THE L^p-THEORY OF THE SPECTRAL SHIFT FUNCTION, THE WEGNER ESTIMATE, AND THE INTEGRATED DENSITY OF STATES FOR SOME RANDOM OPERATORS

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Abstract

We develop the L^p-theory of the spectral shift function, for $p \geq 1$, applicable to pairs of self-adjoint operators whose difference is in the trace ideal $\mathcal{I}_p$, for $0 < p \leq 1$. This result is a key ingredient of a new, short proof of the Wegner estimate applicable to a wide variety of additive and multiplicative random perturbations of deterministic background operators. The proof yields the correct volume dependence of the upper bound. This implies the local Hölder continuity of the integrated density of states at energies in the unperturbed spectral gap. Under an additional condition of the single-site potential, local Hölder continuity is proved at all energies. This new Wegner estimate, together with other, standard results, establishes exponential localization for a new family of models for additive and multiplicative perturbations.

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1 Introduction and Main Results

One of the original challenges in the theory of randomly perturbed media was the demonstration that solutions to the appropriate equations of motion remained localized in space. This is usually interpreted as meaning that the deterministic spectrum of the corresponding random family of self-adjoint generators of the time evolution group, $\{H_\omega \mid \omega \in \Omega\}$, is pure point almost surely in certain energy intervals. A self-adjoint operator H exhibits spectral localization in an energy interval $I \subset I\!\!R$ if the spectrum of H in I is purely pure point. The proof of localization for various random systems on $I\!\!R^d$ begins with an analysis of finite-volume perturbations $H_\Lambda = H_0 + V_\Lambda$, for a bounded region $\Lambda \subset I\!\!R^d$, of a self-adjoint operator H_0, describing the background, unperturbed, situation. Two estimates on H_Λ are needed: 1) a decay estimate on the Green's function of H_Λ at far-separated points, with a probability converging to one as $|\Lambda| \to I\!\!R^d$, and 2) an estimate on the location of the eigenvalues of H_Λ. We refer to [1, 12, 13, 16, 31, 19]. This second estimate is called a *Wegner estimate*. A Wegner estimate is an upper bound on the probability that the spectrum of the local Hamiltonian H_Λ lies within an η-neighborhood of a given energy E. A good Wegner estimate is one for which the upper bound depends linearly on the volume $|\Lambda|$, and vanishes as the size of the energy neighborhood η shrinks to zero. The linear dependence on the volume is essential for the proof of the existence of the integrated density of states (IDS). The rate of vanishing of the upper bound as $\eta \to 0$ determines the continuity of the IDS. In this note, we will present a new proof of a good Wegner estimate modeled on Wegner's original proof [32], and some new estimates on the spectral shift function related to the single-site perturbation. This new result allows us to prove exponential localization and the local Hölder continuity of the integrated density of states for more models than previously known. In order to prove this result, we develop the L^p-theory of the spectral shift function for $p > 1$, that might be of independent interest.

The models that can be treated by this method are described as follows. We can treat both multiplicative (M) and additive (A) perturbations of a background self-adjoint operator H_0^X, for $X = M$ or $X = A$. Additively perturbed operators describe electron propagation, and multiplicatively perturbed operators describe the propagation of acoustic and electromagnetic waves. We refer to [9] for a further discussion of the physical interpretation of these operators. For the Wegner estimate, we are interested in local perturbations V_Λ of a background operator H_0^X, that are local with respect to a bounded region $\Lambda \subset I\!\!R^d$. Multiplicatively perturbed operators H_Λ^M are of the form

$$H_\Lambda^M = A_\Lambda^{-1/2} H_0^M A_\Lambda^{-1/2}, \tag{1.1}$$

where $A_\Lambda = 1 + V_\Lambda$. We assume that $(1 + V_\Lambda)$ is invertible (cf. [9] for a discussion of this condition). Additively perturbed operators H_Λ^A are of the form

$$H_\Lambda^A = H_0^A + V_\Lambda. \tag{1.2}$$

The unperturbed, background medium in the multiplicative case is described by a divergence form operator

$$H_0^M = -C_0 \rho_0^{1/2} \nabla \cdot \rho_0^{-1} \nabla \rho_0^{-1/2} C_0, \tag{1.3}$$

where ρ_0 and C_0 are positive functions that describe the unperturbed density and sound velocity. We assume that ρ_0 and C_0 are sufficiently regular so that $C_0^\infty(I\!\!R^d)$ is an operator core for H_0^M. The unperturbed, background medium in the additive case is described by a Schrödinger operator H_0 given by

$$H_0^A = (-i\nabla - A)^2 + W, \tag{1.4}$$

where A is a vector potential with $A \in L^2_{loc}(I\!\!R^d)$, and $W = W_+ - W_-$ is a background potential with $W_- \in K_d(I\!\!R^d)$ and $W_+ \in K_d^{loc}(I\!\!R^d)$.

The perturbations V_Λ can be of the following two forms. Let $\tilde{\Lambda}$ denote the lattice points in the region Λ, so that $\tilde{\Lambda} \equiv \Lambda \cap Z\!\!\!Z^d$. For any $k \in Z\!\!\!Z^d$, let $\Lambda_L(k)$ denote the cube of side length L centered at point $k \in Z\!\!\!Z^d$. The local perturbation in the Anderson-type model is defined by:

$$V_\Lambda(x) = \sum_{i \in \tilde{\Lambda}} \lambda_i(\omega) u_i(x - i - \xi_i(\omega')), \tag{1.5}$$

provided the random variables $\xi_i(\omega')$, modeling thermal vibrations, are small enough so the positivity condition (H3) (given ahead) holds. The functions u_i are nonnegative and compactly supported in a neighborhood of the origin. They need not be of the form $u_i(x) = u(x)$, for some fixed u, since ergodicity plays no role in the Wegner estimate. For the breather-type model, the local potential is defined by

$$V_\Lambda(x) = \sum_{i \in \tilde{\Lambda}} u_i(\lambda_i(\omega)(x - i)) \tag{1.6}$$

The single-site potentials u_i must satisfy a repulsivity condition stated in Hypothesis (H3). We remark that the dependence of the Anderson-type potential on the random variables $\{\lambda_k(\omega) \mid k \in \tilde{\Lambda}\}$ is linear, and hence analytic, whereas the dependence of the breather-type potential on the random variables $\{\lambda_k(\omega) \mid k \in \tilde{\Lambda}\}$ is nonanalytic when the single-site potentials u_k have compact support. When the sum in (1.5) or (1.6) extends over all the lattice points $\mathbb{Z}^d$, we write V_ω for the potential and H_ω^X, with the operator for $X = A$ given by (1.2) with V_Λ replaced by V_ω, and similarly for $X = M$.

We will put conditions of the random variables $\lambda_k(\omega)$ and the single-site potentials u_k. We note that the Wegner estimate is a local estimate so that for a finite region $\Lambda \subset \mathbb{R}^d$, only a finite number of single-site potentials are involved. We denote the ball of radius $R > 0$ about the origin by $B(R)$.

