sum_j b_{2j-2k}^q.$$

And thus

$$\sum_j \mu * \nu \left(\tilde{I}_j^{(2\varepsilon)}\right)^q \geq \frac{1}{2}\sum_k a_{2k+1}^q \sum_j b_j^q \geq \frac{1}{4}\left(\sum_k a_k^q\right)\left(\sum_j b_j^q\right). \tag{5.9}$$

But by Theorem 2.2, $S_{\mu*\nu}(q, 2\varepsilon) \geq 2^{-(q+1)} \sum_j \mu*\nu\left(\tilde{I}_j^{(2\varepsilon)}\right)^q$. In consequence the estimates (5.8) and (5.9) leads to (5.3) and (5.4) of the Proposition. This ends the proof of the Proposition. □

6 Examples

Example 1.
For $\lambda > 1$ and $\alpha > 0$, let $a_n = a/n^\lambda$, $x_n = 1/n^\alpha$, where $a > 0$ is a normalization constant. We define the pure point probability measure $\mu = \sum_{n=1}^\infty a_n \delta_{x_n}$ on $\mathbb{R}$. This measure has been showed in [3] to have non trivial dimensions, which was of particular interest in the setting [3]. In the context of this paper, this example is particularly interesting to emphasize the difference between the regime $q > 1$ and $q < 1$. We provide here the explicit values of the $D_\mu^\pm(q)$'s for that atomic measure. We shall not supply all the technicals details but restrict ourselves to provide the reader the main significant steps of the calculations. First, as the measure is pure point, $D_\mu^\pm(q) = 0$ for any $q > 1$. One can also note that supp $\mu = \{0, x_1, x_2, ...\}$.
When analyzing, for a fixed ε, the behaviour in x_n of $\mu(x_n - \varepsilon, x_n + \varepsilon)$, three regimes clearly appear. One shall compute

$$I_\mu(q, \varepsilon) = \sum_{n\geq 1} \mu(\{x_n\})\mu^{q-1}(x_n - \varepsilon, x_n + \varepsilon)$$

for $q < 1$ by cutting it in three corresponding pieces I_1, I_2, I_3.
Regime 1: $\frac{1}{n^\alpha} < \varepsilon$, or equivalently $n > (\frac{1}{\varepsilon})^{\frac{1}{\alpha}}$.
Then x_n is close enough to 0 so that $\mu(x_n - \varepsilon, x_n + \varepsilon) = \mu(0, x_n + \varepsilon)$, and hence $\mu(0, \varepsilon) \leq \mu(x_n - \varepsilon, x_n + \varepsilon) \leq \mu(0, 2\varepsilon)$. It thus comes out that $\mu(x_n - \varepsilon, x_n + \varepsilon) \sim \varepsilon^{\frac{\lambda-1}{\alpha}}$. In particular,

$\gamma_\mu^\pm(0) = \frac{\lambda-1}{\alpha}$ (and $\gamma_\mu^\pm(x) = 0$ for other points $x = x_n \in \text{supp } \mu$). The corresponding part I_1 of $I_\mu(q,\varepsilon)$ is equivalent to $\varepsilon^{\frac{q(\lambda-1)}{\alpha}}$.
Regime 2: $(\frac{\alpha}{\varepsilon})^{\frac{1}{\alpha+1}} \leq n \leq (\frac{1}{\varepsilon})^{\frac{1}{\alpha}}$.
Then $x_n - \varepsilon > 0$ and $x_n - x_{n+1} < \varepsilon$. That means that $(x_n - \varepsilon, x_n + \varepsilon)$ contains more than one point of the support but does not reach the edge 0. One then gets $\mu(x_n - \varepsilon, x_n + \varepsilon) \sim \frac{\varepsilon}{n^{\lambda-1-\alpha}}$.
If $\lambda < 1 + \alpha$, then for any $q < 1$ we have

$$I_2 \sim \varepsilon^{\frac{q\lambda-1}{1+\alpha}}. \tag{6.1}$$

If $\lambda > 1 + \alpha$, then (6.1) holds for $q > q_0 := -\frac{\alpha}{\lambda-1-\alpha}$, and for $q < q_0$,

