MULTI-DIMENSIONAL SCHRÖDINGER OPERATORS WITH NO NEGATIVE SPECTRUM

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1. Introduction

In this paper we consider Schrödinger operators

$$-\Delta + V(x), \quad V \in L^\infty(\mathbb{R}^d)$$

acting in the space $L^2(\mathbb{R}^d)$. If $V = 0$ then the operator has purely absolutely continuous spectrum on $(0, +\infty)$. We find conditions on V which guarantee that the absolutely continuous spectrum of both operators $-\Delta + V$ and $-\Delta - V$ is essentially supported by $[0, \infty)$. This means that the spectral projection associated to any subset of positive Lebesgue measure is not zero. Our main result is the following theorem:

Theorem 1.1. *Let W, $V \in L^\infty(\mathbb{R}^d)$ be real functions. Assume that*

$$\int_{\mathbb{R}^d} |W(x)||x|^{1-d} dx < \infty.$$

If the spectra of the operators $-\Delta + V + W$ and $-\Delta - V + W$ are contained in $[0, +\infty)$ then the absolutely continuous spectra of both operators are essentially supported by $[0, +\infty)$.

In particular, this theorem holds for $W = 0$:

Corollary 1.1. *Let $V \in L^\infty(\mathbb{R}^d)$, $d \geq 3$, be a real function. If the spectra of the operators $-\Delta + V$ and $-\Delta - V$ are contained in $[0, +\infty)$ then the absolutely continuous spectra of both operators are essentially supported by $[0, +\infty)$.*

Our methods allow us to prove

Corollary 1.2. *Let $V \in L^\infty(\mathbb{R}^d)$, $d \geq 1$, be a real function. If the spectra of $-\Delta + V$ and $-\Delta - V$ below zero are discrete and consist only of finite number of eigenvalues, then the absolutely continuous spectra of both operators are essentially supported by $[0, +\infty)$.*

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Note, that this is not a consequence of results obtained in [15], since it is still unclear whether potentials whose negative eigenvalues are in $\ell^{1/2}$ can be approximated by compactly supported functions in a proper way.

Typically, slow decaying potentials V must change their sign in order to force the negative eigenvalues of $-\Delta \pm V$ to be in $\ell^{1/2}$. If V is positive then the negative spectrum of the Schrödinger operator $-\Delta + V$ is empty, however the continuous spectrum of $-\Delta - V$ might have a non-trivial part in $(-\infty, 0)$. As a consequence, one can conclude very little about the a.c. spectrum from the fact that $-\Delta + V \geq 0$. Indeed, in the case of periodic potentials the spectrum may have gaps and the theory of random operators says that the spectrum of a Schrödinger operator with a positive potential might be purely point. Therefore one can obtain an interesting information on the a.c. spectrum from the behavior of eigenvalues only by studying V and $-V$ symmetrically. This idea appeared for the first time in [8] in dimension $d = 1$, where the authors proved the following striking

Theorem 1.2. Damanik-Killip [8] *Let* $V \in L^\infty(\mathbb{R}_+)$ *If the spectrum of the operators* $-\frac{d^2}{dx^2} + V$ *and* $-\frac{d^2}{dx^2} - V$ *on the halve line is contained in* $[0, +\infty)$ *then it is purely absolutely continuous and it coincides with* $[0, +\infty)$.

Our main technical lemmas are generalizations of Theorem 1.2 to the multi-dimensional case. Our proofs are based on the estimates of the entropy of the spectral measure, whose importance in the theory of one-dimensional Schrödinger operators was discovered by P.Deift and R.Killip [9].

The main theorem of [9] is a natural culmination of the results obtained by M.Christ, A.Kiselev and C.Remling [4], M.Christ and A.Kiselev [5] and C.Remling [16]. It says that one dimensional Schrödinger operators with square integrable potentials have their a.c. spectra essentially supported by the positive real line.

2. METHOD OF LAPTEV-NABOKO-SAFRONOV [15]

Here we would like to state a better version of the method suggested in [15].

Throughout the paper B_r denotes the ball of radius r around the origin.

Theorem 2.1. *Let* $d \geq 2$ *and let* V_n, U_n, W_n *be sequences of bounded compactly supported functions such that*

$$\sup_n \Big(\int |U_n(x)||x|^{1-d}\, dx + \int |W_n(x)||x|^{1-d}\, dx \Big) < \infty; \tag{2.1}$$

the sums of the roots of the eigenvalues

$$\sum_j |\lambda_j(U_n+V_n)|^{1/2} \le C, \qquad \textit{and} \qquad \sum_j |\lambda_j(W_n-V_n)|^{1/2} \le C$$