(H1a) The self-adjoint operator H_0^X is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$, for $X = A$ and for $X = M$. The operator H_0^X is semi-bounded and has an open spectral gap. That is, there exist constants $-\infty < M_0 \leq C_0 \leq B_- < B_+ < C_1 \leq \infty$ so that $\sigma(H_0) \subset [M_0, \infty)$, and

$$\sigma(H_0) \cap (C_0, C_1) = (C_0, B_-] \cup [B_+, C_1).$$

(H1b) The self-adjoint operator H_0^X is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$, and H_0^X is semi-bounded with $\sigma(H_0^X) \subset [M_0, \infty)$, for some $M_0 > -\infty$.

(H2) The operator H_0^X is locally compact in the sense that for any $\chi \in L^\infty(\mathbb{R}^d)$ with compact support, the operator $\chi(H_0^X - M_1)^{-1}$ is compact for any $M_1 < M_0$.

(H3) The single-site potentials $u_k, k \in \mathbb{Z}^d$, are nonzero. For the Anderson-type model (1.5), we assume that there exists $R > 0$ so that $u_k \in C_0(B(R))$, and that $u_k \geq 0$ for each $k \in \mathbb{Z}^d$. Furthermore, we assume that the family $\{u_k \mid k \in \mathbb{Z}^d\}$ is equicontinuous. For the breather-type model (1.6), we assume that there exists $R > 0$ so that $u_k \in C^1(B(R)\backslash\{0\})$ (there may be a singularity at the origin), and that $u_k \geq 0$ for each $k \in \mathbb{Z}^d$. There exists $\epsilon_0 > 0$ so that $-x \cdot \nabla u_k - \epsilon_0 u_k \geq 0$. Finally, we assume that there exists $\delta > 0$ so that the family $\{x \cdot \nabla u_k \mid k \in \mathbb{Z}^d\}$ is equicontinuous on the set $\{x \mid -x \cdot \nabla u_k \leq \delta\}$.

(H3a) In addition to (H3) for the Anderson-type model, we assume that there exists $\epsilon_1 > 0$ so that $u_k \geq \epsilon_1$ on $\Lambda_1(0)$, where $\Lambda_r(k) \equiv \{x \in \mathbb{R}^d \mid |x_j - k_j| < r/2,\ j = 1 \ldots, d\}$.

(H4) The conditional probability distribution of λ_0, conditioned on ${\lambda_0}^{\perp} \equiv \{\lambda_i \mid i \neq 0\}$, is absolutely continuous with respect to Lebesgue measure. The density h_0 has compact support $[m, M]$, for some constants (m, M) with $-\infty < m < M < \infty$. The density h_0 satisfies $\|h_0\|_\infty < \infty$, where the sup norm is defined with respect to the probability measure $I\!\!P$.

We refer to the review article of Kirsch [17] for a proof of the fact that hypotheses (H3)–(H4) imply the essential self-adjointness of H^A_ω on $C^\infty_0(I\!\!R^d)$ (see [9] for the $X = M$ case). We make two comments about the hypotheses. First, we will assume that the random variables are independent, and identically distributed, but the results hold in the correlated case, and in the case that the supports of the single-site potentials are not necessarily compact (cf. [8, 20]). Second, the open spectral gap condition (H1a) for $\sigma(H^X_0)$ is not needed provided H^X_0 is semibounded from below. In this case, a lower-semibounded operator H^X_0 always has at least one open gap $(-\infty, \inf \sigma(H^X_0))$. If only hypotheses (H1b)–(H4) hold, together with additional assumptions (H5)–(H6) given ahead, localization is proven near the bottom of the spectrum of H^X_0. The main result under these hypotheses on the unperturbed operator H^X_0, and the local perturbation V_Λ, is the following theorem.

Theorem 1.1. *Assume (H1a)–(H4). For any $E_0 \in G = (B_-, B_+)$, for any $q > 1$, and for any $\eta < \frac{1}{2}\,\mathrm{dist}\ (E_0, \sigma(H^X_0))$, there exists a finite constant C_{E_0}, depending on $[\mathrm{dist}\ (\sigma(H^X_0), E_0)]^{-1}$, the dimension d, and $q > 1$, such that:*

$$I\!\!P \left\{ \mathrm{dist}\ (E_0, \sigma(H^X_\Lambda)) \leq \eta \right\} \leq C_{E_0} \eta^{1/q} \, |\Lambda| \ . \tag{1.7}$$

For Anderson-type potentials, if we assume (H1b)–(H4), and, in addition, that the single-site potential satisfies (H3a), then the result (1.7) holds for $H^X_\omega \mid \Lambda$, with Dirichlet boundary conditions on Λ, and for any $E_0 \in I\!\!R$.

This result was proven for H^A_Λ, and for H^M_Λ, with the Anderson-type perturbation (1.5) and $q = 1$, that is, Lipschitz continuity of the integrated density of states, in [1] and [9], respectively. Although localization for $X = A$ and the breather model (1.6) was proven in [7], Theorem 1.1 is new for the breather model for $X = A$, and Theorems 1.1–1.3 are new for $X = M$ with the breather-type perturbation.

There are several prior results on the Wegner estimate for multidimensional, continuous Schrödinger operators. Kotani and Simon [25] proved a

Wegner estimate with a $|\Lambda|$-dependence for Anderson models with overlapping single-site potentials. This condition was removed and extensions were made to the band-edge case in [5] and [1]. An extension to multiplicative perturbations was made in [9, 11, 12, 13]. These methods require a spectral averaging theorem (cf. [7] and references therein). Wegner's original proof [32] for Anderson models did not require spectral averaging. Following Wegner's argument, Kirsch gave a nice, short proof of the Wegner estimate in [18], but obtained a $|\Lambda|^2$-dependence. Recently, Stollmann [30] presented a short, elementary proof of the Wegner estimate for Anderson-type models with singular single-site probability distributions that are assumed to be simply Hölder continuous. He also obtains a $|\Lambda|^2$-dependence. These proofs, and the proof in this paper, do not require spectral averaging. It is not clear, however, how to extend the methods of this paper in order to prove a Wegner estimate for singular distributions with the correct volume dependence.

In order to prove a localization result, we need to assume that the family of operators $\{H_\omega^X \mid \omega \in \Omega\}$ has a deterministic spectrum Σ. For the models described here this is insured if, for example, the single-site potentials satisfy $u_k(x) = u(x-k)$, for all $k \in \mathbb{Z}^d$. Given the Wegner estimate, the existence of localized states near the band edge for these models follows provided we make two other hypotheses.

(H5) The density h_0 decays sufficiently rapidly near m and near M in the following sense:

$$0 < \mathbb{P}\{|\lambda - m| < \varepsilon\} \leq \varepsilon^{3d/2+\beta},$$

$$0 < \mathbb{P}\{|\lambda - M| < \varepsilon\} \leq \varepsilon^{3d/2+\beta},$$

for some $\beta > 0$.