$$I_2 \sim \varepsilon^{\frac{q(\lambda-1)}{\alpha}} \sim I_1.$$

Regime 3: $n < (\frac{\alpha}{\varepsilon})^{\frac{1}{\alpha+1}}$.
In this regime, $(x_n - \varepsilon, x_n + \varepsilon) \cap \text{supp}\mu = \{x_n\}$, thus $\mu(x_n - \varepsilon, x_n + \varepsilon) = \frac{1}{n^\alpha}$. This yields

$$I_3 \sim \text{const}, \quad q \geq 1/\lambda,$$

and

$$I_3 \sim \varepsilon^{\frac{q\lambda-1}{1+\alpha}}, \quad q < 1/\lambda.$$

For any $q < 1$ one can compare the values of I_1, I_2, I_3 and find the dominant term (or terms) which determines the behaviour of $I_\mu(q,\varepsilon)$ as $\varepsilon \downarrow 0$. The result is the following.
1) The case $\lambda < 1 + \alpha$ (in fact, the same result holds also for $\lambda = 1 + \alpha$). One has

$$\tau_\mu^\pm(q) = D_\mu^\pm(q) = 0, \quad q \geq 1/\lambda,$$

$$\tau_\mu^\pm(q) = \frac{1-\lambda q}{1+\alpha}, \; D_\mu^\pm(q) = \frac{1-\lambda q}{(1-q)(1+\alpha)}, \quad q < 1/\lambda.$$

2) The case $\lambda > 1 + \alpha$. One has

$$\tau_\mu^\pm(q) = D_\mu^\pm(q) = 0, \quad q \geq 1/\lambda,$$

$$\tau_\mu^\pm(q) = \frac{1-\lambda q}{1+\alpha}, \; D_\mu^\pm(q) = \frac{1-\lambda q}{(1-q)(1+\alpha)}, \quad q_0 < q \leq 1/\lambda,$$

$$\tau_\mu^\pm(q) = \frac{q(1-\lambda)}{\alpha}, \; D_\mu^\pm(q) = \frac{q(1-\lambda)}{(1-q)\alpha}, \quad q \leq q_0,$$

where $q_0 = -\frac{\alpha}{\lambda-1-\alpha}$. One can note that for all values of parameters, $q_\mu^* = -\infty$ and the functions $\tau_\mu^\pm(q)$, $D_\mu^\pm(q)$ are continuous on $\mathbb{R}$.
Note that for $q = 0$, in both cases, one gets $D_\mu^\pm(0) = 1/(1+\alpha)$ which is equal to the box counting dimension of the support of the measure μ. So in this case one indeed has $D_\mu^\pm(0) = RD_\mu^\pm(0)$.

Example 2. With the same notations as in the previous example, let us take $a_n = \frac{a}{n \log^p n}$, $x_n = \frac{1}{\log n}$, where $n \geq 2$, $p > 1$. One can do the calculations similar to those of example 1. First, one shows that $\gamma_\mu^\pm(0) = p-1$ (and at the same time $\mu -$ ess.sup $\gamma_\mu^\pm(x) = 0$). As to the dimensions $D_\mu^\pm(q)$, the result is the following.

1) The case $1 < p \le 2$.

$$D_\mu^\pm(q) = 0, \quad q > 1, \text{and} \quad D_\mu^\pm(q) = 1, \quad q < 1.$$

2) The case $p > 2$.

$$D_\mu^\pm(q) = 0, \ q > 1; \ D_\mu^\pm(q) = 1, \ q \in (q_0, 1); \ D_\mu^\pm(q) = \frac{q(p-1)}{q-1}, \quad q \le q_0;$$

where $q_0 = -\frac{1}{p-2}$.