of the operators

$$-\Delta + U_n + V_n, \qquad \textit{and} \qquad -\Delta + W_n - V_n$$

with the Dirichlet boundary condition on the unit sphere are bounded uniformly in n *and the sequence* $U_n + V_n$ *converges to the potential* $V \in L^\infty(\mathbb{R}^d)$ *in the class*

$$L^2_{loc}(\mathbb{R}^d \setminus B_1). \tag{2.2}$$

Then the absolutely continuous spectrum of the operator

$$-\Delta + V \qquad \textit{in} \quad L^2(\mathbb{R}^d)$$

is essentially supported by $(0, +\infty)$

Not that this theorem leads to the following result [18] : if $d \ge 3$ and $V \in L^{d+1}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ is a real potential whose Fourier transform is square integrable near the origin, then the a.c. spectrum of $-\Delta + V$ is essentially supported by $\mathbb{R}_+$.

Since the paper [19] is still in preparation we would like to formulate the following assertion as a conjecture. Here we replace (2.1) by a somewhat more local condition.

Conjecture 2.1. *Let* Ω *be a cone in the space* $\mathbb{R}^d$*. The statement of Theorem 2.1 holds true if instead of* (2.1) *we require*

$$\sup_n \int_\Omega \Big(|U_n(x)| + |W_n(x)|\Big) |x|^{1-d}\, dx < \infty.$$

3. The proof of Theorem 1.1

Below B_r denotes the ball of radius r around the origin.

Proposition 3.1. *Let* V *be a bounded real valued function on the space* $\mathbb{R}^d$ *such that the operator* $\Delta + V$ *is non-negative. Then for any* $n > 0$ *the potential* V *is of the form*

$$V = \mathrm{div} A_n(x) + |A_n(x)|^2, \qquad \text{for } 1 < |x| < n, \tag{3.1}$$

where the vector function $A_n(x)$ *belongs to* $L^\infty(B_n)$*.*

Proof. Let χ_n be the characteristic function of the ball B_n of radius n. Let V_0 be a bounded real function supported in B_1 such that the operator

$$-\Delta u + (V_0 + \chi_n V)u + (1-\chi_n)|x|^2 u, \ \text{ in } L^2(\mathbb{R}^d) \tag{3.2}$$

has zero as the lowest eigenvalue. Such V_0 exists if n is large enough. Indeed, $V \le \chi_n V + (1-\chi_n)|x|^2$ for large n and the operator

$$-\Delta + \chi_n V + (1-\chi_n)|x|^2$$

is non negative. The corresponding eigenfunction u_0 of the operator (3.2) is strictly positive almost everywhere (see Reed-Simon [20], Theorem XIII.47). Moreover it satisfies the so called weak Harnack inequality (see [1], [14]) which leads to

$$\inf_{B_r(x)} u_0 > 0$$

for a small ball $B_r(x)$ of radius $r = r(n,x)$ around every x, $|x| < n$. Moreover according to [21] the functions $u_0,\ 1/u_0$ and ∇u_0 are locally bounded.

Thus, we can put

$$A_n(x) = u_0^{-1}\nabla u_0.$$

In to establish (3.1) one has to approximate u_0 by smooth positive functions u_j converging to u_0 so that the functions

$$\operatorname{div} u_j^{-1}\nabla u_j = u_j^{-1}\Delta u_j - u_j^{-2}|\nabla u_j|^2,$$

tend to $V - |A_n|^2$ in $L^2_{loc}(B_n)$ and for any $\phi \in C_0^\infty(B_n)$

$$-\int \operatorname{div} u_j^{-1}\nabla u_j \phi(x)dx = \int u_j^{-1}\nabla u_j \nabla\phi(x)dx \to \int u_0^{-1}\nabla u_0 \nabla \phi(x)dx.$$

The proof is completed. $\square$

Consider positive smooth functions $\zeta_n \le 1$ supported in $B_{n+1}\setminus B_1$ such that

$$\zeta_n(x) = 1, \ \text{ for } 2 < |x| < n, \quad \text{and } |\nabla \zeta_n(x)| \le C$$

Put $\eta_n(x) = |x|^{(1-d)/2}\zeta_n(x)$. Assume that on the support of η_n the potential $V+W$ is of the form $V + W = \operatorname{div} A + |A|^2$, where A is a locally bounded vector function. Since $-\Delta - V + W \ge 0$, we have the estimate

$$\int \operatorname{div} A|\eta_n|^2dx + \int |A|^2|\eta_n|^2 dx \le \int |\nabla \eta_n|^2dx + 2\int W(x)|\eta_n|^2dx.$$

Integration by parts leads to

$$\Big|\int \operatorname{div} A|\eta_n|^2dx\Big| \le \epsilon \int |A|^2|\eta_n|^2dx + \frac{1}{\epsilon}\int |\nabla\eta_n|^2dx.$$

Thus we obtain that

(3.3)

$$(1-\epsilon)\int |A|^2|\eta_n|^2dx \le \big(1+\frac{1}{\epsilon}\big)\int|\nabla\eta_n|^2dx$$

$$+2\int W(x)|\eta_n|^2dx \le \big(1+\frac{1}{\epsilon}\big)\int|\nabla\eta_n|^2dx + 2\int |W(x)||x|^{1-d}dx.$$

Note that the right hand side is bounded uniformly in n and therefore

$$\int_{2<|x|<n} |A|^2|x|^{1-d}dx \leq C, \tag{3.4}$$

where C is independent of a choice of A and n.