(H6) There exist constants $B'_\pm$ satisfying $B_- < B'_- < B'_+ < B_+$ such that

$$\Sigma \cap \{(B_-, B'_-) \cup (B'_+, B_+)\} \neq \emptyset \ .$$

In light of hypothesis (H6), we define the band edges of the almost sure spectrum Σ near the gap G, as follows:

$$\tilde{B}_- \equiv \sup\{E \leq B'_- \mid E \in \Sigma\},$$

and

$$\tilde{B}_+ \equiv \inf\{E \geq B'_+ \mid E \in \Sigma\},$$

Theorem 1.2. *Assume (H1a)–(H6). There exist constants $E_\pm$ satisfying $B_- \leq E_- < \tilde{B}_-$ and $\tilde{B}_+ < E_+ \leq B_+$ such that $\Sigma \cap (E_-, E_+)$ is pure point with exponentially decaying eigenfunctions.*

Theorem 1.3. *Assume (H1a)–(H6), and that the model is ergodic. The integrated density of states is Hölder continuous of order $1/q$, for any $q > 1$, on the interval (B_-, B_+). For Anderson-type potential, if we assume (H1b)–(H6), ergodicity, and, in addition, that the single-site potential satisfies (H3a), then the integrated density of states is locally Hölder continuous of order $1/q$, for any $q > 1$, on Σ.*

The existence of the integrated density of states for additively perturbed, infinite-volume, ergodic models like (1.2) is well-known. A textbook account is found in the lecture notes of Kirsch [17]. The same proof applies to the multiplicatively perturbed model (1.1) with minor modifications. Recently, Nakamura [26] showed the uniqueness of the IDS, in the sense that it is independent of Dirichlet or Neumann boundary conditions, in the case of Schrödinger operators with magnetic fields. The same proof applies to the multiplicatively perturbed model. It is interesting to note that the proof uses the L^1-theory of the spectral shift function.

We mention the recent papers of Kostrykin and Schrader [21, 22, 23] in which they construct a spectral shift density for random Schrödinger operators with Anderson-type potentials. Their idea is to use the spectral shift function $\xi_\Lambda(\lambda)$ for the pair $(H_\Lambda = H_0 + V_\Lambda, H_0)$, and to study the thermodynamic limit

$$\lim_{\Lambda \to \infty} \frac{\xi_\Lambda(\lambda)}{|\Lambda|} \equiv \xi(\lambda). \tag{1.8}$$

They prove that this limit (property defined) exists and is deterministic. Furthermore, they prove that the spectral shift density $\xi(\lambda)$ is related to the IDS through the formula

$$\xi(\lambda) = N_0(\lambda) - N(\lambda), \tag{1.9}$$

where $N_0(\lambda)$ is the IDS for the free Hamiltonian H_0. Their work provides an alternative proof of the existence of the IDS.

The contents of this paper are as follows. The L^p-theory of the spectral shift function for $p > 1$ is developed in section 2. We prove Wegner's estimate in section 3 along the ideas of the original argument. An application of the theory developed in section 2 to the single-site spectral shift function allows us to obtain the correct volume dependence. In section 4, we discuss the localization properties of the eigenfunctions of the operators H_Λ^X corresponding to eigenvalues in the unperturbed spectral gap G. We also present a slightly generalized version of a comparison theorem of Kirsch, Stollmann, and Stolz [19] that allows one to use single-site potentials of arbitrarily small support. Some simple proofs of the trace class estimates used in the proof of Wegner's estimate in section 3 are presented in section 5. Extensions of some of the results of this paper to single-site potentials without fixed sign will appear in [14]. We also mention a recent paper of Kostrykin and Schrader [24] in which they use some of the methods of this paper to study the density of surface states.

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2 The L^p-Theory of the Spectral Shift Function, $p \geq 1$

We develop the L^p-theory of the spectral shift function for $p \geq 1$. Let us recall the L^1-theory, which can be found in the review paper of Birman and Yafaev [3], and the book of Yafaev [33]. Suppose that H_0 and H are two self-adjoint operators on a Hilbert space $\mathcal{H}$ having the property that $V \equiv H - H_0$ is in the trace class. Under these conditions, we can define the Krein spectral shift function (SSF) $\xi(\lambda; H, H_0)$ through the perturbation determinant. Let $R_0(z) = (H_0 - z)^{-1}$, for $Im\ z \neq 0$. We then have

$$\xi(\lambda; H, H_0) \equiv \frac{1}{\pi} \lim_{\epsilon \to 0^+} \arg \det\, (1 + V R_0(\lambda + i\epsilon)). \tag{2.1}$$

It is well-known that

$$\int_{I\!\!R} \xi(\lambda; H, H_0)\, d\lambda = Tr V, \tag{2.2}$$

and that the SSF satisfies the L^1-estimate:

$$\| \xi(\cdot\,; H, H_0) \|_{L^1} \ \leq\ \|V\|_1. \tag{2.3}$$

Let A be a compact operator on $\mathcal{H}$ and let $\mu_j(A)$ denote the j^{th} singular value of A. We say that $A \in \mathcal{I}_{1/p}$, for some $p \geq 1$, if

$$\sum_j \mu_j(A)^{1/p} < \infty. \tag{2.4}$$

Clearly, this means that the singular values of A converge very rapidly to zero. We define a nonnegative functional on the ideal $\mathcal{I}_{1/p}$ by

$$\|A\|_{1/p} \equiv \left(\sum_j \mu_j(A)^{1/p} \right)^p . \tag{2.5}$$

For $p > 1$, this functional is not a norm but satisfies

$$\|A + B\|_{1/p}^{1/p} \leq \|A\|_{1/p}^{1/p} + \|B\|_{1/p}^{1/p}. \tag{2.6}$$

If we define a metric $\rho_{1/p}(A, B) \equiv \|A - B\|_{1/p}^{1/p}$ on $\mathcal{I}_{1/p}$, then the linear space $\mathcal{I}_{1/p}$ is a complete, separable linear metric space. The finite rank operators are dense in $\mathcal{I}_{1/p}$ (cf. [2]).

Since $\mathcal{I}_{1/p} \subset \mathcal{I}_1$, for all $p \geq 1$, we refer to $A \in \mathcal{I}_{1/p}$ as being super-trace class. Consequently, we can define the SSF for a pair of self-adjoint operators H_0 and H for which $V = H - H_0 \in \mathcal{I}_{1/p}$. Our main theorem is the following.