Example 3. Let $a_n = a\exp(-n)$, $x_n = \frac{1}{n}$, $n \ge 1$. Then $\gamma_\mu^\pm(0) = +\infty$ and

$$D_\mu^\pm(q) = 0, \ q > 0; \ \ D_\mu^\pm(q) = +\infty, \ q < 0.$$

In particular, $q_\mu^* = 0$.
One also checks that $D_\mu^\pm(0) = 1/2$, which is equal to $RD_\mu^\pm(0)$.

Example 4. Let $a_n = \exp(-pn)$, $x_n = \exp(-n)$, $p \ge 0$, $n \ge 1$. Then $\gamma_\mu^\pm(0) = p$ and

$$D_\mu^\pm(q) = 0, \ q \ge 0; \ D_\mu^\pm(q) = \frac{pq}{q-1}, \ q < 0.$$

Example 5. For any $a \in (0,1)$, consider the following a.c. measure on $(0,1]$: $d\mu(x) = C(a)x^{-a}dx$. Obviously, supp $\mu = [0,1]$ and $\gamma_\mu^\pm(x) = 1$ for any $x \in (0,1]$. One can easily calculate that $\gamma_\mu^\pm(0) = 1-a$ and $\mu([j\varepsilon, (j+1)\varepsilon)) \sim C\varepsilon^{1-a}j^{-a}$ for $j \ge 1$. Straightforward calculations leads to

$$D_\mu^\pm(q) = 1, \quad q \le 1/a; \ D_\mu^\pm(q) = \frac{q(1-a)}{q-1}, \quad q > 1/a.$$

Example 6. Consider the a.c. measure $d\mu(x) = \frac{1}{x\log^2 x}dx$ on $(0, 1/2]$. Obviously,

$$\mu - \text{ess.inf } \gamma_\mu^\pm(x) = 1.$$

The calculations show that $\gamma_\mu^\pm(0) = 0$ and

$$D_\mu^\pm(q) = 1, \quad q < 1; \ D_\mu^\pm(q) = 0, \quad q > 1.$$

Example 7.
Let $d\mu(x) = x^{-a}(\log x)^{-b}\chi_{(3,+\infty)}(x)\mathrm{d}x$, where $\chi_{(3,+\infty)}$ is the characteristic function of the interval $(3, +\infty)$. Furthermore, we assume that $(a,b) \in ((1,+\infty) \times \mathbb{R}) \cup (\{1\} \times (1,+\infty))$ so that μ is a finite measure. Then for all $q \in \mathbb{R}$ there exists $C_1(q)$ and $C_2(q)$ such that for all $\varepsilon \in (0,1)$,

$$C_1\varepsilon^{q-1}\int_3^{+\infty}\left(x^{-a}(\log x)^{-b}\right)^q \mathrm{d}x \le I_\mu(q,\varepsilon) \le C_2\varepsilon^{q-1}\int_3^{+\infty}\left(x^{-a}(\log x)^{-b}\right)^q \mathrm{d}x \ .$$

This gives $q_\mu^* = \frac{1}{a}$ and

$$D_\mu^\pm(q) = \begin{cases} 1 & \text{for } q > q_\mu^* = 1/a \\ +\infty & \text{for} q < q_\mu^* = 1/a \\ \begin{cases} 1 & \text{if } b/a > 1 \\ +\infty & \text{if } b/a \leq 1 \end{cases} & \text{for } q = q_\mu^* = 1/a \end{cases}$$

This simple example illustrates that $D_\mu^\pm(q)$ at $q = q_\mu^*$ may or may not be right continuous.

Example 8.
Let μ be the Lebesgue measure on $[1,2]$. It is easy to see that for any $q < 0$, one has $(2\varepsilon^{q-1}) \leq I(q,\varepsilon) \leq \varepsilon^{q-1}$, so $D_\mu^\pm(q) = 1$. Let us consider the particular sequence $\varepsilon_n = 2^{-n}(1 + 2^{-n^2})$. One checks that $2^n \varepsilon_n = 1 + 2^{-n^2} > 1$ and that $(2^n - 1)\varepsilon_n < 1$. As a consequence the interval $I_{2^n-1}^{(\varepsilon_n)}$ intersects the support of μ, and in addition $\mu\left(I_{2^n-1}^{(\varepsilon_n)}\right) = 2^{-n^2}$. Therefore

$$S(q,\varepsilon_n) \geq \mu\left(I_{2^n-1}^{(\varepsilon_n)}\right)^q = 2^{-qn^2}.$$