Let $w(x)$ be a positive $C_0^\infty(\mathbb{R}^d)$- function supported by the ball B_2 and which is equal to 1 in a neighborhood of the origin:

$$w(x) = 1, \qquad \text{for } |x| < 1.$$

Define $w_n(x) = w(x/n)$. Let A_N be the vector function such that $W+V = \text{div}A_N + |A_N|^2$ for $1 < |x| < N$ with $N = N(n)$ chosen so that $w_n(x) = 0$ for $|x| > N$. The existence of A_N was proven in the proposition of this section. In order to apply Theorem 2.1 we put

$$V_n(x) = \text{div}\Big(w_n(x)A_N(x)\Big), \quad \text{and } U_n(x) = |w_n(x)A_N(x)|^2.$$

Then clearly,

$$-\Delta \pm V_n + U_n \geq 0, \qquad \sup_n \int |U_n(x)||x|^{1-d}dx < \infty.$$

Here the second bound follows from (3.4). It remains to prove that $V_n + U_n$ converges to $W + V$ in $L^2_{loc}(\mathbb{R}^d \setminus B_1)$. But this follows from the simple fact that

$$W(x) + V(x) = V_n(x) + U_n(x), \qquad \text{for } 1 < |x| < n.$$

4. Second proof

There is a family of non-negative functions $\eta_n \in C_0^\infty(\mathbb{R}^d \setminus B_1)$, $\eta_n \leq 1$, and a positive constant $c_1 > 0$, such that η_n is supported in the set

$$\{x \in \mathbb{R}^d : \ n < |x| < 4n\}$$

and such that

$$\eta_n(x) = 1, \quad \text{for } 2n < |x| < 3n, \quad \text{and } |\nabla \eta_n| \leq c_1 n^{-1}.$$

One can even assume that $\eta_n(x) = \eta(x/n)$. Then there is a positive constant $C > 0$ such that

$$\int |\nabla \eta_n|^2 dx < C\, n^{d-2}. \tag{4.1}$$

Assume that on the support of η_n the potential $V + W$ is of the form $V + W = \text{div}A + |A|^2$, where A is a locally bounded vector function. Since $-\Delta - V + W \geq 0$, we have

$$\int \text{div}A|\eta_n|^2dx + \int |A|^2|\eta_n|^2dx \leq \int |\nabla \eta_n|^2dx + 2\int W(x)|\eta_n|^2dx.$$

Integration by parts leads to

$$\Big|\int \mathrm{div} A|\eta_n|^2dx\Big| \leq \epsilon \int |A|^2|\eta_n|^2dx + \frac{1}{\epsilon}\int |\nabla\eta_n|^2dx$$

So combining (4.1) and two latter inequalities we obtain the estimate

(4.2)

$$(1-\epsilon)\int |A|^2|\eta_n|^2dx \leq \big(1+\frac{1}{\epsilon}\big)\int |\nabla\eta_n|^2dx$$

$$+2\int W(x)|\eta_n|^2dx \leq \frac{1+\epsilon}{\epsilon}C\,n^{d-2} + 2(4n)^{d-1}\int_{n<|x|<4n}|W(x)||x|^{1-d}dx.$$

Therefore

$$\int_{2n<|x|<3n}|A|^2dx \leq 6\,C\,n^{d-2} + 4^dn^{d-1}\int_{n<|x|<4n}|W(x)||x|^{1-d}dx.$$

Since $|x|\geq 2n$ on the set $\{x:\ 2n<|x|<3n\}$, we conclude that

$$\int_{2n<|x|<3n}|x|^{1-d}|A|^2dx \leq (2n)^{1-d}\int_{2n<|x|<3n}|A|^2dx$$

and consequently

$$\int_{2n<|x|<3n}|x|^{1-d}|A|^2dx \leq 2^{1-d}6\,C\,n^{-1} + 2^{d+1}\int_{n<|x|<4n}|W(x)||x|^{1-d}dx.$$