Theorem 2.1. *Suppose that H_0 and H are self-adjoint operators so that $V = H - H_0 \in \mathcal{I}_{1/p}$, for some $p \geq 1$. Then, the SSF $\xi(\lambda; H, H_0) \in L^p(I\!R)$, and satisfies the bound*

$$\|\xi(\cdot\,; H, H_0)\|_{L^p} \;\leq\; \|V\|_{1/p}^{1/p}. \tag{2.7}$$

Proof. The proof follows the standard proof in the L^1-theory, cf. [33]. First, we consider the simple case that V is rank one. Let $V = \lambda(V)|\psi\rangle\langle\psi|$, for some $\psi \in \mathcal{H}$ with $\|\psi\| = 1$, and $\lambda(V) \in I\!R$. Note that the singular value $\mu(V) = |\lambda(V)|$. By a standard calculation, one finds that

$$\|\xi(\cdot\; H, H_0)\|_{L^1} = \int_{I\!R} |\xi(\lambda; H, H_0)|\, d\lambda = \mu(V), \tag{2.8}$$

and the SSF is bounded,

$$|\xi(\lambda; H, H_0)| \leq 1. \tag{2.9}$$

As a consequence, we have

$$\|\xi(\cdot\,; H, H_0)\|_{L^p} \;\leq\; \mu(V)^{1/p}. \tag{2.10}$$

We next consider the case of a finite rank perturbation V of the form

$$V_N = \sum_{j=1}^{N} \mu_j(V_N)\; |\psi_j\rangle\langle\psi_j|, \tag{2.11}$$

where $N < \infty$. Without loss of generality, we also assume that the coefficients $\mu_j(V_N) \geq 0$, and that $\|\psi_j\| = 1$. Let $H_N = H_0 + V_N$. We define a sequence of Hamiltonians H_K by

$$H_K \equiv H_0 + \sum_{k=1}^{K} \mu_k(V_N)\; |\psi_k\rangle\langle\psi_k|,\; j = 1, \ldots, N. \tag{2.12}$$

The factorization of the perturbation determinant into the product of determinants of rank-one perturbations implies that the spectral shift function for H_N can be written as the sum

$$\xi(\lambda; H_N, H_0) = \sum_{j=1}^{N} \xi(\lambda; H_j, H_{j-1}). \tag{2.13}$$

Each term in the sum on the right in (2.13) is the SSF for a rank one perturbation and hence satisfies the bounds (2.8)–(2.10). It follows from those bounds and the Minkowski inequality ($p \geq 1$) that

$$\begin{aligned} \|\xi(\cdot\,; H_N, H_0)\|_{L^p} &= \\ \left\{ \int_{I\!R} |\xi(\lambda; H_N, H_0)|^p \, d\lambda \right\}^{1/p} &\leq \sum_{j=1}^{N} \left\{ \int_{I\!R} |\xi(\lambda; H_j, H_{j-1})|^p \, d\lambda \right\}^{1/p} \\ &\leq \sum_{j=1}^{N} |\mu_j(V_N)|^{1/p}, \end{aligned} \tag{2.14}$$

from which it follows that

$$\|\xi(\cdot\,; H_N, H_0)\|_{L^p} \;\leq\; \|V_N\|_{1/p}^{1/p}. \tag{2.15}$$

As for the general case, let $V \in \mathcal{I}_{1/p}$. Such a potential is the limit of a sequence of finite rank operators V_N, of the form given in (2.6), and satisfying

$$\|V - V_N\|_{1/p} \;\leq\; \left(\sum_{j=N+1}^{\infty} |\mu_j(V)|^{1/p} \right)^{p}. \tag{2.16}$$

Since the series is convergent, the difference vanishes as $N \to \infty$. It follows that

$$\|\xi(\cdot\,; H, H_0)\|_{L^p} \ \leq\ \lim_{N\to\infty} \sum_{j=1}^{N} |\mu_j(V)|^{1/p} = \|V\|_{1/p}^{1/p}, \qquad (2.17)$$

proving the result. □

D. Hundertmark and B. Simon have recently proved an optimal L^p-bound on the spectral shift function [15].

3 A Proof of Wegner's Estimate

In his original article [32], Wegner introduced the clever device of interchanging differentiation with respect to energy with differentiation with respect to the random variables on which the random potential depends. Since an expectation is taken in the course of the proof, these derivatives in the random variables can be removed by an integration with respect to the random variables. This requires some smoothness on the distribution of the random variables. At one point in the standard proof, a Weyl-type estimate is used to bound the trace of smoothed-out spectral projector. We replace this with a finer estimate by first expressing the quantity in terms of a spectral shift function corresponding to a perturbation by a single-site potential, and then by estimating the L^p-norm of this spectral shift function, for $p > 1$, as described in section 2. The proof below uses some of the modifications of Wegner's proof [32] introduced by Kirsch [18]. We note that this proof of the Wegner estimate does not require spectral averaging [7]. It does, however, rely upon some monotonicity of the eigenvalues with respect to the random variables (for comparison, see [4]).

In the case of the Anderson-type potential satisfying (H1b)–(H4), with the single-site potential satisfying (H3a), the proof proceeds along the same lines with the following modification. Instead of working with the local operator $H_\Lambda = H_0 + V_\Lambda$, on the Hilbert space $L^2(I\!R^d)$, we take $H_\Lambda = (H_0 + V_\omega)|\Lambda$, with Dirichlet boundary conditions on $\partial\Lambda$, on $L^2(\Lambda)$. The energy E_0 can be chosen to be any energy and the comments concerning the gap G can be omitted. Finally, the argument in parts 3 and 4 of the proof are significantly simpler. The auxiliary potential defined in (3.12) is bounded below by $\epsilon_1\chi_\Lambda$, the characteristic function on Λ, that is the identity on $L^2(\Lambda)$. The Comparison Theorem of Kirsch, Stollmann, and Stolz [19] is not needed in this case.

Proof of Theorem 1.1.

1. Let $G = (B_-, B_+)$ be the spectral gap of H_0 as in Hypothesis (H1) and fix an energy $E_0 \in G$. Since the local potential V_Λ is a relatively compact perturbation of H_0, its effect is to introduce at most finitely-many eigenvalues $E_j(\Lambda)$ into the gap G. Let $\eta > 0$ be chosen so that $[E_0 - 2\eta, E_0 + 2\eta] \subset G$. We denote by I_η the interval $[E_0 - \eta, E_0 + \eta]$. We want to estimate

$$I\!P\{ \text{ dist } (E_0, \sigma(H_\Lambda)) < \eta\}. \tag{3.1}$$

This probability is expressible in terms of the finite-rank spectral projector for the interval I_η and H_Λ, which we write as $E_\Lambda(I_\eta)$. This projection is a random variable, but we suppress this in the notation. We now apply Chebyshev's inequality to the random variable $Tr(E_\Lambda(I_\eta))$ and obtain

$$\begin{aligned} I\!P\{ \text{ dist } (E_0, \sigma(H_\Lambda)) < \eta\} &= I\!P\{Tr(E_\Lambda(I_\eta)) \geq 1\} \\ &\leq I\!E\{Tr(E_\Lambda(I_\eta))\}. \end{aligned} \tag{3.2}$$