Taking the log, dividing by $\log \varepsilon_n$ and taking the limit, one ends up with

$$RD_\mu^+(q) = \limsup_{\varepsilon \downarrow 0} \frac{\log S(q,\varepsilon)}{(q-1)\log \varepsilon} = +\infty.$$

That illustrates the fact that for negative q the dimensions $RD_\mu^\pm(q)$ are much less stable than for positive q's. One may have $D_\mu^+(q) = 1 < +\infty = RD_\mu^+(q)$ for any $q < 0$.

A Appendix

Let R be a real continuous function such that R decays faster than any polynomials at infinity. We suppose also that $R(t) \geq 0$ for any t and $\inf_{[-1,1]} R(t) = b > 0$. Given $q \in > 0$, $q \neq 1$, and $\varepsilon \in (0,1)$ we define

$$K_\mu^{(R)}(q,\,\varepsilon) = \int_{\mathrm{supp}\mu} \left(b^{(R)}(x,\varepsilon)\right)^{q-1} \,\mathrm{d}\mu(x)\ ,$$

where

$$b^{(R)}(x,\varepsilon) = \int_{\mathrm{supp}\mu} R\left(\frac{x-y}{\varepsilon}\right) \,\mathrm{d}\mu(y)\ .$$

The integral $K_\mu^{(R)}(q,\,\varepsilon)$ is quite close to the integrals $I_\mu(q,\,\varepsilon)$ we studied throughout the present paper. Indeed note that $\mu(x-\varepsilon, x+\varepsilon) = \int \chi_{[-1,1]}((x-y)/\varepsilon)\mathrm{d}y$. In other words $K_\mu^{(R)}(q,\,\varepsilon)$ is a generalization of the integral $I_\mu(q,\,\varepsilon)$ where the characteristic function $\chi_{[-1,1]}$ is replaced by a function R with fast decay at $\pm\infty$: $I_\mu(q,\,\varepsilon) = K_\mu^{(\chi_{[-1,1]})}(q,\,\varepsilon)$. The result is the following.

Theorem A.1. *Pick $q \in (0,1)$. There exists $0 < c(q) < \infty$ such that for all $\varepsilon \in (0,1)$, we have*

$$\frac{1}{c(q)} I_\mu(q,\varepsilon) \leq K_\mu^{(R)}(q,\varepsilon) \leq c(q) I_\mu(q,\varepsilon)\ . \tag{A.1}$$

In particular, $I_\mu(q,\varepsilon)$ and $K_\mu(q,\varepsilon)$ have the same growth exponents: $\tau_\mu^\pm(q)$.

Theorem A.1 is of particular interest in the setting of quantum dynamics. Indeed as it is shown in [2, 3] and [16], a lower bound on the dynamics can be obtained in terms of the integral $K_\mu^{(R)}(q,\,\varepsilon)$ above, $q \in (0,1)$. That leads to a diffusion coefficient which needs not to be equal to $D_\mu^\pm(q)$ *a priori*. Under some assumptions, among them μ with compact support, it has been proved in [3] that both $I_\mu(q,\varepsilon)$ and $K_\mu(q,\varepsilon)$ lead to the same dimension numbers for $q \in (0,1)$, namely $D_\mu^\pm(q)$. Nevertheless it was not clear whether the equality should hold in full generality or not. Theorem A.1 answers positively to that issue.