Taking $n_j=(3/2)^j$ we obtain

$$\int_{2<|x|<3(3/2)^m}|x|^{1-d}|A|^2dx = \sum_{j=0}^{m}\int_{2n_j<|x|<2n_{j+1}}|x|^{(1-d)}|A|^2dx \leq$$

$$\leq C_0 + C_1\int |W(x)||x|^{1-d}dx.$$

where C_0 and C_1 do not depend on $m\in\mathbb{N}$. Thus for any $r>2$

$$\int_{2<|x|<r}|x|^{1-d}|A|^2dx \leq C_0 + C_1\int |W(x)||x|^{1-d}dx. \tag{4.3}$$

Here we used the fact that

$$\sum_j\int_{(3/2)^j<|x|<4(3/2)^j}|W(x)||x|^{1-d}dx \leq \sum_j\int_{(3/2)^j<|x|<(3/2)^{j+4}}|W(x)||x|^{1-d}dx \leq$$

$$\leq 4\sum_j\int_{(3/2)^j<|x|<(3/2)^{j+1}}|W(x)||x|^{1-d}dx$$

5. Generalizations and applications

The following conjecture would be a generalization of Theorem 1.1

Conjecture 5.1. *Let Ω be a cone in $\mathbb{R}^d$, $d \geq 2$. Let W_1, W_2 and V be three bounded real functions on the space $\mathbb{R}^d$ such that*

$$\int_\Omega (|W_1(x)| + |W_2(x)|)|x|^{1-d}dx < \infty \tag{5.1}$$

and the spectra of the operators $-\Delta + V + W_1$ and $-\Delta - V + W_2$ are contained in $[0, +\infty)$. Then the absolutely continuous spectra of both operators are essentially supported by $[0, +\infty)$.

Suppose that the assertion of Conjecture 2.1 holds true. Then the proof would follow closely the second proof of Theorem 1.1 with one detail different. When choosing the sequence η_n, one has to find η with a support in $\Omega \setminus B_1$ and then put $\eta_n(x) = \eta(x/n)$. Then instead of applying Theorem 2.1 one applies the assertion of Conjecture 2.1. $\square$

Remark. Since (5.1) is the only condition on W_1 and W_2, the potentials $V + W_1$ and $-V + W_2$ in the conjecture do not have to decay or oscillate outside of the cone Ω.

We give an example of a potential satisfying the conditions of Theorem 1.1 in the case when $d = 3$. We use a convention that a point in $\mathbb{R}^3$ can be written in the form (x, y, z) with real x, y and z. Let $V(x, y, z) = \frac{\partial}{\partial x}\frac{\cos(x^2)}{(x^2+1)^{1/2}}p(y, z) + \frac{\cos^2(x^2)}{(x^2+1)}p^2(y, z)$, where $p \in L^\infty(\mathbb{R}^2)$ satisfies the bound

$$\int\int\int \frac{|p(y, z)|^2}{(x^2 + 1)(1 + x^2 + y^2 + z^2)}\, dxdydz < \infty$$

Then the absolutely continuous spectrum of $-\Delta+V$ is essentially supported by $[0, +\infty)$. For example, $|p(y, z)| \leq C(1 + y^2 + z^2)^{-\epsilon}$, $\epsilon > 0$ is fine.

More generally, we can state

Theorem 5.1. *Let A be a bounded vector valued function on $\mathbb{R}^d$ with values in $\mathbb{R}^d$ such that*

$$\int_{\mathbb{R}^d} |A|^2|x|^{1-d}dx < \infty, \quad \operatorname{div} A \in L^\infty(\mathbb{R}^d).$$

Then the absolutely continuous spectrum of the operator

$$-\Delta + \operatorname{div} A + |A|^2$$

is essentially supported by the positive real line $(0, +\infty)$.

The potentials of this type with applications to random Schrödinger operators were considered by Denissov [7]. Moreover, the method of [7] allows one to consider potentials div$A$ without the term $|A|^2$. This is a delicate work, in spite that the removal of the second term leads to the assumption that the functions $\operatorname{div} A$ and $|A|$ decay faster than $|x|^{-1/2-\epsilon}$. In applications Denissov presents a class of random potentials which are almost surely of the form $\operatorname{div} A$.

Proof. For any $\phi \in C_0^\infty(\mathbb{R}^d \setminus B_1)$ we have

$$\int |\nabla\phi(x)|^2\,dx \pm \int \operatorname{div}A|\phi(x)|^2dx + \int |A|^2|\phi(x)|^2dx =$$

$$\int |\nabla\phi(x)|^2\,dx \mp 2\mathrm{Re}\int A\nabla\phi(x)\bar{\phi}(x)dx + \int |A|^2|\phi(x)|^2dx =$$

$$= \int |\nabla\phi(x) \mp A\phi(x)|^2\,dx \geq 0.$$

Since the operators $-\Delta \pm \operatorname{div}A + |A|^2$ are essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$, they are positive as operators in $L^2(\mathbb{R}^d)$. Therefore we can apply Theorem 1.1 with $V = \operatorname{div}A$ and $W = |A|^2$. $\square$

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