2. We now proceed to estimate the expectation of the trace, following the original argument of Wegner [32] as modified by Kirsch [18], and using some results of [19]. By our assumptions, there exists a constant $M_0 > -\infty$ so that $\inf \sigma(H_\Lambda) > M_0$, uniformly in Λ. Let ρ be a nonnegative, smooth, monotone decreasing function such that $\rho(x) = 1$, for $x < -\eta/2$, and $\rho(x) = 0$, for $x \geq \eta/2$. Furthermore, we can assume that ρ has compact support: We take $\rho(x) = 0$ for $x << M_0$. By the functional calculus, we have

$$\begin{aligned} &\rho(H_\Lambda - E_0 - 3\eta/2) - \rho(H_\Lambda - E_0 + 3\eta/2) \\ &\quad = \sum_{j \in \boldsymbol{Z}} \{\rho(E_j(\Lambda) - E_0 - 3\eta/2) - \rho(E_j(\Lambda) - E_0 + 3\eta/2)\} P_j(\Lambda), \end{aligned} \tag{3.3}$$

where, due to the support of ρ, the sum is over the eigenvalues $E_j(\Lambda)$ of H_Λ in $[E_0 - 2\eta, E_0 + 2\eta]$. The operators $P_j(\Lambda)$ are the projectors onto the corresponding finite dimensional eigenspaces spanned by normalized eigenfunctions ϕ_j. Consequently, the difference of the two operators on the left in (3.3) is trace class. Note that this difference $\rho(H_\Lambda - E_0 - 3\eta/2) - \rho(H_\Lambda - E_0 + 3\eta/2)$ is, roughly, the number of eigenvalues of H_Λ less than $E_0 + 2\eta$ minus the number of eigenvalues of H_Λ less than $E_0 - 2\eta$. However, the operator H_Λ may have continuous spectrum to the left of $E_0 - 2\eta$, so neither operator $\rho(H_\Lambda - E_0 - 3\eta/2)$, nor $\rho(H_\Lambda - E_0 + 3\eta/2)$, by itself, is in the trace class.

The coefficient of $P_j(\Lambda)$ on the right side of (3.3) is always nonnegative and precisely equal to one for $E_j(\Lambda) \in I_\eta$, so we have

$$Tr(E_\Lambda(I_\eta)) \leq Tr(\rho(H_\Lambda - E_0 - 3\eta/2) - \rho(H_\Lambda - E_0 + 3\eta/2)). \qquad (3.4)$$

Although the counting function for eigenvalues is not differentiable, we will use the fact that this smooth approximation is differentiable with respect to the energy.

3. Returning to the expectation of the trace in (3.4), we can now bound it above by

$$\begin{aligned} I\!E_\Lambda\{Tr(E_\Lambda(I_\eta))\} &\leq I\!E_\Lambda\{Tr[\rho(H_\Lambda - E_0 - 3\eta/2) - \rho(H_\Lambda - E_0 + 3\eta/2)]\} \\ &\leq I\!E_\Lambda\left\{\int_{-3\eta/2}^{3\eta/2} Tr\{-\rho'(H_\Lambda - E_0 - t)\}\, dt\right\}. \qquad (3.5) \end{aligned}$$

We note that by the spectral theorem, the support of $\rho'(x)$ contributing to the trace in (3.5) lies in the interval $[-\eta/2, \eta/2]$, and, by choice of ρ, the derivative is negative there. In order to evaluate the trace, we consider the operator H_Λ as a function of the random coupling constants and rewrite differentiation with respect to the energy t in terms of differentiation with respect to the coupling constants $\lambda_k, k \in \tilde{\Lambda} \equiv \Lambda \cap Z\!\!Z^d$. For this, we use the Gohberg-Krein formula (cf. Simon [27]) giving

$$\sum_{k\in\tilde{\Lambda}} Tr\left\{\frac{\partial}{\partial\lambda_k}\rho(H_\Lambda - E_0 - t)\right\} = \sum_{k\in\tilde{\Lambda}} Tr\left\{\rho'(H_\Lambda - E_0 - t)\left(\frac{\partial H_\Lambda}{\partial\lambda_k}\right)\right\}. \quad (3.6)$$

The proof of this result is similar to the proof in [27]. It uses the fact that for $g \in C_0^\infty(I\!R)$, we have that $(\partial H_\Lambda/\partial\lambda_k)g(H_\Lambda)$ is trace class, as follows from Proposition 5.1. We use the expansion in (3.3) to evaluate the trace on the right in (3.6) and write

$$Tr\left\{\rho'(H_\Lambda - E_0 - t)\left(\frac{\partial H_\Lambda}{\partial\lambda_k}\right)\right\} = \sum_j \rho'(E_j(\Lambda) - E_0 - t)\,\langle\phi_j, \left(\frac{\partial H_\Lambda}{\partial\lambda_k}\right)\phi_j\rangle, \qquad (3.7)$$

where, as above, the the compactness of the support of ρ' implies that the sum is finite and over the eigenvalues of H_Λ in the interval $[E_0 - 2\eta, E_0 + 2\eta]$, including multiplicity, with normalized eigenfunctions ϕ_j. For the case of additive perturbations, the expectation of the derivative of H_Λ on the right side of (3.7) is equal to

$$\langle\phi_j, \left(\frac{\partial V_\Lambda}{\partial\lambda_k}\right)\phi_j\rangle,$$

and for multiplicative perturbations, it is equal to

$$\langle \phi_j, \left(\frac{\partial V_\Lambda}{\partial \lambda_k} \right) (1 + V_\Lambda)^{-1} \phi_j \rangle.$$

We recall that it follows from the hypotheses in the multiplicative case that $(1 + V_\Lambda) > C_1 > 0$, for some constant C_1. Hence, in the next step, we need to estimate the sum

$$\sum_{k \in \tilde{\Lambda}} \langle \phi_j, \left(\frac{\partial V_\Lambda}{\partial \lambda_k} \right) \phi_j \rangle$$

from below so we can solve (3.7) for an upper bound on $-Tr\{\rho'(H_\Lambda - E_0 - t)\} \geq 0$. The Anderson-type potential depends linearly on the coupling constants so that

$$\frac{\partial V_\Lambda}{\partial \lambda_k} = u_k. \tag{3.8}$$

For the breather-type model, the derivative is

$$\frac{\partial V_\Lambda}{\partial \lambda_k} = (x - k) \cdot \nabla u_k(\lambda_k(\omega)(x - k)). \tag{3.9}$$

It follows from this that for the Anderson-type potential, we need to estimate from below the sum

$$\sum_{k \in \tilde{\Lambda}} \langle \phi_j, u_k \phi_j \rangle, \tag{3.10}$$

and, for the breather-type potential, the sum

$$-\sum_{k \in \tilde{\Lambda}} \langle \phi_j, (x - k) \cdot \nabla u_k(\lambda_k(\omega)(x - k)) \phi_j \rangle. \tag{3.11}$$

In light of (3.10)–(3.11), we define an auxiliary potential $\tilde{V}_\Lambda$ for the Anderson-type model by

$$\tilde{V}_\Lambda(x) \equiv \sum_{k \in \tilde{\Lambda}} \frac{\partial V_\Lambda}{\partial \lambda_k}(x) = \sum_{k \in \tilde{\Lambda}} u_k(x - k), \tag{3.12}$$

and, for the breather-type model,

$$\tilde{V}_\Lambda(x) \equiv \sum_{k \in \tilde{\Lambda}} (x - k) \cdot \nabla u_k(\lambda_k(\omega)(x - k)). \tag{3.13}$$

We note that by (H3) and (H4) there exists a finite constant $C_1 > 0$ such that

$$|V_\Lambda(x)| \leq C_1 |\tilde{V}_\Lambda(x)|, \tag{3.14}$$

for any $x \in I\!R^d$, and for any subset $\Lambda \subset I\!R^d$.