Proof of Theorem A.1:
As in Section 2, for all $k \in \mathbb{Z}$ and $\varepsilon \in (0,1)$, we denote $I_k = [\varepsilon k, \varepsilon(k+1))$ and $a_k = \mu(I_k)$. For $n \in \mathbb{Z}$ let $\tilde{R}(n) = \sup_{t\in[n-1,n+1)} R(t)$. Now for all $k, j \in \mathbb{Z}$ and for all $x \in I_j$, we have

$$\begin{aligned} \int_{I_k} R\left(\frac{x-y}{\varepsilon}\right) \mathrm{d}\mu(y) &\leq \int_{I_k} \sup_{(x,y)\in I_j\times I_k} R\left(\frac{x-y}{\varepsilon}\right) \mathrm{d}\mu(y) \\ &\leq \int_{I_k} \tilde{R}(j-k)\,\mathrm{d}\mu(y) = a_k \tilde{R}(j-k)\ . \end{aligned} \tag{A.2}$$

Without loss of generality, we may assume that $\tilde{R}(0) = \sup_{[-1,1]} R(t) \leq 1$. Therefore, for $q \in (0,1)$,

$$\begin{aligned} K_\mu^{(R)}(q,\varepsilon) &= \sum_{j\in\mathbb{Z}} \int_{I_j} \left(\sum_k \int_{I_k} R\left(\frac{x-y}{\varepsilon}\right) \mathrm{d}\mu(y) \right)^{q-1} \mathrm{d}\mu(x) \\ &\geq \sum_{j\in\mathbb{Z}} a_j\,(a_j+b_j)^{q-1}\ , \end{aligned} \tag{A.3}$$

where $b_j := \sum_{k\neq j} \tilde{R}(j-k)a_k$. For fixed $N > 1$, let $A = \{j \in \mathbb{Z} \mid 0 \leq b_j/N < a_j\}$ and $B = \{j \in \mathbb{Z} \mid 0 < a_j \leq b_j/N\}$. Then from (A.3), we get

$$K_\mu^{(R)}(q,\varepsilon) \geq (1+N)^{q-1} \sum_{j\in A} a_j^q\ . \tag{A.4}$$

On the other hand, we have

$$\begin{aligned} \sum_{j\in B} a_j^q &\leq N^{-q} \sum_{j\in B} \left(\sum_{k\neq j} \tilde{R}(j-k)a_k \right)^q \\ &\leq N^{-q} \sum_{j\in Z} \sum_{k\neq j} \left(\tilde{R}(j-k)\right)^q a_k^q \\ &\leq N^{-q} \sum_{k\in Z} a_k^q \sum_{j\neq k} \left(\tilde{R}(j-k)\right)^q \\ &\leq C(q) N^{-q} \sum_{k\in Z} a_k^q\ , \end{aligned} \tag{A.5}$$

where $C(q) := \sum_{k\neq 0} \tilde{R}(k)^q < \infty$ since R decays faster than any polynomials. Taking $N = (2C(q))^{1/q}$ in (A.5) gives, together with (A.4), that

$$K_\mu^{(R)}(q,\varepsilon) \geq \frac{(1+N)^{q-1}}{2} \sum_{j\in Z} a_j^q\ . \tag{A.6}$$

On the other side, we also have the following inequality, for $q < 1$,

$$K_\mu^{(R)}(q,\varepsilon) = \int_{\mathrm{supp}\mu} \left(R\left(\frac{x-y}{\varepsilon}\right) \mathrm{d}\mu(y) \right)^{q-1} \mathrm{d}\mu(x)$$

$$\leq b^{q-1} \int_{\mathrm{supp}\mu} \mu([x-\varepsilon, x+\varepsilon])^{q-1} d\mu(x) = b^{q-1} I_\mu(q,\varepsilon), \tag{A.7}$$

where $b = \inf_{[-1,1]} R(t) > 0$. Inequalities (A.6) and (A.7) and Lemma 2.1 conclude the proof of the Theorem. $\square$

Remark A.1. If $q > 1$ the conclusions of Theorem A.1 still hold. Like for Lemma 2.1 the proof is even simpler, since there is no needs to split the sum over "good" and "bad" a_j's (sets A and B above). Indeed from (A.2) one gets $K_\mu^{(R)}(q,\varepsilon) \leq \sum_{j\in\mathbb{Z}} (\sum_{k\in\mathbb{Z}} a_k \tilde{R}(j-k))^q$. A Jensen inequality ends the proof in this case.

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