4. We estimate from below the quantity $\langle \phi_j, \tilde{V}_\Lambda \phi_j \rangle$ using the Comparison Theorem of Kirsch, Stollmann, and Stolz (KSS) [19] proved in section 4. Referring to Theorem 4.3, we must construct a comparison potential V_0. We do this as follows. We choose some $0 < \epsilon_2$ so small that

$$[E_0 - 2\eta, E_0 + 2\eta] \subset (B_- + \epsilon_2, B_+ - \epsilon_2), \tag{3.15}$$

and so that, for the breather-type model, $\epsilon_2 / C_1 \leq \delta$, with δ as in (H3). We set

$$\mathcal{D} = \left\{ x \in I\!R^d \mid \tilde{V}_\Lambda(x) \geq \epsilon_2 / C_1 \right\}, \tag{3.16}$$

and we define the comparison potential V_0 by

$$\begin{aligned} V_0(x) &= V_\Lambda(x) \text{ for } x \in \mathcal{D}^c \\ &= 0 \text{ for } x \in \mathcal{D}. \end{aligned} \tag{3.17}$$

We then have that $|V_\Lambda(x)| \leq \epsilon_2$, and hence $[E_0 - 2\eta, E_0 + 2\eta] \subset \rho(H_0 + V_0)$. Let $0 < \epsilon_1 < \epsilon_2 / C_1$, and we set

$$F = \left\{ x \in I\!R^d \mid \tilde{V}_\Lambda(x) \leq \epsilon_1 \right\}. \tag{3.18}$$

By the equicontinuity assumption (H3), there exists a finite constant $\theta > 0$, independent of Λ, so that

$$\operatorname{dist}(\mathcal{D}, F) \geq \theta > 0. \tag{3.19}$$

Theorem 4.3 then implies that

$$\begin{aligned} \langle \phi_j, \tilde{V}_\Lambda \phi_j \rangle &\geq \epsilon_1 \langle \phi_j, (1 - \chi_F) \phi_j \rangle \\ &\geq (\epsilon_1 / C_2) \| \phi_j \|^2, \end{aligned} \tag{3.20}$$

for a finite constant $C_2 > 0$ that is independent of Λ and j, and depends only on η, θ, and H_0.

5. Returning to (3.6)–(3.7), we have the lower bound,

$$\sum_{k \in \tilde{\Lambda}} \langle \phi_j, \left(\frac{\partial V_\Lambda}{\partial \lambda_k} \right) \phi_j \rangle \geq C_0, \tag{3.21}$$

uniformly in j, so we obtain from (3.6)–(3.7),

$$\begin{aligned}
&-Tr\{\rho'(H_\Lambda - E_0 - t)\} \\
&\leq \frac{-1}{C_0}\sum_{k\in\tilde{\Lambda}} Tr\left\{\frac{\partial\rho}{\partial\lambda_k}(H_\Lambda - E_0 - t)\right\} \\
&\leq \frac{-1}{C_0}\sum_{k\in\tilde{\Lambda}} Tr\left\{\frac{\partial}{\partial\lambda_k}\{\rho(H_\Lambda - E_0 - t) - \rho(H_0 - E_0 - t)\}\right\}. \quad (3.22)
\end{aligned}$$

We remark that the operator $\{\rho(H_\Lambda - E_0 - t) - \rho(H_0 - E_0 - t)\}$ is trace class since the difference $H_\Lambda - H_0 = V_\Lambda$ has compact support. With this estimate, the right side of (3.5) can be bounded above by

$$\frac{-1}{C_0}\sum_{k\in\tilde{\Lambda}}\int_{-3\eta/2}^{3\eta/2} dt\ I\!\!E\left\{\frac{\partial}{\partial\lambda_k} Tr\{\rho(H_\Lambda - E_0 - t) - \rho(H_0 - E_0 - t)\}\right\}. \quad (3.23)$$

In order to evaluate the expectation, we select one random variable λ_k, with $k \in \tilde{\Lambda}$, and first integrate with respect to this variable using hypothesis (H4). Because of the positivity of $-\rho'$, and of the density h_0, we obtain

$$\begin{aligned}
&-\int d\lambda_k\ h_0(\lambda_k)\frac{\partial}{\partial\lambda_k} Tr\{\rho(H_\Lambda - E_0 - t) - \rho(H_0 - E_0 - t)\} \\
&\leq \|h_0\|_\infty\ \{Tr\{\rho(H_\Lambda^{m,k} - E_0 - t) - \rho(H_\Lambda^{M,k} - E_0 - t)\}, \quad (3.24)
\end{aligned}$$

where $H_\Lambda^{M,k}$ is the local Hamiltonian H_Λ with the coupling constant λ_k at the k^{th}-site fixed at its maximum value. Similarly, the small m denotes the minimum value of the coupling constant. We note once again that the difference $\{\rho(H_\Lambda^{m,k} - E_0 - t) - \rho(H_\Lambda^{M,k} - E_0 - t)\}$ is trace class. Consequently, we are left with the task of estimating

$$\frac{\|h_0\|_\infty}{C_0}\sum_{k\in\tilde{\Lambda}}\int_{-3\eta/2}^{3\eta/2} dt\ \int\prod_{l\neq k} h_0(\lambda_l)\ d\lambda_l\ Tr\{\rho(H_\Lambda^{m,k} - E_0 - t) - \rho(H_\Lambda^{M,k} - E_0 - t)\}. \quad (3.25)$$

6. The expression involving the trace in (3.25) is basically the number of eigenvalues created in the interval $[E_0 - 2\eta, E_0 + 2\eta]$ by decreasing the k^{th}-coupling constant from the maximum value λ_k^M to the minimum value λ_k^m. This trace can be rewritten in terms of a spectral shift function as follows. We consider explicitly the Anderson-type additive case. The other cases

are treated similarly. We let $H_1 \equiv H_\Lambda^{M,k}$ be the unperturbed operator, and write,

$$H_\Lambda^{m,k} = H_1 + (\lambda_k^m - \lambda_k^M) u_k = H_1 + V. \tag{3.26}$$

Although the difference V is not trace class, it does have compact support. We show in section 5 that the difference of sufficiently large powers of the resolvents of H_1 and $H_1 + V$ is not only in the trace class, but is in the super-trace class $\mathcal{I}_{1/p}$, for all $p \geq 1$. Specifically, our assumptions imply that there exists a finite constant M_0 such that $(H_1 + M_0)^{-1}$ and $(H_1 + V + M_0)^{-1}$ are bounded. Let us define the function $g(\lambda) = (\lambda + M_0)^{-k}$, for $\lambda >> M_0$. We prove that for $k > pd/2 + 2$, and $p > 1$,

$$g(H_1 + V) - g(H_1) \in \mathcal{I}_{1/p}. \tag{3.27}$$

Spectral shift function $\xi(\lambda\,; H_1 + V, H_1)$ is defined for the pair $(H_1, H_1 + V)$ by

$$\xi(\lambda\,; H_1 + V, H_1) = sgn(g'(\lambda))\; \xi(g(\lambda)\,; g(H_1 + V), g(H_1)). \tag{3.28}$$

Because the function ρ' has compact support, and the fact that the difference $\{g(H_1 + V) - g(H_1)\}$ is super-trace class, we can apply the Birman-Krein identity [3] to the trace in (3.25). This gives

$$\begin{aligned} & Tr\{\rho(H_\Lambda^{m,k} - E_0 - t) - \rho(H_\Lambda^{M,k} - E_0 - t)\} \\ = & \;\; - \int_{\mathbb{R}} \frac{d}{d\lambda} \rho(\lambda - E_0 - t)\; \xi(\lambda; H_1 + V, H_1)\; d\lambda \\ = & \;\; - \int_{\mathbb{R}} \frac{d}{d\lambda} \rho(\lambda - E_0 - t)\; \xi(g(\lambda);\; g(H_1 + V), g(H_1)). \end{aligned} \tag{3.29}$$

We estimate the integral using the Hölder inequality and the L^p-theory of section 2. Let $\tilde{\xi}(\lambda) = \xi(g(\lambda);\; g(H_1 + V), g(H_1))$, for notational convenience. Let χ be the characteristic function for the interval $[-\eta, \eta]$ so that we can replace ρ' in (3.29) by $\chi\rho'$. We write $\tilde{\chi}(\lambda) \equiv \chi(\lambda - E_0 - t)$. For any $p > 1$, and q such that $\frac{1}{p} + \frac{1}{q} = 1$, the right side of (3.29) can be bounded above by

$$\left\{ \int_{\mathrm{supp}\,\chi} |\rho'|^q \right\}^{1/q} \left\{ \int |\tilde{\xi}(\lambda)\; \tilde{\chi}(\lambda)|^p \right\}^{1/p} \leq C_0 \eta^{(1-q)/q}\; \|\tilde{\xi}\tilde{\chi}\|_{L^p}, \tag{3.30}$$

Here, we integrated one power of ρ' using the fact that $-\rho' > 0$ on the support of χ, and we used the fact that $|\rho'| = \mathcal{O}(\eta^{-1})$, to obtain

$$\begin{aligned} \left\{ \int_{\mathrm{supp}\,\chi} |\rho'|^{q-1}\; |\rho'| \right\}^{1/q} & \leq \;\; \eta^{(1-q)/q} \left(- \int_{\mathrm{supp}\,\chi} \rho' \right)^{1/q} \\ & \leq \;\; C_0 \eta^{(1-q)/q}. \end{aligned} \tag{3.31}$$

By a simple change of variables, we find

$$\begin{aligned}
\|\tilde{\xi}\tilde{\chi}\|_p &= \left\{ \int |\xi(g(\lambda);\; g(H_1+V), g(H_1))|^p \;\chi(\lambda - E_0 - t)\; d\lambda \right\}^{1/p} \\
&\leq C_1 \left\{ \int_{I\!R} |\xi(\lambda; g(H_1+V), g(H_1))|^p \; d\lambda \right\}^{1/p} \\
&\leq C_1 \; \|g(H_1+V) - g(H_1)\|_{1/p}^{1/p}. \qquad (3.32)
\end{aligned}$$

We recall that $V = (\lambda_k^m - \lambda_k^M) u_k$. In particular, the volume of the support of V has order one, and is independent of $|\Lambda|$. We prove in section 5 that the constant $\|g(H_1+V) - g(H_1)\|_{1/p}^{1/p}$ depends only on the single-site potential u_k, and is independent of $|\Lambda|$. Consequently, the right side of (3.32) is bounded above by $C_0 \eta^{(1-q)/q}$, independent of $|\Lambda|$. This estimate, equations (3.25) and (3.5), lead us to the result

$$I\!P\{\text{dist }(E_0, \sigma(H_\Lambda)) < \eta\} \leq C_W \eta^{1/q} \|g\|_\infty |\Lambda|, \qquad (3.33)$$

for any $q > 1$. $\Box$

4 Localization Estimates on Eigenfunctions

The localization properties of the eigenfunctions of the locally-perturbed Hamiltonians H_Λ are essential for the proof of the Wegner estimate given in section 3. We present two results concerning these eigenfunctions in this section. The results here apply to any local perturbation of a background operator H_0 satisfying (H1), and do not use any randomness.

Let us recall a result on the decay of the resolvent $R_0(E) = (H_0 - E)^{-1}$ of the unperturbed operator at an energy $E \in G$. This is a version of the Combes-Thomas result [10] proved in [1].

Proposition 4.1. *Let H_0 satisfy hypothesis (H1a), and let $E \in G$. Let $d_\pm = dist(B_\pm, E)$. There exists a constant, depending on $d_\pm$, so that*

$$|G_0(x,y;E)| \;\leq\; Ce^{-\sqrt{d_- d_+}|x-y|}, \qquad (4.1)$$

for all $x, y \in I\!R^d$ sufficiently separated.

We first present a simple result indicating the exponential decay of the eigenfunctions of H_Λ away from the support of the local perturbation V_Λ. Let H_0 be a background Hamiltonian which is perturbed by a potential V_Λ

supported in a bounded region Λ so that $H_\Lambda = H_0 + V_\Lambda$, acting on $L^2(I\!R^d)$. For any region $\mathcal{O} \subset I\!R^d$, let $\chi_{\mathcal{O}}$ be the characteristic function of the region. Similar results hold for multiplicative perturbations.

Proposition 4.2. *Let ϕ be an L^2-eigenfunction of H_Λ satisfying $H_\Lambda \phi = E\phi$, with $E \in G$, the spectral gap of H_0 as in (H1b). Let $\mathcal{O}$ be an open subset of $\{supp\, V_\Lambda\}^c \subset I\!R^d$ such that dist $(\mathcal{O}, supp\, V_\Lambda) > \delta > 0$. Then, there exist constants $A_E > 0$ and $C_E > 0$ such that*

$$\|\chi_{\mathcal{O}}\phi\| \leq A_E e^{-C_E \delta}. \tag{4.2}$$

Proof. We begin by rewriting the eigenvalue equation as

$$(H_0 - E)\phi = -V_\Lambda \phi. \tag{4.3}$$

Since the energy $E \in \rho(H_0)$, we have

$$\phi = -R_0(E) V_\Lambda \phi. \tag{4.4}$$

We now multiply on the left by the characteristic function of $\mathcal{O}$ to obtain,

$$\|\chi_{\mathcal{O}}\phi\| = \|\chi_{\mathcal{O}} R_0(E) V_\Lambda \phi\|. \tag{4.5}$$

We now use the estimate on the decay of the resolvent of $R_0(E)$ given in Proposition 4.1 to obtain the estimate. $\square$

Kirsch, Stollmann, and Stolz [19] present a nice result, called here the KSS Comparison Theorem, on the localization of the eigenfunctions of the local Hamiltonian H_Λ that provides more precise information about the eigenfunctions in the region Λ. The proof of this theorem is simple and follows the same ideas as in the proof of Proposition 4.2.

Theorem 4.3. *Let H_0 and V_Λ be as above and $H_\Lambda \phi = E\phi$ with $E \in G$ and $\phi \in L^2(I\!R^d)$. Suppose that the following two conditions are satisfied:*

1. *There exists a potential V_0 such that, with $H_\Lambda^0 \equiv H_0 + V_0$, we have $E \in \rho(H_\Lambda^0)$;*

2. *There exists a subset $F \subset \Lambda$ and a constant $\theta > 0$ so that dist $(F \cup \Lambda^c, \{x \mid V_\Lambda(x) \neq V_0(x)\}) > \theta > 0$.*

We then have,

$$\|\phi\| \leq (1 + \|(H_\Lambda^0 - E)^{-1} W_1\|)\|(1 - \chi_F)\phi\|, \tag{4.6}$$

where $W_1 \equiv [H_0, \chi_1]$, with χ_1 is defined in the proof, and χ_F is the characteristic function of F.

Proof. Let $\mathcal{D} \equiv \{x \mid V_\Lambda(x) \neq V_0(x)\}$. Since dist $(\mathcal{D}, F \cup \Lambda^c) > \theta$, we can find a smoothed characteristic function χ_1 such that $\chi_1 \mid \mathcal{D} = 1$, and $\chi_1 = 0$ near ∂F and $\partial\Lambda$, with supp $|\nabla\chi_1| \subset F^c$. Let us define $W_1 \equiv [H_0, \chi_1]$. Note that $(1-\chi_F)W_1 = W_1$. We then have

$$\begin{aligned}(1-\chi_1)(H_0+V_\Lambda)\phi &= (1-\chi_1)(H_0+V_0)\phi \\ &= (H_0+V_0)(1-\chi_1)\phi + W_1\phi \\ &= E(1-\chi_1)\phi. \end{aligned} \tag{4.7}$$

Solving this equation for $(1-\chi_1)\phi$, we find

$$(1-\chi_1)\phi = -(H_0+V_0-E)^{-1}W_1\phi. \tag{4.8}$$

It follows that

$$\|(1-\chi_1)\phi\| \leq \|(H_\Lambda^0 - E)^{-1}W_1\| \, \|(1-\chi_F)\phi\|. \tag{4.9}$$

The result follows directly from this inequality since

$$\begin{aligned}\|\phi\| &\leq \|(1-\chi_1)\phi\| + \|\chi_1\phi\| \\ &\leq (\|(H_\Lambda^0 - E)^{-1}W_1\| + 1) \, \|(1-\chi_F)\phi\|, \end{aligned} \tag{4.10}$$

since $\chi_1 \leq (1-\chi_F)$. $\square$

Of course, the art in applying this theorem lies in the construction of the comparison potential V_0 so that the support and spectral hypotheses are satisfied.

5 Trace Estimates

We present the estimates needed in section 3. We let $K_d(I\!\!R^d)$ denote the Kato class of potentials, and we refer the reader to Simon's article (cf. [29]) for a complete description.

We let H_0 be the Schrödinger operator

$$H_0 = (-i\nabla - A)^2 + W, \tag{5.1}$$

where A is a vector potential with $A \in L^2_{loc}(I\!\!R^d)$, and $W = W_+ - W_-$ is a background potential with $W_- \in K_d(I\!\!R^d)$ and $W_+ \in K_d^{loc}(I\!\!R^d)$. We denote by $H = H_0 + V$, for suitable real-valued functions V. We are interested in a bounded potential V with compact support.

Proposition 5.1. *Let H_0 be as above, and let V_1 be a Kato-class potential such that $\|V_1\|_{K_d} \leq M_1$. Let $H_1 \equiv H_0 + V_1$, and let $M > 0$ be a sufficiently large constant given in the proof. Let V be a Kato-class function supported in $B(R)$, the ball of radius $R > 0$ with center at the origin. Then, for any $p > 0$, we have*

$$V_{eff} \equiv (H_1 + V + M)^{-k} - (H_1 + M)^{-k} \in \mathcal{I}_{1/p}, \tag{5.2}$$

provided $k > dp/2 + 2$. Under these conditions, there exists a constant C_0, depending on $p, k, H_0, M_1, \|V\|_{K_d}$, and R, so that

$$\|V_{eff}\|_{1/p} \ \leq \ C_0. \tag{5.3}$$

Proof. Let $H_2 = H_1 + V$. By the diamagnetic inequality, we have

$$\begin{aligned} |(H_1 + M)^{-1}\phi(x)| \ &\leq \ (-\Delta + W + V_1 + M)^{-1}|\phi|(x) \\ &\leq \ 2(-\Delta + 1)^{-1}|\phi|(x), \end{aligned} \tag{5.4}$$

if M is sufficiently large. We also have

$$|(H_2 + M)^{-1}\phi(x)| \leq 2(-\Delta + 1)^{-1}|\phi|(x), \tag{5.5}$$

if M is sufficiently large. We fix M so large so that these estimates hold for any V_1 satisfying the conditions stated above. We note that

$$\begin{aligned} V_{eff} \ &= \ -\sum_{j=0}^{k-1} (H_2 + M)^{-(k-j)} V (H_1 + M)^{-j-1} \\ &= \ -\sum_{j=0}^{k-1} (J^{k-j}(H_2 + M)^{-(k-j)})^* V (J^{j+1}(H_1 + M)^{-j-1}), \end{aligned} \tag{5.6}$$

where $J \in C_0^\infty(B(R))$ such that $JV = V$. By Proposition 12 of [26], we have

$$J^j (H_a + M)^{-j} = \sum_{\beta=1}^{N} \prod_{\alpha=1}^{j} J_{\alpha\beta}(H_a + M)^{-1} B_{\alpha\beta}, \tag{5.7}$$

for an integer N depending on j, and where $a = 1, 2$, $J_{\alpha\beta} \in C_0^\infty(B(R))$, and $B_{\alpha\beta}$ are uniformly bounded operators. Note that $J_{\alpha\beta}$ are given by linear combinations of the derivatives of J.

Let $q > d/4$. It is then well-known that $J_{\alpha,\beta}(-\Delta + 1)^{-1} \in \mathcal{I}_{2q}$. We now apply Theorem 2.13 of [28], using (5.4) and (5.5), to show that $J_{\alpha,\beta}(H_a + M)^{-1} \in \mathcal{I}_{2q}$, and

$$\|J_{\alpha,\beta}(H_a + M)^{-1}\|_{\mathcal{I}_{2q}} \;\leq\; 2\|J_{\alpha,\beta}(-\Delta + 1)^{-1}\|_{\mathcal{I}_{2q}}. \tag{5.8}$$

Combining this observation with (5.7) and (5.8), we find that $V_{eff} \in \mathcal{I}_{2q/k}$. We choose k so large that $2q/(k-2) \leq 1/p$, i. e. we can take $k > 2pq + 2 > dp/2 + 2$. This proves the theorem. □

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